String Topology, Euler Class and TNCZ free loop fibrations

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Abstract. Let M be a connected, closed oriented manifold. Let $\omega \in H^m(M)$ be its orientation class. Let $\chi(M)$ be its Euler characteristic. Consider the free loop fibration $\Omega M \overset{i}{\hookrightarrow} LM \overset{ev}{\twoheadrightarrow} M$. For any class $a \in H^*(LM)$ of positive degree, we prove that the cup product $\chi(M)a \cup ev^*(\omega)$ is null. In particular, if $i^* : H^*(LM;\mathbb{F}_p) \twoheadrightarrow H^*(\Omega M;\mathbb{F}_p)$ is onto then $\chi(M)$ is divisible by p (or M is a point).

1. Introduction

Denote by $LM := map(S^1, M)$ the free loop space on M. Except where specified, we work over an arbitrary principal ideal domain $\mathbb{k}$.

In String Topology, shriek maps are used to defined operations. But usually shriek maps are used to obtained vanishing results: see for example [2, III.10.1] for an application of the transfer map in group cohomology. In this paper, after defining them carefully, we use the operations in String Topology to obtain the following vanishing result:

Theorem 1. *(Theorem 30 3) and Remark 35 below) Let M be a connected, closed oriented manifold. Let $\omega \in H^m(M)$ be its orientation class. Let $\chi(M)$ be its Euler characteristic. The cohomology of the free loops relative to the constant loops $H^*(LM, M)$ satisfies*

$$H^*(LM, M) \cup \chi(M)ev^*(\omega) = \{0\}.$$

This vanishing result also holds for any generalized cohomology h^* and homotopy fibre product of (the pull-back) of an embedding with itself (Theorem 15 4) below).

Using Leray-Hirsch theorem or Serre spectral sequence, we deduce

Corollary 2. *(Corollary 42 below) Let M be a connected, closed oriented manifold. Suppose that the free loop fibration $\Omega M \overset{i}{\hookrightarrow} LM \overset{ev}{\twoheadrightarrow} M$ is Totally Non-Cohomologous to Zero with respect to a field $\mathbb{F}$, i. e. $H^*(i;\mathbb{F}) : H^*(LM;\mathbb{F}) \twoheadrightarrow H^*(\Omega M;\mathbb{F})$ is onto. Then $\chi(M) = 0$ in $\mathbb{F}$ or M is a point.*

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Again this Corollary is generalized for any generalized cohomology h^* and any fibration with section (Lemmas 18 and 22 below). We deduce then the following theorem.

Theorem 3. *(Theorem 24) Let $g : G \hookrightarrow E$ be the pull-back of an embedding in the sense of definition 11. Under some mild hypothesis, if the fibration p_g associated to g is Totally Non-Cohomologous to Zero and if all the homotopy fibres $p_g^{-1}(*)$ are not acyclic then the Euler class of g is null.*

In the case of the diagonal embedding, Theorem 3 gives Corollary 2.

We now give the plan of the paper.

Part 1. We construct carefully the shriek maps used in String Topology to define the operations. In particular, we give the key property (Proposition 8) that we use in this paper.

Part 2. We give our most general results. In section 5, we define the open string product and open string coproduct of Sullivan [25]. In section 6, we compute the open string coproduct of the homotopy fibre product of an embedding with itself. In section 7, we give general results on Totally Non-Cohomologous to Zero fibrations with sections using Leray-Hirsch (Lemma 18) or Serre spectral sequence (Lemma 22). These general results are used to prove Theorem 3. In section 8, as an example, we consider the case when the embedding is the inclusion of complex projective spaces.

Part 3. We specialize to the case of free loop spaces where the embedding is the diagonal embedding. In section 9, we define the Chas-Sullivan loop product [3] and the loop coproduct. In section 10, we compute the dual of the loop coproduct in term of cup product. In particular, we recover the results of Tamanoi [28] and Sullivan [25] concerning the vanishing of the loop coproduct. In section 11, we give (Theorem 36) a variant of Theorem 1. As application, we prove an homotopy version of a classical result relating fixed point action of the circle and Euler characteristics. And we prove Corollary 2. In section 12, we give many examples showing that Corollary 2 is pertinent. We conjecture that Corollary 2 holds for any simply-connected finite CW-complex. In section 13, we consider the case the case of relative free loop spaces. This is an example where the embedding is the pull-back of the diagonal embedding. As an application, we generalize Theorem 1 to the space $\mathrm{map}(\vee_n S^1, M)$ of maps from the wedge of n circles to M.

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Part 1. The shriek maps

2. The shriek map of an oriented embedding

Let h^* be a generalized cohomology theory which is additive and multiplicative.

Let $\phi : M \hookrightarrow B$ an embedding between two manifolds without boundary of dimensions m and b respectively. Following [21, Corollary 11.2], we suppose that $\phi(M)$ is a closed subset of B, i. e. [18, Proposition A.53 (c), Theorem A.57] Φ is proper. Of course, this is the case if M is compact.

We also suppose that the normal bundle ν is h^*-oriented. If M and B are both h^*-oriented, ν is h^*-oriented since TM and $TB_{|M} = TM \oplus \nu$ are h^*-oriented ([12, Chapter 4, Lemma 4.1] or [7, Theorem 6 p. 45]).

By the tubular neighborhood theorem ([21, 11.1] or [12, Chapter 4, Theorem 5.2]), there exists an open neighborhood V of M in B and a diffeomorphism $exp : \nu \overset{\cong}{\to} V$ such that under this diffeomorphism, the zero section map $M \to \nu$ corresponds to the inclusion map $s : M \hookrightarrow V$.

Consider the associated closed disk bundle $D(\nu)$ and the associated sphere bundle $S(\nu)$. Let $N := exp(D(\nu))$ be a closed tubular neighborhood. Let $\partial N := exp(S(\nu))$ be its boundary. Note that the inclusion map $s : M \overset{\approx}{\hookrightarrow} N$ is a homotopy equivalence.

Since ν is h^*-oriented, there exists a Thom class $u \in h^{b-m}(D(\nu); S(\nu))$ and a Thom isomorphism [21, Theorem 9.1].

Remark 4. Our generalized cohomology h^* does not necessarily satisfies the weak equivalence axiom. Therefore the Thom homomorphism might not be an isomorphism [29, (17.9.1)].

The Thom class u will be thought as an element of $h^{b-m}(N; \partial N)$.

Since M is closed in B, $(B, B-M, N)$ is an excisive triad. The inclusion $\partial N \overset{\approx}{\hookrightarrow} N - M$ is a homotopy equivalence. Therefore the composite

$$i : (N, \partial N) \to (N, N-M) \to (B, B-M)$$

induces an isomorphism in cohomology. Let $j : B \to (B, B-M)$ be the canonical map. By definition [8, p. 419], $\phi^!$ is the composite

$$h^*(M) \xrightarrow[\cong]{s^{*-1}} h^*(N) \xrightarrow[\cong]{-\cup u} h^{*+b-m}(N, \partial N) \xrightarrow[\cong]{i^{*-1}} h^{*+b-m}(B, B-M) \xrightarrow{j^*} h^{*+b-m}(B)$$

3. The shriek map of the pull-back of an embedding

The idea to construct the shriek map of the pull-back of an embedding, is to pull-back the Thom class and the tubular neighborhood. In particular, we will forget the original vector bundle ν and the fact the Thom homomorphism was (may-be see Remark 4) a Thom isormorphism.

Consider a (Serre) fibration $p : E \twoheadrightarrow B$. Consider the pull-back diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\phi}} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow[\phi]{} & B \end{array}$$

The goal of this section is to construct a shriek map for $\tilde{\phi}$.

Let $\tilde{N} := p^{-1}(N)$. Then $p^{-1}(N - M) = p^{-1}(N) - p^{-1}(M) = \tilde{N} - \tilde{M}$. Let $\partial\tilde{N} := p^{-1}(\partial N)$. Consider the two rectangles where all the squares are pull-backs

$$\begin{array}{ccccc} \tilde{M} & \xrightarrow[\tilde{s}]{\simeq} & \tilde{N} & \longrightarrow & E \\ {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow[s]{\approx} & N & \longrightarrow & B \end{array} \qquad \begin{array}{ccccc} \partial\tilde{N} & \xrightarrow{\simeq} & \tilde{N} - \tilde{M} & \longrightarrow & E \\ {\scriptstyle p}\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ \partial N & \xrightarrow{\approx} & N - M & \longrightarrow & B \end{array}$$

Since the inclusion map $s : M \overset{\approx}{\hookrightarrow} N$ is a homotopy equivalence and $p : \tilde{N} \twoheadrightarrow N$ is a (Serre) fibration, $\tilde{s} : \tilde{M} \overset{\simeq}{\hookrightarrow} \tilde{N}$ is a (weak) homotopy equivalence. Similarly, since the inclusion $\partial N \overset{\approx}{\hookrightarrow} N - M$ is a homotopy equivalence, the inclusion $\partial\tilde{N} \overset{\simeq}{\hookrightarrow} \tilde{N} - \tilde{M}$ is also a (weak) homotopy equivalence.

Since the inverse image of an excisive triad is an excisive triad, $(E, E - \tilde{M}, \tilde{N}) = (p^{-1}(B), p^{-1}(B - M), p^{-1}(N))$ is an excisive triad. Therefore the composite

$$\tilde{i} : (\tilde{N}, \partial\tilde{N}) \to (\tilde{N}, \tilde{N} - \tilde{M}) \to (E, E - \tilde{M})$$

induces an isomorphism in cohomology.

Let $\tilde{u}$ be the image of the Thom class u by $p^* : h^*(N, \partial N) \to h^*(\tilde{N}, \partial\tilde{N})$. Let $\tilde{j} : E \to (E, E - M)$ be the canonical map. By definition, $\tilde{\phi}^!$ is the composite

$$h^*(\tilde{M}) \xrightarrow[\cong]{\tilde{s}^{*-1}} h^*(\tilde{N}) \xrightarrow{-\cup\tilde{u}} h^{*+b-m}(\tilde{N}, \partial\tilde{N}) \xrightarrow[\cong]{\tilde{i}^{*-1}} h^{*+b-m}(E, E - \tilde{M}) \xrightarrow{\tilde{j}^*} h^{*+b-m}(E)$$

Comparing with the definition of the shriek map of ϕ given in Section 2, since $\tilde{u} := p^*(u)$, we obviously have the naturality with respect to pull-backs:

$$p^* \circ \phi^! = \tilde{\phi}^! \circ q^*. \tag{5}$$

Till now, our construction of the shriek map $\tilde{\phi}$ follows the construction of Tamanoi in the special case of the loop coproduct [27] and of the loop coproduct [28] in string topology, except that Tamanoi, in each case, construct a specific homotopy equivalence $\tilde{N} \overset{\approx}{\to} \tilde{M}$ replacing our homotopy equivalence $\tilde{s} : \tilde{M} \overset{\approx}{\to} \tilde{N}$. As remarked by Tamanoi [27, p. 8], note that in order to define $\tilde{\phi}$, we don't need to know if the total space $q^*(S(\nu))$ of the bundle induced by pulling-back $S(\nu)$, is diffeomorphic [24, Proposition 5.3], homeomorphic [6, p. 8], or homotopy equivalent to $\partial\tilde{N}$.

Although, we don't need it in this note, let us prove that $q^*(S(\nu))$ is homotopy equivalent to $\partial\tilde{N}$ for completeness:

Proposition 6. *Let $q^*(D(\nu))$ and $q^*(S(\nu))$ the pull-backs of the closed disk bundle $D(\nu)$ and of the sphere disk bundle $S(\nu)$ along the (Serre) fibration $q : \tilde{M} \twoheadrightarrow M$. Then there exist a (weak) homotopy equivalence*

$$\tilde{exp} : q^*(D(\nu)) \overset{\simeq}{\to} \tilde{N}$$

whose restriction to $q^*(S(\nu))$

$$\tilde{exp} : q^*(S(\nu)) \overset{\simeq}{\to} \partial\tilde{N}$$

is also a (weak) homotopy equivalence.

Proof. Since the bundle projection $\nu : D(\nu) \overset{\approx}{\twoheadrightarrow} M$ is a homotopy inverse to the zero section map $M \to D(\nu)$, the following triangle commutes up to an homotopy $H : [0,1] \times D(\nu) \to N$.

$$\begin{array}{ccc} D(\nu) & \xrightarrow[\cong]{exp} & N \\ {\scriptstyle\nu}\downarrow{\scriptstyle\approx} & \nearrow_{s} & \\ M & & \end{array}$$

The restriction of H to $[0,1] \times S(\nu)$ is a homotopy between the composite of *exp* with the inclusion map, $S(\nu) \underset{\cong}{\overset{exp}{\to}} \partial N \hookrightarrow N$, and the composite $S(\nu) \overset{\nu}{\twoheadrightarrow} M \overset{s}{\hookrightarrow} N$. Therefore the two spaces $\tilde{S(\nu)}$ and $q^*(S(\nu))$ obtained by pulling back this two composites along the (Serre) fibration $p : \tilde{N} \twoheadrightarrow N$ are (weakly) homotopy equivalent [23, Chap. 2 Theorem 14],

$$\begin{array}{ccccccc} \widetilde{S(\nu)} & \xrightarrow{\cong} & \widetilde{\partial N} & \longrightarrow & \tilde{N} & \longrightarrow & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ S(\nu) & \xrightarrow[exp]{\cong} & \partial N & \longrightarrow & N & \longrightarrow & B \end{array} \qquad \begin{array}{ccccccc} q^*(S(\nu)) & \longrightarrow & \tilde{M} & \xrightarrow[\simeq]{\tilde{s}} & \tilde{N} & \longrightarrow & E \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ S(\nu) & \xrightarrow[\nu]{} & M & \xrightarrow[s]{\approx} & N & \longrightarrow & B \end{array}$$

Denote by $\tilde{exp} : q^*(S(\nu)) \overset{\simeq}{\to} \widetilde{S(\nu)} \overset{\cong}{\to} \widetilde{\partial N}$ the composite of the weak homotopy equivalence and of the homeomorphism.

Similarly, the homotopy H gives a weak homotopy equivalence $q^*(D(\nu)) \overset{\simeq}{\to} \tilde{N}$. Since $\tilde{exp} : q^*(S(\nu)) \overset{\simeq}{\to} \widetilde{\partial N}$ was defined using the restriction of H to $S(\nu)$, we claim that this weak homotopy equivalence $q^*(D(\nu)) \overset{\simeq}{\to} \tilde{N}$ extends $\tilde{exp} : q^*(S(\nu)) \overset{\simeq}{\to} \widetilde{\partial N}$. Therefore, we call it also $\tilde{exp}$. □

If $p : E \twoheadrightarrow B$ is a fiber bundle, then using [1, 4.6.4], we have that $q^*(S(\nu))$ and $\widetilde{\partial N}$ are homeomorphic, instead of just homotopy equivalent.

Remark 7. Since $q^*(S(\nu)) \twoheadrightarrow S(\nu)$ and $p : \widetilde{S(\nu)} \twoheadrightarrow S(\nu)$ are fiber homotopy equivalent [23, Chap. 2 Theorem 14], the induced isomorphism in cohomology

$$\tilde{exp} : h^*(\tilde{N}, \widetilde{\partial N}) \overset{\cong}{\to} h^*(q^*(D(\nu)), q^*(S(\nu)))$$

fits into the commutative square

$$\begin{array}{ccc} h^*(\tilde{N}, \widetilde{\partial N}) & \xrightarrow[\cong]{\tilde{exp}^*} & h^*(q^*(D(\nu)), q^*(S(\nu))) \\ {\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle q^*} \\ h^*(N, \partial N) & \xrightarrow[\cong]{exp^*} & h^*(D(\nu), S(\nu)) \end{array}$$

Therefore, by naturality of the Thom class [1, 11.7.11],

-$\tilde{u}$, which was defined as $p^*(u)$, coincides with the Thom class of the vector bundle $q^*(\nu)$ induced by pulling-back ν along $q : \tilde{M} \twoheadrightarrow M$ and

-the composite

$$h^*(\tilde{M}) \underset{\cong}{\overset{\tilde{s}^{*-1}}{\to}} h^*(\tilde{N}) \underset{\cong}{\overset{-\cup\tilde{u}}{\to}} h^{*+b-m}(\tilde{N}, \widetilde{\partial N})$$

is a Thom isomorphism (if h^* satisfies the weak equivalence axiom, see Remark 4).

4. The Euler class

Let $\tilde{s}_{rel} : \tilde{M} \to (\tilde{N}, \partial\tilde{N})$ be the relative inclusion map. Since, by Remark 7, $\tilde{u}$ is the Thom class of the vector bundle $q^*(\nu)$ obtained by pull-back, $\tilde{s}^*_{rel}(\tilde{u})$ is its Euler class.

Proposition 8. [26, Theorem 2.1 (5)] *For the shriek map of the pull-back of an embedding, we have the formula for any $x \in h^*(\tilde{M})$:*

$$\tilde{\phi}^* \circ \tilde{\phi}^!(x) = x \cup \tilde{s}^*_{rel}(\tilde{u}).$$

For an embedding $\phi : M \hookrightarrow B$, this formula is well known [14, Theorem 6.1 (5)]. For $\tilde{\phi}$, the pull-back of an embedding, the proof will be similar [14, p. 282].

Proof. Remark that the following square commutes

$$\begin{array}{ccc} (\tilde{N}, \partial\tilde{N}) & \xrightarrow{\tilde{i}} & (E, E - \tilde{M}) \\ \uparrow{\scriptstyle \tilde{s}_{rel}} & & \uparrow{\scriptstyle \tilde{j}} \\ \tilde{M} & \xrightarrow[\tilde{\phi}]{} & E. \end{array}$$

Remark also that

$$\tilde{s}^*_{rel} : h^*(\tilde{N}, \partial\tilde{N}) \to h^*(\tilde{M})$$

is $h^*(\tilde{N})$-linear where $h^*(\tilde{N})$ acts on $h^*(\tilde{M})$ by restriction of scalar with respect to the algebra morphism $\tilde{s}^* : h^*(\tilde{N}) \to h^*(\tilde{M})$. Therefore, by definition of $\tilde{\phi}^!$,

$$\begin{aligned} \tilde{\phi}^* \circ \tilde{\phi}^!(x) &= \tilde{\phi}^* \circ \tilde{j}^* \circ \tilde{i}^{*-1}(\tilde{s}^{*-1}(x) \cup \tilde{u}) \\ &= \tilde{s}^*_{rel} \circ \tilde{i}^* \circ \tilde{i}^{*-1}(\tilde{s}^{*-1}(x) \cup \tilde{u}) \\ &= \left(\tilde{s}^* \circ \tilde{s}^{*-1}(x)\right) \cup \tilde{s}^*_{rel}(\tilde{u}) = x \cup \tilde{s}^*_{rel}(\tilde{u}) \end{aligned}$$

□

Consider the commutative diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{s}_{rel}} & (\tilde{N}, \partial\tilde{N}) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{s_{rel}} & (N, \partial N) \end{array}$$

Since, by definition, $\tilde{u} := p^*(u)$, we obviously have

$$\tilde{s}^*_{rel}(\tilde{u}) = q^* \circ s^*_{rel}(u). \tag{9}$$

Note that, by definition ([14, p. 279] or [21, p. 98]), $s^*_{rel}(u)$ is the Euler class of the normal bundle ν. Since we have remarked that $\tilde{s}^*_{rel}(\tilde{u})$ is the Euler class of the vector bundle obtained by pulling-back ν, this formula is just the naturality of the Euler class [21, Property 9.2].

Corollary 10. *Let $k : F \hookrightarrow E$ be the inclusion of the fiber of the fibration p. If $b > m$ then $k^* \circ \tilde{\phi}^! = 0$ in singular cohomology.*

Proof. Let $l : F \hookrightarrow \tilde{M}$ be the inclusion of the fiber of the induced fibration q. We have the commutative diagram

$$\begin{array}{ccccc} & & \overset{k}{\frown} & & \\ F & \xrightarrow{l} & \tilde{M} & \xrightarrow{\tilde{\phi}} & E \\ \varepsilon\downarrow & & q\downarrow & & \downarrow p \\ \{*\} & \xrightarrow[\eta]{} & M & \xrightarrow[\phi]{} & B \end{array}$$

where the two squares are pull-backs. By Proposition 8 and equation (9),

$$\tilde{\phi}^* \circ \tilde{\phi}^!(x) = x \cup q^*(e_\nu)$$

where $e_\nu \in H^{b-m}(M)$ is the Euler class of the normal bundle ν. Therefore

$$k^* \circ \tilde{\phi}^!(x) = l^* \circ \tilde{\phi}^* \circ \tilde{\phi}^!(x) = l^*(x) \cup l^* \circ q^*(e_\nu) = l^*(x) \cup \varepsilon^* \circ \eta^*(e_\nu) = 0.$$

□

Part 2. The general case

5. The open string (co)products

In this section, we define the joining product $\wedge$ of Sullivan [25] that following Tamanoi [26], we prefer to call the open string product. And more important for us, we define a generalized version of the cutting at time 1/2 product $\vee_{1/2}$ of Sullivan [25] that following Tamanoi [26], we prefer to call the open string coproduct.

Let $f : F \to E$ and $g : G \to E$ be two maps. Let

$${}^fE^g = \{(a, \omega, b) \in F \times E^I \times G / f(a) = \omega(0),\ g(b) = \omega(1)\}$$

denote the *homotopy fibre product* of f and g which is obtained by the following pull-back

$$\begin{array}{ccc} {}^fE^g & \longrightarrow & E^I \\ (ev_0,ev_1)\downarrow & & \downarrow(ev_0,ev_1) \\ F \times G & \xrightarrow[(f\times g)]{} & E \times E. \end{array}$$

Definition 11. A continuous map $g : G \to E$ is the *pull-back of an embedding* if it is equipped with a pull-back diagram

$$\begin{array}{ccc} G & \xrightarrow{g} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow[\phi]{} & B \end{array}$$

where $\phi : M \hookrightarrow B$ a proper embedding between two manifolds of codimension m with h^*-oriented normal bundle and where the map p is a (Serre) fibration (This is exactly the case considered in Section 3).

Remark 12. Suppose that a continous map $g : G \to E$ fits into a pull-back diagram

$$\begin{array}{ccc} G & \xrightarrow{g} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow[\phi]{} & B \end{array}$$

where $\phi : M \hookrightarrow B$ a proper embedding between two manifolds of codimension m with h^*-oriented normal bundle and where the map p is smooth and is transverse to ϕ.

Then $g : G \hookrightarrow E$ a proper embedding between two manifolds of codimension m with h^*-oriented normal bundle.

A common example to both definition 11 and remark 12 is when p is a smooth fibre bundle.

Proof of Remark 12. Since ϕ and p are transverse, $g : G \hookrightarrow E$ is an embedding of codimension m. Note that the normal bundle of g, ν_g, is the pull-back $q^*(\nu_\phi)$ of the normal bundle of ϕ, ν_ϕ, along q. Therefore since ν_ϕ is h^*-oriented, ν_g is also h^*-oriented [1, 11.7.11] without assuming that G or E is h^*-orientable. Since $\phi(M)$ is a closed subset of B, $g(G) = p^{-1}(\phi(M))$ is a closed subset of E. □

Theorem 13. [25] *Let $f : F \to E$, $g : G \to E$ and $h : H \to E$ be three maps.*
1) If G is a smooth h^*-oriented manifold without boundary of dimension n then there is a open string product

$$lp_{f,g,h} : H_*({}^fE^g) \otimes H_*({}^gE^h) \to H_{*-n}({}^fE^h).$$

2) Suppose that g is the pull-back of an embedding in the sense of definition 11. Then there is a open string coproduct

$$lcp_{f,g,h} : h^*({}^fE^g) \otimes h^*({}^gE^h) \to h^{*+m}({}^fE^h).$$

Remark 14. In [25], Sullivan consider the open string product only in the case when g is an embedding.

Proof. 1) Consider the three pull-back squares (Compare with [26, Diagram p.11])

$$\begin{array}{ccccc}
{}^fE^g \times {}^gE^h & \xleftarrow{\tilde{\Delta}} & {}^fE^g \times_E {}^gE^h & \xrightarrow{\mu_{f,g,h}} & {}^fE^h \\
\downarrow{\scriptstyle ev_1 \times ev_0} & & \downarrow{\scriptstyle ev_{1/2}} & & \downarrow{\scriptstyle ev_{1/2}} \\
G \times G & \xleftarrow[\Delta]{} & G & \xrightarrow[g]{} & E \\
 & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
 & & M & \xrightarrow[\phi]{} & B
\end{array}$$

The open string product $lp_{f,g,h}$ is defined as the Chas-Sullivan loop product above using the left pull-back square:

$$H_*{}^fE^g \otimes H_*{}^gE^h \xrightarrow{\times} H_*{}^fE^g \times {}^gE^h \xrightarrow{\tilde{\Delta}_!} H_{*-n}{}^fE^g \times_E {}^gE^h \xrightarrow{\mu_{f,g,h*}} H_{*-n}{}^fE^h.$$

2) Since p is a (Serre) fibration, then the composite

$${}^fE^h \xrightarrow{ev_{1/2}} E \xrightarrow{p} B$$

is also a fibration. Therefore using the total right rectangle, since $\phi : M \hookrightarrow B$ is an embedding, we obtain the shriek map

$$\mu^!_{f,g,h} : h^*({}^fE^g \times_E {}^gE^h) \to h^{*+m}({}^fE^h).$$

The open string coproduct $lcp_{f,g,h}$ is now defined as the composite

$$h^*({}^fE^g) \otimes h^*({}^gE^h) \xrightarrow{\times} h^*({}^fE^g \times {}^gE^h) \xrightarrow{\tilde{\Delta}^*} h^*({}^fE^g \times_E {}^gE^h) \xrightarrow{\mu^!_{f,g,h}} h^{*+m}({}^fE^h).$$

□

6. A simple formula for the open string coproduct

Theorem 15. *Let $f : F \to E$, $g : G \to E$ and $h : H \to E$ be three maps. Suppose that g is the pull-back of an embedding in the sense of definition 11. Let $e_\nu \in h^m(M)$ be the Euler class of the normal bundle of the embedding $\phi : M \hookrightarrow B$.*

Let $ev_0 : {}^gE^h \twoheadrightarrow G$, $(a, w, b) \mapsto a$ be the first projection map. Let $ev_1 : {}^fE^g \twoheadrightarrow G$, $(a, w, b) \mapsto b$ be the second projection map.

Let $\sigma : G \hookrightarrow {}^gE^g$, $b \mapsto (b$, constant path $g(b), b)$ be the section of both projections map ev_0 and ev_1 when $f = g = h$.

Then

1) the open string coproduct

$$lcp_{f,g,g} : h^*({}^fE^g) \otimes h^*({}^gE^g) \to h^{*+m}({}^fE^g)$$

is given by

$$lcp_{f,g,g}(a \otimes b) = a \cup ev_1^* \left(\sigma^*(b) \cup q^*(e_\nu)\right).$$

2) the open string coproduct

$$lcp_{g,g,h} : h^*({}^gE^g) \otimes h^*({}^gE^h) \to h^{*+m}({}^gE^h)$$

is given by

$$lcp_{g,g,h}(b \otimes c) = ev_0^* \circ \sigma^*(b) \cup c \cup ev_0^* \circ q^*(e_\nu).$$

3) In the case $f = g = h$,

$$ev_0^* \circ q^*(e_\nu) = ev_1^* \circ q^*(e_\nu) \in h^*({}^gE^g).$$

4) The ideal $Ker\ \sigma^* : h^*({}^gE^g) \twoheadrightarrow h^*(G)$ satisfies

$$Ker\ \sigma^* \cup ev_0^* \circ q^*(e_\nu) = \{0\}.$$

5) the open string coproduct on $h^*({}^gE^g)$

$$lcp_{g,g,g} : h^*({}^gE^g) \otimes h^*({}^gE^g) \to h^{*+m}({}^gE^g)$$

is given by

$$lcp_{g,g,g}(a \otimes b) = ev_0^* \left(\sigma^*(a) \cup \sigma^*(b) \cup q^*(e_\nu)\right) = ev_1^* \left(\sigma^*(a) \cup \sigma^*(b) \cup q^*(e_\nu)\right).$$

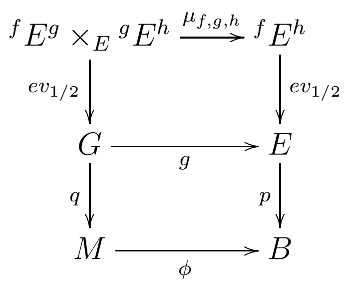
6) Let $\alpha \in {}^gE^g$. Denote by ${}^gE^g_{[\alpha]}$ the path-connected component of α. Denote by ${}^gE^g_0 := \cup_{\alpha \in G} {}^gE^g_{[\sigma(\alpha)]}$, i. e. the subspace of ${}^gE^g$, union of the path-connected components $Im\ \pi_0(\sigma) : \pi_0(G) \hookrightarrow \pi_0({}^gE^g)$ (If G is path-connected, there is only one). Then

the open string coproduct $lcp_{f,g,g}$ is trivial outside of $h^*({}^fE^g) \otimes h^*({}^gE^g_0)$,
the open string coproduct $lcp_{g,g,h}$ is trivial outside of $h^*({}^gE^g_0) \otimes h^*({}^gE^h)$,
the open string coproduct $lcp_{g,g,g}$ is trivial outside of $h^*({}^gE^g_0) \otimes h^*({}^gE^g_0)$.

7) The open string coproduct $lcp_{f,g,g}$ maps $h^*({}^fE^g_{[\alpha]}) \otimes h^*({}^gE^g)$ to $h^{*+m}({}^fE^g_{[\alpha]})$.
The open string coproduct $lcp_{g,g,h}$ maps $h^*({}^gE^g) \otimes h^*({}^gE^h_{[\alpha]})$ to $h^{*+m}({}^gE^h_{[\alpha]})$.
The image of the open string coproduct $lcp_{g,g,g}$ is contained in ${}^gE^g_0$.

8) Suppose that there is an integer $dim\ G$ such that $\forall i > dim\ G$, $h^i(G) = \{0\}$. Then for all $a, b \in h^*({}^gE^g)$ such that $|a| + |b| > dim\ G - m$, $lcp_{g,g,g}(a \otimes b) = 0$.

Proof. 1) and 2) Recall that we have the following two pull-back squares



Denote by $(q\circ ev_{1/2})^*(\nu)$ the pull-back of ν along $q\circ ev_{1/2}$. Let $e_{(q\circ ev_{1/2})^*(\nu)}$ by its Euler class. By Proposition 8, for any $x\in h^*({}^fE^g\times_E {}^gE^h)$

$$\mu^*_{f,g,h}\circ\mu^!_{f,g,h}(x)=x\cup e_{(q\circ ev_{1/2})^*(\nu)}.$$

By formula (9) (Naturality of Euler class),

$$e_{(g\circ q)^*(TM)}=(q\circ ev_{1/2})^*(e_\nu)$$

where e_ν is the Euler class of the normal bundle.

Putting everything together, we have that

$$\mu^*_{f,g,h}\circ\mu^!_{f,g,h}(x)=x\cup ev^*_{1/2}\circ q^*(e_\nu). \tag{16}$$

Note that till here, the same discussion for the open string product, shows that

$$\tilde{\Delta}^*\circ\tilde{\Delta}^!(x)=x\cup ev^*_{1/2}(e_{TG}). \tag{17}$$

But now, we will see that both $\mu^*_{f,g,g}$ and $\mu^*_{g,g,h}$ admit a retract. In particular, $\mu^*_{g,g,g}$ admits two different retracts i_1^* and i_2^*. Let $i_1:{}^fE^g\to {}^fE^g\times_E {}^gE^g$ be the inclusion map on the first factor defined by

$$i_1(a,w,b)=((a,w,b),(b,\text{constant path } g(b),b))$$

for $a\in F$, $w\in E^I$ and $b\in G$ such that $f(a)=w(0)$ and $g(b)=w(1)$. Let $i_2:{}^gE^h\to {}^gE^g\times_E {}^gE^h$ be the inclusion map on the second factor defined by

$$i_2(a,w,b)=((a,\text{constant path } g(a),a),(a,w,b)))$$

for $a\in G$, $w\in E^I$ and $b\in H$ such that $(g(a),h(b))=(w(0),w(1))$. The first inclusion i_1 is a section up to homotopy of $\mu_{f,g,g}$. Therefore i_1^* is a retract of $\mu^*_{f,g,g}$.

So by formula (16), since $ev_{1/2}\circ i_1$ is the projection map $ev_1:{}^fE^g\twoheadrightarrow G$, $(a,w,b)\mapsto b$, for any $x\in h^*({}^fE^g\times_E {}^gE^g)$,

$$\begin{aligned}\mu^!_{f,g,g}(x)=i_1^*\circ\mu^*_{f,g,g}\circ\mu^!_{f,g,g}(x)&=i_1^*(x\cup ev^*_{1/2}\circ q^*(e_\nu))\\ &=i_1^*(x)\cup i_1^*\circ ev^*_{1/2}\circ q^*(e_\nu))=i_1^*(x)\cup ev_1^*\circ q^*(e_\nu).\end{aligned}$$

Similarly since i_2^* is a retract of $\mu^*_{g,g,h}$ and since $ev_{1/2}\circ i_2$ is the projection map $ev_0:{}^gE^h\twoheadrightarrow G$, $(a,w,b)\mapsto a$, for any $x\in h^*({}^gE^g\times_E {}^gE^h)$,

$$\mu^!_{g,g,h}(x)=i_2^*(x)\cup ev_0^*\circ q^*(e_\nu).$$

Since the following two squares commute

$$\begin{array}{ccc} {}^fE^g & \xrightarrow{i_1} & {}^fE^g \times_N {}^gE^g \\ \Delta\downarrow & & \downarrow\tilde{\Delta} \\ {}^fE^g \times {}^fE^g & \xrightarrow[id\times(\sigma\circ ev_1)]{} & {}^fE^g \times {}^gE^g \end{array} \qquad \begin{array}{ccc} {}^gE^h & \xrightarrow{i_2} & {}^gE^g \times_N {}^gE^h \\ \Delta\downarrow & & \downarrow\tilde{\Delta} \\ {}^gE^h \times {}^gE^h & \xrightarrow[(\sigma\circ ev_0)\times id]{} & {}^gE^g \times {}^gE^h, \end{array}$$

for any $a \in h^*({}^fE^g)$ and $b \in h^*({}^gE^g)$,

$$\begin{aligned} \mu^!_{f,g,g} \circ \tilde{\Delta}^*(a \times b) &= i_1^* \circ \tilde{\Delta}^*(a \times b) \cup ev_1^* \circ q^*(e_\nu) \\ &= a \cup (\sigma \circ ev_1)^*(b) \cup ev_1^* \circ q^*(e_\nu) \end{aligned}$$

and for any $b \in h^*({}^gE^g)$ and $c \in h^*({}^gE^h)$,

$$\mu^!_{g,g,h} \circ \tilde{\Delta}^*(b \times c) = (\sigma \circ ev_0)^*(b) \cup c \cup ev_0^* \circ q^*(e_\nu).$$

3) Using 1),

$$lcop_{g,g,g}(1 \otimes 1) = ev_1^* \circ q^*(e_\nu).$$

Using 2),

$$lcop_{g,g,g}(1 \otimes 1) = ev_0^* \circ q^*(e_\nu).$$

4) Let ε be 0 or 1. Since $\sigma^* \circ ev_\varepsilon^* = id$, using the split short exact sequence

$$0 \to \text{Ker}\ \sigma^* \overset{i}{\hookrightarrow} h^*({}^gE^g) \overset{\sigma^*}{\twoheadrightarrow} h^*(G) \to 0,$$

we obtain that Ker $\sigma^* =$ Im r_ε where $r_\varepsilon : h^*({}^gE^g) \twoheadrightarrow \text{Ker}\ \sigma^*$ is the retract of the inclusion defined by $r_\varepsilon(a) = a - ev_\varepsilon^* \circ \sigma^*(a)$.

By 1) and 3),

$$lcop_{g,g,g}(a \otimes 1) = a \cup ev_1^* \circ q^*(e_\nu) = a \cup ev_0^* \circ q^*(e_\nu).$$

By 2),

$$lcop_{g,g,g}(a \otimes 1) = ev_0^* \circ \sigma^*(a) \cup ev_0^* \circ q^*(e_\nu).$$

Therefore $r_0(a) \cup ev_0^* \circ q^*(e_\nu) = 0$.

5) By 4),

$$b \cup ev_0^* \circ q^*(e_\nu) = ev_0^* \circ \sigma^*(b) \cup ev_0^* \circ q^*(e_\nu).$$

Therefore by 2)

$$lcop_{g,g,g}(a \otimes b) = ev_0^* \circ \sigma^*(a) \cup ev_0^* \circ \sigma^*(b) \cup ev_0^* \circ q^*(e_\nu).$$

By 4) and 3)

$$a \cup ev_1^* \circ q^*(e_\nu) = ev_1^* \circ \sigma^*(a) \cup ev_1^* \circ q^*(e_\nu).$$

Therefore by 1) and the graded commutativity of the cup product

$$lcop_{g,g,g}(a \otimes b) = ev_1^* \circ \sigma^*(a) \cup ev_1^* \circ \sigma^*(b) \cup ev_1^* \circ q^*(e_\nu).$$

6) Since by definition, σ arrives inside ${}^gE_0^g$, σ^* factorizes through the projection $h^*({}^gE^g) \twoheadrightarrow h^*({}^gE_0^g)$. So using 1) and 2), 6) is proved.
7) Let $a \in h^*({}^fE^g_{[\alpha]})$ and $b \in h^*({}^gE^g)$. The cohomology of a space is isomorphic to the product of the cohomology algebras of its path-connected components. Therefore the cup product with $ev_1^*(\sigma^*(b) \cup q^*(e_\nu))$ defines a linear application from $h^*({}^fE^g_{[\alpha]})$ to itself. So by 1), $lcop_{f,g,g}(a \otimes b) \in h^*({}^fE^g_{[\alpha]})$.

In the case $f = g$, by 6), if $lcop_{g,g,g}(a \otimes b)$ is non-zero then the path-connected component of α belongs to the image of $\pi_0(\sigma)$ and in this case $lcop_{g,g,g}(a \otimes b) \in h^*({}^fE_0^g)$.
8) The element $\sigma^*(a) \cup \sigma^*(b) \cup q^*(e_\nu) \in h^*(G)$ is of degree $|a|+|b|+m >$ dim G and so is null. Using 5), we obtain that $lcop_{g,g,g}(a \otimes b) = 0$. □

7. TNCZ fibrations

Lemma 18. *Suppose that B is path-connected. Suppose also that B is a CW-complex or that h^* satisfies the weak equivalence axiom. Let $F \overset{i}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} B$ be a (Serre) fibration with base B which admits a section $\sigma : B \to E$ up to homotopy, i. e. $p \circ \sigma \approx id_B$ and an element $e \in h^*(B)$ such that*

$$Ker\ \sigma^* \cup p^*(e) = \{0\}.$$

If the fibration p is Totally Non-Cohomologous to Zero, i. e. $h^*(i) : h^*(E) \twoheadrightarrow h^*(F)$ is onto and if $\tilde{h}^*(F)$ is a finitely generated free graded h^*-module then $e = 0$ or $\tilde{h}^*(F) = \{0\}$.

Remark 19. Let $p : E \twoheadrightarrow B$ be a fibration. Let $\omega : I \to B$ be a path from b to b'. Let $w_\# : p^{-1}(b) \overset{\approx}{\to} p^{-1}(b')$ be the induced homotopy equivalence between the fibers [23, Theorem 2.8.12]. By definition, $w_\#$ commutes up to homotopy with the inclusions of fibers $i_b : p^{-1}(b) \hookrightarrow E$ and $i_{b'} : p^{-1}(b') \hookrightarrow E$. In cohomology, $i_b^* = w_\#^* \circ i_{b'}^*$ [29, Proof of (17.9.3)]. So i_b^* is surjective if and only if $i_{b'}^*$ is surjective. And given a family of vectors $c_j \in h^*(E)$, the $i_b^*(c_j)$'s form a h^*-basis of $h^*(p^{-1}(b))$ if and only if the $i_{b'}^*(c_j)$'s form a h^*-basis of $h^*(p^{-1}(b'))$.

Proof. Let $b_0 \in B$. Let $\varepsilon_F : F \to *$ be the unique map to a point. Since B is path-connected and $\tilde{h}^*(F) :=$ coker $\varepsilon_F^* : h^* \to h^*(F)$ does not depend of a base point and is homotopy invariant [29, 17.1.3], we can chose any fibre F. We take $F := p^{-1}(\{p \circ \sigma(b_0)\})$. Since $h^*(i) : h^*(E) \twoheadrightarrow h^*(F)$ is surjective, $\tilde{h}^*(i) : \tilde{h}^*(E) \twoheadrightarrow \tilde{h}^*(F)$ is also surjective.

Since $\tilde{h}^*(\sigma)\circ\tilde{h}^*(p) = id$, $\tilde{h}^*(E) = \text{Ker }\tilde{h}^*(\sigma)\oplus\text{Im }\tilde{h}^*(p)$. Since Im $\tilde{h}^*(p)\subset$ Ker $\tilde{h}^*(i)$, the restriction of $\tilde{h}^*(i)$ to Ker $\tilde{h}^*(\sigma)$ is also surjective.

Suppose that $\tilde{h}^*(F)\neq\{0\}$. Then there exists classes $c_j\in$ Ker $\tilde{h}^*(\sigma)\subset\tilde{h}^*(E)$ such that the $\tilde{h}^*(i)(c_j)$'s form a h^*-basis of $\tilde{h}^*(F)$.

Now let $b_0\in B$, $\sigma(b_0)\in E$ and $\sigma(b_0)\in F$ be the chosen base points $*$ of B, E and F. Denote by $\eta^X_{x_0}:\{x_0\}\hookrightarrow X$ be the inclusion of the base point x_0 into a based space X. Then we have the canonical identification for a based space X between $\tilde{h}^*(X)$ and $h^*(X,x_0)\cong$ Ker $\eta^{X*}_{x_0}: h^*(X)\rightarrow h^*$. Since $\sigma:(B,*)\hookrightarrow(E,*)$ and $i:(F,*)\hookrightarrow(E,*)$ are based maps, we can consider that the classes $c_j\in h^*(E,*)$, that the $i^*(c_j)$'s form a h^*-basis of $h^*(F,*)$, that Ker $\tilde{h}^*(\sigma)=$ Ker σ^* and so that $c_j\cup p^*(e)=0$.

Since $h^*(F)\cong h^*(F,*)\oplus h^*$, the classes $c_j\in h^*(E,*)$ and the unit $1\in h^0(E)$ are send by i^* to a h^*-basis of $h^*(F)$. So by the Leray-Hirsch theorem for generalized cohomology ([29, (17.8.4)] using Remark 19), $h^*(E)$ is a free $h^*(B)$-modules with basis 1 and the c_j's. So $e=0$. □

Remark 20. In Lemma 18, when the generalized cohomology h^* is a singular cohomology H^*, it is enough to suppose that $\tilde{H}^q(F)$ is $\mathbb{k}$-free module of finite type for each degree $q\geq 0$. Indeed in this case, we can apply the Leray-Hirsch theorem for singular cohomology [11, Exercise 3 p. 51] (see also [29, (17.8.1)] using Remark 19 or [22, Theorem 4.4] where the fibre of the fibration is assumed to be path-connected). In Lemma 22 below, we improve further Lemma 18 for singular cohomology. The proof of Lemma 22 relies on the following interesting Lemma:

Lemma 21. *Let $F\overset{i}{\hookrightarrow}E\overset{p}{\twoheadrightarrow}B$ be a (Serre) fibration with path-connected base B which admits a section $\sigma:B\rightarrow E$ up to homotopy, i. e. $p\circ\sigma\approx id_B$ and an element $e\in H^m(B)$ such that*

$$Ker\ \sigma^*\cup p^*(e)=\{0\}.$$

Consider the cohomological Serre spectral sequence $(E_r^{*,*},d_r)$ associated to the fibration p. Suppose that the action of $\pi_1(B)$ on $H^*(F)$ is trivial.

Let f be an element of $\tilde{H}^*(F)$. Suppose that f is in the image of $i^*:H^*(E)\rightarrow H^*(F)$.

Denote by $e\otimes 1$, the image of e by the canonical morphism [22, III.2.10.10]

$$i_*:H^m(B)\rightarrow H^m(B;H^0(F))=E_2^{m,0}$$

Denote by $1\otimes f$ *the image of* f *by the inverse of the canonical morphism* [22, III.2.10.9]

$$i_0^*: E_2^{0,*}=H^0(B;H^*(F))\xrightarrow{\cong} H^0(*;H^*(F))=H^*(F).$$

Then the product $e\otimes 1\cup 1\otimes f\in E_2^{m,*}$ *must be killed: there exist* $r\geq 2$ *and* $x\in E_r^{*,*}$ *such that* $d_r(x)=e\otimes 1\cup 1\otimes f$.

Proof. Since B is path-connected, the triviality of the local coefficients $H^*(F)$ implies that i_0^* is an isomorphism (and conversely by [22, III.1.18 (3)]).

For degree reasons, $\forall r\geq 2, d_r(e\otimes 1)=0$ and $e\otimes 1$ in $E_\infty^{m,0}=F^mH^m(E)$ is $p^*(e)$. Since $f\in \text{Im } i^*$, $\forall r\geq 2, d_r(1\otimes f)=0$. Let q be the degree of f. Let $c\in\tilde{H}^q(E)$ such that $\tilde{H}^*(i)(c)=f$. As explained in the proof of Lemma 18, c can be chosen in Ker σ^*. Then $1\otimes f$ in $E_\infty^{0,q}=H^q(E)/F^1H^q(E)$ is the class of c. Therefore $e\otimes 1\cup 1\otimes f$ in $E_\infty^{m,q}=F^mH^{m+q}(E)/F^{m+1}H^{m+q}(E)$ is the class of $p^*(e)\cup c$. Since Ker $\sigma^*\cup p^*(e)=\{0\}$, $p^*(e)\cup c=0$. Therefore $e\otimes 1\cup 1\otimes f$ must be killed. □

Lemma 22. *Let* $F\overset{i}{\hookrightarrow} E\overset{p}{\twoheadrightarrow} B$ *be a (Serre) fibration with path-connected base* B *which admits a section* $\sigma: B\to E$ *up to homotopy, i. e.* $p\circ\sigma\approx id_B$ *and an element* $e\in H^m(B)$ *such that*

$$Ker\ \sigma^*\cup p^*(e)=\{0\}.$$

Suppose that $\forall n\in\mathbb{N}$, $H_n(B)$ *is a finitely generated* $\Bbbk$*-module or* $\forall q\in\mathbb{N}$, $H^q(F)$ *is a finitely generated* $\Bbbk$*-module. Suppose also that* $\forall q\geq 2$, $H^q(F)$ *is a torsion free* $\Bbbk$*-module.*

If the fibration p *is Totally Non-Cohomologous to Zero, i. e.* $H^*(i): H^*(E)\twoheadrightarrow H^*(F)$ *is onto and if* $H^m(B)$ *is a free* $\Bbbk$*-module then* $e=0$ *or* $\tilde{H}^*(F)=\{0\}$.

Proof. Since $H^*(i)$ is onto, the action of $\pi_1(B)$ on $H^*(F)$ is trivial [22, III.Theorem 4.4].

By hypothesis, $H^q(F)$ is a finitely generated $\Bbbk$-module or $\forall n\geq 0$ $H_n(B)$ is a finitely generated $\Bbbk$-module. So since $\Bbbk$ is a principal ideal domain by [23, Theorem 5.5.10], we have a short exact sequence

$$0\to H^p(B)\otimes H^q(F)\xrightarrow{\mu} E_2^{p,q}\to \text{Tor}^{\Bbbk}(H^{p+1}(B),H^q(F))\to 0$$

where μ is a morphism of algebras. Therefore, since $\forall q\geq 0$ $H^q(F)$ is torsion free, $\forall p,q\in\mathbb{N}$ $E_2^{p,q}\cong H^p(B)\otimes H^q(F)$ as algebras. Since i^* is onto, d_r is null on $E^{0,q}$. Therefore the Serre spectral sequence collapses on the E_2-term (Here we have reproved the well-known [22, III.Theorem 4.4] with weaker hypothesis).

So by Lemma 21, for any $f \in \tilde{H}^q(F)$, the element $e \otimes f \in E^{m,q} = H^m(B) \otimes H^q(F)$ must be zero. Since $H^m(F)$ is free, $e \otimes f = 0$ implies that $e = 0$ or f has torsion. □

The following lemma is a generalization of Lemma 22 if the base B of the fibration is not path-connected.

Lemma 23. *Let $B = \cup_{\beta\in\pi_0(B)} B_\beta$ the decomposition of B into its path-connected components. Let $p : E \twoheadrightarrow B$ be a (Serre) fibration with base B which admits a section $\sigma : B \to E$ up to homotopy, i. e. $p\circ\sigma \approx id_B$ and an element $e = (e_\beta)_{\beta\in\pi_0(B)} \in H^m(B) = \Pi_{\beta\in\pi_0(B)} H^m(B_\beta)$ such that*

$$Ker\ \sigma^* \cup p^*(e) = \{0\}.$$

Denote by F_β, the fibre $p^{-1}(b)$ when $b \in B_\beta$. For all $\beta \in \pi_0(B)$, suppose that $\forall n \in \mathbb{N}$, $H_n(B_\beta)$ is a finitely generated $\mathbb{k}$-module or $\forall q \in \mathbb{N}$, $H^q(F_\beta)$ is a finitely generated $\mathbb{k}$-module. Suppose also that $\forall\beta \in \pi_0(B)$, $\forall q \geq 2$, $H^q(F_\beta)$ is a torsion free $\mathbb{k}$-module.

If the fibration p is Totally Non-Cohomologous to Zero, i. e. $\forall\beta \in \pi_0(B)$ $H^*(i_\beta) : H^*(E) \twoheadrightarrow H^*(F_\beta)$ is onto and if $\forall\beta \in \pi_0(B)$ $H^m(B_\beta)$ is a free $\mathbb{k}$-module then $\forall\beta \in \pi_0(B)$ ($e_\beta = 0$ or $\tilde{H}^*(F_\beta) = \{0\}$).

Proof. For all $\beta \in \pi_0(B)$, we apply Lemma 22 to the fibration

$$F_\beta \overset{i_\beta}{\hookrightarrow} E_{\pi_0(p)^{-1}(\beta)} \overset{p_\beta}{\twoheadrightarrow} B_\beta$$

obtained by restricting p to the union $E_{\pi_0(p)^{-1}(\beta)}$ of path-connected components α of E such that $\pi_0(p)(\alpha) = \beta$. The fibration p_β admits the restriction of σ to B_β, $\sigma_\beta : B_\beta \to E_{\pi_0(p)^{-1}(\beta)}$ as section up to homotopy. The product of maps

$$\Pi_{\beta\in\pi_0(B)}\sigma^*_\beta : \Pi_{\beta\in\pi_0(B)} H^*(E_{\pi_0(p)^{-1}(\beta)}) \to \Pi_{\beta\in\pi_0(B)} H^*(B_\beta)$$

can be identified with $\sigma^* : H^*(E) \to H^*(B)$. Therefore Ker $\sigma^* \cup p^*(e)$ can be identified with $\Pi_{\beta\in\pi_0(B)}(\text{Ker } \sigma^*_\beta \cup p^*_\beta(e_\beta))$. □

Of course, the following theorem generalizes the fundamental Corollary 11.4 of [21].

Theorem 24. *Let $g : G \hookrightarrow E$ be the pull-back of an embedding in the sense of definition 11. Let $e_\nu \in h^m(M)$ be the Euler class of the normal bundle of the embedding $\phi : M \hookrightarrow B$. Let $p_g : E^I \times_g G \twoheadrightarrow E$ be the fibration associated to g defined by $p_g((\omega, b)) = \omega(0)$ for any $b \in G$ and any path $\omega : I \to E$ such that $\omega(1) = g(b)$. Let $p_\phi : B^I \times_\phi M \twoheadrightarrow B$ be the fibration associated to ϕ.*

Suppose that $\forall n \in \mathbb{N}$, $H_n(G)$ is a finitely generated $\mathbb{k}$-module or $\forall y \in B$, $\forall q \in \mathbb{N}$, $H^q(p_\phi^{-1}(y))$ is a finitely generated $\mathbb{k}$-module. Suppose also that $\forall y \in B$, $\forall q \geq 2$, $H^q(p_\phi^{-1}(y))$ is a torsion free $\mathbb{k}$-module.

If the fibration p_g is Totally Non-Cohomologous to Zero, i. e. $\forall x \in E$ $H^*(i_x) : H^*(E^I \times_g G) \twoheadrightarrow H^*(p_g^{-1}(x))$ is onto and if $H^m(G)$ is a free $\mathbb{k}$-module then $\forall b \in G$ either $\tilde{H}^*(p_\phi^{-1}(p \circ g(b))) = \{0\}$ or the component of $q^*(e_\nu)$ in $H^m(G_{[b]})$ the cohomology of the path-connected component of b in G is trivial.

Proof. By definition of the homotopy fibre product ${}^fE^g$, for any $a \in F$, we have the following commutative diagram of spaces

$$\begin{array}{ccccccc}
ev_0^{-1}(a) & \xrightarrow{\cong} & p_g^{-1}(f(a)) & \xrightarrow{\approx} & p_\phi^{-1}(p \circ f(a)) & & \\
\downarrow & & \downarrow{\scriptstyle i_{f(a)}} & & \downarrow & & \\
{}^fE^g & \longrightarrow & E^I \times_g G & \xrightarrow{p^I \times q} & B^I \times_\phi M & \xleftarrow[\approx]{s} & M \\
{\scriptstyle ev_0}\downarrow & & {\scriptstyle p_g}\downarrow & & {\scriptstyle p_\phi}\downarrow & \swarrow{\scriptstyle \phi} & \\
F & \xrightarrow[f]{} & E & \xrightarrow[p]{} & B & &
\end{array}$$

where the bottom left square is a pull-back, s is a homotopy equivalence and the bottom right square is a homotopy pull-back (i. e. the induced map $p_g^{-1}(f(a)) \to p_\phi^{-1}(p \circ f(a))$ between the homotopy fibre of g and ϕ is a homotopy equivalence). Since $\forall a \in F$, $H^*(i_{f(a)}) : H^*(E^I \times_g G) \to H^*(p_g^{-1}(f(a)))$ is onto, $H^*({}^fE^g) \to H^*(ev_0^{-1}(a))$ is also onto, i. e. ev_0 is Totally Non-Cohomologous to Zero.

Suppose now that $f = g$ (and $F = G$). By part 4) of Theorem 15,

$$\text{Ker } \sigma^* \cup ev_0^* \circ q^*(e_\nu) = \{0\}.$$

By applying Lemma 23 to the fibration $ev_0 : {}^gE^g \twoheadrightarrow G$ we obtain that for all $b \in G$, the component of $q^*(e_\nu)$ in $H^*(G_{[b]})$ is trivial or $\tilde{H}^*(ev_0^{-1}(b)) = \{0\}$. □

8. AN EXAMPLE

Corollary 25. *Consider the following pull-back diagram*

$$\begin{array}{ccc}
G & \xrightarrow{g} & E \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\
\mathbb{CP}^q & \xrightarrow[\phi]{} & \mathbb{CP}^n
\end{array}$$

where $p : E \twoheadrightarrow \mathbb{CP}^n$ is a (Serre) fibration over the n-th complex projective space $\mathbb{CP}^n$ and $\phi : \mathbb{CP}^q \hookrightarrow \mathbb{CP}^n$ is the inclusion, $0 \leq q < n$.

If the fibration p_g associated to g is Totally Non-Cohomologous to Zero and if $H^{2n-2q}(G)$ is a free $\mathbb{k}$-module then $q^*(a^{n-q}) = 0$. Here a is a generator of $H^2(\mathbb{CP}^q)$.

Proof. By [21, Theorem 14.10], $c(T\mathbb{CP}^n)$, the total Chern class of the tangent bundle of $\mathbb{CP}^n$ is equal to $(1+a)^{n+1}$ in $H^{\Pi}(\mathbb{CP}^n)$. Since $T\mathbb{CP}^q \oplus \nu = T\mathbb{CP}^n_{|\mathbb{CP}^q}$, in $H^{\Pi}(\mathbb{CP}^q)$,

$$c(\nu) = c(T\mathbb{CP}^n_{|\mathbb{CP}^q})/c(T\mathbb{CP}^q) = (1+a)^{n+1}/(1+a)^{q+1} = (1+a)^{n-q}.$$

Therefore $e(\nu) = c_{n-q}(\nu) = a^{n-q}$.

We have a morphism of S^1-principal fibre bundles

$$\begin{array}{ccc} S^{2q+1} & \xrightarrow{\tilde{\phi}} & S^{2n+1} \\ \downarrow & & \downarrow \\ \mathbb{CP}^q & \xrightarrow{\phi} & \mathbb{CP}^n \end{array}$$

where $\tilde{\phi} : S^{2q+1} \hookrightarrow S^{2n+1}$ is the inclusion. Since this square is a pull-back, the homotopy fibre of ϕ, $p_{\phi}^{-1}(*)$, is homotopy equivalent to the homotopy fibre of $\tilde{\phi}$. Since $\pi_{2q+1}(S^{2n+1}) = \{0\}$, $\tilde{\phi}$ is homotopically trivial and its homotopy fibre is homotopy equivalent to $S^{2q+1} \times \Omega S^{2n+1}$. Therefore by Theorem 24, all the components of $q^*(e_\nu)$ are trivial. □

Corollary 26. *Let $f : \mathbb{CP}^p \hookrightarrow \mathbb{CP}^n$ and $g : \mathbb{CP}^q \hookrightarrow \mathbb{CP}^n$ be the inclusions, $0 \leq p < n$, $0 \leq q < n$. If the fibration $ev_0 : {}^fE^g \twoheadrightarrow \mathbb{CP}^p$ is Totally Non-Cohomologous to Zero then $q \geq p$ and $n > p + q$.*

Proof. Suppose that $q < p$. Since $ev_0^* \circ f^* = ev_1^* \circ g^*$ and f^* is surjective, for all $q < i \leq p$,

$$ev_0^*(a^i) = ev_0^* \circ f^*(a^i) = ev_1^* \circ g^*(a^i) = ev_1^*(a^i) = 0.$$

Therefore ev_0^* is not injective and so by [22, III.Theorem 4.4], ev_0 is not Totally Non-Cohomologous to Zero.

Consider the Serre spectral sequence associated to the fibration $F \hookrightarrow {}^fE^g \overset{ev_1}{\twoheadrightarrow} \mathbb{CP}^q$. We saw in the proof of Corollary 25 that its fibre F is homotopy equivalent to $S^{2p+1} \times \Omega S^{2n+1}$. Therefore $H^+(F)$ is concentrated in degree $\geq 2p+1$. And so $E_r^{s,t} \neq \{0\} \Rightarrow t = 0$ or $t \geq 2p+1$. Therefore ev_1^* is an isomorphism in degree $\leq 2p$. (In particular, if $p \geq q$, ev_1^* is injective).

Suppose now that ev_0 is Totally Non-Cohomologous to Zero and that $n - q \leq p$. Then $H^{2n-2q}(ev_1) : H^{2n-2q}(\mathbb{CP}^q) \overset{\cong}{\to} H^{2n-2q}({}^fE^g)$ is an isomorphism. Since $H^{2n-2q}(\mathbb{CP}^q)$ is $\mathbb{k}$-free, by Corollary 25, $H^{2n-2q}(ev_1)(a^{n-q}) = 0$. And so $a^{n-q} = 0$ in $H^*(\mathbb{CP}^q)$. Therefore $n - q > q$. In particular, $p > q$. □

Remark 27. In the case $p = q$ of Corollary 26, parts 4) and 8) of Theorem 15 give that

$$\text{Ker } \sigma^* \cup ev_0^*(a^{n-q}) = \{0\}$$

and that for all $b \in H^*({}^gE^g)$ of degree $> 4q - 2n$, $b \cup ev_0^*(a^{n-q}) = 0$.

Remark 28. (Over $\mathbb{Q}$, the converse of Corollary 26 is true) Over $\mathbb{Q}$, a relative Sullivan model of ev_0 is given by the inclusion of differential graded algebras

$$(\Lambda(x_2, z_{2p+1}), d) \hookrightarrow (\Lambda(x_2, z_{2p+1}, t_{2q+1}, sy_{2n+1}), d)$$

with $d(z_{2p+1}) = x_2^{p+1}$, $d(t_{2q+1}) = x_2^{q+1}$ and $d(sy_{2n+1}) = t_{2q+1}x_2^{n-q} - z_{2p+1}x_2^{n-p}$(Compare with [20, Example 7.3]).

If $n > p+q$, by replacing sy by $sy - ztx^{n-p-q-1}$, we can assume that $d(sy) = 0$. If $q \geq p$, by replacing t by $t - zx^{q-p}$, we can assume that $d(t) = 0$. Therefore if $n > p+q$ and $q \geq p$ then over $\mathbb{Q}$, ev_0 is Totally Non-Cohomologous to Zero.

Part 3. **the free loops case**

In this part, we consider our main example of homotopy fibre product, the space LM of free loops on a manifold.

9. The loop product and the loop coproduct

Let M be a smooth oriented manifold without boundary. In this section, M is not necessarily compact. The diagonal map $\Delta : M \hookrightarrow M \times M$ is an embedding. Since M is Hausdorff, $\Delta(M)$ is a closed subset of $M \times M$. As we have explained in Section 2, we can define the shriek map of Δ, $\Delta_!$ in homology.

By definition, the *intersection product* in homology, is the composite

$$H_*(M) \otimes H_*(M) \overset{\times}{\to} H_*(M \times M) \overset{\Delta_!}{\to} H_{*-m}(M).$$

We have the following push-out squares

$$\begin{array}{ccccc} S^1 \coprod S^1 & \longrightarrow & S^1 \vee S^1 & \overset{c}{\longleftarrow} & S^1 \\ \uparrow & & \uparrow & & \uparrow \\ \star \coprod \star & \longrightarrow & \star & \longleftarrow & S^0 \end{array}$$

where $c : S^1 \to S^1 \vee S^1$ is the comultiplication or pinch map of S^1. Note that all the vertical maps are cofibrations. Since the functor $map(-, M)$ transforms push-out squares in pull-back squares, we have the following pull-back squares where all the vertical maps are fibrations

$$\begin{array}{ccccc} LM\times LM & \xleftarrow{\tilde{\Delta}} & LM\times_M LM & \xrightarrow{\mu} & LM \\ \downarrow{\scriptstyle ev\times ev} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle (ev,ev_{1/2})} \\ M\times M & \xleftarrow[\Delta]{} & M & \xrightarrow[\Delta]{} & M\times M \end{array}$$

and $\mu := map(c, M)$ is the composition or multiplication of loops. Since $\Delta : M \hookrightarrow M \times M$ is an embedding, as we have explained in Section 3, we can define the shriek map of $\tilde{\Delta}$, $\tilde{\Delta}_!$ in homology, and the shriek maps of μ, $\mu_!$ in homology, $\mu^!$ in cohomology.

By definition, the Chas-Sullivan loop product in homology, is the composite

$$H_*(LM)\otimes H_*(LM) \overset{\times}{\to} H_*(LM\times LM) \overset{\tilde{\Delta}_!}{\to} H_{*-m}(LM\times_M LM) \overset{\mu_*}{\to} H_{*-m}(LM).$$

By definition, the loop coproduct in homology is the composite

$$H_*(LM) \overset{\mu_!}{\to} H_{*-m}(LM\times_M LM) \overset{\tilde{\Delta}_*}{\to} H_{*-m}(LM\times LM).$$

In this note, we work over an arbitrary principal ideal domain $\mathbb{k}$ and so the cross product is not in general an isomorphism. Therefore, we will consider the loop coproduct in cohomology. By definition, the loop coproduct in cohomology is the product defined by the composite

$$H^*(LM)\otimes H^*(LM) \overset{\times}{\to} H^*(LM\times LM) \overset{\tilde{\Delta}^*}{\to} H^*(LM\times_M LM) \overset{\mu^!}{\to} H^{*+m}(LM)$$

Remark 29. Let $k : \Omega M \hookrightarrow LM$ be the inclusion of the pointed loops into the free loops. If the dimension of M is positive, from Corollary 10, we obtain that the composite

$$H_*(\Omega M)\otimes H_*(\Omega M) \overset{k_*\otimes k_*}{\to} H_*(LM)\otimes H_*(LM) \overset{\text{loop product}}{\to} H_{*-m}(LM)$$

is trivial.

10. A simple formula for the loop coproduct

Denote by $LM_{[1]}$ the path-connected component of LM of freely contractile loops. Recall that $ev : LM \twoheadrightarrow M$ is the evaluation map. Let $\sigma : M \hookrightarrow LM$, $m \mapsto$ constant loop m, be its trivial section.

Theorem 30. *Let M be a connected, closed $\mathbb{k}$-oriented manifold of dimension m. Let $\omega \in H^m(M)$ be its orientation class. Let $\chi(M)$ be its Euler characteristic. Then*
1) The loop coproduct, $\mu^!\circ\tilde{\Delta}^*$ on $H^*(LM)$ is given for $a, b \in H^*(LM)$, by

$$\mu^!\circ\tilde{\Delta}^*(a\otimes b) = \chi(M)a\cup ev^*(\sigma^*(b)\cup\omega).$$

Here $\cup$ *is the cup product on* $H^*(LM)$.
2) The loop coproduct, $\mu^! \circ \tilde{\Delta}^*$ *on* $H^*(LM)$ *is graded commutative with respect to the usual degrees: that is, for* $a \in H^p(LM)$, $b \in H^q(LM)$
$\mu^! \circ \tilde{\Delta}^*(a \otimes b) = (-1)^{pq}\mu^! \circ \tilde{\Delta}^*(b \otimes a)$.
3) The ideal $Ker\ \sigma^* : H^*(LM) \twoheadrightarrow H^*(M)$ *satisfies*

$$Ker\ \sigma^* \cup \chi(M)ev^*(\omega) = \{0\}.$$

4) The loop coproduct, $\mu^! \circ \tilde{\Delta}^*$ *on* $H^*(LM)$ *is given for* $a, b \in H^*(LM)$, *by*

$$\mu^! \circ \tilde{\Delta}^*(a \otimes b) = \chi(M)ev^*(\sigma^*(a) \cup \sigma^*(b) \cup \omega).$$

5) the loop coproduct, $\mu^! \circ \tilde{\Delta}^*$ *on* $H^*(LM)$ *is trivial outside of* $H^0(LM_{[1]}) \otimes H^0(LM_{[1]}) \cong \mathbb{k} \otimes \mathbb{k}$.
6) On $H^0(LM_{[1]}) \otimes H^0(LM_{[1]})$, *the loop coproduct is given by*

$$\mu^! \circ \tilde{\Delta}^*(1 \otimes 1) = \chi(M)ev^*(\omega).$$

7) The image of the loop coproduct $\mu^! \circ \tilde{\Delta}^*$ *is contained in* $H^*(LM_{[1]})$.

Remark 31. Over a field, parts 2), 5) and 7) of this Theorem are not new. Indeed over a field, the commutativity of the loop coproduct was proved by Cohen and Godin [5] and parts 5) and 7) are the duals of [28, Theorem B (2)].

Lemma 32. *Consider* ${}^{\Delta}M^{\Delta}$, *the self homotopy fibre product along the diagonal. Explicitly* ${}^{\Delta}M^{\Delta}$ *is just the subspace*

$$\{(\omega, \omega') \in M^I \times M^I / \omega(0) = \omega'(0),\ \omega(1) = \omega'(1)\}.$$

Let $\Theta : {}^{\Delta}M^{\Delta} \overset{\cong}{\to} LM$ *be the homeomorphism mapping* (ω, ω') *to the free loop* $\omega * \omega'^{-1}$ *obtained by composing the path* ω *with the inverse of the path* ω'. *Then*

1) [25, Example iii) free loop space] *With respect to the loop product and the open string product,*

$$H_*(\Theta) : H_*({}^{\Delta}M^{\Delta}) \overset{\cong}{\to} H_*(LM)$$

is an isomorphism of algebras.

2) With respect to the loop coproduct and the open string coproduct,

$$H^*(\Theta) : H^*(LM) \overset{\cong}{\to} H^*({}^{\Delta}M^{\Delta})$$

is an isomorphism of algebras.

Proof. Denote by $\rho_\alpha : LM \overset{\cong}{\to} LM$ the homeomorphism mapping a free loop l to the rotated free loop $t \mapsto l(t + \alpha)$. Up to the homeomorphism

Θ, the two pull-back squares defining the open string (co)product on

$$\begin{array}{ccccc}
{}^{\Delta}M^{\Delta} \times {}^{\Delta}M^{\Delta} & \xleftarrow{\tilde{\Delta}} & {}^{\Delta}M^{\Delta} \times_{M\times M} {}^{\Delta}M^{\Delta} & \xrightarrow{\mu_{\Delta,\Delta,\Delta}} & {}^{\Delta}M^{\Delta} \\
\downarrow{\scriptstyle ev_1\times ev_0} & & \downarrow{\scriptstyle ev_{1/2}} & & \downarrow{\scriptstyle ev_{1/2}} \\
M\times M & \xleftarrow[\Delta]{} & M & \xrightarrow[\Delta]{} & M\times M
\end{array}$$

coincide with the following two vertical rectangles defining the loop (co)product since ρ_α is homotopic to the identity map.

$$\begin{array}{ccccc}
LM\times LM & \longleftarrow & LM_{1/2}\times_M LM_0 & \longrightarrow & LM \\
\downarrow{\scriptstyle \rho_{1/2}\times id}\ \cong & & \cong\downarrow & & \cong\downarrow{\scriptstyle \rho_{1/4}} \\
LM\times LM & \xleftarrow{\tilde{\Delta}} & LM\times_M LM & \xrightarrow{\mu} & LM \\
\downarrow{\scriptstyle ev\times ev} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle (ev,ev_{1/2})} \\
M\times M & \xleftarrow[\Delta]{} & M & \xrightarrow[\Delta]{} & M\times M
\end{array}$$

□

Proof of Theorem 30. We apply Theorem 15 in the case where $f=g=h=\phi$ is the diagonal embedding $\Delta : M \hookrightarrow M\times M$.

The normal bundle ν of Δ is isomorphic to the tangent bundle of M, TM [21, Lemma 11.5]. Since M is compact and connected, the Euler class of the tangent bundle is the fundamental class multiplied by the Euler characteristics [21, Corollary 11.12]:

$$e_\nu = e_{TM} = \chi(M)\omega.$$

Using part 2) of Lemma 32, we have proved Theorem 30. □

Remark 33. Let M be a connected, non-compact $\mathbb{k}$-oriented manifold of dimension m and suppose that $\mathbb{k}$ is a field. Then its loop coproduct $\mu^! \circ \tilde{\Delta}^*$ on $H^*(LM)$ is trivial.

Proof of Remark 33. Since M is non-compact then $H_m(M)=0$. Since $\mathbb{k}$ is a field, $H^m(M)=\mathrm{Hom}_{\mathbb{k}}(H_m(M),\mathbb{k})=0$. So e_{TM} is trivial. Therefore the same proof as the proof of Theorem 30 shows that the loop coproduct is trivial.

Alternatively, for any $x\in H^*(LM\times_M LM)$

$$\mu^*\circ\mu^!(x) = x\cup q^*(e_{TM}) = 0.$$

Since the composition of loops μ admits a section, μ^* is injective and so $\mu^!$ is null. □

Corollary 34. *The loop coproduct is trivial if and only if $\chi(M)=0$ in $\mathbb{k}$.*

This corollary follows also from [28, (3-1) and (3-2)] (Compare also with [28, Corollary 3.2] or [25, Bottom p. 7]). In [4], Chataur and Thomas gave the first example of manifold with non-trivial loop coproduct.

Proof. Since $ev \circ \sigma = id$, ev^* is injective. Therefore since w is a basis of $H^m(M)$,

$$\chi(M)ev^*(\omega) = 0 \Longleftrightarrow \chi(M)\omega = 0 \Longleftrightarrow \chi(M) = 0 \text{ in } \mathbb{k}.$$

□

Remark 35. Since $ev \circ \sigma = id_M$, M is a subspace of LM and we can consider the relative cohomology $H^*(LM, M)$. Using the long exact sequence associated, $H^*(LM, M)$ can be identified with Ker $\sigma^* : H^*(LM) \twoheadrightarrow H^*(M)$. From part 3) of Theorem 30, we have that the loop coproduct vanishes on $H^*(LM, M)$. In [25], Sullivan introduced a non trivial product on $H^*(LM, M)$ of different degree that he called the cutting at any time $\vee$. In [9], Goresky and Hingston rediscover this non trivial product that they denote $\circledast$.

11. Applications

Theorem 36. *Let M be a connected, closed $\mathbb{k}$-oriented manifold of dimension m. Let $\omega \in H^m(M)$ be its orientation class. Let $\chi(M)$ be its Euler characteristic. Then*
1) $\chi(M)ev^*(\omega) \in H^m(LM_{[1]})$.
2) For any $a \in H^*(LM)$ of positive degree,

$$\chi(M)a \cup ev^*(\omega) = 0.$$

Proof. Comparing 6) and 7) in Theorem 30, we get 1).
By 5) and 1) in Theorem 30,

$$0 = \mu^! \circ \tilde{\Delta}^*(a \otimes 1) = \chi(M)a \cup ev^*(\omega).$$

□

If $\mathbb{k}$ is a field then 1) means that for all non contractile free loop α and for all $a \in H_m(LM_{[\alpha]})$,

$$\chi(M)H_m(ev)(a) = 0.$$

Remark 37. In general, $ev^*(\omega)$ does not belong to $H^*(LM_{[1]})$ and $a \cup ev^*(\omega)$ is not trivial: Suppose that $\mathbb{k}$ is a field. Let G be a connected compact Lie group. Note that $\chi(G) = 0$. For any $[\alpha] \in \pi_1(G)$, let Θ_α be the usual isomorphism from the tensor product $H_*(\Omega_{[\alpha]}G) \otimes H_*(G)$ to $H_*(LG_{[\alpha]})$. Here $\Omega_{[\alpha]}G$ denotes the pointed loops of G homotopic to

α. Let ε be the augmentation of $H_*(\Omega_{[\alpha]}G)$. The previous isomorphism Θ_α fits into the commutative triangle of graded vector spaces

$$\begin{array}{ccc} H_*(\Omega_{[\alpha]}G)\otimes H_*(G) & \xrightarrow[\cong]{\Theta_\alpha} & H_*(LG_{[\alpha]}) \\ & {\scriptstyle \varepsilon\otimes Id}\searrow \quad \swarrow{\scriptstyle H_*(ev)} & \\ & \mathbb{k}\otimes H_*(G) & \end{array}$$

Let $[G]$ be the fundamental class of G. Recall that $[\alpha]$ is a generator of $H_0(\Omega_{[\alpha]}G)$. Then

$$H_{\dim G}(ev)\circ\Theta_\alpha([\alpha]\otimes[G])=[G]\neq 0$$

Therefore $ev^*(\omega)$ does not belong to $H^{\dim G}(LG_{[1]})$ for any non simply-connected, connected compact Lie group G (e. g. S^1).

Let $\eta:\mathbb{k}\to H^*(\Omega G)$ be the unit map of $H^*(\Omega G)$. The usual isomorphism of algebras Θ from the tensor product of graded algebras $H^*(\Omega G)\otimes H^*(G)$ to $H^*(LG)$ fits similarly into the commutative triangle of graded algebras

$$\begin{array}{ccc} H^*(\Omega G)\otimes H^*(G) & \xrightarrow[\cong]{\Theta} & H^*(LG) \\ & {\scriptstyle \eta\otimes Id}\nwarrow \quad \nearrow{\scriptstyle H^*(ev)} & \\ & \mathbb{k}\otimes H^*(G) & \end{array}$$

Therefore for any non-zero element b of $H^*(\Omega G)$,

$$\Theta(b\otimes 1)\cup H^*(ev)(\omega)=\Theta(b\otimes w)\neq 0.$$

If G is a connected compact Lie group such that $H^*(\Omega G)$ is not concentrated in degree 0 (e. g. S^3), we have obtained an element a of positive degree such that $a\cup ev^*(\omega)$ is non zero.

Corollary 38. *Let M be a connected, closed $\mathbb{k}$-oriented manifold of dimension m such that in $\mathbb{k}$, $\chi(M)\neq 0$. Let $\mu:S^1\times M\to M$ be a continuous map such that the composite*

$$\{1\}\times M\to S^1\times M\overset{\mu}{\to} M$$

is homotopic to the identity map. Then there exists an map $\nu:S^1\times M\to M$ homotopic to μ who has at least a fixed point m_0, i. e. $\nu(S^1\times\{m_0\})=m_0$.

Proof. Let $\sigma_\mu:M\to LM$ be the map sending $m\in M$ to its orbit $\mu(-,m):S^1\to M$. Since for all $m\in M$, $ev\circ\sigma_\mu(m)=\mu(1,m)$, σ_μ is a section up to homotopy of ev. Therefore in cohomology,

$$\sigma_\mu^*\circ ev^*(\chi(M)\omega)=\chi(M)\omega.$$

Since M is path-connected, σ_μ arrives in the path-connected component $LM_{[\alpha]}$ of a free loop α. So σ_μ^* is trivial outside of $H^*(LM_{[\alpha]})$. By 1) of Theorem 36, $\chi(M)ev^*(\omega) \in H^m(LM_{[1]})$. Since $\chi(M)$ is not zero in $\Bbbk$, $\chi(M)\omega = \sigma_\mu^* \circ ev^*(\chi(M)\omega)$ is not trivial. Therefore α is contractile, i. e. $[\alpha] = [1]$.

Let $i : \{m_0\} \hookrightarrow M$ be the inclusion of a non-degenerated base point into M. Since α is contractile, $\sigma_\mu(m_0)$ is homotopic to the constant loop $\hat{m_0}$ and so the following triangle commutes up to homotopy.

$$\begin{array}{ccc} S^1 \times M & \xrightarrow{\mu} & M \\ {\scriptstyle S^1\times i}\uparrow & \nearrow_{\hat{m_0}} & \\ S^1 \times \{m_0\} & & \end{array}$$

By the homotopy extension property of the cofibration $S^1 \times i : S^1 \times \{m_0\} \hookrightarrow S^1 \times M$, we can change up to homotopy μ into a map ν such that the triangle commutes now exactly. □

Corollary 38 should be considered as an homotopy version of the following classic result:

Theorem 39. *(*[23, Theorem 4.7.12]*. Compare also with* [14, Theorem 5.39 or Corollary 6.17]*) Let M be a compact Euclidean Neighborhood Retract (e. g. a compact topological manifold* [10, A.9]*) such that $\chi(M) \neq 0$. Let $\mu : S^1 \times M \to M$ be an action of the circle on M. Then M has at least a fixed point.*

Remark 40. If a map $\mu : S^1 \times M \to M$ is only an action up to homotopy then it may happen that M has no fixed point. Therefore the conclusion of Corollary 38 cannot be improved in general: Consider the sphere $M = S^2$ in $\mathbb{R}^3$. Let $\nu : S^1 \times S^2 \to S^2$ be the action given by rotation of axis z. Let $f : S^2 \to S^2$ be the rotation of angle π and of axis y. Since f is homotopic to the identity map, the composite $f \circ \nu$ is an action up to homotopy without any fixed point.

Corollary 41. *Let M be a connected, closed $\Bbbk$-oriented manifold of dimension m. Consider the cohomological Serre spectral sequence $(E_r^{*,*}, d_r)$ associated to the free loop fibration $\Omega M \overset{i}{\hookrightarrow} LM \overset{ev}{\twoheadrightarrow} M$. Suppose that the (conjugation) action of $\pi_1(M)$ on $H^*(\Omega M)$ is trivial.*

Let f be an element of $\tilde{H}^*(\Omega M)$. Suppose that f is in the image of $i^* : H^*(LM) \to H^*(\Omega M)$. Then $\chi(M)\omega \otimes f \in H^m(M) \otimes H^*(\Omega M) = E_2^{m,*}$ must be killed: there exist $r \geq 2$ and $x \in E_r^{*,*}$ such that $d_r(x) = \chi(M)\omega \otimes f$.

Proof. Since M is compact, $\forall n \geq 0$, $H_n(M)$ is a finitely generated $\Bbbk$-module. So since $\Bbbk$ is a principal ideal domain by [23, Theorem 5.5.10], we have a short exact sequence

$$0 \to H^p(M) \otimes H^q(\Omega M) \overset{\mu}{\to} E_2^{p,q} \to \mathrm{Tor}^{\Bbbk}(H^{p+1}(M), H^q(\Omega M)) \to 0$$

where μ is a morphism of algebras. Since $H^0(\Omega M)$, $H^1(M) \cong \mathrm{Hom}(H_1(M), \Bbbk)$ and $H^{m+1}(M) = 0$ are torsion free, $E_2^{p,q} \cong H^p(M) \otimes H^q(\Omega M)$ if $p = 0$ or $p = m$ or $q = 0$.

By part 3) of Theorem 30, Ker $\sigma^* \cup \chi(M)ev^*(\omega) = \{0\}$. So by Lemma 21, $e \otimes 1 \cup 1 \otimes f = \chi(M)\omega \otimes f$ must be killed. □

Corollary 42. *Let M be a connected, closed $\Bbbk$-oriented manifold. Suppose that the free loop fibration $\Omega M \overset{i}{\hookrightarrow} LM \overset{ev}{\twoheadrightarrow} M$ is Totally Non-Cohomologous to Zero, i. e. $H^*(i)$ is onto and that $H^k(\Omega M)$ is a torsion free $\Bbbk$-module for each $k \geq 1$. Then $\chi(M) = 0$ in $\Bbbk$ or M is a point.*

Proof. Suppose that $\chi(M)$ is not equal to zero in $\Bbbk$. By part 3) of Theorem 30, Ker $\sigma^* \cup \chi(M)ev^*(\omega) = \{0\}$. So by Lemma 22, $\tilde{H}^*(\Omega M) = \{0\}$. This means that $H^{>0}(\Omega M) = \{0\}$ and that $\pi_1(M) = \{0\}$. So $H^*(M) \cong \Bbbk$. Since $H^{\dim M}(M) = \Bbbk\omega$, M must be of dimension 0, i. e. M is a point. □

Interpretation and proofs of Corollaries 41 and 42 in term of integration along the basis. Let $F \overset{i}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} M$ be a fibration. Suppose that $\pi_1(M)$ acts trivially on $H^q(F)$. Let $\int i : H^q(F) \to H^{q+m}(E)$ be the composite

$$H^q(F) \underset{\cong}{\overset{\omega\otimes Id}{\to}} H^m(M)\otimes H^q(F) = E_2^{m,q} \twoheadrightarrow E_\infty^{m,q} = F^m H^{m+q}(E) \subset H^{m+q}(E).$$

If M is a sphere, this *integration along the basis* $\int i$ appears in Wang exact sequence and the following two properties are well known [31, Theorems (3.1) and (3.5) Chapter VII]: for $x \in H^*(F)$ and $y \in H^*(E)$,

$$(\int i)(i^*(y) \cup x) = y \cup (\int i)(x)$$

and $p^*(\omega) = (\int i)(1)$. In general, these properties are easy to deduce from the multiplicativity and the naturality of the Serre spectral sequence. In particular, we have $(\int i) \circ i^*(y) = y \cup p^*(\omega)$. Since the inclusion of the fibre $i : F \hookrightarrow E$ is the pull-back along p of the embedding $* \hookrightarrow M$, following Section 3, one can define a shriek map $i^!$ for i. In this paper, we will not use that $i^!$ coincides with $(\int i)$ although this should follows from the diagram in [17, (2) p. 12].

Consider the free loop fibration $\Omega M \overset{i}{\hookrightarrow} LM \overset{ev}{\twoheadrightarrow} M$. In this case, $i_!$ is the intersection morphism $H_{*+\dim M}(LM) \to H_*(\Omega M)$ defined by Chas and Sullivan.

Let f be an element as in Corollary 41. By Theorem 36,

$$\chi(M)\int i(f) = \chi(M)(\int i)\circ i^*(c) = \chi(M)c \cup ev^*(\omega) = 0.$$

So we have recover Corollary 41.

Suppose that i^* is onto and that $H^*(\Omega M)$ is torsion free, the Serre spectral sequence collapses at the E_2-term. So $\int i$ is injective. Therefore $\chi(M)f = 0$. And we have recover Corollary 42.

12. TNCZ free loop fibrations

Recall our result on TNCZ free loop fibration.

Corollary 43. *(Corollary 42) Let M be a connected, closed $\Bbbk$-oriented manifold. Suppose that the free loop fibration $\Omega M \overset{i}{\hookrightarrow} LM \overset{ev}{\twoheadrightarrow} M$ is Totally Non-Cohomologous to Zero, i. e. $H^*(i)$ is onto and that $H^k(\Omega M)$ is a torsion free $\Bbbk$-module for each $k \geq 1$. Then $\chi(M) = 0$ in $\Bbbk$ or M is a point.*

1) The first examples to have in mind are connected compact Lie groups.

2) Let M be a sphere S^d or the complex or quaternionic projective space $\mathbb{CP}^n$, $\mathbb{HP}^n$. Since $\chi(S^d) = 1 + (-1)^d$ and $\chi(\mathbb{CP}^n) = \chi(\mathbb{HP}^n) = n+1$, it follows from our calculations in[19] that over any commutative ring $\Bbbk$, $H^*(i;\Bbbk)$ is onto if and only if $\chi(M) = 0$ in $\Bbbk$ (when $\Bbbk$ is a field, this follows easily from the formality of M using the Jones isomorphism between Hochschild homology and free loop space cohomology).

3) The converse of Corollary 42 is not true: if $n + 1$ is not equal to 0 in $\Bbbk$, take for example $M = \mathbb{CP}^n \times S^3$.

Over $\mathbb{Q}$, Vigué-Poirrier has characterised which free loop fibrations are TNCZ.

Theorem 44. [30] *Let X be a simply-connected topological space such that for all n, $H^n(X;\mathbb{Q})$ is finite dimensional. Then*

$$H^*(i;\mathbb{Q}) : H^*(LX;\mathbb{Q}) \to H^*(\Omega X;\mathbb{Q})$$

is onto if and only if $H^*(X;\mathbb{Q})$ is a free graded commutative algebra.

In [15, Theorem 2], Kuribayashi studied TNCZ free loop fibrations for some homogeneous spaces over a prime field $\mathbb{F}_p$.

Let $V_k(\mathbb{R}^n)$ denotes the real Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$. Using the fibration $S^{n-k} \hookrightarrow V_k(\mathbb{R}^n) \twoheadrightarrow V_{k-1}(\mathbb{R}^n)$, we see that if $k \geq 2$, $\chi(V_k(\mathbb{R}^n)) = 0$. Similarly for the complex or quartenionic Stiefel manifold, $\chi(V_k(\mathbb{C}^n)) = \chi(V_k(\mathbb{H}^n)) = 0$. The Euler characteristic $\chi(G/H)$ of the quotient of a compact connected Lie group G by a connected closed subgroup H of same rank is the quotient $|W(G)|/|W(H)|$ of the cardinals of their Weyl groups [22, VII.Theorem 3.13]. Therefore $\chi(Sp(n)/U(n) = 2^n$ and for Grassmannians $\chi(G_k(\mathbb{C}^n)) = \chi(G_k(\mathbb{H}^n)) = \binom{n}{k}$. We can now rewrite the main theorem of [15] in term of Euler characteristics.

Theorem 45. *(*[15, Theorem 2]*)*

1) Let M be $Sp(n)/U(n)$ or $V_k(\mathbb{C}^n))$ or $V_k(\mathbb{H}^n)$. Then $H^*(i;\mathbb{F}_p) : H^*(LM;\mathbb{F}_p) \to H^*(\Omega M;\mathbb{F}_p)$ is onto if and only if $\chi(M) = 0$ modulo p.

2) Let M be $G_m(\mathbb{C}^{m+n})$ or $G_m(\mathbb{H}^{m+n})$ with m and $n \geq 2$ and p any prime. Then $H^*(i;\mathbb{F}_p) : H^*(LM;\mathbb{F}_p) \to H^*(\Omega M;\mathbb{F}_p)$ is not onto.

3) Let p an odd prime. Then $H^*(i;\mathbb{F}_p) : H^*(LV_m(\mathbb{R}^{m+n});\mathbb{F}_p) \to H^*(\Omega V_m(\mathbb{R}^{m+n});\mathbb{F}_p)$ is onto if and only if n is odd.

So over $\mathbb{F}_p$, it is not clear when the converse of Corollary 42 holds or not.

Let X be a topological space. Suppose that $\forall n \geq 0$, $H_n(\Omega X;\mathbb{Z})$ is a finitely generated free abelian group, that $H_*(LX;\mathbb{Z})$ has no p-torsion and that $H^*(i;\mathbb{F}_p) : H^*(LX;\mathbb{F}_p) \to H^*(\Omega X;\mathbb{F}_p)$ is onto. Then by the universal coefficient theorem for homology, $H^*(i;\mathbb{Q}) : H^*(LX;\mathbb{Q}) \to H^*(\Omega X;\mathbb{Q})$ is onto.

We now give a last result on TNCZ free loop fibration due to Iwase in the context of classifying space BG of finite loop spaces.

Theorem 46. [13, Theorem 2.2] *Let X be a pointed topological space. If $H_*(i;\Bbbk) : H_*(\Omega X;\Bbbk) \to H_*(LX;\Bbbk)$ is injective then the Pontryagin algebra $H_*(\Omega X;\Bbbk)$ is graded commutative (in particular $\pi_1(X)$ is abelian).*

Proof. We have the following two strictly commutative squares and the following triangle commutting up to a homotopy H.

$$\begin{array}{ccc} \Omega X \times \Omega X & \xrightarrow{j} & LX \times_X LX \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} \\ \Omega X & \xrightarrow[i]{} & LX \end{array} \qquad \begin{array}{ccccc} \Omega X \times \Omega X & \xrightarrow{j} & LX \times_X LX & & \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} & \searrow{\scriptstyle\mu} & \\ \Omega X \times \Omega X & \xrightarrow[j]{} & LX \times_X LX & \xrightarrow[\mu]{} & LX \end{array}$$

where the maps μ are the composition of loops and the maps τ are the exchange isomorphisms. The homotopy H is the restriction to $[0,1/2] \times LX \times_X LX$ of the composite of

$$Id \times \mu : S^1 \times LX \times_X LX \to S^1 \times LX$$

and of the action of the circle on free loops $S^1 \times LX \to LX$. So finally, $i \circ \mu$ is homotopic to $i \circ \mu \circ \tau$. Since $H_*(i)$ is injective, $H_*(\mu) \circ H_*(\tau) = H_*(\mu)$. □

Note that our homotopy between $i \circ \mu$ and $i \circ \mu \circ \tau$ is much simpler than the one arriving in $EG \times_G G^{ad}$ given by Iwase in [13, Proof of Lemma 3.1].

Example 47. (Suspension) Suppose that $\Bbbk$ is a field. Let X be a path-connected space such that $H_*(X)$ is not concentrated in degree 0. By Bott-Samelson theorem, the Pontryagin algebra $H_*(\Omega\Sigma X)$ is isomorphic to the tensor algebra $TH_+(X)$ on the homology of X in positive degrees. Suppose that $H^*(i;\Bbbk) : H^*(L\Sigma X;\Bbbk) \to H^*(\Omega\Sigma X;\Bbbk)$ is surjective. Then $H_*(i;\Bbbk) : H_*(\Omega\Sigma X;\Bbbk) \to H_*(L\Sigma X;\Bbbk)$ is injective. So by Theorem 46, the Pontryagin ring $H_*(\Omega\Sigma X)$ is graded commutative. So $H_+(X)$ is of dimension 1 and is concentrated in even degree if the characteristic of $\Bbbk$ is different from 2 (See [16, Example 2.6] for a proof using Hochschild homology). In particular $\chi(\Sigma X) = 0$ modulo the characteristic of $\Bbbk$.

Conjecture 48. *Let X be a simply-connected finite CW-complex and suppose that $\Bbbk$ is a field. If $H^*(i;\Bbbk) : H^*(LX;\Bbbk) \to H^*(\Omega X;\Bbbk)$ is onto then $\chi(X)$ is zero modulo the characteristic of $\Bbbk$ or $H_*(X) \cong \Bbbk$.*

It follows from the theorem of Vigue-Poirrier recalled above (Theorem 44) that the conjecture is true over the rationals. In Example 47, we have checked the conjecture for suspensions. In this paper, we proved the conjecture when X is a smooth connected, closed $\Bbbk$-oriented manifold M (Corollary 42). We believe that conjecture 48 can be proved easily using Spanier-Whitehead duality.

13. The relative free loops case

Let $g : N \to M$ be a map. Let g^*LM denote the *relative free loops space* of g which is obtained by the following pull-back

$$\begin{array}{ccc} g^*LM & \longrightarrow & LM \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle ev} \\ N & \xrightarrow[g]{} & M. \end{array}$$

Theorem 49. *Let $\sigma : N \hookrightarrow g^*LM$, $n \mapsto (n, \text{constant loop } g(n))$ be the section of the projection map $p : g^*LM \to N$, $(n,w) \mapsto n$. Let M be a smooth h^*-oriented manifold of dimension m. Let $e_{TM} \in h^m(M)$*

be the Euler class of the tangent bundle of M. *Suppose that* g *is the composite*

$$N \overset{g_1}{\twoheadrightarrow} L \overset{g_2}{\rightarrow} M$$

where $\begin{cases} a) \ N \text{ is a smooth manifold without boundary and } g_2 \text{ is smooth and} \\ b) \ g_1 \text{ is a (Serre) fibration.} \end{cases}$

Then for any $b \in h^*(g^*LM)$,

$$b \cup p^* \circ g^*(e_{TM}) = p^* \circ \sigma^*(b) \cup p^* \circ g^*(e_{TM}).$$

In particular if there is an integer dim N *such that* $\forall i >$ *dim* N, $h^i(N) = \{0\}$ *then for all* $b \in h^*(g^*LM)$ *of degree* $|b| >$ *dim* $G - m$, $b \cup p^* \circ g^*(e_{TM}) = 0$.

Example 50. An interesting example is when the fibration $g = g_1 :$ map$(S, M) \twoheadrightarrow M$ is the evaluation at the base point of a well-pointed space S. In this case, the pull-back g^*LM of LM along g is the space map$(S \vee S^1, M)$ of maps from the wedge of S and the circle to M.

Example 51. A generalization of the preceding example is when $g_2 :$ $L \to M$ is any smooth map and the fibration $g_1 :$ map$(S, L) \twoheadrightarrow L$ is the evaluation at the base point of a well-pointed space S. In this case, the pull-back g^*LM of LM along g is the space map$((S \vee S^1, S), (M, L))$ of couples of maps (φ, ψ) such that the square

$$\begin{array}{ccc} S & \xrightarrow{\psi} & L \\ \downarrow & & \downarrow{\scriptstyle g_2} \\ S \vee S^1 & \xrightarrow{\varphi} & M. \end{array}$$

commutes.

Corollary 52. *Let* M *be a smooth* h^**-oriented manifold of dimension* m. *Let* $e_{TM} \in h^m(M)$ *be the Euler class of the tangent bundle of* M. *Denote by* $\vee_n S^1$ *the wedge of* $n \geq 0$ *circles. Let* $\sigma_n : M \hookrightarrow$ *map*$(\vee_n S^1, M)$, $m \mapsto$ *constant map* m *be the section of the evaluation map* $ev_n :$ *map*$(\vee_n S^1, M) \twoheadrightarrow M$. *Then for any* $b \in h^*($*map*$(\vee_n S^1, M))$,

$$b \cup ev_n^*(e_{TM}) = ev_n^* \circ \sigma_n^*(b) \cup ev_n^*(e_{TM}).$$

Proof. When $n = 0$, the formula is true since ev_0 and σ_0 are just the identity map of M. By induction on n, suppose that for any $a \in h^*(\text{map}(\vee_{n-1} S^1, M))$,

$$a \cup ev_{n-1}^*(e_{TM}) = ev_{n-1}^* \circ \sigma_{n-1}^*(a) \cup ev_{n-1}^*(e_{TM}).$$

By example 50 in the case $S = \vee_{n-1} S^1$ and $g = ev_{n-1}$, for any $b \in h^*(\text{map}(S \vee S^1, M))$,

$$b \cup p^* \circ ev_{n-1}^*(e_{TM}) = p^* \left(\sigma^*(b) \cup ev_{n-1}^*(e_{TM})\right).$$

By taking $a = \sigma^*(b)$, the latter is equal to

$$p^*\left(ev_{n-1}^* \circ \sigma_{n-1}^* \circ \sigma^*(b) \cup ev_{n-1}^*(e_{TM})\right)$$

Since $p^* \circ ev_{n-1}^* = ev_n^*$ and $\sigma_{n-1}^* \circ \sigma^* = \sigma_n^*$, the conclusion follows. □

Remark 53. Again, let S be a well-pointed space. Suppose that the fibration $\text{map}_*(S \vee S^1, M) \overset{i}{\hookrightarrow} \text{map}(S \vee S^1, M) \overset{ev}{\twoheadrightarrow} M$ is Totally Non-Cohomologous to Zero, i. e. $H^*(i)$ is onto. Let $\pi : S \vee S^1 \twoheadrightarrow S^1$ be the canonical projection. Then we have the commutative triangle

$$\begin{array}{ccc} LM & \xrightarrow{\text{map}(\pi,M)} & \text{map}(S \vee S^1, M) \\ & \searrow^{ev} & \downarrow{\scriptstyle ev} \\ & & M \end{array}$$

Since the induced map between the fibers in cohomology, $H^*(\text{map}_*(\pi, M))$: $H^*(\text{map}_*(S \vee S^1, M)) \to H^*(\Omega M)$ is surjective, the free loop fibration $\Omega M \overset{i}{\hookrightarrow} LM \overset{ev}{\twoheadrightarrow} M$ is also Totally Non-Cohomologous to Zero, i. e. $H^*(i)$ is onto.

Conclusion: the preceding corollary together with Lemma 22 is not really interesting to see if the fibration $(\Omega M)^{\times n} \overset{i}{\hookrightarrow} \text{map}(\vee_n S^1, M) \overset{ev_n}{\twoheadrightarrow} M$ is Totally Non-Cohomologous to Zero or not.

Let $f : N \to M$ and $g : N \to M$ be two maps. Let $N \times_f M^I \times_g N$ denote the *homotopy coincidence point space* of f and g which is obtained by the following pull-back

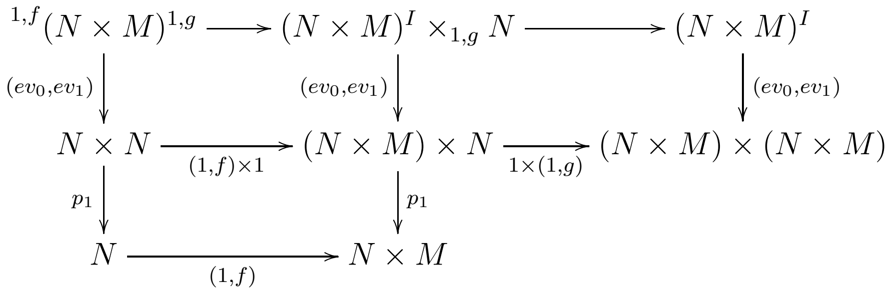
$$\begin{array}{ccc} N \times_f M^I \times_g N & \longrightarrow & M^I \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle (ev_0, ev_1)} \\ N & \xrightarrow[(f,g)]{} & M \times M. \end{array}$$

Lemma 54. *Let $\bar{\xi} : N \times_f M^I \times_g N \hookrightarrow {}^{1,f}(N \times M)^{1,g}$ be the map from the homotopy coincidence point space of f and g to the homotopy fibre product of $(1, f) : N \to N \times M$ and $(1, g) : N \to N \times M$ defined by*

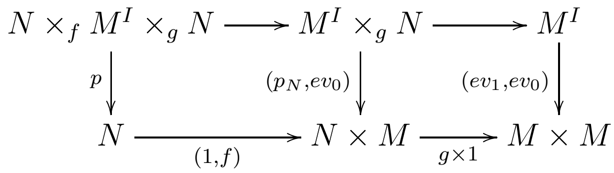
$$\bar{\xi}(n, \omega) = (n, \textit{the path } t \mapsto (n, \omega(t)), n)$$

for any $n \in N$ and any path $\omega : I \to M$ such that $\omega(0) = f(n)$ and $\omega(1) = g(n)$. Then $\bar{\xi}$ is a homotopy equivalence

Proof. Consider the three pull-backs squares



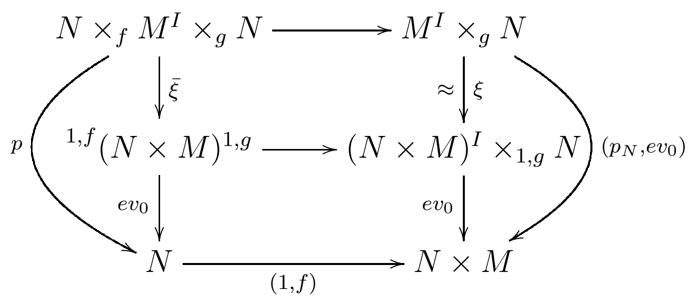
Here p_1 are the projections on the first factor. Consider also the two pull-back squares



Let $\xi : M^I \times_g N \hookrightarrow (N \times M)^I \times_{1,g} N$ be the map defined by

$$\xi(n, \omega) = (\text{the path } t \mapsto (n, \omega(t)), n)$$

for any $n \in N$ and any path $\omega : I \to M$ such that $\omega(1) = g(n)$. Obviously ξ is a homotopy equivalence. We obtain the following commutative diagram where the two squares are pull-backs according to the two previous diagrams.



By decomposing $(1, f)$ into the composite of a homotopy equivalence and of a fibration, we show using the structure of model category on topological spaces, that $\bar{\xi}$ is a homotopy equivalence since both ev_0 and $(p_N, ev_0) = ev_0 \circ \xi$ are fibrations. □

Proof of Theorem 49. Consider the two pull-back squares

$$\begin{array}{ccc} N & \xrightarrow{(1,g)} & N\times M \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_1\times 1} \\ L & \xrightarrow{(1,g_2)} & L\times M \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle g_2\times 1} \\ M & \xrightarrow[\Delta]{} & M\times M. \end{array}$$

Since g_2 and $1 : M \to M$ are transverse, $g_2 \times 1$ is transverse to the diagonal embedding Δ. And so by Remark 12, $(1, g_2) : L \to L\times M$ is a proper embedding of codimension m with h^*-oriented normal bundle.

Since g_1 is a (Serre) fibration, $g_1\times 1$ is also a (Serre) fibration. Therefore $(1, g) : N \to N \times M$ is the pull-back of an embedding in the sense of definition 11. So we can apply part 4) of Theorem 15. Since the homotopy equivalence $\bar{\xi} : N \times_f M^I \times_g N \overset{\approx}{\hookrightarrow} {}^{1,f}(N\times M)^{1,g}$ of Lemma 54 in the case $f = g$ commutes with the projection maps, i. e. $ev_0 \circ \bar{\xi} = p$ and also with the two sections σ, the ideal Ker $\sigma^* : h^*(g^*LM) \twoheadrightarrow h^*(N)$ satisfies

$$\text{Ker } \sigma^* \cup p^* \circ g^*(e_{TM}) = \{0\}.$$

□

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