Approximation of backward stochastic differential equations using Malliavin weights and least-squares regression

Emmanuel Gobet, Plamen Turkedjiev

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E. Gobet† and P. Turkedjiev‡
Centre de Mathématiques Appliquées
Ecole Polytechnique and CNRS
Route de Saclay
91128 Palaiseau cedex, France

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Abstract: We design a numerical scheme for solving a Dynamic Programming equation with Malliavin weights arising from the time-discretization of backward stochastic differential equations with the integration by parts-representation of the Z-component by [18]. When the sequence of conditional expectations is computed using empirical least-squares regressions, we establish, under general conditions, tight error bounds as the time-average of local regression errors only (up to logarithmic factors). We compute the algorithm complexity by a suitable optimization of the parameters, depending on the dimension and the smoothness of value functions, in the limit as the number of grid times goes to infinity. The estimates take into account the regularity of the terminal function.

Keywords: Backward stochastic differential equations, Malliavin calculus, dynamic programming equation, empirical regressions, non-asymptotic error estimates.

MSC 2010: 49L20, 60H07, 62Jxx, 65C30, 93E24.

1 Introduction

1.1 Setting

Let $T > 0$ be a fixed terminal time and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a filtered probability space whose filtration is augmented with the $\mathbb{P}$-null sets. Let $\pi = \{0 =: t_0 < t_1 < \ldots < t_{N-1} < t_N := T\}$ be a given time-grid on $[0, T]$ and $\Delta_i := t_{i+1} - t_i$. Additionally, for a fixed $q \in \mathbb{N}\backslash\{0\}$, we are given a set $\{H_j^{(i)} \ : \ 0 \leq i < j \leq N\}$ of $(\mathbb{R}^q)^\top$-valued random variables in $\mathbf{L}_2(\mathcal{F}_T, \mathbb{P})$ (i.e. square integrable and $\mathcal{F}_T$-measurable) that we call Malliavin weights. Here $^\top$ stands for the transpose.

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†The author's research is part of the Chair Financial Risks of the Risk Foundation and of the FiME Laboratory.
Email: emmanuel.gobet@polytechnique.edu
‡Corresponding Author. A significant part of the author's research has been done while at Humboldt University. Email: turkedjiev@cmap.polytechnique.fr

In this paper, we introduce the Malliavin Weights Least Squares algorithm, abreviated **MWLS**, to approximate discrete time stochastic processes (Y, Z) defined by

$$\begin{cases} Y_i = \mathbb{E}_i[\xi + \sum_{j=i}^{N-1} f_j(Y_{j+1}, Z_j)\Delta_j], \quad 0 \le i \le N, \\ Z_i = \mathbb{E}_i[\xi H_N^{(i)} + \sum_{j=i+1}^{N-1} f_j(Y_{j+1}, Z_j) H_j^{(i)}\Delta_j], \quad 0 \le i \le N-1, \end{cases} \tag{1.1}$$

where $\mathbb{E}_i[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_{t_i}]$, ξ is a $\mathbb{R}$-valued random variable in $\mathbf{L}_2(\mathcal{F}_T, \mathbb{P})$, and $(\omega, y, z) \mapsto f_j(\omega, y, z)$ is $\mathcal{F}_{t_j} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}((\mathbb{R}^q)^\top)$-measurable. This system is solved backward in time in the order $Y_N, Z_{N-1}, Y_{N-1} \dots$ and it takes the form of a dynamic programming equation with Malliavin weights. We call it the Malliavin Weights Dynamic Programming equation (MWDP for short).

The main application of (1.1) is to approximate continuous-time, decoupled Forward-Backward SDEs (FBSDEs) of the form

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s \tag{1.2}$$

where $(W_s)_{s \ge 0}$ is a Brownian motion in $\mathbb{R}^q$, $(X_s)_{s \ge 0}$ is a diffusion in $\mathbb{R}^d$ and ξ is of the form $\Phi(X_T)$. Indeed, the MWDP (1.1) was inspired by [18, Theorem 4.2], which states that there is a version of the continuous-time process $(Z_t)_{0 \le t < T}$ given by

$$Z_t = \mathbb{E}_t[\xi H_T^{(t)} + \int_t^T f(s, X_s, Y_s, Z_s) H_s^{(t)} ds] \tag{1.3}$$

where the processes $(H_s^{(t)})_{0 \le t < s \le T}$ are Malliavin weights defined by

$$H_s^{(t)} = \frac{1}{s-t} \Big(\int_t^s (\sigma^{-1}(r, X_r) D_t X_r)^\top dW_r \Big)^\top, \quad 0 \le t < s \le T, \tag{1.4}$$

for $(D_t X_r)_t$ being the Malliavin derivative of X_r and $\sigma(\cdot)$ is the diffusion coefficient of X. The representation (1.3) is obtained via a Malliavin calculus integration by parts formula, see [19] for a general account on the subject. A discretization procedure to approximate the FBSDE (1.2-1.3) with (1.1), including explicit definitions of the random variables $H_j^{(i)}$ based on (1.4), is given in [21][22], where the author also computes the discretization error in terms of N. In honour of the connection between (1.1) and (1.2-1.3), we call the random variables $H_j^{(i)}$ Malliavin weights, ξ the terminal condition, and $(i, \omega, y, z) \mapsto f_i(y, z)$ the driver. We say that the pair (Y, Z) satisfying (1.1) solves a MWDP with terminal condition ξ and driver $f_i(y, z)$.

1.2 Contributions

In this paper, we are not concerned with the discretization procedure, rather with the analysis of the MWDP equation (1.1) itself and its numerical resolution via the MWLS algorithm, in which one uses empirical least-squares regressions (approximations on finite basis of functions using simulations) to compute conditional expectations. Since the system (1.1) may be relevant to problems beyond the FBSDE system (1.2-1.3), we allow the framework and assumptions to accomodate as much generality as possible. However, MWLS is, to the best of our knowledge, the first direct implementation of formula (1.3) in a fully implementable numerical scheme. For other applications of Malliavin calculus in numerical simulations, with rather different perspectives and results to ours, see for instance [8][17][3][12][1][16][4].

We adapt the recent theoretical analysis of the Least Squares Multi-step Forward Dynamical Programming algorithm (LSMDP) of [13] for discrete BSDEs (without Malliavin weights) to the setting of MWDP. As in the aforementioned reference, we consider locally Lipschitz driver $f_i(y, z)$ that is locally bounded at $(y, z) = (0, 0)$ - see Section 1.4. This allows the algorithm of the current paper to be applied for the approximation of some quadratic BSDEs and for some proxy/variance reduction methods. See [13, Section 2.2] for more details on these applications. Moreover, we make use of analogous stability results and conditioning arguments in the proof of the main result, Theorem 3.10, as in the proof of [13, Theorem 4.11]. However, the Malliavin weights lead to significantly differences in the main theorem and stability results, both in the technical elements of the proofs and the results. We develop seemingly novel Gronwall type inequalities to handle the technical differences; these results are outlined in Section 2.1 and proved in Appendix A.1. Furthermore, the stability results are rather more powerful and the complexity of the MWLS is rather better than the LSMDP, as will be discussed in what follows.

We would like to mention that the class of quadratic problems we can treat with these assumptions is quite different to the recent [5]. Here we are treating the non-degenerate setting where the terminal condition may be Hölder continuous, whereas the other reference allows degeneracy at the expense of requiring locally Lipschitz terminal conditions.

We prove stability results on the MWDP in Section 2. Much effort is made to keep the constants explicit. These results are instrumental throughout the paper. The stability estimates on Z are at the individual time points (coherently with the representation theorem of [18]) rather than the time-averaged estimates of [13, Proposition 3.2]. This allows for finer and more precise computations. The time-dependency in our estimates also takes better into account the regularity of the terminal condition, similarly to the continuous-time case [7].

Section 3 is the core of the paper: it is dedicated to the MWLS algorithm in the Markovian context $Y_i = y_i(X_i)$ and $Z_i = z_i(X_i)$ for some Markov chain X_i in $\mathbb{R}^d$ and unknown functions $(y_i(\cdot), z_i(\cdot))$. In MWLS, the conditional expectations in (1.1) are replaced by Monte-Carlo least-squares regressions. For each point of the time-grid, we use a cloud of independent paths of the Markov chain X and the Malliavin weights H, and some approximation spaces for representing the value functions $(y_i(\cdot), z_i(\cdot))$. The algorithm is detailed in Section 3.2, and a full error analysis in terms of the number of simulations and the approximation spaces is performed in Sections 3.3 and 3.4. The final error estimates (Theorem 3.10) are similar to [13, Theorem 4.11] in that they are the time-averaged local regression errors of the discrete BSDE, but the results are in a stronger norm and the time-dependency is better. The constants are completely explicit. Although the norms are stronger than in to [13], the estimates do not deteriorate; rather, they are significantly improved. This is intrinsically due to the MWDP representation, which avoids the usual $1/\Delta_i$-factor in front of all controls on Z. This improvement can be tracked by inspecting the a.s. bounds (compare (2.10) and [13, Eq. (14)]) and the statistical error bounds (compare the $\frac{K_{Z,k}}{M_k}$-terms in (3.10) of Theorem 3.10 and the $\frac{K_{Z,k}}{\Delta_k M_k}$-terms of [13, Theorem 4.11]). These error estimates are optimal with respect to the convergence rates (up to logarithmic factors) under rather great generality regarding the distribution of the stochastic model for X and H, even if the constants may be rather conservative. This is because the local regression errors are optimal under model-free estimates (Proposition 3.9).

With the error estimates of Theorem 3.10 in hand, we perform a complexity analysis in Section 3.5. We propose a choice of basis functions and use it to calibrate the number of simulations in order to achieve a specified error level. This then allows us to compute the complexity of the algorithm for that error level. The methodology for doing this is analogous to [13, Section 4.4] in that we use the same basis functions - which also enable us to study the benefit of smoothness properties of the underlying Markov functions $(y_i(\cdot), z_i(\cdot))$ - and also in that we set the ensure the global error level by calibrating the local regression errors. However, the conclusion of this section is that MWLS yields better performance in terms of complexity than LSMDP. The main reason for this is the improved time-dependancy of the error estimates, which is a systemic improvement that allows one to make generate fewer simulations to obtain certain error levels. Unfortunately, the complexity reduction does not reduce the dependence on the dimension compared to the LSMDP. The curse of dimensionality still appears, and the rates depend on the dimension of the Markov chain X (i.e. d). Nevertheless, the reduction of complexity is substantial and, since one is able to make fewer simulations to obtain the same error level, will help alleviate the pressure on memory resources that is typical with least-squares Monte Carlo algorithms.

This paper is theoretically oriented, and is aimed at paving the way for new such numerical approaches based on Malliavin calculus. Future works will be devoted to a deeper investigation about the numerical performance of the MWLS algorithm compared to other known numerical schemes.

1.3 Notation used throughout the paper

- $|x|$ stands for the Euclidean norm of the vector x, $^\top$ denotes the transpose operator.
- $|U|_p := (\mathbb{E}[|U|^p])^{\frac{1}{p}}$ stands for the $\mathbf{L}_p(\mathbb{P})$-norm ($p \geq 1$) of a random variable U. The $\mathcal{F}_{t_k}$-conditional version is denoted by $|U|_{p,k} := (\mathbb{E}_k[|U|^p])^{\frac{1}{p}}$. To indicate that U is additionally measurable w.r.t. the σ-algebra $\mathcal{Q}$, we may write $U \in \mathbf{L}_p(\mathcal{Q}, \mathbb{P})$.
- For a multidimensional process $U = (U_i)_{0\leq i\leq N}$, its l-th component is denoted by $U_l = (U_{l,i})_{0\leq i\leq N}$.
- For any finite $L > 0$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define the truncation function $\mathcal{T}_L(x) := (-L \vee x_1 \wedge L, \ldots, -L \vee x_n \wedge L)$.
- For finite $x > 0$, $\log(x)$ is the natural logarithm of x.

1.4 Assumptions

First set of hypotheses. The following assumptions hold throughout the entirety of the paper. Let $R_\pi > 0$ be a fixed parameter: this constant determines which time-grid can be used. The larger R_π, the larger the class of admissible time-grids. All subsequent error estimates depend on R_π.

$(\mathbf{A}_\xi)$ ξ is in $\mathbf{L}_2(\mathcal{F}_T, \mathbb{P})$,

$(\mathbf{A_F})$ i) $(\omega, y, z) \mapsto f_i(y, z)$ is $\mathcal{F}_{t_i} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}((\mathbb{R}^q)^\top)$-measurable for every $i < N$, and there exist deterministic parameters $\theta_L \in (0, 1]$ and $L_f \in [0, +\infty)$ such that

$$|f_i(y, z) - f_i(y', z')| \leq \frac{L_f}{(T - t_i)^{(1-\theta_L)/2}} (|y - y'| + |z - z'|),$$

for any $(y, y', z, z') \in \mathbb{R} \times \mathbb{R} \times (\mathbb{R}^q)^\top \times (\mathbb{R}^q)^\top$.

ii) There exist deterministic parameters $\theta_c \in (0, 1]$ and $C_f \in [0, +\infty)$ such that

$$|f_i(0, 0)| \leq \frac{C_f}{(T - t_i)^{1-\theta_c}}, \quad \forall 0 \leq i < N.$$

iii) The time-grid $\pi := \{0 = t_0 < \ldots < t_N = T\}$ satisfies

$$\max_{0\leq i\leq N-2} \frac{\Delta_{i+1}}{\Delta_i} \leq R_\pi.$$

$(\mathbf{A_H})$ For all $0 \leq i < j \leq N$, the Malliavin weights satisfy

$$\mathbb{E}[H_j^{(i)}|\mathcal{F}_{t_i}] = 0, \quad \left[\mathbb{E}[|H_j^{(i)}|^2|\mathcal{F}_{t_i}]\right]^{1/2} \leq \frac{C_M}{(t_j - t_i)^{1/2}}$$

for a finite constant $C_M \geq 0$.

Comments. We remark that assumptions $(\mathbf{A}_\xi)$ and $(\mathbf{A_F}$-**i-ii**$)$ are the same as their equivalents in [13, Section 2]. The usual case of "Lipschitz" BSDE is covered by $\theta_L = \theta_c = 1$. As explained in [13], the case of locally Lipschitz driver ($\theta_L < 1$ and/or $\theta_c < 1$) is interesting because it allows a large variety of applications, such as solving BSDEs using proxy methods or variance reduction methods, and solving quadratic BSDEs. We refer the reader to [13, Section 2.2] for details.

Assumption $(\mathbf{A_F}$-**iii**$)$ is much simpler compared to [13]. If $R_\pi \geq 1$, $(\mathbf{A_F}$-**iii**$)$ is satisfied by any time grid with non-increasing time-step, such as the grids of [11, 20, 9]. This may be valuable for future work on time-grid optimization.

Assumption $(\mathbf{A_H})$ is specific to the dynamic programming equation with Malliavin weights. It is satisfied for the weights derived in [18], and this can remain true after discretization (see [21] or [12]). It is also satisfied if X takes the form $X_t = g(t, W_t)$ (like multi-dimensional geometric Brownian motion), by a simple change of variables one can use the Malliavin weights $H_s^{(t)} = \frac{(W_s - W_t)^\top}{s-t}$ (note the process X may be degenerate).

Second set of hypotheses: Markovian assumptions. The following assumptions will mostly be used in Section 3. $(\mathbf{A_X})$, $(\mathbf{A'_F})$, and $(\mathbf{A'_H})$ give us a Markov representation for solutions of the discrete BSDEs (see Equation (3.1) later). $(\mathbf{A}'_\xi)$ is used for obtaining (model free) estimates on regression errors. We also include additional optional assumptions, $(\mathbf{A}''_\xi)$, on the terminal condition to obtain tighter estimates on Z_i (see Corollary 2.6 and subsequent remarks).

$(\mathbf{A_X})$ X is a Markov chain in $\mathbb{R}^d$ ($1 \leq d < +\infty$) adapted to $(\mathcal{F}_{t_i})_i$. For every $i < N$ and $j > i$, there exist $\mathcal{G}_i \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions $V_j^{(i)} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ where $\mathcal{G}_i \subset \mathcal{F}_T$ is independent of $\mathcal{F}_{t_i}$, such that $X_j = V_j^{(i)}(X_i)$.

$(\mathbf{A}'_\xi)$ i) ξ is a bounded $\mathcal{F}_T$-measurable random variable: $C_\xi := \mathbb{P} - \operatorname{ess\,sup}_\omega |\xi(\omega)| < +\infty$.

ii) ξ is of form $\xi := \Phi(X_N)$ for a bounded, measurable function Φ.

$(\mathbf{A}''_\xi)$ In addition to $(\mathbf{A}'_\xi)$, for some $\theta_\Phi \in [0,1]$ and a finite constant $C_\Phi \geq 0$, we have $|\xi - \mathbb{E}_i\xi|_{2,i} \leq C_\Phi(T - t_i)^{\theta_\Phi/2}$ for any $i \in \{0, \ldots, N\}$.

$(\mathbf{A'_F})$ For every $i < N$, the driver is of the form $f_i(\omega, y, z) = f_i(X_i(\omega), y, z)$, and $(x, y, z) \mapsto f_i(x, y, z)$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}((\mathbb{R}^q)^\top)$-measurable and $(\mathbf{A_F})$ is satisfied.

$(\mathbf{A'_H})$ In addition to $(\mathbf{A_H})$, for every $i < N$ and $j > i$, there is a function $\hbar_j^{(i)} : \Omega \times \mathbb{R}^d \to (\mathbb{R}^q)^\top$ that is $\mathcal{G}_i \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, where $\mathcal{G}_i \subset \mathcal{F}_T$ is independent of $\mathcal{F}_{t_i}$, such that $H_j^{(i)} := \hbar_j^{(i)}(X_i)$.

Comments. $(\mathbf{A_X})$ is usually satisfied when X is the solution of SDE or its Euler scheme built on the time grid π.

The statement of $(\mathbf{A}''_\xi)$ is inspired by the fractional smoothness condition of [11], although somewhat stronger. It is satisfied, for instance, if the terminal function Φ is θ_Φ-Hölder continuous and if the Markov chain satisfies $\mathbb{E}_i[|X_N - X_i|^2] \leq C_X(T - t_i)$. This is a reasonable assumption on the Markov chain, since it is satisfied by a diffusion process (possibly including bounded jumps) with bounded coefficients and also by its Euler approximation. Indeed, we have

$$|\xi - \mathbb{E}_i\xi|_{2,i} \le |\Phi(X_N) - \Phi(X_i)|_{2,i} \le C_\Phi(C_X(T-t_i))^{\frac{\theta_\Phi}{2}}.$$

($\mathbf{A'_H}$) is satisfied by the Malliavin weights (1.4) under various conditions. It is valid for the example $X_t = g(t, W_t)$ mentioned before. Consider now the more complex case of the SDE with coefficients $b(t,x)$ for the drift and $(\sigma_1(t,x), \dots, \sigma_d(t,x))$ for the diffusion ($q = d$) both having first space-derivatives that are uniformly bounded. Recall the relation for the Malliavin derivative of a SDE given by

$$D_t X_r = \nabla X_r \nabla X_t^{-1} \sigma(t, X_t) \mathbf{1}_{t \le r} = \nabla_x X_r^{t,x}|_{x=X_t} \sigma(t, X_t) \mathbf{1}_{t \le r}$$

where $X^{t,x}$ denotes the SDE solution starting from x at time t, and $\nabla X_t := \nabla_x X_t^{0,x}$ for $\nabla_x X_t^{0,x}$ solving the $(d \times d)$-dimensional, matrix valued linear SDE

$$\nabla_x X_r^{t,x} = I_d + \int_t^r \nabla_x b(u, X_u^{(t,x)}) \nabla_x X_u^{t,x} du + \sum_{j=1}^d \int_t^r \nabla_x \sigma_j(u, X_u^{(t,x)}) \nabla_x X_u^{t,x} dW_{j,u}.$$

Then, it is an easy exercice to prove that if σ and its inverse are uniformly bounded, then ($\mathbf{A'_H}$) is fulfilled.

2 Stability

2.1 Gronwall type inequalities

Here we gather deterministic inequalities frequently used throughout the paper. These inequalities are crucial due to novel technical problems caused by the Malliavin weights. They show how linear inequalities with singular coefficients propagate. They take the form of unusual Gronwall type inequalities. Their proofs are postponed to Appendix A.1. We assume that π is in the class of time-grids satisfying ($\mathbf{A_F}$-**iii**).

Lemma 2.1. *Let $\alpha, \beta > 0$ be finite. There exists a finite constant $B_{\alpha,\beta} \ge 0$ depending only on R_π, α and β (but not on the time-grid) such that, for any $0 \le i < k \le N$,*

$$\sum_{j=i}^{k-1} \frac{\Delta_j}{(t_k - t_j)^{1-\alpha}} \le B_{\alpha,1}(t_k - t_i)^\alpha, \qquad \sum_{j=i+1}^{k-1} \frac{\Delta_j}{(t_k - t_j)^{1-\alpha}(t_j - t_i)^{1-\beta}} \le B_{\alpha,\beta}(t_k - t_i)^{\alpha+\beta-1}.$$

Lemma 2.2 (exponent improvement in recursive equations)**.** *Let $\alpha \ge 0, \beta \in (0, \frac{1}{2}]$ and $k \in \{0, \dots, N-1\}$. Suppose that, for a finite constant $C_u \ge 0$, the finite non-negative real-valued sequences $\{u_l\}_{l \ge k}$ and $\{w_l\}_{l \ge k}$ satisfy*

$$u_j \le w_j + C_u \sum_{l=j+1}^{N-1} \frac{u_l \Delta_l}{(T - t_l)^{\frac{1}{2}-\beta}(t_l - t_j)^{\frac{1}{2}-\alpha}}, \quad k \le j \le N. \tag{2.1}$$

Then, for two finite constants $\mathcal{C}_{(2.2a)} \ge 0$ and $\mathcal{C}_{(2.2b)} \ge 0$ that depend only on C_u, T, α, β and R_π,

$$u_j \le \mathcal{C}_{(2.2a)} w_j + \mathcal{C}_{(2.2a)} \sum_{l=j+1}^{N-1} \frac{w_l \Delta_l}{(T - t_l)^{\frac{1}{2}-\beta}(t_l - t_j)^{\frac{1}{2}-\alpha}} + \mathcal{C}_{(2.2b)} \sum_{l=j+1}^{N-1} \frac{u_l \Delta_l}{(T - t_l)^{\frac{1}{2}-\beta}}, \quad k \le j \le N. \tag{2.2}$$

Lemma 2.3 (intermediate a priori estimates)**.** *Let $\alpha \geq 0, \beta \in (0, \frac{1}{2}]$ and $k \in \{0, \dots, N-1\}$. Assume that the finite non-negative real-valued sequences $\{u_l\}_{l\geq k}$ and $\{w_l\}_{l\geq k}$ satisfy* (2.2) *for finite constants $\mathcal{C}_{(2.2a)} \geq 0$ and $\mathcal{C}_{(2.2b)} \geq 0$. Then, for any finite $\gamma > 0$, there is a finite constant $\mathcal{C}^{(\gamma)}_{(2.3)} \geq 0$ (depending only on $\mathcal{C}_{(2.2a)}$, $\mathcal{C}_{(2.2b)}$, T, α, β, R_π and γ) such that*

$$\sum_{l=j+1}^{N-1} \frac{u_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}(t_l-t_j)^{1-\gamma}} \leq \mathcal{C}^{(\gamma)}_{(2.3)} \sum_{l=j+1}^{N-1} \frac{w_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}(t_l-t_j)^{1-\gamma}}, \quad k \leq j \leq N. \tag{2.3}$$

Plugging (2.3) with $\gamma = \frac{1}{2} + \alpha$ into (2.1) gives a ready-to-use result.

Proposition 2.4 (final a priori estimates)**.** *Under the assumptions of Lemma 2.2,* (2.1) *implies*

$$u_j \leq w_j + \mathcal{C}^{(\frac{1}{2}+\alpha)}_{(2.3)} C_u \sum_{l=j+1}^{N-1} \frac{w_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}(t_l-t_j)^{\frac{1}{2}-\alpha}}, \quad k \leq j \leq N.$$

2.2 Stability of discrete BSDEs with Malliavin weights

Suppose that (Y_1, Z_1) (resp. (Y_2, Z_2)) solves a MWDP with terminal condition/driver $(\xi_1, f_{1,i})$ (resp. $(\xi_2, f_{2,i})$). We are interested in obtaining estimates on the differences $(Y_1 - Y_2, Z_1 - Z_2)$. To give a notion of how stability estimates are used, the processes (Y_1, Z_1) are typically obtained by construction. For example, in Section 2.3, they are $(0, 0)$, whereas in the proof of Theorem 3.10, they are a set of processes determined from a series of arguments based on conditioning w.r.t. the Monte Carlo samples. One then applies the stability estimates based on a priori knowledge that what stands on the right hand side is beneficial to the computations. In Corollary 2.6, for example, the right hand side yields almost sure bounds for the processes (Y, Z). We note that the assumptions on the drivers in this section are rather weaker than the general assumptions of Section 1.4. The driver $f_{1,i}(y, z)$ does not have to be Lipschitz continuous, but we assume that each $f_{1,i}(Y_{1,i+1}, Z_{1,i})$ is in $\mathbf{L}_2(\mathcal{F}_T)$ so that $Y_{1,i}$ and $Z_{1,i}$ are also square integrable for any i (thanks to $(\mathbf{A_H})$). The driver $f_{2,i}(y, z)$ is locally Lipschitz continuous w.r.t. (y, z) as in ($\mathbf{A_F}$-**i**), which is crucial for the validity of the a priori estimates. Additionally, we do not insist that the drivers be adapted, which will be needed in the setting of sample-dependant drivers. Define

$$\Delta Y := Y_1 - Y_2, \quad \Delta Z := Z_1 - Z_2, \quad \Delta\xi := \xi_1 - \xi_2,$$
$$\Delta f_i := f_{1,i}(Y_{1,i+1}, Z_{1,i}) - f_{2,i}(Y_{1,i+1}, Z_{1,i}).$$

Let $k \in \{0, \dots, N-1\}$ be fixed: throughout this subsection, $\mathcal{F}_{t_k}$-conditional $\mathbf{L}_2$-norms are considered and we recall the notation $|U|_{2,k} := \sqrt{\mathbb{E}_k[|U|^2]}$ for any square integrable random variable U. For $j \geq k$, define

$$|\Theta_j|_{2,k} := |\Delta Y_{j+1}|_{2,k} + |\Delta Z_j|_{2,k}.$$

Using $(\mathbf{A_H})$, we obtain $\mathbb{E}_i[\Delta\xi H_N^{(i)}] = \mathbb{E}_i[(\Delta\xi - \mathbb{E}_i\Delta\xi)H_N^{(i)}]$ and

$$|\mathbb{E}_i[\Delta\xi H_N^{(i)}]|^2 \leq \mathbb{E}_i[|\Delta\xi - \mathbb{E}_i\Delta\xi|^2]\frac{C_M^2}{(t_N - t_i)}, \qquad |\mathbb{E}_i[\Delta f_j H_j^{(i)}]|^2 \leq \frac{C_M^2 \mathbb{E}_i[|\Delta f_j|^2]}{t_j - t_i} \quad j \geq i+1. \tag{2.4}$$

Combining this kind of estimates with ($\mathbf{A_F}$-**i**) and the triangle inequality, our stability equations (for $k \leq i$) are

$$|\Delta Y_i|_{2,k} \leq |\Delta\xi|_{2,k} + \sum_{j=i}^{N-1} |\Delta f_j|_{2,k}\Delta_j + \sum_{j=i}^{N-1} \frac{L_{f_2}|\Theta_j|_{2,k}}{(T-t_j)^{(1-\theta_L)/2}}\Delta_j, \tag{2.5}$$

$$|\Delta Z_i|_{2,k} \le \frac{C_M|\Delta\xi - \mathbb{E}_i\Delta\xi|_{2,k}}{\sqrt{T-t_i}} + \sum_{j=i+1}^{N-1} \frac{C_M|\Delta f_j|_{2,k}}{\sqrt{t_j - t_i}}\Delta_j + \sum_{j=i+1}^{N-1} \frac{L_{f_2} C_M |\Theta_j|_{2,k}}{(T-t_j)^{(1-\theta_L)/2}\sqrt{t_j-t_i}}\Delta_j. \quad (2.6)$$

Proposition 2.5. *Taking $\alpha = 0$, $\beta = \theta_L/2$ and $C_u = L_{f_2}(C_M + \sqrt{T})$ in Lemmas 2.2 and 2.3, recall the constant $\mathcal{C}^{(\gamma)}_{(2.3)}$. Assume that ξ_j is in $\mathbf{L}_2(\mathcal{F}_T)$. Moreover, for each $i \in \{0, \dots, N-1\}$, assume that $f_{1,i}(Y_{1,i+1}, Z_{1,i})$ is in $\mathbf{L}_2(\mathcal{F}_T)$ and $f_{2,i}(y,z)$ is locally Lipschitz continuous w.r.t. y and z as in ($\mathbf{A_F}$-$\mathbf{i}$), with a constant L_{f_2}. Then, under ($\mathbf{A_H}$), we have*

$$|\Delta Y_i|_{2,k} \le C_y^{(1)}|\Delta\xi|_{2,k} + C_y^{(2)}\sum_{j=i}^{N-1}|\Delta f_j|_{2,k}\Delta_j, \quad 0 \le k \le i \le N,$$

$$|\Delta Z_i|_{2,k} \le C_z^{(1)}\frac{|\Delta\xi - \mathbb{E}_i\Delta\xi|_{2,k}}{\sqrt{T-t_i}} + C_z^{(2)}\sum_{j=i+1}^{N-1}\frac{|\Delta f_j|_{2,k}}{\sqrt{t_j-t_i}}\Delta_j + C_z^{(3)}|\Delta\xi|_{2,k}(T-t_i)^{\frac{\theta_L}{2}}, \quad 0 \le k \le i < N,$$

where the above constants can be written explicitly:

$$\begin{aligned}
C_y^{(1)} &:= 1 + L_{f_2}\mathcal{C}^{(1)}_{(2.3)}(C_M B_{\frac{\theta_L}{2},1} + B_{\frac12+\frac{\theta_L}{2},1}\sqrt{T})T^{\frac{\theta_L}{2}},\\
C_y^{(2)} &:= 1 + L_{f_2}\mathcal{C}^{(1)}_{(2.3)}(C_M + \sqrt{T})B_{\frac{\theta_L}{2},1}T^{\frac{\theta_L}{2}},\\
C_z^{(1)} &:= C_M(1 + L_{f_2}\mathcal{C}^{(\frac12)}_{(2.3)}C_M B_{\frac{\theta_L}{2},\frac12}T^{\frac{\theta_L}{2}}),\\
C_z^{(2)} &:= C_M(1 + L_{f_2}\mathcal{C}^{(\frac12)}_{(2.3)}(C_M + \sqrt{T})B_{\frac{\theta_L}{2},\frac12}T^{\frac{\theta_L}{2}}),\\
C_z^{(3)} &:= C_M L_{f_2}\mathcal{C}^{(\frac12)}_{(2.3)}B_{\frac12+\frac{\theta_L}{2},\frac12}.
\end{aligned}$$

Proof. Using (2.5) and (2.6), we obtain

$$\begin{aligned}
|\Theta_j|_{2,k} \le{}& C_M\frac{|\Delta\xi - \mathbb{E}_j\Delta\xi|_{2,k}}{\sqrt{T-t_j}} + |\Delta\xi|_{2,k} + (C_M+\sqrt{T})\sum_{l=j+1}^{N-1}\frac{|\Delta f_l|_{2,k}\Delta_l}{\sqrt{t_l-t_j}}\\
&+ (C_M+\sqrt{T})\sum_{l=j+1}^{N-1}\frac{L_{f_2}|\Theta_l|_{2,k}\Delta_l}{(T-t_l)^{(1-\theta_L)/2}\sqrt{t_l-t_j}}, \quad j \ge k. \qquad (2.7)
\end{aligned}$$

Upper bound for (2.7). We apply Lemmas 2.2 and 2.3 under the setting $u_j = |\Theta_j|_{2,k}$, $w_j = C_M\frac{|\Delta\xi - \mathbb{E}_j\Delta\xi|_{2,k}}{\sqrt{T-t_j}} + |\Delta\xi|_{2,k} + (C_M+\sqrt{T})\sum_{l=j+1}^{N-1}\frac{|\Delta f_l|_{2,k}\Delta_l}{\sqrt{t_l-t_j}}$, $\alpha = 0$, $\beta = \frac{\theta_L}{2}$, $C_u = L_{f_2}(C_M+\sqrt{T})$. To make results fully explicit, we first need to upper bound quantities of the form ($\gamma > 0$)

$$\mathcal{I}^{(\gamma)}_{j+1} := \sum_{l=j+1}^{N-1}\frac{w_l\Delta_l}{(T-t_l)^{\frac12-\frac{\theta_L}{2}}(t_l-t_j)^{1-\gamma}}.$$

Using that $|\Delta\xi - \mathbb{E}_l\Delta\xi|_{2,k}$ is non-increasing in l and Lemma 2.1, we obtain

$$\begin{aligned}
\mathcal{I}^{(\gamma)}_{j+1} &= \sum_{l=j+1}^{N-1}\frac{C_M\frac{|\Delta\xi-\mathbb{E}_l\Delta\xi|_{2,k}}{\sqrt{T-t_l}} + |\Delta\xi|_{2,k} + (C_M+\sqrt{T})\sum_{r=l+1}^{N-1}\frac{|\Delta f_r|_{2,k}\Delta_r}{\sqrt{t_r-t_l}}}{(T-t_l)^{\frac12-\frac{\theta_L}{2}}(t_l-t_j)^{1-\gamma}}\Delta_l\\
&\le C_M B_{\frac{\theta_L}{2},\gamma}\frac{|\Delta\xi - \mathbb{E}_{j+1}\Delta\xi|_{2,k}}{(T-t_j)^{1-\frac{\theta_L}{2}-\gamma}} + B_{\frac12+\frac{\theta_L}{2},\gamma}\frac{|\Delta\xi|_{2,k}}{(T-t_j)^{\frac12-\frac{\theta_L}{2}-\gamma}}
\end{aligned}$$

$$+ (C_M + \sqrt{T}) B_{\frac{\theta_L}{2},\gamma} \sum_{l=j+2}^{N-1} \frac{|\Delta f_l|_{2,k} \Delta_l}{(t_l - t_j)^{1-\frac{\theta_L}{2}-\gamma}}. \tag{2.8}$$

Upper bound for $|\Delta Y_i|_{2,k}$. Starting from (2.5) and applying Lemma 2.3, we get

$$|\Delta Y_i|_{2,k} \leq |\Delta \xi|_{2,k} + \sum_{j=i}^{N-1} |\Delta f_j|_{2,k} \Delta_j + L_{f_2} \mathcal{C}^{(1)}_{(2.3)} \mathcal{I}^{(1)}_i;$$

then using the estimate (2.8) and $|\Delta \xi - \mathbb{E}_i \Delta \xi|_{2,k} \leq |\Delta \xi|_{2,k}$, we obtain the announced inequality.
Upper bound of $|\Delta Z_i|_{2,k}$. Starting from (2.6) and applying Lemma 2.3, we have

$$|\Delta Z_i|_{2,k} \leq \frac{C_M |\Delta \xi - \mathbb{E}_i \Delta \xi|_{2,k}}{\sqrt{T - t_i}} + \sum_{j=i+1}^{N-1} \frac{C_M |\Delta f_j|_{2,k}}{\sqrt{t_j - t_i}} \Delta_j + L_{f_2} C_M \mathcal{C}^{(\frac{1}{2})}_{(2.3)} \mathcal{I}^{(\frac{1}{2})}_{i+1};$$

therefore using the estimate (2.8), we derive the advertised upper bound on $|\Delta Z_i|_{2,k}$. □

2.3 Almost sure bounds

In order to obtain error estimates for the Monte Carlo scheme, we use the model-free estimates of Proposition 3.9. Typically, these estimates require that the object one is trying to approximate is bounded. Therefore, the following almost sure bounds are crucial.

Corollary 2.6. *Assume* ($\mathbf{A}'_\xi$**-i**), ($\mathbf{A_F}$) *and* ($\mathbf{A_H}$) *and recall the constants* $C^{(\cdot)}_y$ *and* $C^{(\cdot)}_z$ *from Proposition 2.5 where* L_{f_2} *is replaced by* L_f. *Then, we have*

$$|Y_i| \leq C_{y,i} := C^{(1)}_y C_\xi + C^{(2)}_y C_f B_{\theta_c,1} (T - t_i)^{\theta_c}, \tag{2.9}$$

$$|Z_i| \leq C_{z,i} := C^{(1)}_z \frac{\operatorname{ess\,sup}_\omega |\xi - \mathbb{E}_i \xi|_{2,i}}{\sqrt{T - t_i}} + \frac{C^{(2)}_z C_f B_{\theta_c,\frac{1}{2}}}{(T - t_i)^{\frac{1}{2}-\theta_c}} + C^{(3)}_z C_\xi (T - t_i)^{\frac{\theta_L}{2}}. \tag{2.10}$$

The above upper bounds are able to handle the rather general terminal values ξ admitted by ($\mathbf{A}'_\xi$**-i**). Without any further information on ξ, we can derive the simple bounds

$$|Y_i| + \sqrt{T - t_i} |Z_i| \leq C_{y,z} \tag{2.11}$$

for an explicit, time uniform constant $C_{y,z}$. It may, however, be useful to take advantage of additional information on ξ, to obtain finer estimates on $C_{y,i}$ and $C_{z,i}$ with the aim of better tuning the parameters of the MWLS method (see Section 3.5). Two situations are of particular interest.

- For zero terminal condition, Y and Z get smaller and smaller as t_i goes to T as expected: $|Y_i| + \sqrt{T - t_i}|Z_i| \leq C(T - t_i)^{\theta_c}$ for a constant C depending only on $C^{(2)}_y$, $C^{(2)}_z$, C_f, θ_c and R_π. This result is useful for variance reduction methods like the proxy method of [13, Section 2.2], the method of Martingale basis [2], and the multilevel method of [21].

- Under ($\mathbf{A}''_\xi$), we have $|\xi - \mathbb{E}_i \xi|_{2,i} \leq C_\Phi (T - t_i)^{\theta_\Phi/2}$, which leads to an improved estimate for Z: $|Z_i| \leq C(T - t_i)^{-\frac{1}{2}+\theta_c \wedge \frac{\theta_\Phi}{2}}$ for some constant C depending only on $C^{(1)}_z$, $C^{(2)}_z$, $C^{(3)}_z$, C_f, θ_c, R_π, T, C_ξ and C_Φ.

This is why in the subsequent analysis, we keep track on the general dependence on i of the constants $C_{y,i}$ and $C_{z,i}$.

Proof of Corollary 2.6. $(0,0)$ is the solution of the MWDP with data $(\xi_1 \equiv 0, f_{1,i} \equiv 0)$. Applying Proposition 2.5 with $(Y_1, Z_1) = (0,0)$ and $(Y_2, Z_2) = (Y, Z)$ yields

$$|Y_i|_{2,k} \le C_y^{(1)} |\xi|_{2,k} + C_y^{(2)} \sum_{j=i}^{N-1} |f_j(0,0)|_{2,k} \Delta_j,$$

$$|Z_i|_{2,k} \le \frac{C_z^{(1)} |\xi - \mathbb{E}_i \xi|_{2,k}}{\sqrt{T - t_i}} + C_z^{(2)} \sum_{j=i+1}^{N-1} \frac{|f_j(0,0)|_{2,k}}{\sqrt{t_j - t_i}} \Delta_j + C_z^{(3)} |\xi|_{2,k} (T - t_i)^{\frac{\theta_L}{2}},$$

for $i = 0, \ldots, N-1$. Taking $k = i$, plugging in the almost sure bounds on $|\xi|$ from ($\mathbf{A}'_\xi$-**i**) and $|f_j(0,0)|$ from ($\mathbf{A_F}$-**ii**), and using Lemma 2.1 then yields the result. □

3 Monte-Carlo regression scheme

Throughout this section, the Markovian assumptions ($\mathbf{A_X}$), ($\mathbf{A}'_\xi$), ($\mathbf{A'_F}$) and ($\mathbf{A'_H}$) are in force. The notation and preliminary results used in this section overlap with [13, Section 4], and we recall and adapt them to the setting of MWLS in Section 3.1 for completeness.

3.1 Preliminaries

An elegant property of the Markovian assumptions is there are measurable, deterministic (but unknown) functions $y_i(\cdot) : \mathbb{R}^d \to \mathbb{R}$ and $z_i(\cdot) : \mathbb{R}^d \to (\mathbb{R}^q)^\top$ for each $i \in \{0, \ldots, N-1\}$ such that the solution $(Y_i, Z_i)_{0 \le i \le N-1}$ of the discrete BSDE (1.1) is given by

$$(Y_i, Z_i) := \big(y_i(X_i), z_i(X_i)\big). \tag{3.1}$$

In this section, we estimate these functions. One needs to apply Lemma 3.1 below combined with $\mathcal{G} = \mathcal{G}_i$ – defined in the assumptions ($\mathbf{A_X}$) and ($\mathbf{A'_H}$) – $U = X_i$, and

$$F(x) := \Phi(V_N^{(i)}(x)) + \sum_{k=i}^{N-1} f_k\big(V_k^{(i)}(x), y_{k+1}(V_{k+1}^{(i)}(x)), z_k(V_k^{(i)}(x))\big)\Delta_k \qquad \text{for } y_i(\cdot),$$

$$\text{and } F(x) := \Phi(V_N^{(i)}(x))\hbar_N^{(i)}(x) + \sum_{k=i+1}^{N-1} f_k\big(V_k^{(i)}(x), y_{k+1}(V_{k+1}^{(i)}(x)), z_k(V_k^{(i)}(x))\big)\hbar_k^{(i)}(x)\Delta_k \quad \text{for } z_i(\cdot).$$

Lemma 3.1 ([13, Lemma 4.1]). *Suppose that $\mathcal{G}$ and $\mathcal{H}$ are independent sub-σ-algebras of $\mathcal{F}$. For $l \ge 1$, let $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^l$ be bounded and $\mathcal{G} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, and $U : \Omega \to \mathbb{R}^d$ be $\mathcal{H}$-measurable. Then, $\mathbb{E}[F(U)|\mathcal{H}] = j(U)$ where $j(h) = \mathbb{E}[F(h)]$ for all $h \in \mathbb{R}^d$.*

Least-squares regression has its traditional implementation in nonparametric statistics and signal processing [15]. In the traditional setting, the random object is a pair of random variables (O, R) termed the "observation" O and the "response" R. R is considered to be some function of O, with the possible addition of noise, and one needs recover this function. There are three important differences in the use of least-squares regression methods in our setting, and for this reason we give a definition of (ordinary) least-squares regression (OLS) that enables us to approach our problems. Firstly, the response we consider is a nonlinear transformation of the *paths* of the Markov chain X

and the Malliavin Weights H. We are able to simulate observations and responses (active learning) and we know the nonlinear function; what is unknown is the regression function, i.e. the conditional expectation. Therefore, OLS is defined in a way that easily enables path-dependence and joint laws by defining the path of the Markov chain and Malliavin weights as a single random variable, $\mathcal{X}$, with law ν. Secondly, since we are in a dynamical setting, least-squares regressions will be computed using independent clouds of simulations on each point of the time-grid. This causes a dependence on an additional source of randomness in the observations, namely the cloud of simulations from the preceding computations. Therefore, OLS is defined to depend on *two* probability spaces: one for the preceding clouds $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and one for the current cloud distribution $(\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l), \nu)$. Finally, we will make use of both general probability measures (associated to the joint-law of the Markov chain and Malliavin weights) and empirical measures. The use of simulations to generate the empirical measure creates dependency issues that are avoided when using laws, whence we make two distinct definitions depending on which measure is in use. We recall the general notation of [13, Section 4.1] for ordinary least-squares regression problems:

Definition 3.2 (Ordinary least-squares regression)**.** *For $l, l' \geq 1$ and for probability spaces $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $(\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l), \nu)$, let S be a $\tilde{\mathcal{F}} \otimes \mathcal{B}(\mathbb{R}^l)$-measurable $\mathbb{R}^{l'}$-valued function such that $S(\omega, \cdot) \in \mathbf{L}_2(\mathcal{B}(\mathbb{R}^l), \nu)$ for $\tilde{\mathbb{P}}$-a.e. $\omega \in \tilde{\Omega}$, and $\mathcal{K}$ a linear vector subspace of $\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l), \nu)$ spanned by deterministic $\mathbb{R}^{l'}$-valued functions $\{p_k(.), k \geq 1\}$. The least-squares approximation of S in the space $\mathcal{K}$ with respect to ν is the ($\tilde{\mathbb{P}} \times \nu$-a.e.) unique, $\tilde{\mathcal{F}} \otimes \mathcal{B}(\mathbb{R}^l)$-measurable function $S^\star$ given by*

$$S^\star(\omega, \cdot) := \arg \inf_{\phi \in \mathcal{K}} \int |\phi(x) - S(\omega, x)|^2 \nu(dx). \tag{3.2}$$

We say that $S^\star$ solves $\mathbf{OLS}(S, \mathcal{K}, \nu)$.

On the other hand, suppose that $\nu_M = \frac{1}{M} \sum_{m=1}^M \delta_{\mathcal{X}^{(m)}}$ is a discrete probability measure on $(\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$, where δ_x is the Dirac measure on x and $\mathcal{X}^{(1)}, \ldots, \mathcal{X}^{(M)} : \tilde{\Omega} \to \mathbb{R}^l$ are i.i.d. random variables. For an $\tilde{\mathcal{F}} \otimes \mathcal{B}(\mathbb{R}^l)$-measurable $\mathbb{R}^{l'}$-valued function S such that $\left|S\big(\omega, \mathcal{X}^{(m)}(\omega)\big)\right| < \infty$ for any m and $\tilde{\mathbb{P}}$-a.e. $\omega \in \tilde{\Omega}$, the least-squares approximation of S in the space $\mathcal{K}$ with respect to ν_M is the ($\tilde{\mathbb{P}}$-a.e.) unique, $\tilde{\mathcal{F}} \otimes \mathcal{B}(\mathbb{R}^l)$–measurable function $S^\star$ given by

$$S^\star(\omega, \cdot) := \arg \inf_{\phi \in \mathcal{K}} \frac{1}{M} \sum_{m=1}^M |\phi\big(\mathcal{X}^{(m)}(\omega)\big) - S\big(\omega, \mathcal{X}^{(m)}(\omega)\big)|^2. \tag{3.3}$$

We say that $S^\star$ solves $\mathbf{OLS}(S, \mathcal{K}, \nu_M)$.

From (3.1), the MWDP (1.1) can be reformulated in terms of Definition 3.2: taking for $\mathcal{K}_i^{(l')}$ any dense subset in the $\mathbb{R}^{l'}$-valued functions belonging to $\mathbf{L}_2(\mathcal{B}(\mathbb{R}^d), \mathbb{P} \circ (X_i)^{-1})$, for each $i \in \{0, \ldots, N-1\}$,

$$\begin{cases} y_i(\cdot) \text{ solves } \mathbf{OLS}(\ S_{Y,i}(\underline{\mathbf{x}}^{(i)})\ ,\ \mathcal{K}_i^{(1)}\ ,\ \nu_i\ ), \\ \qquad \text{for } S_{Y,i}(\underline{\mathbf{x}}^{(i)}) := \Phi(x_N) + \displaystyle\sum_{k=i}^{N-1} f_k\big(x_k, y_{k+1}(x_{k+1}), z_k(x_k)\big)\Delta_k, \\ z_i(\cdot) \text{ solves } \mathbf{OLS}(\ S_{Z,i}(\underline{\mathbf{h}}^{(i)}, \underline{\mathbf{x}}^{(i)})\ ,\ \mathcal{K}_i^{(q)}\ ,\ \nu_i\ ), \\ \qquad \text{for } S_{Z,i}(\underline{\mathbf{h}}^{(i)}, \underline{\mathbf{x}}^{(i)}) := \Phi(x_N)h_N + \displaystyle\sum_{k=i+1}^{N-1} f_k\big(x_k, y_{k+1}(x_{k+1}), z_k(x_k)\big)h_k\Delta_k \end{cases} \tag{3.4}$$

$$\begin{aligned} &\nu_i := \mathbb{P} \circ (H_{i+1}^{(i)}, \ldots, H_N^{(i)}, X_i, \ldots, X_N)^{-1}, \\ &\underline{\mathbf{h}}^{(i)} := (h_{i+1}, \ldots, h_N) \in ((\mathbb{R}^q)^\top)^{N-i}, \qquad \underline{\mathbf{x}}^{(i)} := (x_i, \ldots, x_N) \in (\mathbb{R}^d)^{N-i+1}. \end{aligned} \tag{3.5}$$

However, the above least-squares regressions encounter two computational problems:

(CP1) $\mathbf{L}_2(\mathcal{B}(\mathbb{R}^d), \mathbb{P} \circ (X_i)^{-1})$ is usually infinite dimensional;

(CP2) (3.2) is computed using the presumably untractable law of $(H_{i+1}^{(i)}, \ldots, H_N^{(i)}, X_i, \ldots, X_N)$.

Therefore, the functions $y_i(\cdot)$ and $z_i(\cdot)$ are to be approximated on finite dimensional function spaces with the sample-based empirical version of the law, as described in the next subsection.

3.2 Algorithm

The first computational problem **(CP1)** is handled using a pre-selected *finite dimensional vector spaces.*

Definition 3.3 (Finite dimensional approximation spaces)**.** *For $i \in \{0, \ldots, N-1\}$, we are given two finite functional linear spaces of dimension $K_{Y,i}$ and $K_{Z,i}$*

$$\begin{cases} \mathcal{K}_{Y,i} := \operatorname{span}\{p_{Y,i}^{(1)}, \ldots, p_{Y,i}^{(K_{Y,i})}\}, \ \textit{for } p_{Y,i}^{(k)} : \mathbb{R}^d \to \mathbb{R} \ \textit{s.t.} \ \mathbb{E}[|p_{Y,i}^{(k)}(X_i)|^2] < +\infty, \\ \mathcal{K}_{Z,i} := \operatorname{span}\{p_{Z,i}^{(1)}, \ldots, p_{Z,i}^{(K_{Z,i})}\}, \ \textit{for } p_{Z,i}^{(k)} : \mathbb{R}^d \to (\mathbb{R}^q)^\top \ \textit{s.t.} \ \mathbb{E}[|p_{Z,i}^{(k)}(X_i)|^2] < +\infty. \end{cases}$$

The function $y_i(\cdot)$ (resp. $z_i(\cdot)$) will be approximated in the linear space $\mathcal{K}_{Y,i}$ (resp. $\mathcal{K}_{Z,i}$). The best approximation errors are defined by

$$\mathcal{E}^Y_{\mathrm{App.},i} := \sqrt{\inf_{\phi \in \mathcal{K}_{Y,i}} \mathbb{E}\Big[|\phi(X_i) - y_i(X_i)|^2\Big]}, \qquad \mathcal{E}^Z_{\mathrm{App.},i} := \sqrt{\inf_{\phi \in \mathcal{K}_{Z,i}} \mathbb{E}\Big[|\phi(X_i) - z_i(X_i)|^2\Big]}.$$

The second computational problem **(CP2)** is solved using the *empirical* measure built from independent simulations with distribution ν_i. The number of simulations is large enough to avoid having under-determined systems of equations to solve.

Definition 3.4 (Simulations and empirical measures)**.**
For $i \in \{0, \ldots, N-1\}$, generate $M_i \geq K_{Y,i} \vee K_{Z,i}$ independent copies $\mathcal{C}_i := \{(H^{(i,m)}, X^{(i,m)}) : m = 1, \ldots, M_i\}$ of $(H^{(i)}, X^{(i)}) := (H_{i+1}^{(i)}, \ldots, H_N^{(i)}, X_i, \ldots, X_N)$: $\mathcal{C}_i$ forms a cloud of simulations *used for the regression at time i. Denote by $\nu_{i,M}$ the empirical probability measure of the $\mathcal{C}_i$-simulations, i.e.*

$$\nu_{i,M} := \frac{1}{M_i} \sum_{m=1}^{M_i} \delta_{(H_{i+1}^{(i,m)}, \ldots, H_N^{(i,m)}, X_i^{(i,m)}, \ldots, X_N^{(i,m)})}. \tag{3.6}$$

Furthermore, we assume that the clouds of simulations $(\mathcal{C}_i : 0 \leq i < N)$ are independently generated. All these random variables are defined on a probability space $(\Omega^{(M)}, \mathcal{F}^{(M)}, \mathbb{P}^{(M)})$.

Observe that allowing time-dependency in the number of simulations M_i and in the vector spaces $\mathcal{K}_{Y,i}$ and $\mathcal{K}_{Z,i}$ is coherent with our setting of time-dependent local Lipschitz driver.

Denoting by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space supporting $(H^{(0)}, \ldots, H^{(N-1)}, X)$, which serves as a generic element for the clouds of simulations, the full probability space used to analyze our algorithm is the product space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) = (\Omega, \mathcal{F}, \mathbb{P}) \otimes (\Omega^{(M)}, \mathcal{F}^{(M)}, \mathbb{P}^{(M)})$. By a slight abuse of notation, we write $\mathbb{P}$ (resp. $\mathbb{E}$) to mean $\bar{\mathbb{P}}$ (resp. $\bar{\mathbb{E}}$) from now on.

In what follows, extensive use will be made of conditioning on the clouds of simulations. This is much in the spirit of the proof of [13, Theorem 4.11], and the arguments are based on the following definition of σ-algebras.

Definition 3.5. *Define the σ-algebras*

$$\mathcal{F}_i^{(*)} := \sigma(\mathcal{C}_{i+1}, \dots, \mathcal{C}_{N-1}), \qquad \mathcal{F}_i^{(M)} := \mathcal{F}_i^{(*)} \vee \sigma(X_i^{(i,m)} \; : \; 1 \le m \le M_i).$$

For every $i \in \{0, \dots, N-1\}$, let $\mathbb{E}_i^M[\cdot]$ (resp. $\mathbb{P}_i^M$) with respect to $\mathcal{F}_i^{(M)}$.

We now come to the definition of the MWLS algorithm: this is merely the finite-dimensional version of (3.4) plus a soft truncation of the solutions using the truncation function $\mathcal{T}_{\cdot}(.)$ (defined in Section 1.3).

Definition 3.6 (MWLS algorithm)**.** *Set $y_N^{(M)}(\cdot) := \Phi(\cdot)$. For each $i = N-1, N-2, \dots, 0$, set the random functions $y_i^{(M)}(\cdot)$ and $z_i^{(M)}(\cdot)$ recursively as follows.*

1. *First, define $z_i^{(M)}(\cdot) := \mathcal{T}_{C_{z,i}}\big(\psi_{Z,i}^{(M)}(\cdot)\big)$ where $C_{z,i}$ is the almost sure bound of Corollary 2.6 and where*

$$\begin{cases} \psi_{Z,i}^{(M)}(\cdot) \text{ solves } \mathbf{OLS}(\; S_{Z,i}^{(M)}(\underline{\mathbf{h}}^{(i)}, \underline{\mathbf{x}}^{(i)}) \;,\; \mathcal{K}_{Z,i} \;,\; \nu_{i,M}) \\ \qquad \text{for } S_{Z,i}^{(M)}(\underline{\mathbf{h}}^{(i)}, \underline{\mathbf{x}}^{(i)}) := \Phi(x_N) h_N + \displaystyle\sum_{k=i+1}^{N-1} f_k\big(x_k, y_{k+1}^{(M)}(x_{k+1}), z_k^{(M)}(x_k)\big) h_k \Delta_k, \end{cases} \tag{3.7}$$

   where $\underline{\mathbf{h}}^{(i)}, \underline{\mathbf{x}}^{(i)}, \nu_{i,M}$ are defined in (3.5) and (3.6).

2. *Second and similarly, define $y_i^{(M)}(\cdot) := \mathcal{T}_{C_{y,i}}\big(\psi_{Y,i}^{(M)}(\cdot)\big)$ where*

$$\begin{cases} \psi_{Y,i}^{(M)}(\cdot) \text{ solves } \mathbf{OLS}(S_{Y,i}^{(M)}(\underline{\mathbf{x}}^{(i)}) \;,\; \mathcal{K}_{Y,i} \;,\; \nu_{i,M}) \\ \qquad \text{for } S_{Y,i}^{(M)}(\underline{\mathbf{x}}^{(i)}) := \Phi(x_N) + \displaystyle\sum_{k=i}^{N-1} f_k\big(x_k, y_{k+1}^{(M)}(x_{k+1}), z_k^{(M)}(x_k)\big) \Delta_k. \end{cases} \tag{3.8}$$

Before performing the error analysis, we state the following uniform (resp. conditional variance) bounds on the functions $S_{Y,i}^{(M)}(\cdot)$ (resp. the l-th coordinate of $S_{Z,i}^{(M)}(H^{(i,m)}, X^{(i,m)})$ for each m and l). These bounds are used repeatedly in Section 3.3 in conjunction with Propostion 3.11 in order to obtain estimates on the conditional variance of the regressions. The proof is postponed to Appendix A.2.

Lemma 3.7. *For all $i \in \{0, \dots, N-1\}$, there are finite constants $\bar{C}_{y,i} \ge 0$ and $\bar{C}_{z,i} \ge 0$ such that*

$$|S_{Y,i}^{(M)}(\underline{\mathbf{x}}^{(i)})| \le \bar{C}_{y,i}, \quad \forall \underline{\mathbf{x}}^{(i)},$$
$$\sum_{l=1}^{q} \mathbb{V}\mathrm{ar}\Big[S_{l,Z,i}^{(M)}(H^{(i,m)}, X^{(i,m)}) \;\big|\; \mathcal{F}_i^{(M)}\Big] \le \bar{C}_{z,i}^2, \quad \forall m \in \{1, \dots, M_i\}.$$

We can write a precise time-dependency of the constants $\bar{C}_{y,i}$ and $\bar{C}_{z,i}$:

$$\bar{C}_{y,i} := c_1 C_\xi + c_2 C_f (T - t_i)_c^{\theta}, \qquad \bar{C}_{z,i} := c_3 C_\xi (T - t_i)^{-1/2} + c_4 C_f (T - t_i)^{\theta_c - \frac{1}{2}}, \tag{3.9}$$

where $(c_j)_{1 \le j \le 4}$ depend only on $(L_f, C_M, q, C_y^{(1)}, C_y^{(2)}, C_z^{(1)}, C_z^{(2)}, C_z^{(3)}, T, R_\pi, \theta_L, \theta_c)$ (computed explicitely in the proof).

The above time-dependency is to be used to derive convergence rates for the complexity analysis.

3.3 Main result: error analysis

We precise the random norms used to quantify the error of MWLS.

Definition 3.8. *Let* $\varphi : \Omega^{(M)} \times \mathbb{R}^d \to \mathbb{R}$ *or* $(\mathbb{R}^q)^\top$ *be* $\mathcal{F}^{(M)} \otimes \mathcal{B}(\mathbb{R}^d)$*-measurable. For each* $i \in \{0, \ldots, N-1\}$*, define the random norms*

$$\|\varphi\|_{i,\infty}^2 := \int_{\mathbb{R}^d} |\varphi(x)|^2 \ \mathbb{P} \circ X_i^{-1}(dx), \qquad \|\varphi\|_{i,M}^2 := \frac{1}{M_i} \sum_{m=1}^{M_i} |\varphi(X_i^{(i,m)})|^2.$$

The accuracy of the MWLS algorithm is measured as follows:

$$\bar{\mathcal{E}}(Y, M, i) := \sqrt{\mathbb{E}\Big[\|y_i^{(M)}(\cdot) - y_i(\cdot)\|_{i,\infty}^2\Big]}, \qquad \bar{\mathcal{E}}(Z, M, i) := \sqrt{\mathbb{E}\Big[\|z_i^{(M)}(\cdot) - z_i(\cdot)\|_{i,\infty}^2\Big]},$$
$$\mathcal{E}(Y, M, i) := \sqrt{\mathbb{E}\Big[\|y_i^{(M)}(\cdot) - y_i(\cdot)\|_{i,M}^2\Big]}, \qquad \mathcal{E}(Z, M, i) := \sqrt{\mathbb{E}\Big[\|z_i^{(M)}(\cdot) - z_i(\cdot)\|_{i,M}^2\Big]}.$$

In our analysis, we will have to switch from errors in true measure $\bar{\mathcal{E}}(\ldots)$ to errors in empirical measure $\mathcal{E}(\ldots)$, and vice-versa: this is not trivial since $(y_i^{(M)}(.), z_i^{(M)}(.))$ and the empirical norm $\|.\|_{i,M}$ depend on the same sample. However, the switch can be performed using concentration-of-measure estimates uniformly on a class of functions [15, Chapter 9]. We directly state the ready-to-use result, which is a straightforward adaptation of [13, Proposition 4.10] to our context.

Proposition 3.9. *Recall the constants* $C_{y,i}$ *(resp.* $C_{z,i}$*) from Corollary 2.6, and define the* interdependence errors

$$\mathcal{E}_{\mathrm{Dep.},i}^Y := C_{y,i}\sqrt{\frac{2028(K_{Y,i}+1)\log(3M_i)}{M_i}}, \qquad \mathcal{E}_{\mathrm{Dep.},i}^Z := C_{z,i}\sqrt{\frac{2028(K_{Z,i}+1)q\log(3M_i)}{M_i}}.$$

For each $i \in \{0, \ldots, N-1\}$*, we have*

$$\bar{\mathcal{E}}(Y, M, i) \le \sqrt{2}\mathcal{E}(Y, M, i) + \mathcal{E}_{\mathrm{Dep.},i}^Y, \qquad \bar{\mathcal{E}}(Z, M, i) \le \sqrt{2}\mathcal{E}(Z, M, i) + \mathcal{E}_{\mathrm{Dep.},i}^Z.$$

The aim is to determine a rate of convergence for $\mathcal{E}(Y, M, k) := (\mathbb{E}[\|y_k - y_k^M\|_{k,M}^2])^{\frac{1}{2}}$ and $\mathcal{E}(Z, M, k) := (\mathbb{E}[\|z_k - z_k^M\|_{k,M}^2])^{\frac{1}{2}}$ using the local error terms $(\mathcal{E}(k))_k$ defined below.

Theorem 3.10 (global error of the MWLS algorithm)**.** *For* $0 \le k \le N-1$*, define*

$$\mathcal{E}(k) := \mathcal{E}_{\mathrm{App.},k+1}^Y + \bar{C}_{y,k+1}\sqrt{\frac{K_{Y,k+1}}{M_{k+1}}} + \mathcal{E}_{\mathrm{App.},k}^Z + \bar{C}_{z,k}\sqrt{\frac{K_{Z,k}}{M_k}} + L_f\big(\mathcal{E}_{\mathrm{Dep.},k+1}^Y + \mathcal{E}_{\mathrm{Dep.},k}^Z\big). \tag{3.10}$$

For every $k \in \{0, \ldots, N-1\}$*,*

$$\Big(\mathbb{E}[\|y_k - y_k^M\|_{k,M}^2]\Big)^{1/2} \le \mathcal{E}_{\mathrm{App.},k}^Y + \bar{C}_{y,k}\sqrt{\frac{K_{Y,k}}{M_k}} + C_y^{(M)} \sum_{j=k}^{N-1} \frac{\mathcal{E}(j)\Delta_j}{(T-t_j)^{(1-\theta_L)/2}}, \tag{3.11}$$

$$\Big(\mathbb{E}[\|z_k - z_k^M\|_{k,M}^2]\Big)^{1/2} \le \mathcal{E}_{\mathrm{App.},k}^Z + \bar{C}_{z,k}\sqrt{\frac{K_{Z,k}}{M_k}} + C_z^{(M)} \sum_{j=k+1}^{N-1} \frac{\mathcal{E}(j)\Delta_j}{(T-t_j)^{(1-\theta_L)/2}\sqrt{t_j - t_k}}, \tag{3.12}$$

where, recalling the constant $\mathcal{C}^{(\gamma)}_{(2.3)}$ *from Lemma 2.3 (with* $\alpha = 0$*,* $\beta = \frac{\theta_L}{2}$*,* $\gamma \in \{\frac{1}{2}, 1\}$ *and* $C_u = L_f(\sqrt{2}C_M + 4\sqrt{T})$*),*

$$
\begin{aligned}
C^{(M)}_y &:= 2 + 4L_f\mathcal{C}^{(1)}_{(2.3)}(1 + B_{\frac{\theta_L}{2},1}T^{\frac{\theta_L}{2}}(C_M + 2\sqrt{T})),\\
C^{(M)}_z &:= C_M + \sqrt{2}C_M L_f\mathcal{C}^{(\frac{1}{2})}_{(2.3)}(1 + B_{\frac{\theta_L}{2},\frac{1}{2}}T^{\frac{\theta_L}{2}}(C_M + 2\sqrt{T})).
\end{aligned}
$$

Discussion. Observe that owing to Proposition 3.9, similar estimates (with modified constants) are valid for $\bar{\mathcal{E}}(Y, M, k) = (\mathbb{E}[\|y_k - y^M_k\|^2_{k,\infty}])^{\frac{1}{2}}$ and $\bar{\mathcal{E}}(Z, M, k) = (\mathbb{E}[\|z_k - z^M_k\|^2_{k,\infty}])^{\frac{1}{2}}$. The global error (3.11-3.12) is a weighted time-average of three different errors.

1) The contributions $\mathcal{E}_{\text{App.},.}^{\cdot}$ are the best approximation errors using the vector spaces of functions: this accuracy is achieved asymptotically with an infinite number of simulations (take $M_k \to +\infty$ in our estimates).

2) The contributions $\sqrt{\frac{K_{\cdot,\cdot}}{M_\cdot}}$ are the usual statistical error terms: the larger the number of simulations or the smaller the dimensions of the vector spaces, the better the estimation error.

3) The contributions $\mathcal{E}_{\text{Dep.},.}^{\cdot}$ are related to the interdependencies between regressions at different times. This is intrinsic to the dynamic programming equation with N nested empirical regressions.

However, due to Proposition 3.9, the latter contributions are of same magnitude as statistical error terms (up to logarithmic factors). Therefore roughly speaking, the global error is of order of the best approximation errors plus statistical errors, as if there were a single regression problem [15, Theorem 11.1]. In this sense, these error bounds are optimal: it is not possible to improve the above estimates with respect to the convergence rates (but only possibly with respect to the constants). An optimal tuning of parameters is proposed in Section 3.5.

In comparison to [13], where a different Monte-Carlo regression scheme is analyzed, the upper bound for the global error has a similar shape, but with two important differences.

- *Norm on Z.* In [13], one uses the time averaged squared $\mathbf{L}_2$-norm $\sum_i \mathbb{E}[\|\cdot\|^2_{i,M}]\Delta_i$ to estimate the error in Z, whereas here the norm used is time-wise. This leads to more informative error bounds. This is an advantage of the discrete BSDE with Malliavin weights against the MDP of [13].

- *Time-dependency.* The MWDP yields better estimates on $y(.)$ and $z(.)$ w.r.t. time in the local error estimates, which allows better parameters tuning and therefore better convergence rates (see Section 3.5).

3.4 Proof of Theorem 3.10

3.4.1 Preliminary results

The following proposition lists rather standard key tools from the theory of regression. They will be used repeatedly in the proof of Theorem 3.10. This proposition was also used in [13], and we refer the reader to that paper for the proof. The two first properties are of deterministic nature, the two last are probabilistic. Item (iv) is stated in high generality; this readily allows its further use in other regression-based Monte Carlo algorithms.

Proposition 3.11 ([13, Proposition 4.12])**.** *With the notation of Definition 3.2, suppose that $\mathcal{K}$ is finite dimensional and spanned by the functions $\{p_1(.), \ldots, p_K(.)\}$. Let $S^\star$ solve $\mathbf{OLS}(S, \mathcal{K}, \nu)$ (resp. $\mathbf{OLS}(S, \mathcal{K}, \nu_M)$), according to (3.2) (resp. (3.3)). The following properties are satisfied:*

(i) *linearity: the mapping $S \mapsto S^\star$ is linear.*

(ii) *stability property: $\|S^\star\|_{\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l),\mu)} \leq \|S\|_{\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l),\mu)}$, where $\mu = \nu$ (resp. $\mu = \nu_M$).*

(iii) *conditional expectation solution: in the case of the discrete probability measure ν_M, assume additionally that the sub-σ-algebra $\mathcal{Q} \subset \tilde{\mathcal{F}}$ is such that $\big(p_j(\mathcal{X}^{(1)}), \ldots, p_j(\mathcal{X}^{(M)})\big)$ is $\mathcal{Q}$-measurable for every $j \in \{1, \ldots, K\}$. Setting $S_{\mathcal{Q}}(\mathcal{X}^{(m)}) := \tilde{\mathbb{E}}[S(\mathcal{X}^{(m)})|\mathcal{Q}]$ for each $m \in \{1, \ldots, M\}$, then $\tilde{\mathbb{E}}[S^\star|\mathcal{Q}]$ solves $\mathbf{OLS}\big(S_{\mathcal{Q}}, \mathcal{K}, \nu_M\big)$.*

(iv) *bounded conditional variance: in the case of the discrete probability measure ν_M, suppose that $S(\omega, x)$ is $\mathcal{G} \otimes \mathcal{B}(\mathbb{R}^l)$-measurable, for $\mathcal{G} \subset \tilde{\mathcal{F}}$ independent of $\sigma(\mathcal{X}^{(1)}, \ldots, \mathcal{X}^{(M)})$, there exists a Borel measurable function $g : \mathbb{R}^l \to \mathcal{E}$, for some Euclidean space $\mathcal{E}$, such that the random variables $\{p_j(\mathcal{X}^{(m)}) \ : \ m = 1, \ldots, M, \ j = 1, \ldots, K\}$ are $\mathcal{H} := \sigma(g(\mathcal{X}^{(m)}) \ : \ m = 1, \ldots, M)$-measurable, and there is a finite constant $\sigma^2 \geq 0$ that uniformly bounds the conditional variances $\tilde{\mathbb{E}}\big[|S(\mathcal{X}^{(m)}) - \tilde{\mathbb{E}}(S(\mathcal{X}^{(m)})|\mathcal{G} \vee \mathcal{H})|^2 \mid \mathcal{G} \vee \mathcal{H}\big] \leq \sigma^2$ $\tilde{\mathbb{P}}$-a.s. and for all $m \in \{1, \ldots, M\}$. Then*

$$\tilde{\mathbb{E}}\Big[\|S^\star(\cdot) - \tilde{\mathbb{E}}[S^\star(\cdot)|\mathcal{G} \vee \mathcal{H}]\|^2_{\mathbf{L}_2(\mathcal{B}(\mathbb{R}^l),\nu_M)} \mid \mathcal{G} \vee \mathcal{H}\Big] \leq \sigma^2 \frac{K}{M}.$$

Intermediate processes and local error terms. Another technique we borrow from [13] is to introduce intermediate, *fictional* regressions based on the true solutions: one replaces the full $\mathbf{L}_2$ space for the approximation space and the true measure for the empirical measure in (3.4). For each $k \in \{0, \ldots, N-1\}$, recall the functions $S_{Y,k}(\underline{\mathbf{x}}^{(i)})$ and $S_{Z,k}(\underline{\mathbf{h}}^{(i)}, \underline{\mathbf{x}}^{(i)})$ from (3.4), the linear spaces $\mathcal{K}_{Y,k}$ and $\mathcal{K}_{Z,k}$ from Definition 3.3, and the empirical measure $\nu_{k,M}$ from (3.6), and set

$$\psi_{Y,k}(\cdot) \quad \text{solves} \quad \mathbf{OLS}(\ S_{Y,k}(\underline{\mathbf{x}}^{(i)})\ ,\ \mathcal{K}_{Y,k}\ ,\ \nu_{k,M}\ ),$$

$$\psi_{Z,k}(\cdot) \quad \text{solves} \quad \mathbf{OLS}(\ S_{Z,k}(\underline{\mathbf{h}}^{(i)}, \underline{\mathbf{x}}^{(i)})\ ,\ \mathcal{K}_{Z,k}\ ,\ \nu_{k,M}\ ).$$

Note that these regressions are not numerically accessible, because they require knowledge of the true solution to be applied. After a series of conditioning arguments, based on Lemma 3.12 below, the fictional regressions will eventually allow the use of the stability estimates of Section 2.2, and (after a somewhat complex application of the Gronwall inequalities of Section 2.1) this will yield final result. From Lemma 3.1 and our Markovian assumptions, observe that

$$(\mathbb{E}^M_k[S_{Y,k}(X^{(k,m)})]\ , \mathbb{E}^M_k[S_{Z,k}(H^{(k,m)}, X^{(k,m)})]) = \big(y_k(X^{(k,m)}_k), z_k(X^{(k,m)}_k)\big)$$

for each $m \in \{1, \ldots, M_k\}$ where $\big(y_k(\cdot), z_k(\cdot)\big)$ are the unknown functions defined in (3.1). Proposition 3.11(iii) implies the first statement of the following lemma. The second statement results from a direct interchange of inf and $\mathbb{E}$, and from the identical distribution of $(X^{(k,m)}_k)$ for all m.

Lemma 3.12. *For each $k \in \{0, \ldots, N-1\}$,*

$$\mathbb{E}^M_k[\psi_{Y,k}(\cdot)] \quad \textit{solves} \quad \mathbf{OLS}(\ y_k(.)\ ,\ \mathcal{K}_{Y,k}\ ,\ \nu_{k,M}\ ),$$

$$\mathbb{E}^M_k[\psi_{Z,k}(\cdot)] \quad \textit{solves} \quad \mathbf{OLS}(\ z_k(.)\ ,\ \mathcal{K}_{Z,k}\ ,\ \nu_{k,M}\ ).$$

In addition, recalling the local error terms $\mathcal{E}^Y_{\mathrm{App.},k}$ and $\mathcal{E}^Z_{\mathrm{App.},k}$ from Definition 3.3,

$$\mathbb{E}\big[\|\mathbb{E}^M_k[\psi_{Y,k}(\cdot)] - y_k(\cdot)\|^2_{k,M}\big] = \mathbb{E}\big[\inf_{\phi\in\mathcal{K}_{Y,k}} \|\phi(\cdot) - y_k(\cdot)\|^2_{k,M}\big] \le (\mathcal{E}^Y_{\mathrm{App.},k})^2,$$
$$\mathbb{E}\big[\|\mathbb{E}^M_k[\psi_{Z,k}(\cdot)] - z_k(\cdot)\|^2_{k,M}\big] = \mathbb{E}\big[\inf_{\phi\in\mathcal{K}_{Z,k}} \|\phi(\cdot) - z_k(\cdot)\|^2_{k,M}\big] \le (\mathcal{E}^Z_{\mathrm{App.},k})^2.$$

3.4.2 Proof of Theorem 3.10

Step 1: *decomposition of the error on Y.* Recall the soft truncation function $\mathcal{T}_L(x) := (-L \vee x_1 \wedge L, \dots, -L \vee x_n \wedge L)$ for $x \in \mathbb{R}^n$. From the almost sure bounds of Corollary 2.6, $\mathcal{T}_{C_{y,k}}(y_k) = y_k$. Then, the Lipschitz continuity of $\mathcal{T}_{C_{y,k}}$ yields $\|y_k(\cdot) - y^{(M)}_k(\cdot)\|_{k,M}$ is less than or equal to $\|y_k(\cdot) - \psi^{(M)}_{Y,k}(\cdot)\|_{k,M}$. Using the triangle inequality for the $\|.\|_{k,M}$-norm, it follows that

$$\|y_k(\cdot) - y^{(M)}_k(\cdot)\|_{k,M} \le \|y_k(\cdot) - \mathbb{E}^M_k[\psi_{Y,k}(\cdot)]\|_{k,M} + \|\mathbb{E}^M_k[\psi_{Y,k}(\cdot)] - \psi^{(M)}_{Y,k}(\cdot)\|_{k,M}. \tag{3.13}$$

Because $S^{(M)}_{Y,k}(\cdot)$ depends on $z^{(M)}_k(\cdot)$ computed with the same cloud of simulations $\mathcal{C}_k$ as that used to define the OLS solution $\psi^{(M)}_{Y,k}(\cdot)$, it raises some interdependency issues that we solve by making a small perturbation to the intermediate processes as follows (compare with (3.4) and (3.8)): for $\underline{\mathbf{x}}^{(k)} = (x_k, \dots, x_N)$, define

$$\tilde{S}^{(M)}_{Y,k}(\underline{\mathbf{x}}^{(k)}) := \Phi(x_N) + f_k\big(x_k, y^{(M)}_{k+1}(x_{k+1}), z_k(x_k)\big)\Delta_k + \sum_{i=k+1}^{N-1} f_i\big(x_i, y^{(M)}_{i+1}(x_{i+1}), z^{(M)}_i(x_i)\big)\Delta_i,$$
$$\tilde{\psi}^{(M)}_{Y,k}(\cdot) \quad \text{solves} \quad \mathbf{OLS}(\tilde{S}^{(M)}_{Y,k}(\underline{\mathbf{x}}^{(k)}) \ , \ \mathcal{K}_{Y,k} \ , \ \nu_{k,M}).$$

This perturbation is not needed for the Z-component, because $S^{(M)}_{Z,k}(\underline{\mathbf{h}}^{(k)}, \underline{\mathbf{x}}^{(k)})$ depends only on the subsequent clouds of simulations $\{\mathcal{C}_j, j \ge k+1\}$. Applying the $\mathbf{L}_2$-norm $|\cdot|_2$, the triangle inequality in (3.13), and the first part of Lemma 3.12 yields

$$\begin{aligned}\mathcal{E}(Y, M, k) \le \mathcal{E}^Y_{\mathrm{App.},k} &+ |\|\mathbb{E}^M_k[\tilde{\psi}^{(M)}_{Y,k}(\cdot) - \psi_{Y,k}(\cdot)]\|_{k,M}|_2 + |\|\tilde{\psi}^{(M)}_{Y,k}(\cdot) - \mathbb{E}^M_k[\tilde{\psi}^{(M)}_{Y,k}(\cdot)]\|_{k,M}|_2 \\ &+ |\|\tilde{\psi}^{(M)}_{Y,k}(\cdot) - \psi^{(M)}_{Y,k}(\cdot)\|_{k,M}|_2.\end{aligned} \tag{3.14}$$

Let us handle each term in the above inequality separately.
$\triangleright$ Term $|\|\mathbb{E}^M_k[\tilde{\psi}^{(M)}_{Y,k}(\cdot) - \psi_{Y,k}(\cdot)]\|_{k,M}|_2$. Set

$$\tilde{\xi}^*_{Y,k}(x) := \mathbb{E}(\tilde{S}^{(M)}_{Y,k}(X^{(k)}) - S_{Y,k}(X^{(k)}) | X^{(k)}_k = x, \mathcal{F}^{(M)}).$$

Recalling that $\tilde{S}^{(M)}_{Y,k}(\underline{\mathbf{x}}^{(k)}) - S_{Y,k}(\underline{\mathbf{x}}^{(k)})$ is built only using the clouds $\{\mathcal{C}_j, j \ge k+1\}$, it follows from Lemma 3.1 that $\mathbb{E}^M_k[\tilde{S}^{(M)}_{Y,k}(X^{(k,m)}) - S_{Y,k}(X^{(k,m)})]$ is equal to $\tilde{\xi}^*_{Y,k}(X^{(k,m)}_k)$ for every $m \in \{1, \dots, M_k\}$. Then, using Proposition 3.11(i)(iii), $\mathbb{E}^M_k[\tilde{\psi}^{(M)}_{Y,k}(\cdot) - \psi_{Y,k}(\cdot)]$ solves $\mathbf{OLS}(\tilde{\xi}^*_{Y,k}(\cdot), \ \mathcal{K}_{Y,k} \ , \ \nu_{k,M})$. By Proposition 3.11(ii),

$$\mathbb{E}\big[\|\mathbb{E}^M_k[\tilde{\psi}^{(M)}_{Y,k}(\cdot) - \psi_{Y,k}(\cdot)]\|^2_{k,M}\big] \le \mathbb{E}\big[\|\tilde{\xi}^*_{Y,k}(\cdot)\|^2_{k,M}\big] = \mathbb{E}\big[(\tilde{\xi}^*_{Y,k}(X_k))^2\big],$$

where the final equality follows from the fact that $\tilde{\xi}^*_{Y,k}(\cdot)$ is generated only using the simulations in the clouds $\{\mathcal{C}_j \ : \ j > k\}$ and $\{X_k, X^{(k,1)}_k, \dots, X^{(k,M_k)}_k\}$ are identically distributed. Defining

$$\xi^*_{Y,k}(x) := \mathbb{E}[S^{(M)}_{Y,k}(X^{(k)}) - S_{Y,k}(X^{(k)}) | X^{(k)}_k = x, \mathcal{F}^{(M)}], \tag{3.15}$$

the triangle inequality yields

$$
\begin{aligned}
|\tilde{\xi}^*_{Y,k}(X_k)|_2 &\le |\tilde{S}^{(M)}_{Y,k}(X^{(k)}) - S^{(M)}_{Y,k}(X^{(k)})|_2 + |\xi^*_{Y,k}(X_k)|_2 \\
&\le |f_k(X_k, y^{(M)}_{k+1}(X_{k+1}), z^{(M)}_k(X_k)) - f_k(X_k, y^{(M)}_{k+1}(X_{k+1}), z_k(X_k))|_2 \Delta_k + |\xi^*_{Y,k}(X_k)|_2 \\
&\le \frac{L_f \Delta_k}{(T-t_k)^{\frac{1}{2}-\frac{\theta_L}{2}}} \bar{\mathcal{E}}(Z,M,k) + |\xi^*_{Y,k}(X_k)|_2.
\end{aligned}
$$

$\triangleright$ Term $|\|\tilde{\psi}^{(M)}_{Y,k}(\cdot) - \mathbb{E}^M_k[\tilde{\psi}^{(M)}_{Y,k}(\cdot)]\|_{k,M}|_2$. Since $\tilde{S}^{(M)}_{Y,k}(.)$ depends only on the clouds $\{\mathcal{C}_j, j \ge k+1\}$ and is bounded from above by $\bar{C}_{y,k}$ (like $S^{(M)}_{Y,k}(.)$, see Lemma 3.7), it follows from Proposition 3.11(iv) that $|\|\tilde{\psi}^{(M)}_{Y,k}(\cdot) - \mathbb{E}^M_k[\tilde{\psi}^{(M)}_{Y,k}(\cdot)]\|_{k,M}|_2$ is bounded from above by $\bar{C}_{y,k}\sqrt{K_{Y,k}/M_k}$. This is similar to the statistical error term in usual regression theory.

$\triangleright$ Term $|\|\tilde{\psi}^{(M)}_{Y,k}(\cdot) - \psi^{(M)}_{Y,k}(\cdot)\|_{k,M}|_2$. Owing to Proposition 3.11(i)(ii), $\|\tilde{\psi}^{(M)}_{Y,k}(\cdot) - \psi^{(M)}_{Y,k}(\cdot)\|^2_{k,M}$ is bounded from above by $\|\tilde{S}^{(M)}_{Y,k}(\cdot) - S^{(M)}_{Y,k}(\cdot)\|^2_{k,M}$, which equals

$$
\begin{aligned}
&\frac{\Delta_k^2}{M_k}\sum_{m=1}^{M_k} |f_k(X^{(k,m)}_k, y^{(M)}_{k+1}(X^{(k,m)}_{k+1}), z^{(M)}_k(X^{(k,m)}_k)) - f_k(X^{(k,m)}_k, y^{(M)}_{k+1}(X^{(k,m)}_{k+1}), z_k(X^{(k,m)}_k))|^2 \\
&\le \frac{L_f^2 \Delta_k^2 \|z_k(\cdot) - z^{(M)}_k(\cdot)\|^2_{k,M}}{(T-t_k)^{1-\theta_L}}.
\end{aligned}
$$

Collecting the bounds on the three terms, substituting them into (3.14) and applying Proposition 3.9 yields

$$
\mathcal{E}(Y,M,k) \le \mathcal{E}^Y_{\mathrm{App.},k} + |\xi^*_{Y,k}(X_k)|_2 + \bar{C}_{y,k}\sqrt{\frac{K_{Y,k}}{M_k}} + \frac{L_f \Delta_k}{(T-t_k)^{\frac{1}{2}-\frac{\theta_L}{2}}}\Big\{(1+\sqrt{2})\mathcal{E}(Z,M,k) + \mathcal{E}^Z_{\mathrm{Dep.},k}\Big\}. \tag{3.16}
$$

Step 2: *decomposition of the error on Z.* Analogously to (3.14), one obtains the upper bound

$$
\mathcal{E}(Z,M,k) \le \mathcal{E}^Z_{\mathrm{App.},k} + |\|\mathbb{E}^M_k[\psi^{(M)}_{Z,k}(\cdot) - \psi_{Z,k}(\cdot)]\|_{k,M}|_2 + |\|\psi^{(M)}_{Z,k}(\cdot) - \mathbb{E}^M_k[\psi^{(M)}_{Z,k}(\cdot)]\|_{k,M}|_2.
$$

Since $S^{(M)}_{Z,k}(.)$ depends only on the clouds $\{\mathcal{C}_j, j \ge k+1\}$ and the $\mathcal{F}^{(M)}_k$-conditional variance of $S^{(M)}_{Z,k}(H^{(k,m)}, X^{(k,m)})$ is bounded from above by $\bar{C}^2_{z,k}$ for all m (see Lemma 3.7), it follows from Proposition 3.11(iv) that $|\|\psi^{(M)}_{Z,k}(\cdot) - \mathbb{E}^M_k[\psi^{(M)}_{Z,k}(\cdot)]\|_{k,M}|_2$ is bounded from above by $\bar{C}_{z,k}\sqrt{K_{Z,k}/M_k}$. Defining

$$
\xi^*_{Z,k}(x) := \mathbb{E}[S^{(M)}_{Z,k}(H^{(k)}, X^{(k)}) - S_{Z,k}(H^{(k)}, X^{(k)}) | X^{(k)}_k = x, \mathcal{F}^{(M)}], \tag{3.17}
$$

it follows that $\mathbb{E}^M_k[\psi^{(M)}_{Z,k}(\cdot) - \psi_{Z,k}(\cdot)]$ solves $\mathbf{OLS}(\xi^*_{Z,k}(\cdot),\ \mathcal{K}_{Z,k}\ ,\ \nu_{k,M})$. Therefore,

$$
\mathcal{E}(Z,M,k) \le \mathcal{E}^Z_{\mathrm{App.},k} + |\xi^*_{Z,k}(X_k)|_2 + \bar{C}_{z,k}\sqrt{\frac{K_{Z,k}}{M_k}}. \tag{3.18}
$$

Step 3: *error propagation and a priori estimates.* Observe that $(\xi^*_{Y,k}(X_k), \xi^*_{Z,k}(X_k))$ defined in (3.15,3.17) solves a MWDP with terminal condition 0 and driver $f_{\xi^*,k}(y,z) := f_k(X_k, y^{(M)}_{k+1}(X_{k+1}), z^{(M)}_k(X_k)) - f_k(X_k, y_{k+1}(X_{k+1}), z_k(X_k))$. Applying Proposition 2.5 with $(Y_2, Z_2) \equiv 0$ (so that $L_{f_2} = 0$) and using the Lipschitz continuity of $f_j(.)$ yields

$$
|\xi^*_{Y,k}(X_k)|_2 \le L_f \sum_{j=k}^{N-1} \frac{\bar{\mathcal{E}}(Y,M,j+1) + \bar{\mathcal{E}}(Z,M,j)}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}}\Delta_j,
$$

$$|\xi^*_{Z,k}(X_k)|_2 \le C_M L_f \sum_{j=k+1}^{N-1} \frac{\bar{\mathcal{E}}(Y,M,j+1)+\bar{\mathcal{E}}(Z,M,j)}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}\sqrt{t_j-t_k}}\Delta_j.$$

Next, introducing the notation $\Theta_j := \mathcal{E}(Y,M,j+1)+\mathcal{E}(Z,M,j)$ and applying Proposition 3.9, it follows that

$$\begin{aligned}
|\xi^*_{Y,k}(X_k)|_2 &\le \sqrt{2}L_f \sum_{j=k}^{N-1} \frac{\Theta_j\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}} + L_f \sum_{j=k}^{N-1} \frac{(\mathcal{E}^Y_{\text{Dep.},j+1}+\mathcal{E}^Z_{\text{Dep.},j})\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}},\\
|\xi^*_{Z,k}(X_k)|_2 &\le \sqrt{2}C_M L_f \sum_{j=k+1}^{N-1} \frac{\Theta_j\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}\sqrt{t_j-t_k}} + C_M L_f \sum_{j=k+1}^{N-1} \frac{(\mathcal{E}^Y_{\text{Dep.},j+1}+\mathcal{E}^Z_{\text{Dep.},j})\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}\sqrt{t_j-t_k}}.
\end{aligned}$$

Substituting the above into (3.16) and (3.18), and merging together the terms in Z, it follows that

$$\begin{aligned}
\mathcal{E}(Y,M,k) &\le \mathcal{E}^Y_{\text{App.},k} + \bar{C}_{y,k}\sqrt{\frac{K_{Y,k}}{M_k}} + 2L_f \sum_{j=k}^{N-1} \frac{(\mathcal{E}^Y_{\text{Dep.},j+1}+\mathcal{E}^Z_{\text{Dep.},j})\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}} + 4L_f \sum_{j=k}^{N-1} \frac{\Theta_j\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}}\\
&\le \mathcal{E}^Y_{\text{App.},k} + \bar{C}_{y,k}\sqrt{\frac{K_{Y,k}}{M_k}} + 2\sum_{j=k}^{N-1} \frac{\mathcal{E}(j)\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}} + 4L_f \sum_{j=k}^{N-1} \frac{\Theta_j\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}}, \qquad (3.19)\\
\mathcal{E}(Z,M,k) &\le \mathcal{E}^Z_{\text{App.},k} + \bar{C}_{z,k}\sqrt{\frac{K_{Z,k}}{M_k}} + C_M \sum_{j=k+1}^{N-1} \frac{\mathcal{E}(j)\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}\sqrt{t_j-t_k}}\\
&\quad + \sqrt{2}C_M L_f \sum_{j=k+1}^{N-1} \frac{\Theta_j\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}\sqrt{t_j-t_k}}. \qquad (3.20)
\end{aligned}$$

Step 4: *final estimates.* Now, summing (3.20) and (3.19), one obtains an estimate for Θ_k:

$$\begin{aligned}
\Theta_k \le \mathcal{E}(k) &+ (C_M+2\sqrt{T}) \sum_{j=k+1}^{N-1} \frac{\mathcal{E}(j)\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}\sqrt{t_j-t_k}}\\
&+ L_f(\sqrt{2}C_M+4\sqrt{T}) \sum_{j=k+1}^{N-1} \frac{\Theta_j\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}\sqrt{t_j-t_k}}.
\end{aligned}$$

Thus, using Lemmas 2.2 and 2.3 with $\alpha = 0$, $\beta = \frac{\theta_L}{2}$, $C_u = L_f(\sqrt{2}C_M + 4\sqrt{T})$, $w_k := \mathcal{E}(k) + (C_M + 2\sqrt{T})\sum_{j=k+1}^{N-1} \frac{\mathcal{E}(j)\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}\sqrt{t_j-t_k}}$, we can control weighted sums involving $(\Theta_k)_k$ using weighted sums of $(w_k)_k$, which is exactly what we need to complete the upper bounds (3.19-3.20) for $\mathcal{E}(Y,M,k)$ and $\mathcal{E}(Z,M,k)$. Namely, let $\gamma > 0$:

$$\begin{aligned}
&\sum_{j=k+1}^{N-1} \frac{w_j\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}(t_j-t_k)^{1-\gamma}}\\
&\le \sum_{j=k+1}^{N-1} \frac{\mathcal{E}(j)\Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}(t_j-t_k)^{1-\gamma}} + (C_M+2\sqrt{T}) \sum_{l=k+2}^{N-1} \frac{\mathcal{E}(l)\Delta_l}{(T-t_l)^{\frac{1}{2}-\frac{\theta_L}{2}}} \sum_{j=k+1}^{l-1} \frac{\Delta_j}{(t_l-t_j)^{1-\frac{\theta_L}{2}}(t_j-t_k)^{1-\gamma}}\\
&\le (1+B_{\frac{\theta_L}{2},\gamma}T^{\frac{\theta_L}{2}}(C_M+2\sqrt{T})) \sum_{l=k+1}^{N-1} \frac{\mathcal{E}(l)\Delta_l}{(T-t_l)^{\frac{1}{2}-\frac{\theta_L}{2}}(t_l-t_k)^{1-\gamma}},
\end{aligned}$$

where we have applied Lemma 2.1. Thus,

$$\sum_{j=k+1}^{N-1} \frac{\Theta_j \Delta_j}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}(t_j-t_k)^{1-\gamma}} \le \mathcal{C}^{(\gamma)}_{(2.3)}(1+B_{\frac{\theta_L}{2},\gamma}T^{\frac{\theta_L}{2}}(C_M+2\sqrt{T}))\sum_{l=k+1}^{N-1} \frac{\mathcal{E}(l)\Delta_l}{(T-t_l)^{\frac{1}{2}-\frac{\theta_L}{2}}(t_l-t_k)^{1-\gamma}}.$$

and plugging the above inequality into (3.19) and (3.20) yields (3.11) and (3.12). □

3.5 Complexity analysis

As usual in empirical regression theory, appropriately tuning numerical paramaters is crucial for finding the right trade-off between statistical errors and estimation errors. This analysis allows to express the error magnitude as a function of computational work (complexity analysis). We discuss the complexity in different cases according to the regularity of the value functions $(y_i(\cdot), z_i(\cdot))$ and the choice of the grid π. In order to have a fair comparison with other numerical schemes, we revisit the setting of [13, Section 4.4], which we partly recall for completeness, and extend the analysis to include more general settings.

- We perform an asymptotic complexity analysis as the number N of grid times goes to $+\infty$. We are concerned with time-dependent bounds: thus in the following, the order convention, $O(.)$ or $o(.)$, is uniform in t_i.

- The grids under consideration are of the form $\pi^{(\theta_\pi)} := \{t_i = T - T(1-\frac{i}{N})^{\frac{1}{\theta_\pi}}\}$ for $\theta_\pi \in (0,1]$ (inspired by [11, 9]). Observe that their time-step Δ_i is not-increasing in i, hence they all satisfy ($\mathbf{A_F}$**-iii**) with the same parameter $R_\pi = 1$.

- The magnitude of the final accuracy is denoted by $N^{-\theta_{\text{conv}}}$ for some parameter $\theta_{\text{conv}} > 0$. This is usually related to time-discretization errors between the continuous-time BSDE and the discrete-time one, θ_{conv} may range from 0^+ (for non smooth data [11, Theorem 1.1]) to 1 (in the case of smooth data [10, Theorems 7 and 8]).

- The approximation spaces are given by local polynomials of degree n ($n \ge 0$) defined on hypercubes with edge length $\delta > 0$, covering the set $[-R, R]^d$ ($R > 0$): we denote it by $\mathcal{P}^{n,\delta,R}_{\text{loc.}}$. The functions in $\mathcal{P}^{n,\delta,R}_{\text{loc.}}$ take values in $\mathbb{R}$ for the y-component and in $(\mathbb{R}^q)^\top$ for z (using local polynomials component-wise), but we omit this in the notation. The best-approximation errors are easily controled (using the Taylor formula):

$$\inf_{\varphi \in \mathcal{P}^{n,\delta,R}_{\text{loc.}}} |\varphi(X_i) - u(X_i)|_2 \le |u|_\infty (\mathbb{P}(|X_i|_\infty > R))^{1/2} + c_n |D^{n+1}u|_\infty \delta^{n+1} \tag{3.21}$$

  for any function u that is bounded, $n+1$-times continuously differentiable with bounded derivatives, and where the constant c_n does not depend on (R, u, δ). The dimension of the vector space $\mathcal{P}^{n,\delta,R}_{\text{loc.}}$ is bounded by $\tilde{c}_n(2R/\delta)^d$ where $\tilde{c}_n$ is the number of polynomials on each hypercube (it depends on d and n).

  A significant computational advantage of local polynomial basis is that the cost of computing the regression coefficients associated to a sample of size $M \ge \dim(\mathcal{P}^{n,\delta,R}_{\text{loc.}})$ is $O(M)$ flops. The cost of the regression in the l-th hypercube is of order $M^{(l)} \times \tilde{c}_n^2$ using SVD least squares minimization [14, Chapter 5], where $M^{(l)}$ is the number of simulations that land in the hypercube. Therefore, the total cost of the regressions at any time-point is of order $\tilde{c}_n^2 \sum_l M^{(l)} = \tilde{c}_n^2 M = O(M)$.

On the other hand, the cost of generating the clouds of simulations and computing the simulated functionals $(S^{(M)}_{Y,i}(X^{(i,m)}), S^{(M)}_{Z,i}(H^{(i,m)}, X^{(i,m)}))_{i,m}$ is $O(\sum_{i=0}^{N-1} NM_i)$, which is clearly dominant in the computational cost $\mathcal{C}$ of the MWLS algorithm. To summarize, the computational cost is

$$\mathcal{C} = O(\sum_{i=0}^{N-1} NM_i).$$

Another advantage of the local polynomial basis is that there is substantial potential for parallel computing.

- To make the tail contributions (outside $[-R, R]^d$) small enough, we assume that X_i has exponential moments (uniformly in i), i.e. $\sup_{N\geq 1}\sup_{0\leq i\leq N} \mathbb{E}(e^{\lambda |X_i|_\infty}) < +\infty$ for some $\lambda > 0$, so that the choice $R := 2\theta_{\text{conv}}\lambda^{-1}\log(N + 1)$ is sufficient to ensure $(\mathbb{P}(|X_i|_\infty > R))^{1/2} = O(N^{-\theta_{\text{conv}}})$.

To simplify the discussion, we assume $\theta_L = \theta_c = 1$.

Smooth functions. Assume that $y_i(\cdot), z_i(\cdot)$ are respectively of class $\mathcal{C}^{l+1}_b(\mathbb{R}^d, \mathbb{R})$ and $\mathcal{C}^l_b(\mathbb{R}^d, (\mathbb{R}^q)^\top)$ (bounded with bounded derivatives) for some $l \in \mathbb{N}\backslash\{0\}$: this is similar to the discussion of [13, Section 4.4]. In fact, this is usually valid for the continuous-time limit (a priori estimates on the semi-linear PDE, see [7, 6]) provided that the data are smooth enough. In particular, we may assume $(\mathbf{A}''_\xi)$ with $\theta_\Phi = 1$. This leads to time-uniform bounds on the quantities $C_{y,i}, C_{z,i}, \bar{C}_{y,i}, \sqrt{T - t_i}\bar{C}_{z,i}$.

Set

$$\delta_{y,i} := N^{-\frac{\theta_{\text{conv}}}{l+1}}, \quad \delta_{z,i} := N^{-\frac{\theta_{\text{conv}}}{l}}, \quad M_i := (\log(N + 1))^{d+1}N^{\theta_{\text{conv}}(2+\frac{d}{l})},$$

take $\mathcal{K}_{Y,i} := \mathcal{P}^{l,\delta_{y,i},R}_{\text{loc.}}$ and $\mathcal{K}_{Z,i} := \mathcal{P}^{l-1,\delta_{z,i},R}_{\text{loc.}}$. From Proposition 3.9, Theorem 3.10 and the inequality (3.21), it is easy to check that

$$\mathcal{E}^Y_{\text{App.},i} = O(N^{-\theta_{\text{conv}}}), \quad \mathcal{E}^Y_{\text{Dep.},i} = o(N^{-\theta_{\text{conv}}}), \quad \bar{C}_{y,i}\sqrt{\frac{K_{Y,i}}{M_i}} = o\Big(N^{-\theta_{\text{conv}}}/\sqrt{\log(N + 1)}\Big),$$

$$\mathcal{E}^Z_{\text{App.},i} = O(N^{-\theta_{\text{conv}}}), \quad \mathcal{E}^Z_{\text{Dep.},i} = O(N^{-\theta_{\text{conv}}}), \quad \bar{C}_{z,i}\sqrt{\frac{K_{Z,i}}{M_i}} = (T - t_i)^{-\frac{1}{2}}O\Big(N^{-\theta_{\text{conv}}}/\sqrt{\log(N + 1)}\Big).$$

Consequently, using Lemma 2.1, we finally obtain

$$\Big(\mathbb{E}[\|y_i - y^M_i\|^2_{i,M}]\Big)^{1/2} = O(N^{-\theta_{\text{conv}}}), \quad \Big(\mathbb{E}[\|z_i - z^M_i\|^2_{i,M}]\Big)^{1/2} = O(N^{-\theta_{\text{conv}}})\Big(1 + \frac{(T - t_i)^{-\frac{1}{2}}}{\sqrt{\log(N + 1)}}\Big).$$

For any time-grid $\pi = \pi^{(\theta_\pi)}$, we get $\sup_{0\leq i\leq N} \mathbb{E}[\|y_i-y^M_i\|^2_{i,M}]+\sum_{i=0}^{N-1} \Delta_i\mathbb{E}[\|z_i-z^M_i\|^2_{i,M}] = O(N^{-2\theta_{\text{conv}}})$. The computational cost is $\mathcal{C} = O\big(\log(N + 1))^{d+1}N^{\theta_{\text{conv}}(2+\frac{d}{l})+2}\big)$. Ignoring the logarithmic factors, we obtain a final accuracy in terms of the computational cost:

$$\mathcal{C}^{-\frac{1}{(2+\frac{d}{l})+\frac{2}{\theta_{\text{conv}}}}}.$$

It should be compared with the rate $\mathcal{C}^{-\frac{1}{(2+\frac{d}{l})+\frac{3}{\theta_{\text{conv}}}}}$ which is valid for the Least Squares Multi-step forward Dynamical Programming algorithm (LSMDP) [13]. This shows an improvement on the rate, although there is no change in the dependence on dimension. The ratio d/l is the usual balance between dimension and smoothness, arising when approximating a multidimensional function. The controls of MWLS are stated in stronger norms than the controls of LSMDP, and despite that, the estimates improve. The convergence improvement is due to better MWDP-intrinsic estimates on Z, which avoid the $1/\Delta_i$-factor of the LSMDP. This results in better local error bounds, whence better global estimates. The reader can easily check that this happens already in the simple case with null driver.

Hölder terminal condition. We investigate the case of non-smooth terminal condition, where nevertheless there is a smoothing effect of the conditional expectation yielding smooth value functions $(y_i(\cdot), z_i(\cdot))$. Namely, assume that Φ is bounded and θ_Φ-Hölder continuous (in particular with $(\mathbf{A}''_\xi)$), and that, for all i, the function $y_i(\cdot)$ (resp. $z_i(\cdot)$) is $(l+1)$-times (resp. l-times) continuously differentiable with highest derivatives bounded by

$$|D_x^{l+1} y_i|_\infty \leq C(T-t_i)^{(\theta_\Phi - l)/2}, \qquad |D_x^l z_i|_\infty \leq C(T-t_i)^{(\theta_\Phi-(l+1))/2}. \tag{3.22}$$

These qualitative assumptions are related to the works of [7, 6], who have determined similar estimates for the gradients of quasi-linear PDEs under quite general conditions on the driver, terminal condition and differential operator. Their estimates cover the case $l = 0$ [7, Theorem 2.1] or $\theta_\Phi = 0$ and $l \geq 1$ [6, Theorem 1.4], but the Hölder continuous setting with high order derivatives is not investigated. We therefore extrapolate these results in the assumptions (3.22) for the purposes of this discussion.

In this setting, we have time-uniform bounds on the quantities $C_{y,i}, (T-t_i)^{\frac{1-\theta_\Phi}{2}} C_{z,i}, \bar{C}_{y,i}, \sqrt{T-t_i}\bar{C}_{z,i}$. Set

$$\delta_{y,i} := \sqrt{T-t_i} N^{-\frac{\theta_{\mathrm{conv}}}{l+1}}, \quad \delta_{z,i} := \sqrt{T-t_i} N^{-\frac{\theta_{\mathrm{conv}}}{l}}, \quad M_i := (\log(N+1))^{d+1} N^{\theta_{\mathrm{conv}}(2+\frac{d}{l})}(T-t_i)^{-d/2},$$

take $\mathcal{K}_{Y,i} := \mathcal{P}_{\mathrm{loc.}}^{l,\delta_{y,i},R}$ and $\mathcal{K}_{Z,i} := \mathcal{P}_{\mathrm{loc.}}^{l-1,\delta_{z,i},R}$. Similarly to before, using in particular (3.21), we eventually obtain

$$\begin{aligned}
&\mathcal{E}^Y_{\mathrm{App.},i} = O(N^{-\theta_{\mathrm{conv}}}), \quad \mathcal{E}^Y_{\mathrm{Dep.},i} = o(N^{-\theta_{\mathrm{conv}}}), \quad \bar{C}_{y,i}\sqrt{\frac{K_{Y,i}}{M_i}} = o\Big(N^{-\theta_{\mathrm{conv}}}/\sqrt{\log(N+1)}\Big),\\
&\mathcal{E}^Z_{\mathrm{App.},i} = (T-t_i)^{\frac{\theta_\Phi-1}{2}} O(N^{-\theta_{\mathrm{conv}}}), \quad \mathcal{E}^Z_{\mathrm{Dep.},i} = (T-t_i)^{\frac{\theta_\Phi-1}{2}} O(N^{-\theta_{\mathrm{conv}}}),\\
&\bar{C}_{z,i}\sqrt{\frac{K_{Z,i}}{M_i}} = (T-t_i)^{-\frac{1}{2}} O\Big(N^{-\theta_{\mathrm{conv}}}/\sqrt{\log(N+1)}\Big).
\end{aligned}$$

Consequently, using Lemma 2.1, we finally obtain

$$\begin{aligned}
\Big(\mathbb{E}[\|y_i - y_i^M\|^2_{i,M}]\Big)^{1/2} &= O(N^{-\theta_{\mathrm{conv}}}),\\
\Big(\mathbb{E}[\|z_i - z_i^M\|^2_{i,M}]\Big)^{1/2} &= O(N^{-\theta_{\mathrm{conv}}})\Big((T-t_i)^{\frac{\theta_\Phi-1}{2}} + \frac{(T-t_i)^{-\frac{1}{2}}}{\sqrt{\log(N+1)}}\Big).
\end{aligned}$$

The computation cost is given by (under the assumption $\pi = \pi^{(\theta_\pi)}$)

$$\mathcal{C} = O\Big(\sum_{i=0}^{N-1} N M_i\Big) = O\Big(\log(N+1))^{d+1} N^{1+\theta_{\mathrm{conv}}(2+\frac{d}{l})}\Big)\sum_{i=0}^{N-1}(1-\frac{i}{N})^{-\frac{d}{2\theta_\pi}}.$$

Up to possibly a $\log(N)$-factor, the last sum is $O(N^{\frac{d}{2\theta_\pi}\vee 1})$. Ignoring the logarithmic factors, we obtain $\mathcal{C} = O(N^{1+\frac{d}{2\theta_\pi}\vee 1+\theta_{\mathrm{conv}}(2+\frac{d}{l})})$. Equivalently, as a function of the computational cost, the convergence rate of the final accuracy equals

$$\mathcal{C}^{-\frac{1}{(2+\frac{d}{l})+\frac{1}{\theta_{\mathrm{conv}}}(1+\frac{d}{2\theta_\pi}\vee 1)}}.$$

Following [11] (under suitable assumptions), two time-grid choices are possible for solving the same BSDE.

- The uniform grid $\pi = \pi^{(1)}$ gives $\theta_{\mathrm{conv}} = \theta_\Phi/2$ (at least). The convergence order becomes $(2 + \frac{d}{l} + \frac{2}{\theta_\Phi}(1+\frac{d}{2}\vee 1))^{-1}$.

- The grid $\pi = \pi^{(\theta)}$ (for $\theta < \theta_\Phi$) gives $\theta_{\text{conv}} = 1/2$. Taking $\theta \uparrow \theta_\Phi$, the convergence order is $(2 + \frac{d}{l} + \frac{2}{\theta_\Phi}(\theta_\Phi + \frac{d}{2} \vee \theta_\Phi))^{-1}$.

The grid $\pi^{(\theta)}$ exhibits a better convergence rate compared to the uniform grid. This corroborates the interest in time grids that are well adapted to the regularity of the data. These features will be investigated in subsequent more experimental works.

A Appendix

A.1 Proof of Lemmas 2.1, 2.2 and 2.3

A.1.1 Proof of Lemmas 2.1

The first inequality, for $\alpha \le 1$, follows by bounding the sum by $\int_{t_i}^{t_k} (t_k - t)^{\alpha-1} dt$, whence $B_{\alpha,1} = 1/\alpha$. The case $\alpha > 1$ is obvious with $B_{\alpha,1} = 1$. For the second inequality, there are two main cases:
$\triangleright$ If $\alpha \ge 1$ and $\beta \ge 1$, the advertised inequality is obvious with $B_{\alpha,\beta} = 1$.
$\triangleright$ Now, assume the complementary case, i.e. $\alpha < 1$ and/or $\beta < 1$, and first consider the case $t_i = 0$ and $t_k = 1$. We set $\varphi(s) = (1-s)^{\alpha-1} s^{\beta-1}$ and we use the integral $\int_0^1 \varphi(s) ds$ (equivalent to the usual beta function with parameters (α, β)) to bound the sum. A simple but useful property (due to $\alpha < 1$ and/or $\beta < 1$) is that φ is either monotone or has a unique minimum on (0,1), whence

$$(1-t_j)^{\alpha-1} t_j^{\beta-1} \Delta_j \le R_\pi \int_{t_{j-1}}^{t_j} \varphi(s) ds + \int_{t_j}^{t_{j+1}} \varphi(s) ds.$$

Summing up over j and defining $B_{\alpha,\beta} = (1 + R_\pi) \int_0^1 \varphi(s) ds$ concludes the proof for the simple case. For general t_i and t_k one can use the bounds on the simple case by rearranging the j-sum which is equal to

$$(t_k - t_i)^{\alpha+\beta-1} \sum_{j=i+1}^{k-1} (1 - \frac{t_j - t_i}{t_k - t_i})^{\alpha-1} (\frac{t_j - t_i}{t_k - t_i})^{\beta-1} \frac{\Delta_j}{t_k - t_i} \le B_{\alpha,\beta} (t_k - t_i)^{\alpha+\beta-1}.$$

$\square$

A.1.2 Proof of Lemma 2.2

If $\alpha \ge \frac{1}{2}$, the result trivially holds with $\mathcal{C}_{(2.2a)} = 1$ and $\mathcal{C}_{(2.2b)} = C_u T^{\alpha - \frac{1}{2}}$.
Now, assume $\alpha < \frac{1}{2}$: if (2.1) holds, of course we also have

$$u_j \le w_j + \sum_{l=j+1}^{N-1} \frac{w_l \Delta_l}{(T - t_l)^{\frac{1}{2}-\beta} (t_l - t_j)^{\frac{1}{2}-\alpha}} + C_u \sum_{l=j+1}^{N-1} \frac{u_l \Delta_l}{(T - t_l)^{\frac{1}{2}-\beta} (t_l - t_j)^{\frac{1}{2}-\alpha}}. \tag{A.1}$$

By substituting (A.1) into the last sum, and using Lemma 2.1 we observe

$$\begin{aligned}
&\sum_{l=j+1}^{N-1} \frac{u_l \Delta_l}{(T - t_l)^{\frac{1}{2}-\beta} (t_l - t_j)^{\frac{1}{2}-\alpha}} \\
&\le \sum_{l=j+1}^{N-1} \frac{w_l \Delta_l}{(T - t_l)^{\frac{1}{2}-\beta} (t_l - t_j)^{\frac{1}{2}-\alpha}} + \sum_{l=j+1}^{N-1} \frac{\sum_{r=l+1}^{N-1} \frac{w_r \Delta_r}{(T-t_r)^{\frac{1}{2}-\beta} (t_r - t_l)^{\frac{1}{2}-\alpha}} \Delta_l}{(T - t_l)^{\frac{1}{2}-\beta} (t_l - t_j)^{\frac{1}{2}-\alpha}}
\end{aligned}$$

$$
\begin{aligned}
&\quad + C_u \sum_{l=j+1}^{N-1} \frac{\sum_{r=l+1}^{N-1} \frac{u_r \Delta_r}{(T-t_r)^{\frac{1}{2}-\beta}(t_r-t_l)^{\frac{1}{2}-\alpha}} \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}(t_l-t_j)^{\frac{1}{2}-\alpha}} \\
&\le \sum_{l=j+1}^{N-1} \frac{w_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}(t_l-t_j)^{\frac{1}{2}-\alpha}} + B_{\alpha+\beta,\frac{1}{2}+\alpha} \sum_{r=j+2}^{N-1} \frac{w_r \Delta_r}{(T-t_r)^{\frac{1}{2}-\beta}(t_r-t_j)^{\frac{1}{2}-2\alpha-\beta}} \\
&\quad + C_u B_{\alpha+\beta,\frac{1}{2}+\alpha} \sum_{r=j+2}^{N-1} \frac{u_r \Delta_r}{(T-t_r)^{\frac{1}{2}-\beta}(t_r-t_j)^{\frac{1}{2}-2\alpha-\beta}}.
\end{aligned}
$$

Substituting into (A.1), we observe that we have an equation of similar form to (A.1), except that, in the sum involving u, $\alpha \mapsto 2\alpha + \beta$ and $C_u \mapsto C_u^2 B_{\alpha+\beta,\frac{1}{2}+\alpha}$, and, in the sum involving w, $w \mapsto (1 + C_u(1 + T^{\alpha+\beta} B_{\alpha+\beta,\frac{1}{2}+\alpha}))w$.

After κ iterations of the previous step, we obtain $\alpha \mapsto 2^\kappa(\alpha+\beta) - \beta =: \alpha_\kappa$. Hence, for κ sufficiently large so that $\alpha_\kappa \ge \frac{1}{2}$, i.e. $\kappa \ge \log_2\left(\frac{\frac{1}{2}+\beta}{\alpha+\beta}\right)$, we obtain the bound advertised in the Lemma statement. □

A.1.3 Proof of Lemma 2.3

W.l.o.g we can assume that $\mathcal{C}_{(2.2a)} = 1$ in (2.2); if it is not, one can redefine w as $\mathcal{C}_{(2.2a)} w$. We first prove the case $\gamma = 1$. Define

$$
\zeta_s := 2\mathcal{C}_{(2.2b)} \int_0^s \frac{dr}{(T-r)^{\frac{1}{2}-\beta}} \le \frac{2}{1+2\beta} 2\mathcal{C}_{(2.2b)} T^{(1+2\beta)/2}, \tag{A.2}
$$

and write $\zeta_j = \zeta_{t_j}$ for brevity. We first multiply (2.2) by $\frac{e_j^\zeta \Delta_j}{(T-t_j)^{\frac{1}{2}-\beta}}$, then sum the outcome equation over $j \in \{i+1, \dots, N-1\}$, and finally switch the order of summation on the right hand side as follows:

$$
\begin{aligned}
\sum_{j=i+1}^{N-1} \frac{u_j e^{\zeta_j} \Delta_j}{(T-t_j)^{\frac{1}{2}-\beta}} &\le \sum_{j=i+1}^{N-1} \frac{w_j e^{\zeta_j} \Delta_j}{(T-t_j)^{\frac{1}{2}-\beta}} + \sum_{j=i+1}^{N-1} \frac{\sum_{l=j+1}^{N-1} \frac{w_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}(t_l-t_j)^{\frac{1}{2}-\alpha}} e^{\zeta_j} \Delta_j}{(T-t_j)^{\frac{1}{2}-\beta}} \\
&\quad + \mathcal{C}_{(2.2b)} \sum_{j=i+1}^{N-1} \frac{\sum_{l=j+1}^{N-1} \frac{u_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}} e^{\zeta_j} \Delta_j}{(T-t_j)^{\frac{1}{2}-\beta}} \\
&\le e^{\zeta_T} \sum_{j=i+1}^{N-1} \frac{w_j \Delta_j}{(T-t_j)^{\frac{1}{2}-\beta}} + e^{\zeta_T} B_{\alpha+\beta,1} \sum_{l=i+2}^{N-1} \frac{w_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}(t_l-t_i)^{-\alpha-\beta}} \\
&\quad + \mathcal{C}_{(2.2b)} \sum_{l=i+2}^{N-1} \frac{u_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}} \sum_{j=i+1}^{l-1} \frac{e^{\zeta_j} \Delta_j}{(T-t_j)^{\frac{1}{2}-\beta}} \\
&\le e^{\zeta_T}(1 + B_{\alpha+\beta,1} T^{\alpha+\beta}) \sum_{l=i+1}^{N-1} \frac{w_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}} + \frac{1}{2} \sum_{l=i+1}^{N-1} \frac{u_l e^{\zeta_l} \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}}
\end{aligned}
$$

where we have used (because ζ is non-decreasing and $\beta \le \frac{1}{2}$)

$$
\mathcal{C}_{(2.2b)} \sum_{j=i+1}^{l-1} \frac{e^{\zeta_j} \Delta_j}{(T-t_j)^{\frac{1}{2}-\beta}} \le \int_{t_{i+1}}^{t_l} \frac{\mathcal{C}_{(2.2b)} e^{\zeta_s}}{(T-s)^{\frac{1}{2}-\beta}} ds \le \frac{e^{\zeta_l}}{2}.
$$

By subtracting the term with factor $\frac{1}{2}$, the result for $\gamma = 1$ follows. Moreover, plugging the result into (2.2), and returning to general $\mathcal{C}_{(2.2a)}$, gives

$$u_j \leq \mathcal{C}_{(\mathrm{A}.3)} w_j + \mathcal{C}_{(\mathrm{A}.3)} \sum_{l=j+1}^{N-1} \frac{w_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}(t_l-t_j)^{\frac{1}{2}-\alpha}} + \mathcal{C}_{(\mathrm{A}.3)} \sum_{l=j+1}^{N-1} \frac{w_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}} \tag{A.3}$$

for a constant $\mathcal{C}_{(\mathrm{A}.3)} := 2\mathcal{C}_{(2.2a)} e^{\zeta_T}(1 + B_{\alpha+\beta,1} T^{\alpha+\beta})$. Now for the general case $\gamma > 0$, observe that, for any $\delta \geq 0$, one obtains by change of the order of summation that

$$\sum_{j=i+1}^{N-1} \frac{\sum_{l=j+1}^{N-1} \frac{w_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}(t_l-t_j)^{\frac{1}{2}-\delta}} \Delta_j}{(T-t_j)^{\frac{1}{2}-\beta}(t_j-t_i)^{1-\gamma}} \leq B_{\beta+\delta,\gamma} \sum_{l=i+2}^{N-1} \frac{w_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}(t_l-t_i)^{1-\beta-\delta-\gamma}}. \tag{A.4}$$

Thus, (A.3) yields

$$\begin{aligned}
\sum_{j=i+1}^{N-1} \frac{u_j \Delta_j}{(T-t_j)^{\frac{1}{2}-\beta}(t_j-t_i)^{1-\gamma}} &\leq \mathcal{C}_{(\mathrm{A}.3)} \sum_{j=i+1}^{N-1} \frac{w_j \Delta_j}{(T-t_j)^{\frac{1}{2}-\beta}(t_j-t_i)^{1-\gamma}} \\
&\quad + \mathcal{C}_{(\mathrm{A}.3)} B_{\beta+\alpha,\gamma} \sum_{l=i+2}^{N-1} \frac{w_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}(t_l-t_i)^{1-\beta-\alpha-\gamma}} \\
&\quad + \mathcal{C}_{(\mathrm{A}.3)} B_{\beta+\frac{1}{2},\gamma} \sum_{l=i+2}^{N-1} \frac{w_l \Delta_l}{(T-t_l)^{\frac{1}{2}-\beta}(t_l-t_i)^{\frac{1}{2}-\beta-\gamma}} \\
&\leq \mathcal{C}_{(\mathrm{A}.3)}(1 + B_{\beta+\alpha,\gamma} T^{\alpha+\beta} + B_{\beta+\frac{1}{2},\gamma} T^{\frac{1}{2}+\beta}) \sum_{j=i+1}^{N-1} \frac{w_j \Delta_j}{(T-t_j)^{\frac{1}{2}-\beta}(t_j-t_i)^{1-\gamma}}.
\end{aligned}$$

□

A.2 Proof of Lemma 3.7

Using the bounds $C_{y,i}$ and $C_{z,i}$ on $y_i^{(M)}(\cdot)$ and $z_i^{(M)}(\cdot)$, respectively, one applies the local Lipschitz continuity and boundedness properties of f_j given in $(\mathbf{A_F})$ to obtain the bound

$$|f_j(x_j, y_{j+1}^{(M)}(x_{j+1}), z_j^{(M)}(x_j))| \leq \frac{L_f(C_{y,j+1} + C_{z,j})}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}} + \frac{C_f}{(T-t_j)^{1-\theta_c}}. \tag{A.5}$$

Substituting this into the definition $S_{Y,i}^{(M)}(\underline{\mathbf{x}}^{(i)})$ (see (3.8)), it follows from $(\mathbf{A_F})$ that

$$|S_{Y,i}^{(M)}(\underline{\mathbf{x}}^{(i)})| \leq C_\xi + \sum_{j=i}^{N-1} \Big(\frac{L_f(C_{y,j+1} + C_{z,j})}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}} + \frac{C_f}{(T-t_j)^{1-\theta_c}} \Big) \Delta_j.$$

Substituting the value of $C_{y,j}$ and $C_{z,j}$ given in Equations (2.9) and (2.10), respectively, using the crude bound $|\xi - \mathbb{E}_i \xi|_{2,i} \leq C_\xi$ and Lemma 2.1, we obtain the bound $\bar{C}_{y,i}$, with the form (3.9).

To obtain the bound $\bar{C}_{z,i}$, apply first the triangle inequality on the conditional standard deviation of $S_{l,Z,i}^{(M)}(H^{(i,m)}, X^{(i,m)})$; second use the bound (A.5) on the driver, and the bound $(\mathbf{A_H})$ to obtain

$$\sqrt{\mathbb{V}\mathrm{ar}\Big[S_{l,Z,i}^{(M)}(H^{(i,m)}, X^{(i,m)}) \ \Big| \ \mathcal{F}_i^{(M)} \Big]}$$

$$\leq \frac{C_\xi C_M}{\sqrt{T-t_i}} + \sum_{j=i+1}^{N-1} \Big(\frac{L_f(C_{y,j+1}+C_{z,j})}{(T-t_j)^{\frac{1}{2}-\frac{\theta_L}{2}}} + \frac{C_f}{(T-t_j)^{1-\theta_c}} \Big) \frac{C_M}{\sqrt{t_j-t_i}} \Delta_j.$$

Then, the computation of $\bar{C}_{z,i}$ follows again from Equations (2.9) and (2.10), and Lemma 2.1. The form (3.9) is also derived. We skip details. □

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