Some Complements about Reachability and Pole Placement for Implicit Systems

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Abstract: One of the most important features of the reachability of a state space representation is the complete assignability of the closed loop spectrum by means of a state feedback. This equivalence is no longer the case when dealing with implicit representations. For the reachable implicit description case, a geometric condition has to be added in order to guarantee such a *pole assignment ability.* In this paper, we give geometric conditions, which enable us to assign the closed loop spectrum of: *(i)* a reachable implicit description, and *(ii)* a reachable and observable implicit description with output equation.

Notation

Script capitals $\mathscr{V}$, $\mathscr{W}$, ..., denote linear spaces with elements v, w, ...; the dimension of a space $\mathscr{V}$ is denoted $\dim(\mathscr{V})$; $\mathscr{V} \approx \mathscr{W}$ stands for $\dim(\mathscr{V}) = \dim(\mathscr{W})$; when $\mathscr{V} \subset \mathscr{W}$, $\frac{\mathscr{W}}{\mathscr{V}}$ or $\mathscr{W}/\mathscr{V}$ stands for the quotient space $\mathscr{W}$ modulo $\mathscr{V}$; the direct sum of independent spaces is written as $\oplus$. $X^{-1}\mathscr{V}$, stands for the inverse image of the subspace $\mathscr{V}$ by the linear transformation X. Given a linear transformation $X : \mathscr{V} \to \mathscr{W}$, $\operatorname{Im} X = X\mathscr{V}$ denotes its image, and $\operatorname{Ker} X$ denotes its kernel. The special subspaces: $\operatorname{Im} B$, $\operatorname{Ker} E$ and $\operatorname{Ker} C$, are denoted by: $\mathscr{B}$, $\mathscr{K}_E$ and $\mathscr{K}_C$, respectively. The zero dimension subspace is denoted $\{0\}$, and the identity operator is denoted I, namely: $\mathrm{I}x = x$. Given the linear transformations, $X : \mathscr{V} \to \mathscr{V}$ and $Y : \mathscr{W} \to \mathscr{V}$, $\langle X \mid \operatorname{Im} Y \rangle$ stands for the subspace of $\mathscr{V}$: $\operatorname{Im} Y + X \operatorname{Im} Y + \cdots + X^{\dim(\mathscr{V})-1} \operatorname{Im} Y$. A_{F_p} and E_{F_d} stand for $(A + BF_p)$ and $(E - BF_d)$, respectively. $\mathbb{R}^+$, $\mathbb{R}^{+*}$ and $\mathbb{Z}^+$, stand for the sets of: non negative real numbers, positive real numbers and non negative integers, respectively. $\mathcal{C}^\infty(\mathbb{R}^+, \mathscr{V})$ is the space of infinitely differentiable functions.

Geometric Algorithms

Given the linear transformations, $X : \mathscr{V} \to \mathscr{W}$, $Y : \mathscr{T} \to \mathscr{W}$, and $Z : \mathscr{V} \to \mathscr{W}$, and the subspace, $\mathscr{K} \subset \mathscr{V}$, we have the two following popular geometric algorithms (see mainly Verghese [1981], Özçaldiran [1986], Malabre [1987, 1989], Lewis [1992]):

Algorithm for computing *the supremal* (X, Z, Y) *invariant subspace* contained in $\mathscr{K}$:

$$\begin{aligned} \mathscr{V}^0_{[\mathscr{K}:\,X,Z,Y]} &= \mathscr{V}, \\ \mathscr{V}^{\mu+1}_{[\mathscr{K}:\,X,Z,Y]} &= \mathscr{K} \cap X^{-1}\left(Z\mathscr{V}^{\mu}_{[X,Z,Y]} + \operatorname{Im} Y \right). \end{aligned} \qquad \text{(ALG–V)}$$

which limit is $\mathscr{V}^*_{[\mathscr{K}:\,X,Z,Y]} = \sup\{\mathscr{S} \subset \mathscr{K} \mid X\mathscr{S} \subset Z\mathscr{S} + \operatorname{Im} Y\}$.

Algorithm for computing *the infimal* (Z, X, Y) *invariant subspace related to* $\operatorname{Im} Y$:

$$\mathscr{S}^0_{[Z,X,Y]} = \{0\}, \quad \mathscr{S}^{\mu+1}_{[Z,X,Y]} = Z^{-1}\left(X\mathscr{S}^{\mu}_{[Z,X,Y]} + \operatorname{Im} Y \right). \qquad \text{(ALG–S)}$$

which limit is $\mathscr{S}^*_{[Z,X,Y]} = \inf\left\{ \mathscr{S} \subset \mathscr{V} \mid \mathscr{S} = Z^{-1}(X\mathscr{S} + \operatorname{Im} Y) \right\}$.

In the case where: *i)* $\mathscr{K} = \mathscr{X}$, $X = A : \mathscr{X} \to \mathscr{X}$, $Y = B : \mathscr{U} \to \mathscr{X}$, and $Z = \mathrm{I}$, we write $\mathscr{V}^*_{\mathscr{X}}$ and $\mathscr{S}^*_{\mathscr{X}}$; *ii)* $X = A : \mathscr{X}_d \to \underline{\mathscr{X}}_{eq}$, $Y = B : \mathscr{U} \to \underline{\mathscr{X}}_{eq}$, and $Z = E : \mathscr{X}_d \to \underline{\mathscr{X}}_{eq}$, we write $\mathscr{V}^*_{\mathscr{X}_d}$ and $\mathscr{S}^*_{\mathscr{X}_d}$, for $\mathscr{K} = \mathscr{X}_d$, and we write $\mathscr{V}^*$, for $\mathscr{K} = \mathscr{K}_C$; and *iii)* $\mathscr{K} = \mathscr{K}_C$, $X = A_{F_p}$, $Y = 0$ and $Z = E_{F_p}$, we write $\mathscr{N}_{(F_p,F_d)}$ (this is the closed loop unobservable space). Let us note that: *(i)* $\mathscr{V}^*_{[\mathscr{K}:\,A,E,B]} = \mathscr{V}^*_{[\mathscr{K}:\,A_{F_p},E_{F_d},B]}$, *(ii)* $\mathscr{S}^*_{[\mathscr{K}:\,E,A,B]} = \mathscr{S}^*_{[\mathscr{K}:\,E_{F_d},A_{F_p},B]}$, and *(iii)* for any F_d, there exists F_p such that: $A_{F_p}\mathscr{V}^*_{[\mathscr{K}:\,A_{F_p},E_{F_d},B]} \subset E_{F_d}\ \mathscr{V}^*_{[\mathscr{K}:\,A_{F_p},E_{F_d},B]}$, the set of such pairs (F_p, F_d) is identified by $\mathbf{F}(\mathscr{V}^*_{[\mathscr{K}:\,A,E,B]})$.

1. INTRODUCTION

As a generalization of proper linear systems, described by state space representations, $\mathfrak{R}^{ss}(A, B)$,

$$\mathrm{d}x/\mathrm{d}t = Ax + Bu, \qquad (1.1)$$

Rosenbrock [1970] introduced the *implicit representations*, $\mathfrak{R}^{imp}(E, A, B)$, which are a set of differential and algebraic equations [Brenan *et al*, 1996] of the following form (see also Lewis [1992]):

$$E\mathrm{d}x/\mathrm{d}t = Ax + Bu, \qquad (1.2)$$

where: $E : \mathscr{X}_d \to \underline{\mathscr{X}}_{eq}$, $A : \mathscr{X}_d \to \underline{\mathscr{X}}_{eq}$ and $B : \mathscr{U} \to \underline{\mathscr{X}}_{eq}$ are linear maps. The linear spaces $\mathscr{X}_d \approx \mathbb{R}^{n_d}$, $\underline{\mathscr{X}}_{eq} \approx \mathbb{R}^{\underline{n}_{eq}}$, and $\mathscr{U} \approx \mathbb{R}^m$ are called the descriptor, the equation, and the input spaces, respectively. In order to avoid redundant components in the input variable, u, and linear dependance on the descriptor equations, (1.2), it is usual to assume: **[H1]** $\operatorname{Ker} B = \{0\}$ and **[H2]** $\operatorname{Im} E + \operatorname{Im} A + \mathscr{B} = \underline{\mathscr{X}}_{eq}$.

One of the most studied concepts in System Theory is the one of the *reachability.* This concept is normally associated with "*the set of vectors which can be reached from the origin, in a finite time, following trajectories, solutions of the system, generated by an exogenous input*".

For the case of regular implicit representations [Gantmacher, 1977], *i.e.* representations where the linear transformations E and A are square and the pencil $[\lambda E - A]$ is invertible, the *reachability* was studied by Verghese, Lévy and Kailath [1981], from a transfer function point of view, Yip and Sincovec [1981], in the time domain, Cobb [1984], from a distributional point of view, and by Özçaldiran [1985], from a geometric point of view.

In the case of implicit representations, where the linear transformations E and A are square and the pencil $[\lambda E - A]$ is not necessarily invertible, Özçaldiran [1986] extended his *reachability* geometric characterization [Özçaldiran, 1985], for the case of regular implicit representations, by means of the supremal (A, E, B) reachability subspace contained in $\mathscr{X}_d$:

$$\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{S}^*_{\mathscr{X}_d}. \tag{1.3}$$

This is a nice generalization of the classical case, $\mathfrak{R}^{ss}(A,\, B) = \mathfrak{R}^{imp}(\mathrm{I},\, A,\, B)$, where the reachable space, $\mathscr{R}^*_{\mathscr{X}_d}$, is equal to $\langle A \mid \mathscr{B}\rangle)$. Thus, for representations $\mathfrak{R}^{imp}(E,\, A,\, B)$, with E and A not necessarily square, it was natural to associate its reachability with $\mathscr{R}^*_{\mathscr{X}_d}$.

Frankowska [1990] firmly established the pertinence of this reachability concept, using differential inclusions to relate it with behavioral properties.

In this paper, we study the reachability notion in the sense of Frankowska [1990], showing some connections with the work of Geerts [1993], and we consider the relationships, between the reachability property and the complete pole assignment ability.

2. BACKGROUND

2.1 Implicit Systems

In this Subsection, we formalize the notion of *implicit systems*, following the behavioral point of view. For this, let us first state the following definition:

Definition 1. An implicit representation, $\mathfrak{R}^{imp}(E,\, A,\, B)$, is called an *input/descriptor system*, when for all initial condition, $x_0 \in \mathscr{X}_d$, there exists at least one solution, $(u, x) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}_d)$, such that: $x(0) = x_0$. The *input/descriptor system* is defined by the triple: [1] $\Sigma_{i/d} = (\mathbb{R}^+, \mathscr{U} \times \mathscr{X}_d, \mathfrak{B}_{[E,A,B]})$, with behavior:

$$\mathfrak{B}_{[E,A,B]} = \left\{ (u,x) \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{U} \times \mathscr{X}) \,\middle|\, \begin{bmatrix} (E\frac{\mathrm{d}}{\mathrm{d}t} - A) & -B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \right\} \tag{2.4}$$

At this point, it is important to clarify what exactly means the sentence "*there exists at least one solution*". For this, we are going to recall hereafter the notions of *existence of solution* introduced by Geerts [1993] and Aubin & Frankowska [1991].

A. Existence of solution for every initial condition Following [Hautus, 1976] and [Hautus & Silverman, 1983], Geerts [1993] generalized the solvability results of [Geerts & Mehrmann, 1990]. One advantage of this generalization is that the solvability is introduced in a very natural way, passing from the distributional framework [Schwartz, 1978] to the usual time domain with ordinary differential equations.

Geerts [1993] considered the linear combinations of impulsive and smooth distributions, with μ coordinates, denoted by $\mathscr{C}^\mu_{\mathrm{imp}}$, as the signal sets. The set $\mathscr{C}^\mu_{\mathrm{imp}}$ is a subalgebra and is also decomposed as, $\mathscr{C}^\mu_{\mathrm{p-imp}} \oplus \mathscr{C}^\mu_{\mathrm{sm}}$, where $\mathscr{C}^\mu_{\mathrm{p-imp}}$ and $\mathscr{C}^\mu_{\mathrm{sm}}$ denote the subalgebras of pure impulses [2] and smooth distributions [3], respectively [Schwartz, 1978]. He introduced the following definitions for the distributional version of the *implicit representation* (1.2): [4] $\mathfrak{R}^{imp}_{dist}(E, A, B)$: $p\,Ex = Ax + Bu + Ex_0$ (*c.f.* [Definitions 3.1 and 4.1, Geerts, 1993])

Definition 2. [Geerts, 1993] Given the *solution set*, $S_C(x_0, u) := \left\{ x \in \mathscr{C}^{n_d}_{\mathrm{imp}} \,\middle|\, [p\,E - A]x = Bu + Ex_0 \right\}$, the *implicit representation* $\mathfrak{R}^{imp}_{dist}(E, A, B)$ is:

C-solvable if $\forall\, x_0 \in \mathscr{X}_d \;\exists\; u \in \mathscr{C}^m_{\mathrm{imp}} : S_C(x_0,\, u) \neq \emptyset$,

C-solvable in the function sense if $\forall\, x_0 \in \mathscr{X}_d \;\exists\; u \in \mathscr{C}^m_{\mathrm{sm}} : S_C(x_0,\, u) \cap \mathscr{C}^n_{\mathrm{sm}} \neq \emptyset$.

Given the *"consistent initial conditions set"*, $I_C := \left\{ z_0 \in \mathscr{X}_d \,\middle|\, \exists\, u \in \mathscr{C}^m_{\mathrm{sm}}\; \exists\, x \in\; S_C(z_0,\, u) \cap \mathscr{C}^{n_d}_{\mathrm{sm}} :\; x(0^+) = z_0 \right\}$, and the *"weakly consistent initial conditions set"*, $I^w_C := \left\{ z_0 \in \mathscr{X}_d \,\middle|\, \exists\, u \in \mathscr{C}^m_{\mathrm{sm}}\;\; \exists\, x \in S_C(z_0,\, u) \cap \mathscr{C}^{n_d}_{\mathrm{sm}} \neq \emptyset \right\}$, a point $x_0 \in \mathscr{X}_d$ is called *C-consistent* if $x_0 \in I_C$ and *weakly C-consistent* if $x_0 \in I^w_C$.

Geerts [1993] characterized the existence of solution for every initial condition in his Corollary 3.6, Proposition 4.2 and Theorem 4.5, hereafter we summarize these results with their geometric equivalences:

Theorem 3. [Geerts, 1993] If [**H2**] is fulfilled, then:

$\mathfrak{R}^{imp}_{dist}(E, A, B)$ is *C-solvable* if and only if:

$$\begin{array}{c} [(\lambda E - A) \;\; -B]] \text{ is right inv. as a rational matrix} \\ \left(E\mathscr{V}^*_{\mathscr{X}_d} + A\mathscr{S}^*_{\mathscr{X}_d} + \mathscr{B} = \underline{\mathscr{X}}_{eq} \right) \end{array} \tag{2.5}$$

$\mathfrak{R}^{imp}_{dist}(E, A, B)$ is *C-solvable in the function sense* if and only if $I^w_C = \mathscr{X}_d$, namely, if and only if:

$$\begin{array}{c} \mathrm{Im}\, E + A\mathscr{K}_E + \mathscr{B} = \underline{\mathscr{X}}_{eq} \\ \left(E\mathscr{V}_{\mathscr{X}^*_d} = \mathrm{Im}\, E \right) \end{array} \tag{2.6}$$

$I_C = \mathscr{X}_d$ if and only if:

$$\begin{array}{c} \mathrm{Im}\, E + \mathscr{B} = \underline{\mathscr{X}}_{eq} \\ \left(E\mathscr{V}_{\mathscr{X}^*_d} + \mathscr{B} = \underline{\mathscr{X}}_{eq} \right) \end{array} \tag{2.7}$$

B. Existence of a viable solution In order to study the reachability for implicit systems, Frankowska [1990]

[1] See also Polderman & Willems [1998] and Kuijper [1992].

[2] The unit element is the Dirac delta distribution, δ.

[3] The set of regular distributions are distributions that are functions; namely, piecewise continuous, integrable, or measurable functions. In those papers, they assume that the regular distributions $u(t)$ are smooth on $[0, \infty)$, *i.e.*, that a function $v : [0, \infty) \to \mathbb{R}$ exists, arbitrarily often differentiable including at $t = 0$, such that: $u(t) = 0$ for $t < 0$ and $u(t) = v(t)$ for $t \geq 0$ [Hautus & Silverman, 1983]. These distributions are identified as ordinary functions with support on $\mathbb{R}^+$.

[4] Ex_0 stands for $Ex_0\,\delta$, being $x_0 \in \mathscr{X}_d$ the initial condition, and $p\,Ex$ stands for $\delta^{(1)} * Ex$ ($*$ denotes convolution); if $p\,Ex$ is smooth and $E\dot{x}$ stands for the distribution that can be identified with the ordinary derivative, $E\mathrm{d}x/\mathrm{d}t$, then $p\,Ex = E\dot{x} + Ex_{0+}$.

introduced the set–valued map (the set of all admissible velocities), $\mathbf{F}: \mathscr{X}_d \rightsquigarrow \mathscr{X}_d$, $\mathbf{F}(x) = E^{-1}\big(Ax + \mathscr{B}\big) = \big\{v \in \mathscr{X} \,|\, Ev \in Ax + \mathscr{B}\big\}$, and the differential inclusion:

$$\mathrm{d}x/\mathrm{d}t \in \mathbf{F}(x), \quad \text{where: } x(0) = x_0, \tag{2.8}$$

Frankowska [1990] showed that the solutions of (1.2) and the ones of (2.8) are the same. She also clarified the meaning of a viable solution and she characterized the largest subspace of such viable solutions:

Definition 4. [Frankowska, 1990, Aubin & Frankowska, 1991] An absolutely continuous function, $x: \mathbb{R}^+ \to \mathscr{X}_d$, is called a *trajectory* of (2.8), if: $x(0) = x_0$ and $\mathrm{d}x/\mathrm{d}t \in \mathbf{F}(x)$ for almost every $t \in \mathbb{R}^+$, that is to say, if there exists a measurable function, $u: \mathbb{R}^+ \to \mathscr{U}$, such that: $x(0) = x_0$ and $E\mathrm{d}x/\mathrm{d}t = Ax + Bu$ for almost every $t \in \mathbb{R}^+$.

Let $\mathscr{K}$ be a subspace [5] of $\mathscr{X}_d$. A trajectory x of (2.8) is called *viable in* $\mathscr{K}$, if $x(t) \in \mathscr{K}$ for all $t \geq 0$. The set of such trajectories is called *the set of solutions viable in* $\mathscr{K}$. The subspace $\mathscr{K}$ is called *a viability domain of* $\mathbf{F}$, if for all $x \in \mathscr{K} : \mathbf{F}(x) \cap \mathscr{K} \neq \emptyset$. The subspace $\mathscr{K}$ is called *the viability kernel of* (2.8) when it is the largest viability domain of $\mathbf{F}$.

Theorem 5. [Aubin & Frankowska, 1991] The supremal (A, E, B)–invariant subspace contained in $\mathscr{X}_d$, $\mathscr{V}^*_{\mathscr{X}_d}$, is the viability kernel of $\mathscr{X}_d$ for the set-valued map, $\mathbf{F}: \mathscr{X}_d \rightsquigarrow \mathscr{X}_d$, $\mathbf{F}(x) = E^{-1}(Ax + \mathscr{B})$. Moreover, for all $x_0 \in \mathscr{V}^*_{\mathscr{X}_d}$, there exists a trajectory, $x \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{V}^*_{\mathscr{X}_d})$, solution of (1.2), satisfying $x(0) = x_0$.

Frankowska [1990] called a singular system, *"strict"*, when the viability kernel coincides with the whole descriptor space, $\mathscr{X}_d$, namely:

$$\mathscr{V}^*_{\mathscr{X}_d} = \mathscr{X}_d \tag{2.9}$$

An importance contribution of Geerts [1993], is that it gives conditions under which the distributional and time-domain frameworks lead to the same conclusions with respect to the shape of the resulting system's solution trajectories (*c.f.* (2.7) and (2.6)), namely the resulting distributions are identified as ordinary functions, with support on $\mathbb{R}^+$, and the generalized derivatives can be identified with ordinary derivatives. Also, it is well connected with the viability discussion of Frankowska [1990] and Aubin & Frankowska [1991]; indeed, a singular system is strict if and only if the consistent initial condition set, I_C, coincides with the whole descriptor variable space, $\mathscr{X}_d$ (*c.f.* (2.9) and (2.7), and recall Assumption **[H2]**).

2.2 Reachability

For the case of implicit systems, Frankowska [1990] extended the classical reachability definition as follows:

Definition 6. [Frankowska, 1990] The implicit representation (1.2) is called *reachable* if for any pair of vectors x_0, $x_1 \in \mathscr{X}_d$ and for any pair of real numbers $t_1 > t_0 \geq 0$, there exists a trajectory $x(\cdot)$, solution of (1.2), such that $x(t_0) = x_0$ and $x(t_1) = x_1$.

[5] We restrict our discussion to subspaces of finite dimensional vector spaces. In [Frankowska, 1990] and in [Aubin & Frankowska, 1991] this definitions are stated in the more general framework of closed sets of normed vector spaces.

Frankowska [1990] has established in her Theorem 4.4 that $\mathscr{R}^*_{\mathscr{X}_d}$ (see (1.3)) is the reachable space of implicit systems like (1.2), with E and A not necessarily square. Hereafter, we recall Corollary 2.4 of Aubin and Frankowska [1991], which is more *ad hoc* for our paper:

Theorem 7. [Aubin & Frankowska, 1991] For any $t_1 > 0$ and for a system like (1.2), with E and A not necessarily square, the reachable space of (1.2) at time t_1, from the initial descriptor variable $x(0)$, is equal to $\mathscr{R}^*_{\mathscr{X}_d}$. Moreover, $\mathscr{R}^*_{\mathscr{X}_d}$ is the supremal subspace such that for all, $x_0, x_1 \in \mathscr{R}^*_{\mathscr{X}_d}$ and $t_1 > 0$, there exists a trajectory $x \in \mathcal{C}^\infty(\mathbb{R}^+, \mathscr{R}^*_{\mathscr{X}_d})$, solution of (1.2), satisfying $x(0) = x_0$ and $x(t_1) = x_1$.

Lemma 8. When $\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{X}_d$, the *implicit representation* (1.2) can be restricted to $\mathscr{R}^*_{\mathscr{X}_d}$ in the domain, and to $A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}$ in the codomain.

Moreover, the *implicit representation* (1.2), restricted to $\mathscr{R}^*_{\mathscr{X}_d}$ in the domain, and to $A\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{B}$ in the codomain, takes the following form:

$$\underbrace{\begin{bmatrix} \mathrm{I}_{\mathcal{C}} & 0 \\ 0 & 0 \end{bmatrix}}_{\overline{E}} \mathrm{d}x/\mathrm{d}t = \underbrace{\begin{bmatrix} \overline{A}_{1,1} & \overline{A}_{1,2} \\ \overline{A}_{2,1} & \overline{A}_{2,2} \end{bmatrix}}_{\overline{A}} x + \underbrace{\begin{bmatrix} \overline{B}_1 & 0 \\ 0 & \mathrm{I}_{\mathscr{U}_{\mathcal{C}}} \end{bmatrix}}_{\overline{B}} u, \tag{2.10}$$

where $\mathrm{I}_{\mathcal{C}}: \ \mathscr{R}^*_{\mathscr{X}_d}\big/(\mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E) \ \leftrightarrow \ E\mathscr{R}^*_{\mathscr{X}_d}$, and $\mathrm{I}_{\mathscr{U}_{\mathcal{C}}}: \mathscr{U}\big/B^{-1}E\mathscr{R}^*_{\mathscr{X}_d} \leftrightarrow \mathscr{B}\big/(\mathscr{B} \cap E\mathscr{R}^*_{\mathscr{X}_d})$ are isomorphisms.

3. POLE ASSIGNMENT

One of the most important features of the reachability of a state space representation, (1.1), is the complete assignability of the closed loop spectrum by means of a state feedback. This equivalence is no longer the case when dealing with implicit representations, (1.2). For the implicit description case, a geometric condition has to be added in order to guarantee such a *pole assignment ability.* In the sequel we give geometric conditions, which enable us to assign the closed loop spectrum of: *(i)* a reachable implicit description, (1.2), and *(ii)* a reachable and observable implicit description with output equation, $\mathfrak{R}^{imp}(E, A, B, C)$:

$$E\mathrm{d}x/\mathrm{d}t = Ax + Bu, \quad y = Cx, \tag{3.11}$$

where: $C: \mathscr{X}_d \to \mathscr{Y}$ is a linear map, and the linear space, $\mathscr{Y}$, is the output space.

3.1 Pole Assignment for a Reachable Implicit Description

Theorem 9. [Bonilla & Malabre, 1993] Given an implicit system, represented by (1.2), for every symmetric set of complex numbers, Λ, of cardinality, $\dim(\mathscr{R}^*_{\mathscr{X}_d})$, there exists a proportional and derivative descriptor variable feedback, $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t$, such that the spectrum of $(\lambda E_{F_d} - A_{F_p})$ is Λ, if and only if:

$$\mathscr{R}^*_{\mathscr{X}_d} = \mathscr{X}_d, \tag{3.12}$$

$$\dim(E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B}) \geq \dim(\mathscr{V}^*_{\mathscr{X}_d}). \tag{3.13}$$

Bonilla & Malabre [1993] named this property as *external reachability.* In that paper, the condition (3.13) is expressed in its equivalent form:

$$\dim(\mathscr{B}/(\mathscr{B} \cap E\mathscr{V}^*_{\mathscr{X}_d})) \geq \dim(\mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{K}_E) \tag{3.14}$$

Let us note that the geometric condition (3.12) is the *reachability* condition of Frankowska [1990] (*c.f.* Theorem 7) and the geometric condition (3.13) is the descriptor variable uniqueness of Lebret [1991], namely the closed loop left invertibility property, which enables us to assign the poles by means of a proportional and derivative feedback:

Lemma 10. [Lebret, 1991] There exists a proportional and derivative descriptor variable feedback, $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t + v$, such that the fed-back implicit representation, $\mathfrak{R}^{imp}(E_{F_d}, A_{F_p}, B)$, satisfies $\operatorname{Ker}\left(\lambda E_{F_d} - A_{F_p}\right) = \{0\}$ iff (3.13) is satisfied.

Let us also note that in the case of a strict singular system, the geometric condition (3.13) is translated to (*c.f.* (2.9) and (2.7)). $\dim(\underline{\mathscr{X}}_{eq}) \geq \dim(\mathscr{X}_d)$. In other words, it is not possible to assign all the spectrum of an implicit system having a degree of freedom, as for example the ones considered in [Bonilla & Malabre, 2003].

We have the following Corollary of Theorem 9:

Corollary 11. Let the implicit representation (1.2) satisfy the geometric conditions (3.12) and (3.13). We then have the following two cases:

For the case $\mathscr{U}_{\mathcal{C}} = \{0\}$, the implicit representation (2.10) reduces to the following reachable state space representation:

$$\mathrm{d}x/\mathrm{d}t = \overline{A}_{1,1}x + \overline{B}_1 u \quad \text{with} \quad \left\langle \overline{A}_{1,1} \,\middle|\, \operatorname{Im}\overline{B}_1 \right\rangle = \mathscr{X}_d\,. \tag{3.15}$$

For the case $\mathscr{U}_{\mathcal{C}} \neq \{0\}$, there exists a map $\overline{V}_\ell : \mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E \to \mathscr{B}_{\mathcal{C}}$ such that $\operatorname{Ker}\overline{V}_\ell = \{0\}$.

Then, applying the proportional feedback:

$$u = \begin{bmatrix} 0 & 0 \\ -\mathrm{I}^{-1}_{\mathscr{U}_{\mathcal{C}}}\overline{A}_{2,1} & -\mathrm{I}^{-1}_{\mathscr{U}_{\mathcal{C}}}(\overline{A}_{2,2} + \overline{V}_\ell) \end{bmatrix} x + v \tag{3.16}$$

we get:

$$\begin{bmatrix} \mathrm{I}_{\mathcal{C}} & 0 \\ 0 & 0 \end{bmatrix} \mathrm{d}x/\mathrm{d}t = \begin{bmatrix} \overline{A}_{1,1} & 0 \\ 0 & -\mathrm{I} \end{bmatrix} x + \begin{bmatrix} \overline{B}_1 & \overline{A}_{1,2}\overline{V}^g_\ell \mathrm{I}_{\mathscr{U}_{\mathcal{C}}} \\ 0 & \overline{V}^g_\ell \mathrm{I}_{\mathscr{U}_{\mathcal{C}}} \end{bmatrix} v, \tag{3.17}$$

where $\overline{V}^g_\ell : \mathscr{B}_{\mathcal{C}} \to \mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E$ is some left inverse of $\overline{V}_\ell$, and:

$$\begin{array}{l} \mathrm{I}_{\mathcal{C}}\mathscr{R}_{\mathcal{C}} = E\mathscr{R}^*_{\mathscr{X}_d} = \left\langle \overline{A}_{1,1} \,\middle|\, \begin{bmatrix} \overline{B}_1 & \overline{A}_{1,2}\overline{V}^g_\ell \mathrm{I}_{\mathscr{U}_{\mathcal{C}}} \end{bmatrix} \right\rangle \quad \text{and} \\ \mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E = \overline{V}^g_\ell \mathrm{I}_{\mathscr{U}_{\mathcal{C}}}\mathscr{U}_{\mathcal{C}}. \end{array} \tag{3.18}$$

Furthermore, applying the proportional and derivative feedback:

$$u = \begin{bmatrix} 0 & 0 \\ -\mathrm{I}^{-1}_{\mathscr{U}_{\mathcal{C}}}\overline{A}_{2,1} & -\mathrm{I}^{-1}_{\mathscr{U}_{\mathcal{C}}}(\overline{A}_{2,2} + \overline{V}_\ell) \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & -\mathrm{I}^{-1}_{\mathscr{U}_{\mathcal{C}}}\overline{V}_\ell \end{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} + v \tag{3.19}$$

we get:

$$\mathrm{d}x/\mathrm{d}t = \begin{bmatrix} \overline{A}_{1,1} & \overline{A}_{1,2} \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} \overline{B}_1 & 0 \\ 0 & \overline{V}^g_\ell \mathrm{I}_{\mathscr{U}_{\mathcal{C}}} \end{bmatrix} v, \tag{3.20}$$

with:

$$\begin{array}{r} \left\langle \begin{bmatrix} \overline{A}_{1,1} & \overline{A}_{1,2} \\ 0 & 0 \end{bmatrix} \,\middle|\, \operatorname{Im} \begin{bmatrix} \overline{B}_1 & 0 \\ 0 & \overline{V}^g_\ell \mathrm{I}_{\mathscr{U}_{\mathcal{C}}} \end{bmatrix} \right\rangle \\ = \left\langle \overline{A}_{1,1} \,\middle|\, \operatorname{Im}\begin{bmatrix} \overline{B}_1 & \overline{A}_{1,2} \end{bmatrix} \right\rangle \oplus \mathscr{U}_{\mathcal{C}} = \mathscr{X}_d \end{array} \tag{3.21}$$

From this Corollary, we realize that with a proportional feedback, we can only modify the spectrum of: $\overline{A}_{1,1} = \left(E\mathscr{R}^*_{\mathscr{X}_d}\right) \left|\overline{A}\right| \left(\mathscr{R}^*_{\mathscr{X}_d} / \mathscr{R}^*_{\mathscr{X}_d} \cap \mathscr{K}_E\right)$; to assign all the spectrum of $\overline{A}$, we need a proportional and derivative feedback.

3.2 Pole Assignment for a Reachable and Observable Implicit Description

In this Section we are going to consider the reachability of the observable part, after feedback, of the implicit representation (3.11). For this, let us recall the supremal (A, E, B)–invariant subspace contained in $\operatorname{Ker} C$, $\mathscr{V}^* = \sup\{\mathscr{V} \subset \mathscr{K}_C |\ A\mathscr{V} \subset E\mathscr{V} + \operatorname{Im} B\}$, which characterizes the biggest part of a given implicit representation, $\mathfrak{R}^{imp}(E, A, B, C)$, which can be made unobservable by means of a suitable proportional and derivative descriptor variable feedback (*c.f.* the early Geometric Algorithms Section).

Given a proportional and derivative descriptor variable feedback, $u = F^*_p x + F^*_d \mathrm{d}x/\mathrm{d}t$, where $(F^*_p, F^*_d) \in \mathbf{F}(\mathscr{V}^*)$, let us consider the *quotient implicit representation*, $\mathfrak{R}^{imp}(E_*, A_*, B_*, C_*)$, where the linear applications, E_*, A_*, B_*, C_*, are the induced maps uniquely defined by:

$$E_*\Phi = \Pi E_{F^*_d}, \quad A_*\Phi = \Pi A_{F^*_p}, \quad B_* = \Pi B \quad \text{and} \quad C = C_*\Phi \tag{3.22}$$

where $\Phi : \mathscr{X}_d \to \mathscr{X}_d \big/ \mathscr{V}^*$ and $\Pi : E\mathscr{X}_d \to E\mathscr{X}_d \big/ E_{F^*_d}\mathscr{V}^*$ are the canonical projections.

Theorem 12. Given an implicit system, represented by (3.11), for every symmetric set of complex numbers, Λ, of cardinality, $\dim\left((\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{V}^*)\big/\mathscr{V}^*\right)$, there exists a proportional and derivative descriptor variable feedback, $u = F^*_p x + F^*_d \mathrm{d}x/\mathrm{d}t + v$, with $(F^*_p, F^*_d) \in \mathbf{F}(\mathscr{V}^*)$, such that the spectrum of $(\lambda E_* - A_*)$ is Λ, where E_* and A_* are the induced maps (3.22), if and only if:

$$(\mathscr{R}^*_{\mathscr{X}_d} + \mathscr{V}^*)\big/\mathscr{V}^* = \mathscr{X}_d\big/\mathscr{V}^*, \tag{3.23}$$

$$\dim\left(\frac{(E\mathscr{V}^*_{\mathscr{X}_d} + \mathscr{B})}{(E\mathscr{V}^* + \mathscr{B})}\right) + \dim(\mathscr{B}) \geq \dim\left(\frac{\mathscr{V}^*_{\mathscr{X}_d}}{\mathscr{V}^*}\right). \tag{3.24}$$

Let us note that (3.24) is equivalent to:

$$\dim\left(\frac{\mathscr{B}}{(\mathscr{B} \cap E\mathscr{V}^*_{\mathscr{X}_d})}\right) \geq \dim\left(\mathscr{V}^*_{\mathscr{X}_d} \cap \mathscr{K}_E\right) - \dim\left(\mathscr{V}^* \cap E^{-1}\mathscr{B}\right) \tag{3.25}$$

For the implicit representations (3.11), satisfying Theorem 12, we will say that they have the *externally reachable output dynamics property* [6]. Theorem 12 is important because it enables us to tackle systems having an internal variable structure (see for example Bonilla & Malabre [1991], Bonilla & Malabre [2003], and Bonilla & Malabre [2008]). Let us also note that the geometric condition (3.24) is the *descriptor variable uniqueness property* notion of Lebret [1991], namely the closed loop left invertibility property of the observable part of the system:

Lemma 13. [Lebret, 1991] There exists a proportional and derivative descriptor variable feedback, $u = F_p x + F_d \mathrm{d}x/\mathrm{d}t + v$, such that the fed-back implicit representation, $\mathfrak{R}^{imp}(E_{F_d}, A_{F_p}, B)$, satisfies $\operatorname{Ker}\left(\lambda E_{F_d} - A_{F_p}\right) \subset \mathscr{N}_{(F_p, F_d)}$ iff (3.24) is satisfied.

Let us finally note that, when comparing (3.24) with (3.13), we realize that Theorem 12 is indeed establishing the external reachabilty of the observable part, after feedback. Also note that in the case: $\mathscr{V}^* = \{0\}$, (3.24) and (3.13) are the same; and in the case: $\mathscr{V}^* = \mathscr{V}^*_{\mathscr{X}_d}$, we get the trivial condition: $\dim(\mathscr{B}) \geq 0$.

[6] The *externally reachable output dynamics* notion is a simplification of the one of *reachable with output dynamics assignment* [see Bonilla *et al*, 1994, Definition 6].

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