α-STABLE DENSITIES ARE HYPERBOLICALLY COMPLETELY MONOTONE FOR $\alpha \in]0, 1/4] \cup [1/3, 1/2]$

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Abstract. We investigate the problem raised by L. Bondesson in [1], about the hyperbolic complete monotonicity of α-stable densities. We prove that densitites of subordinators of order α are HCM for $\alpha \in]0, 1/4] \cup [1/3, 1/2]$.

1. Introduction

Hyperbolically completely monotone functions (HCM in short) were introduced by L. Bondesson [1] in order to analyze infinitely divisible distributions, and particularly the so-called generalized gamma convolutions introduced by O. Thorin [4]. We recall their definition in section 1 below.

Bondesson showed that the densities of α-stable positive random variables are HCM for $\alpha = n^{-1}$, for any integer $n \geq 2$. Furthermore, he conjectured that the HCM property actually holds for all $\alpha \in]0, 1/2]$. Recently, Wissem Jedidi and Thomas Simon [2] investigated some aspects of the problem. I thank them for pointing out this question to me.

In this paper we prove this conjecture for values of α in $]0, 1/4] \cup [1/3, 1/2]$.

For this we introduce the functions

$$G_\alpha(x) = x^{-\frac{1}{\alpha}} g_\alpha(x^{-\frac{1-\alpha}{\alpha}})$$

where g_α is the density of the positive α-stable distribution. We show that G_α extends to an analytic function on the slit plane $\mathbf{C}\setminus]-\infty, 0]$. By analyzing its behaviour at infinity and near the cut, we are able to prove that it has the following form

$$G_\alpha(z) = ce^{-\delta z} \exp\left(\int_0^{+\infty} \left[\frac{1}{z+t} - \frac{1}{1+t}\right] \theta(t) dt\right) \tag{1.1}$$

where c, δ are positive constants and θ takes values in $]0, 1[$.

In order that G_α be HCM it is then enough that the function θ be increasing, which we prove for $\alpha \in [1/3, 1/2]$. The HCM property for the remaining values of α is obtained by a multiplicative convolution argument.

This paper is organized as follows. In the section 2 we recall some results of Zolotarev on densities of stable distributions. These are used in the next section to obtain the asymptotic behaviour of the function G_α in the complex plane. In section four we establish the integral representation (1.1). Finally, in section five, we prove that θ is increasing for $1/3 < \alpha < 1/2$, and we finish the proof of this part of the conjecture.

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2. Hyperbolically completely monotone functions

We recall here the basic definition and properties of the class of hyperbolically completely monotone functions, and refer to [1] for more details.

A real valued function H defined on $]0,+\infty[$ is called hyperbolically completely monotone (HCM) if for every $u>0$ the function $H(uv)H(uv^{-1})$ is a completely monotone function of the variable $v+v^{-1}$. Bondesson [1] introduced this property in order to analyze infinitely divisible distributions, and particularly the so-called generalized gamma convolutions introduced by O. Thorin.

Proposition 2.1.
(i) H is HCM if and only if $H(x^{-1})$ is HCM.
(ii) H is HCM if and only if it admits the following representation

(2.1)
$$H(x)=cx^{\beta-1}\exp\left(-a_1x-\int_1^\infty\log\frac{x+t}{1+t}\mu_1(dt)-a_2x^{-1}-\int_1^\infty\log\frac{x^{-1}+t}{1+t}\mu_2(dt)\right)$$

where β is real, a_1,a_2 are positive constants and μ_1,μ_2 positive measures.
(iii) If H is HCM then $H(x^\beta)$ is HCM for all $\beta\le 1$
(iv) H is HCM if and only if the functions $x^\gamma H(x)$ are HCM for all values of $\gamma\in\mathbf{R}$.
(v) If X and Y are independent positive random variables both with an HCM density then the random variable XY also has an HCM density.

In particular, from (iii) and (iv) we deduce that if X is a positive random variable with HCM density, then X^γ has HCM density for all $\gamma\ge 1$.

3. Stable random variables

Let $\alpha\in]0,1[$ and $\rho\in]0,1[$, we denote $g_{\alpha,\rho}$ the density of the strictly α-stable distribution with asymmetry parameter ρ (cf [5]). For $\rho=1$ (and only for this value) this distribution is supported on the half axis $]0,+\infty[$, and we simply put $g_\alpha=g_{\alpha,1}$.

The following result is an integral representation for the functions $g_{\alpha,\rho}$ on the positive axis, due to Zolotarev.

Theorem 3.1. *(Zolotarev,* [5]*, Theorem 2.4.2)*
For all $x>0,\alpha,\rho\in]0,1[$

$$g_{\alpha,\rho}(x)=(2i\pi)^{-1}\int_0^\infty\frac{e^{-e^{-i\pi\rho\alpha}y^\alpha x^{-\alpha}}-e^{-e^{i\pi\rho\alpha}y^\alpha x^{-\alpha}}}{x}e^{-y}dy \tag{3.1}$$

The following result which is easily obtained by a subordination argument, plays an important role in the following.

Lemma 3.2. *Let X and Y be independent positive stable random variables, with respective parameters (α,ρ) and $(\beta,1)$, then $XY^{1/\alpha}$ is a stable random variable with parameter $(\alpha\beta,\rho)$.*

We deduce from the preceding lemma and Proposition 2.1 that

Proposition 3.3. *The set of $\alpha\in]0,1[$ such that g_α is HCM is a semigroup under multiplication.*

4. The function G_α

Denote G_α the function

$$G_\alpha(z) = (2i\pi)^{-1}\int_0^\infty \frac{e^{-e^{-i\pi\alpha}y^\alpha z^{1-\alpha}} - e^{-e^{i\pi\alpha}y^\alpha z^{1-\alpha}}}{z} e^{-y}dy \tag{4.1}$$

where we take (as in the rest of the paper) for z^h, the determination of the power function which is positive on $]0,+\infty[$ and analytic on $\mathbf{C}\backslash]-\infty,0]$).

This function G_α is analytic in $\mathbf{C}\backslash]-\infty,0]$. In fact $z^\alpha G_\alpha(z) = F_\alpha(z^{1-\alpha})$ where F_α is an entire function. One has, for all $x>0$,

$$g_\alpha(x) = x^{-\frac{1}{1-\alpha}} G_\alpha(x^{-\frac{\alpha}{1-\alpha}})$$

and for all $z \in \mathbf{C}\backslash]-\infty,0]$

$$G_\alpha(\bar z) = \overline{G_\alpha(z)}$$

For $r>0$ we denote

$$G_\alpha(-r^+) = \lim_{z\to -r, \Im(z)>0} G_\alpha(z) \qquad G_\alpha(-r^-) = \lim_{z\to -r,\Im(z)<0} G_\alpha(z) = \overline{G_\alpha(-r^+)}$$

the boundary values of G_α.

5. Behaviour near 0.

It follows from (4.1) that, as $z\to 0$,

$$G_\alpha(z) = \Gamma(\alpha+1)\frac{\sin(2\pi\alpha)}{\pi} z^{-\alpha}(1+O(|z|^{1-\alpha})) \tag{5.1}$$

6. Bounds at infinity

Theorem 6.1. *Let $\theta \in]-1,1[$ be fixed, and*

$$\delta = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} \qquad c = (1-\alpha)^{-\frac{1}{2}}\alpha^{\frac{1}{2(1-\alpha)}}$$

then, as $r\to+\infty$, for $z = re^{i\pi\theta}$, one has

$$G_\alpha(z) \sim cz^{-\frac{1}{2}}e^{-\delta z} \tag{6.1}$$

As $r\to+\infty$

$$G_\alpha(-r^+) \sim -icr^{-\frac{1}{2}}e^{\delta r} \qquad G_\alpha(-r^-) \sim icr^{-\frac{1}{2}}e^{\delta r} \tag{6.2}$$

Furthermore, for some $R>0$, the function $G_\alpha(z)z^{1/2}e^{\delta z}$ is uniformly bounded on $\mathbf{C}\backslash]-\infty,0]\cap\{|z|>R\}$.

In order to obtain this asymptotic result, observe that one can rewrite the integral defining G_α as a contour integral:

$$G_\alpha(z) = (2i\pi)^{-1}\int_\Gamma \frac{e^{y-y^\alpha z^{1-\alpha}}}{z}dy$$

where Γ is a contour which starts from $-\infty$, following the negative axis, taking the lower branch of y^α, encircles 0 then goes back to $-\infty$ along the negative axis, this time picking up the upper branch of y^α.

In order to obtain the asymptotics we take $z = re^{i\theta}$ and rewrite the integral as

$$G_\alpha(z) = (2i\pi)^{-1}\int_\Gamma e^{r(y-y^\alpha e^{i\pi(1-\alpha)\theta})}e^{-i\pi\theta}dy$$

This integral is subject to the steepest descent method (see [3] for example) using the unique saddle point at $y = \alpha^{\frac{1}{1-\alpha}} e^{i\pi\theta}$ of the function $y - y^\alpha e^{i\pi(1-\alpha)\theta}$. This gives the point wise convergence for a fixed θ. In order to obtain the uniform convergence, first notice that uniform property is clear for θ in any compact subset of $]-1,+1[$, say for $\theta \in [-7/8, 7/8]$. then, for $\theta \in]7/8, 1[$, the saddle point is over the half line $]-\infty, 0[$ and close to it, then one can use another determination of y^α with a cut say on the half-line $\arg(y) = -3\pi/4$, and a contour encircling the cut and going back to a neighborhood of $-\infty$ by an arc with a ray going to infinity. Then again one can deform this contour to go through the saddle point and then conclude of the uniform convergence for $\theta \in]7/8, 1[$. A symmetrical argument gives the uniformity for $\theta \in]-1, -\frac{7}{8}[$.

7. Behaviour of G_α on the cut

Lemma 7.1. *For any $r > 0$*

$$G_\alpha(-r^+) = (2i\pi)^{-1} \int_0^\infty \frac{e^{r^{1-\alpha}y^\alpha} - e^{e^{-2i\pi\alpha}r^{1-\alpha}y^\alpha}}{r} e^{-y} dy \tag{7.1}$$

$$G_\alpha(-r^+) = (2i\pi)^{-1} \sum_1^\infty \frac{\Gamma(n\alpha+1)}{\Gamma(n+1)} (1 - e^{-2i\pi n\alpha}) r^{n(1-\alpha)-1} \tag{7.2}$$

Proof. The first formula follows at once from (4.1) by letting $z \to -r$, the second one comes from expanding the exponentials in the numerator of (4.1) and integrating term by term. □

Lemma 7.2. *For any $r > 0$ one has $\Im(G_\alpha(-r^+)) < 0$. Furthermore, $-r^\alpha \Im(G_\alpha(-r^+))$ is an increasing function of r.*

Proof. By (7.2) we get

$$-\Im(G_\alpha(-r^+)) = (2\pi)^{-1} \sum_1^\infty \frac{\Gamma(n\alpha+1)}{\Gamma(n+1)} (1 - \cos(2\pi n\alpha)) r^{n(1-\alpha)-1}$$

in which all terms in the sum are positive; the two claims are clear. □

In the sequel, denote

$$G_\alpha(-r^+) =: R(r) e^{-i\pi\theta(r)}$$

the polar decomposition of $G_\alpha(-r^+)$. Since $\Im(G_\alpha(-r^+)) < 0$, one can chose $\theta(r)$ in $]0, 1[$.

Observe also that $\theta(r) \to 1/2$ as $r \to +\infty$ (by (6.2)).

Remark 7.3. in fact one could also obtain from the integral representation that $\theta(t) - 1/2 = o(e^{-\epsilon r})$ as $r \to +\infty$ for some $\epsilon > 0$, but we will not use this).

8. Integral representation

Proposition 8.1. *For all $z \in \mathbf{C} \backslash]-\infty, 0]$*

$$G_\alpha(z) = a e^{-\delta z} \exp \int_0^\infty \left[\frac{1}{z+t} - \frac{1}{1+t} \right] \theta(t) dt \tag{8.1}$$

for some $a > 0$.

Proof. Let

$$L_\alpha(z) = \exp \int_0^\infty \left[\frac{1}{z+t} - \frac{1}{1+t}\right]\theta(t)dt$$

This is an analytic function on $\mathbf{C}\setminus]-\infty,0]$, and it satifies, by well known properties of Stieltjes transforms,

$$\frac{L_\alpha(-r^+)}{L_\alpha(-r^-)} = e^{-2i\pi\theta(r)}$$

Furthermore, as $z\to\infty$, since $\theta(t)\to_{t\to+\infty} 1/2$, one has

$$L_\alpha(z) = z^{-1/2}\exp(o(\log(|z|))$$

Near zero, one has $\theta(t) = \alpha + O(t^{1-\alpha})$ by (5.1), which implies

$$L_\alpha(z) \sim z^\alpha \qquad z\to 0$$

On the other hand, for $r>0$,

$$\frac{G_\alpha(-r^+)}{G_\alpha(-r^-)} = e^{-2i\pi\theta(r)}$$

therefore the function

$$E_\alpha(z) = e^{\delta z}G_\alpha(z)/L_\alpha(z)$$

is analytic on $\mathbf{C}\setminus]-\infty,0]$, and can be extended continuouly to $\mathbf{C}\setminus\{\mathbf{0}\}$. Since it is bounded near 0 it can be extended to an entire function and it satifies

$$E_\alpha(z) = \exp(o(\log(|z|))$$

at infinity thus it is constant. Since both functions G_α, L_α take positive values on $]0,+\infty[$, this constant is positive. □

9. The function θ is monotone for $\alpha\in[1/3,1/2]$

Lemma 9.1. *For $0\le\rho\le\inf(1,\frac{1}{2\alpha})$ the function $\tilde g_{\alpha,\rho}(x) = x^{-1-\alpha}g_{\alpha,\rho}(x^{-1})$ is decreasing on $]0,+\infty[$.*

Proof. Recall that if X is a stable variable with parameters $(2\alpha,\rho)$, and Y an independent stable variable with parameters $(1/2,1)$, then $Z = XY^{\frac{1}{2\alpha}}$ is a stable variable with parameters (α,ρ) . Since the density of Y is $\frac{e^{-\frac{1}{2t}}}{\sqrt{2\pi t^3}}$ one has

$$g_{\alpha,\rho}(x) = 2\alpha\int_0^\infty g_{2\alpha,\rho}(y)\frac{e^{-\frac12(y/x)^{2\alpha}}y^\alpha}{\sqrt{2\pi}x^{\alpha+1}}dy$$

Therefore

$$x^{-1-\alpha}g_{\alpha,\rho}(x^{-1}) = 2\alpha\int_0^\infty g_{2\alpha,\rho}(y)\frac{e^{-\frac12(yx)^{2\alpha}}y^\alpha}{\sqrt{2\pi}}dy$$

which is clearly decreasing in x.

□

Lemma 9.2. *For $\alpha\in[1/3,1/2]$ the function $r^\alpha\Re G_\alpha(-r^+)$ is decreasing.*

Proof. Note that, by formulas (3.1) and (7.1) one has

$$\Re G_\alpha(-r^+) = r^{-1/\alpha}g_{\alpha,\frac{1}{\alpha}-2}(r^{-\frac{1-\alpha}{\alpha}})$$

for $\alpha\in[1/3,1/2]$. it follows that

$$r^\alpha\Re G_\alpha(-r^+) = r^{\alpha-1/\alpha}g_{\alpha,\frac{1}{\alpha}-2}(r^{\frac{1-\alpha}{\alpha}}) = x^{-1-\alpha}g_{\alpha,\frac{1}{\alpha}-2}(x^{-1})$$

with $x = r^{1-\frac{1}{\alpha}}$. The result follows from the preceding lemma.

□

Theorem 9.3. *For $\alpha \in]1/3, 1/2|$, the function θ increases from the value α to the value $1/2$, and*

$$G_\alpha(z) = \Gamma(\alpha+1)e^{-\delta z}z^{-1/2}\exp - \int_0^\infty \log(1+t/z)\theta'(t)dt$$

Proof. One has

$$\tan(\pi\theta(r)) = \frac{-r^\alpha \Im(G_\alpha(-r^+))}{r^\alpha \Re(G_\alpha(-r^+))}$$

and the numerator and denominator of this formula are positive and respectively increasing and decreasing. This implies that θ is increasing. The other claim follows by integrating by parts. □

10. The HCM property of stable distribution

For $\alpha \in [\frac{1}{3}, \frac{1}{2}]$ one has $\frac{1-\alpha}{\alpha} \geq 1$ and G_α is HCM. This implies that g_α is HCM. By Proposition (3.3) the set of α such that g_α is HCM thus contains the multiplicative semigroup generated by $[1/3, 1/2]$, which is $]0, 1/4] \cup [1/3, 1/2]$. □

Remark 10.1. Following the same arguments than above, one can prove that for $\alpha \geq 1/2$, the function θ decreases from α to $1/2$ and consequently G_α enjoys the next decomposition :

$$G_\alpha(z) = \Gamma(\alpha+1)e^{-\delta z}z^{-1/2}\exp \int_0^\infty \log(1+t/z)|\theta'(t)|dt$$

In other words, $e^{-\delta z}\frac{1}{G_\alpha(z)}$ is an HCM function.

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