Linear multifractional multistable motion: LePage series representation and modulus of continuity
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Abstract - In this paper, we obtain an upper bound of the modulus of continuity of linear multifractional multistable random motions. Such processes are generalizations of linear multifractional $\alpha$-stable motions for which the stability index $\alpha$ is also allowed to vary in time. In the case of linear multifractional $\alpha$-stable motions, we improve the recent result of [2]. The main idea is to consider some conditionnally sub-Gaussian LePage series representations to fit the framework of [5].

Key words and phrases : stable and multistable random fields, modulus of continuity.

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1 Introduction

Self-similar random fields are required to model persistent phenomena in internet traffic, hydrology, geophysics or financial markets, e.g. [1, 22]. The fractional Brownian motion ([15, 9]) provides the most famous self-similar model. Nevertheless, in image modeling, in finance or in biology for example, the phenomena under study are rarely Gaussian. Then, $\alpha$-stable random processes have been proposed as an alternative to Gaussian modeling, since they allow to model data with heavy tails, such as in internet traffic [16]. The linear fractional stable motion, which has been proposed in [21, 14], is one of the numerous stable extensions of the fractional Brownian motion. Let us recall how this self-similar random motion can be defined through a stochastic integral representation. To this way, let us consider $H_1 \in (0, 1)$, $\alpha_1 \in (0, 2)$ and $M_{\alpha_1}$ a real-valued symmetric $\alpha_1$-stable random measure with Lebesgue control measure (see [17] p.281 for details on such measures). Then, a linear fractional stable motion is defined by

$$X_{\alpha_1,H_1}(t) = \int_{\mathbb{R}} f_+(\alpha_1, H_1, t, \xi) M_{\alpha_1}(d\xi), \quad t \in \mathbb{R}$$ (1.1)

where $f_+$ is defined by

$$f_+(\alpha_1, H_1, t, \xi) = (t - \xi)^{H_1 - 1/\alpha_1} - (-\xi)^{H_1 - 1/\alpha_1}$$ (1.2)
with for \( c \in \mathbb{R} \),
\[
(x)_+^c = \begin{cases} 
 x^c & \text{if } x > 0 \\
 0 & \text{if } x \leq 0.
\end{cases}
\]

Since the self-similarity property is a global property which can be too restrictive for applications, a multifractional generalization \( X_{\alpha, h} \) of this process has also been introduced by [18] to model internet traffic, by replacing \( H_1 \) by a real function \( h \) with values on \((0, 1)\). Some necessary and sufficient conditions for the stochastic continuity of the linear multifractional stable motion \( X_{\alpha, h} \) have been given in [18] and its Hölder sample path regularity has been studied in [19]. The Hölder sample path properties have also been improved in [2] by establishing upper and lower bounds for the modulus of continuity. In the following, we will improve the upper bound, using the results we established in [5]. Let us mention that in the case where \( h \equiv H_1 \) is constant, that is when \( X_{\alpha, h} \) is a linear fractional stable motion, sample path regularity properties have previously been studied in [17, 20, 10].

Moreover, the framework of [5] allows to study \( X_{\alpha, h} \) as well as some multistable generalizations for which the stability index \( \alpha \) is also allowed to vary with \( t \). Multistable processes have been defined in [7] using sums over Poisson processes or in [6] using a Klass-Ferguson LePage series.

In this paper we consider a random field \( S_m \) defined using a Lépaege series representation of the linear fractional \( \alpha_1 \)-stable motion and such that
\[
S_m(\alpha(t), h(t), t), \quad t \in \mathbb{R}
\]
is a linear multifractional multistable motion. This auxiliary random field \( S_m \) allows to study the variations due to the functions \( \alpha \), \( h \) and to the position \( t \) separately. Then, to study sample path regularity of linear multistable motions, our first step is to establish an upper bound for the modulus of continuity of the field \( S_m \) considering a conditionally sub-Gaussian representation and applying [5]. The main property of sub-Gaussian random variables, which have been introduced by [8], is that their tail distributions decrease exponentially as the Gaussian ones. This property is one of the main tool used in [5] to study the sample path regularity property of conditionally sub-Gaussian random series.

The paper is organized as follows. Section 2 introduces LePage series random fields under study. An upper bound of their modulus of continuity and a rate of convergence are stated in Section 3. Section 4 focuses on linear multifractional multistable motions. Some technical proofs are postponed to the appendix for reader convenience.

### 2 LePage series models

In order to define LePage series, let us introduce some notation.
**Hypothesis 2.1** Let \((g_n)_{n \geq 1}, (\xi_n)_{n \geq 1}\) and \((T_n)_{n \geq 1}\) be three independent sequences of random variables satisfying the following conditions.

1. \((g_n)_{n \geq 1}\) is a sequence of independent identically distributed (i.i.d.) real-valued symmetric sub-Gaussian random variables, that is such that there exists \(s \in [0, +\infty)\) for which
   \[
   \forall \lambda \in \mathbb{R}, \ E(e^{\lambda g_n}) \leq e^{s^2 \lambda^2}. \tag{2.3}
   \]

2. \((\xi_n)_{n \geq 1}\) is a sequence of i.i.d. random variables with common law
   \[\mu(d\xi) = m(\xi)d\xi\]
   equivalent to the Lebesgue measure (that is such that \(m(\xi) > 0\) for almost every \(\xi\)).

3. \(T_n\) is the \(n\)th arrival time of a Poisson process with intensity 1.

Let us now introduce the random field \((S_{m}(\alpha, H, t))_{(\alpha, H, t) \in (0, 2) \times (0, 1) \times \mathbb{R}}\) we study in this paper.

**Proposition 2.1 (LePage series representation)** Assume that Hypothesis 2.1 is fulfilled and let \(f_+\) be defined by (1.2). Then, for any \((\alpha, H, t) \in (0, 2) \times (0, 1) \times \mathbb{R}\), the sequence

\[
S_{m,N}(\alpha, H, t) = \sum_{n=1}^{N} T_n^{-1/\alpha} f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha} g_n, \quad N \geq 1 \tag{2.4}
\]

converges almost surely and its limit is denoted by

\[
S_m(\alpha, H, t) := \sum_{n=1}^{+\infty} T_n^{-1/\alpha} f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha} g_n. \tag{2.5}
\]

**Proof.** Let \((\alpha, H, t) \in (0, 2) \times (0, 1) \times \mathbb{R}\). Then, since Hypothesis 2.1 holds, the variables

\[W_n := f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha} g_n, \quad n \geq 1,
\]

are i.i.d., symmetric and such that

\[E(|W_1|^\alpha) = E(|g_1|^\alpha) \int \left| f_+(\alpha, H, t, \xi) \right|^\alpha d\xi < +\infty,
\]

since \(g_1\) and \(\xi_1\) are independent (see e.g. [17]). Therefore, by Theorem 5.1 of [13], the sequence

\[
\left( \sum_{n=1}^{N} T_n^{-1/\alpha} W_n \right)_{N \geq 1}
\]
converges almost surely as \( N \to +\infty \), that is \((S_{m,N}(\alpha, H, t))_{N \geq 1}\) converges almost surely.

\[ \square \]

Let us conclude this section by some remarks.

**Remark 2.1** According to Proposition 5.1 of [5], the finite dimensional distributions of \( S_m \) do not depend on \( m \) as soon as Condition 2 of Hypothesis 2.1 holds. Moreover, when studying the sample path regularity of \( S_m \), Proposition 5.1 of [5] allows us to change \( m \) by a more convenient function \( \tilde{m} \) if necessary.

**Remark 2.2** When \( \alpha = \alpha_1 \in (0, 2) \) is fixed, \((S_m(\alpha_1, H, t))_{(H,t) \in (0,1) \times \mathbb{R}}\) is an \( \alpha_1 \)-stable symmetric random field, which can also be represented as an integral under an \( \alpha_1 \)-stable random measure \( M_{\alpha_1} \) with Lebesgue control measure. More precisely, for every \( \alpha_1 \in (0, 2) \),

\[
(S_m(\alpha_1, H, t))_{(H,t) \in (0,1) \times \mathbb{R}} \overset{fdd}{=} d_{\alpha_1}(Y_{\alpha_1}(H, t))_{(H,t) \in (0,1) \times \mathbb{R}} \quad (2.6)
\]

where \( fdd \) means equality of finite distributions and

\[
Y_{\alpha_1}(H, t) := \int_{\mathbb{R}} f_+(\alpha_1, H, t, \xi) M_{\alpha_1}(d\xi), \quad (H, t) \in (0, 1) \times \mathbb{R}, \quad (2.7)
\]

for \( M_{\alpha_1} \) a real-valued symmetric \( \alpha_1 \)-stable random measure with Lebesgue control measure and

\[
d_{\alpha_1} := \mathbb{E}(|g_1|^{\alpha_1})^{1/\alpha_1} \left( \int_0^{+\infty} \frac{\sin x}{x^{\alpha_1}} dx \right)^{1/\alpha_1}. \quad (2.8)
\]

One can check Equation (2.6) following the proof of Proposition 5.1 of [5] or Proposition 4.2 of [4], which is a consequence of Lemma 4.1 of [11].

### 3 Sample path properties

Several papers [20, 10, 18, 19, 2] have already investigated sample path properties of the linear fractional stable motion \( X_{\alpha_1, H_1} \) defined by Equation (1.1) or of its multifractional generalization \( X_{\alpha_1, h} \) defined on \( \mathbb{R} \) by

\[
X_{\alpha_1, h}(t) := Y_{\alpha_1}(h(t), t), \quad t \in \mathbb{R} \quad (3.9)
\]

where \( \alpha_1 \in (0, 2) \), \( Y_{\alpha_1} \) is given by (2.7) and \( h \) is a function with values in \((0, 1)\). In the following, we improve the upper bound of the global modulus of continuity of \( X_{\alpha_1, h} \) stated in [2]. Our first step is to establish an upper bound for the global modulus of continuity of the field \( S_m \) defined by (2.5) on a compact set \( K \) of \((0, 2) \times (0, 1) \times \mathbb{R}\). To obtain our upper bound, we use
the results we established in [5] on conditionally sub-Gaussian random series.

Let us first recall (see [17] for example) that the \( \alpha \)-stable random process
\( X_{\alpha_1, H_1} = (Y_{\alpha_1}(H_1, t))_{t \in \mathbb{R}} \) is unbounded almost surely on each compact set with non-empty interior when \( H_1 < 1/\alpha_1 \). A similar result holds for \( S_m \) as stated in the following proposition.

**Proposition 3.1** Assume that \( K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [a, b] \subset (0, 2) \times (0, 1) \times \mathbb{R} \) with \( 0 < \alpha_1 \leq \alpha_2 < 2, 0 < H_1 \leq H_2 < 1 \) and \( a < b \).

1. If \( H_1 < 1/\alpha_1 \), then the random field \( S_m \) is almost surely unbounded on \( K \).

2. If \( H_1 = 1/\alpha_1 \), then \( S_m \) does not have almost surely continuous sample paths on the compact set \( K \).

**Proof.** By Equation (2.6)
\[
(S_m(\alpha_1, H_1, t))_{t \in \mathbb{R}} \overset{fdd}{=} d_{\alpha_1}(X_{\alpha_1, H_1}(t))_{t \in \mathbb{R}}, \tag{3.10}
\]
where \( d_{\alpha_1} \) is defined by Equation (2.8) and \( X_{\alpha_1, H_1} \) is the linear fractional stable motion given by (1.1).

Let us first assume that \( H_1 < 1/\alpha_1 \). Then, since \( a < b \), by Corollary 10.2.4 of [17], \( (S_m(\alpha_1, H_1, t))_{t \in \mathbb{R}} \) is unbounded almost surely on the compact set \([a, b] \). It follows that
\[
\sup_{(\alpha, H, t) \in K} |S_m(\alpha, H, t)| = +\infty \text{ a.s.}
\]
since \( \sup_{(\alpha, H, t) \in K} |S_m(\alpha, H, t)| \geq \sup_{t \in [a, b]} |S_m(\alpha_1, H_1, t)| \).

Let us now assume that \( H_1 = 1/\alpha_1 \) (which implies that \( \alpha_1 > 1 \)). Then,
\[
X_{\alpha_1, H_1} = (M_{\alpha_1}([0, t])1_{t>0} + M_{\alpha_1}((t, 0])1_{t<0})_{t \in \mathbb{R}}
\]
is a Lévy \( \alpha_1 \)-stable motion and by Equation (3.10), so is the process \( (S_m(\alpha_1, H_1, t))_{t \in \mathbb{R}} \). Since \( \alpha_1 < 2 \), the stable motion \( (S_m(\alpha_1, 1/\alpha_1, t))_{t \in \mathbb{R}} \) is not a Brownian motion and then does not have almost surely continuous sample paths (see Exercise 2.7 p.64 of [12] for instance). This concludes the proof. \( \square \)

Therefore, it remains to study the sample paths on a compact set
\[
K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A] \subset (0, 2) \times (0, 1) \times \mathbb{R}
\]
such that \( H_1 > 1/\alpha_1 \), which implies that \( \alpha_1 \in (1, 2) \) and \( H_1 > 1/2 \).

The main result of this paper is the following theorem, which states an upper bound for the modulus of continuity of \( S_m \) on \( K \), and for some \( m \) a rate of uniform convergence on \( K \) for the series \( S_{m,N} \) defined by (2.4).
Theorem 3.1 Assume that Hypothesis 2.1 is fulfilled. Let $S_{m,N}$ and $S_m$ be defined by (2.4) and (2.5) and let us consider the compact set

$$K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A] \subset (1, 2) \times (1/2, 1) \times \mathbb{R}$$

with $A > 0$ and $H_1 > 1/\alpha_1$.

1. As $N \to +\infty$, the series $(S_{m,N})_{N \geq 1}$ converges uniformly on $K$ to $S_m$ and almost surely

$$\sup_{x, x' \in K, x \neq x'} \frac{|S_m(x) - S_m(x')|}{\tau(x - x') \sqrt{\log (\tau(x - x'))} + 1} < +\infty$$

with $\tau(z) = |\alpha| + |H| + |t|^{H_1 - 1/\alpha_1}$ for $z = (\alpha, H, t) \in \mathbb{R}^3$.

2. For $\eta > 0$, let us consider $m = m_\eta$ defined by

$$m_\eta(\xi) = c_\eta |\xi|^{-1} (1 + |\log(|\xi|)|)^{-1-\eta}, \quad (3.11)$$

with $c_\eta > 0$ such that $\int_{\mathbb{R}} m_\eta(\xi) d\xi = 1$. Then, almost surely

$$\sup_{N \geq 1} N^\varepsilon \sup_{x \in K} |S_{m,N}(x) - S_m(x)| < +\infty$$

for any $\varepsilon \in (0, 1/\alpha_2 - 1/2)$.

Proof. For all $x = (\alpha, H, t) \in (0, 2) \times (0, 1) \times \mathbb{R}$ and all integer $n \geq 1$, we consider

$$V_{m,n}(x) := f_+(\alpha, H, t, \xi_n)m(\xi_n)^{-1/\alpha}, \quad (3.12)$$

so that

$$S_{m,N}(x) = \sum_{n=1}^{N} T_n^{-1/\alpha} V_{m,n}(x) g_n \quad \text{and} \quad S_m(x) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} V_{m,n}(x) g_n.$$ 

Let us also remark that for all $x = (\alpha, H, t) \in (0, 2) \times (0, 1) \times \mathbb{R}$,

$$\mathbb{E}(|V_{m,n}(x)|^{\alpha}) = \int_{\mathbb{R}} |f_+(\alpha, H, t, \xi)|^{\alpha} d\xi < +\infty.$$ 

Note that if in Equation (2.3) the sub-Gaussian parameter $s$ of $g_n$ is less than 1, Equation (2.3) also holds for $s = 1$. Moreover, if $s$ is greater than 1 we may write $V_{m,n}(x) g_n = (s V_{m,n}(x)) g_n / s$ so that $g_n / s$ is sub-Gaussian with parameter 1. Hence without loss of generality we may and will assume that $s = 1$. It follows that $(g_n)_{n \geq 1}$, $(T_n)_{n \geq 1}$ and $(V_{m,n})_{n \geq 1}$ are three independent sequences that satisfy Assumption 4 in [5] on $(0, 2) \times (0, 1) \times \mathbb{R}$. Then, by Theorem 4.2 of [5], the result follows once we prove $\mathbb{E}(|V_{m,1}(x_0)|^{\alpha}) < +\infty$.
for some \( x_0 \in K \) and Equation (15) of [5] for \( p = 1 \), namely (in our setting) if there exists \( r > 0 \) such that

\[
\mathbb{E} \left( \sup_{x,x' \in K} \left[ \frac{|V_{m,1}(x) - V_{m,1}(x')|}{\tau(x - x')} \right]^2 \right) < +\infty. \tag{3.13}
\]

The following proposition, whose proof is postponed to the appendix, allows to find some \( m \) satisfying such conditions.

**Proposition 3.2** There exists a finite deterministic constant \( c_{3,1}(K) > 0 \) such that a.s. for all \( x, x' \in K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A] \),

\[
|V_{m,1}(x) - V_{m,1}(x')| \leq c_{3,1}(K) \tau(x - x') h_{m,K}(\xi_1),
\]

with, for almost every \( \xi \in \mathbb{R} \),

\[
h_{m,K}(\xi) = \max \left( m(\xi)^{-1/\alpha_1}, m(\xi)^{-1/\alpha_2} \right) (1 + |\log m(\xi)|) \tag{3.14}
\]

\[
\times \left( 1_{|\xi| \leq e} + |\xi|^{-1+H_2-1/\alpha_2} \log |\xi| 1_{|\xi| > e} \right).
\]

Let us first consider \( m = m_\eta \) given by (3.11) for some \( \eta > 0 \). In view of Proposition 3.2, since \( V_{m,1}(\alpha, H, 0) = 0 \) for all \((\alpha, H, 0) \in K\), up to use a finite covering of \( K \), it is enough to prove that there exists \( r > 0 \) with

\[
\mathbb{E} \left( h_{m_\eta,K}(\xi_1)^2 \right) < +\infty, \tag{3.15}
\]

for \( K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A] \) with \( \alpha_2 - \alpha_1 \leq r \). One has

\[
\mathbb{E}(h_{m_\eta,K}(\xi_1)^2) = \int_{\mathbb{R}} h_{m_\eta,K}(\xi)^2 m_\eta(\xi) d\xi
\]

\[
= \int_{|\xi| \leq e} + \int_{|\xi| > e} := I_1 + I_2.
\]

On the one hand,

\[
I_1 = \int_{|\xi| \leq e} m_\eta(\xi) \max(m_\eta(\xi)^{-2/\alpha_1}, m_\eta(\xi)^{-2/\alpha_2})(1 + |\log(m_\eta(\xi))|)^2 d\xi
\]

\[
\leq c_{3,2}(\eta, K) \int_{|\xi| \leq e} |\xi|^{-1+2/\alpha_2} (1 + |\log(|\xi|)|)^{(1+\eta)(2/\alpha_1-1)} (1 + |\log(m_\eta(\xi))|)^2 d\xi,
\]

with \( c_{3,2}(\eta, K) \) a positive finite constant. It follows that \( I_1 < +\infty \) since \( \alpha_2 > 0 \). On the other hand,

\[
I_2 = \int_{|\xi| > e} m_\eta(\xi) \max(m_\eta(\xi)^{-2/\alpha_1}, m_\eta(\xi)^{-2/\alpha_2})(1 + |\log(m_\eta(\xi))|)^2 |\xi|^{2(H_2-1/\alpha_2)-2} \log(|\xi|)^2 d\xi
\]

\[
\leq c_{3,3}(\eta, K) \int_{|\xi| > e} |\xi|^{2(H_2+1/\alpha_1-1/\alpha_2)-3} \log(|\xi|)^{(1+\eta)(2/\alpha_1-1)+2} (1 + |\log(m_\eta(\xi))|)^2 d\xi,
\]
with \( c_{3,3}(\eta, K) \) a positive finite constant. Since \( \alpha_1 > 1 \), note that \( \alpha_2 - \alpha_1 < 1 - H_2 \) implies that \( H_2 + 1/\alpha_1 - 1/\alpha_2 < H_2 + \alpha_2 - \alpha_1 < 1 \) and thus \( I_2 < +\infty \). Therefore choosing \( r \in (0, 1 - H_2) \), Equation (3.15) and then (3.13) hold for \( m = m_\eta \). By Theorem 4.2 of [5], \((S_{m_\eta, N})_{N \geq 1}\) and \( S_{m_\eta} \) satisfy 1. and 2. of the theorem. Since for almost every \( \xi \in \mathbb{R} \) the map \((\alpha, H, t) \mapsto f_+ (\alpha, H, t, \xi) \) is continuous on \( K \), by Assertion 2. of Proposition 5.1 of [5], \( S_m \) satisfies Assertion 1. whatever \( m \) is.

\[ \square \]

**Remark 3.1** Assertion 2. in Theorem 3.1 holds for any \( m \) satisfying Equation (3.15) instead of \( m_\eta \).

### 4 Linear multifractional multistable and stable motions

From now on let us consider \( \alpha : \mathbb{R} \mapsto (0, 2) \) and \( h : \mathbb{R} \mapsto (0, 1) \) two continuous functions. Under Hypothesis 2.1, by Proposition 2.1, we may consider the linear multifractional multistable motion defined on \( \mathbb{R} \) by

\[ \tilde{S}_m(t) := S_m(\alpha(t), h(t), t), \quad (4.16) \]

with \( S_m \) given by (2.5).

#### 4.1 Regularity and rate of convergence

We may also define \( \tilde{S}_{m,N}(t) := S_{m,N}(\alpha(t), h(t), t) \), for all \( N \geq 1 \). The following theorem is a direct consequence of Theorem 3.1.

**Theorem 4.1** Let us consider \( \alpha : \mathbb{R} \mapsto (0, 2) \) and \( h : \mathbb{R} \mapsto (0, 1) \) two continuous functions and two real numbers \( a < b \). Then let us set

\[ \alpha_1 = \min_{t \in [a, b]} \alpha(t), \quad \alpha_2 = \max_{t \in [a, b]} \alpha(t) \quad \text{and} \quad H_1 = \min_{t \in [a, b]} h(t). \]

Assume that \( H_1 > 1/\alpha_1 \) and that \( \alpha \) and \( h \) are \((H_1 - 1/\alpha_1)\)-Hölder continuous functions on \([a, b]\).

1. Then, as \( N \to +\infty \), the series \( \left( \tilde{S}_{m,N} \right)_{N \geq 1} \) converges uniformly on \([a, b]\) to \( \tilde{S}_m \) and almost surely

\[ \sup_{t, t' \in [a, b], t \neq t'} \left| \frac{\tilde{S}_m(t) - \tilde{S}_m(t')}{|t - t'|^{H_1 - 1/\alpha_1} \sqrt{\log |t - t'|} + 1} \right| < +\infty. \]
2. Moreover if \( m = m_\eta \) is defined by (3.11) with \( \eta > 0 \), then, almost surely
\[
\sup_{N \geq 1} \sup_{t \in [a, b]} \left| \tilde{S}_{m_\eta, N}(t) - \tilde{S}_{m_\eta}(t^{'}) \right| < +\infty
\]
for any \( \varepsilon \in (0, 1/\alpha_2 - 1/2) \).

Note that one can use \( \tilde{S}_{m_\eta, N} \) to simulate \( \tilde{S}_{m_\eta} \). The error of approximation is then given by \( N^\varepsilon \).

### 4.2 Stochastic integral and series representation

Assuming that \( \alpha \) is a constant function equal to \( \alpha_1 \), we have already seen that \( \tilde{S}_m \overset{fdd}{=} d_{\alpha_1} X_{\alpha_1, h} \) where \( X_{\alpha_1, h} \) is the linear multifractional \( \alpha_1 \)-stable motion defined by (3.9) and \( d_{\alpha_1} \) is given by (2.8). Using the previous theorem we will prove the following one.

**Theorem 4.2** Let \( \alpha_1 \in (0, 2) \) and \( h : \mathbb{R} \rightarrow (0, 1) \) be a continuous function. Let us also consider \( X_{\alpha_1, h} \) the linear multifractional \( \alpha_1 \)-stable motion defined by (3.9) and two real numbers \( a < b \). If \( H_1 := \min_{t \in [a, b]} h(t) > 1/\alpha_1 \) and if \( h \) is \( (H_1 - 1/\alpha_1) \)-Hölder continuous on \([a, b]\), then there exists a continuous modification \( X_{\alpha_1, h}^* \) of \( X_{\alpha_1, h} \) such that almost surely
\[
\sup_{t, t' \in [a, b]} \left| X_{\alpha_1, h}^*(t) - X_{\alpha_1, h}^*(t^{'}) \right| \frac{|t - t'|^{H_1 - 1/\alpha_1} \sqrt{|\log |t - t'||} + 1}{|t - t'|^{H_1 - 1/\alpha_1} \sqrt{|\log |t - t'||} + 1} < +\infty.
\]

**Proof.** Let \( \alpha : \mathbb{R} \rightarrow (0, 2) \) be the constant function equal to \( \alpha_1 \) and let \( \tilde{S}_m \) be defined by (4.16). Since \( \tilde{S}_m \overset{fdd}{=} d_{\alpha_1} X_{\alpha_1, h} \) with \( d_{\alpha_1} \neq 0 \) defined by (2.8), by Theorem 4.1, we already know that a.s.
\[
\sup_{t, t' \in [a, b] \cap D} \frac{|X_{\alpha_1, h}(t) - X_{\alpha_1, h}(t^{'})|}{|t - t'|^{H_1 - 1/\alpha_1} \sqrt{|\log |t - t'||} + 1} < +\infty,
\]
where \( D \) is the dense set of dyadic real numbers. Moreover, since \( h \) is continuous with values in \((0, 1)\), the stochastic continuity of the linear multifractional \( \alpha_1 \)-stable motion \( X_{\alpha_1, h} \) has been established in [19]. This implies that there exists a modification \( X_{\alpha_1, h}^* \) of \( X_{\alpha_1, h} \) such that
\[
\sup_{t, t' \in [a, b]} \frac{|X_{\alpha_1, h}^*(t) - X_{\alpha_1, h}^*(t^{'})|}{|t - t'|^{H_1 - 1/\alpha_1} \sqrt{|\log |t - t'||} + 1} < +\infty.
\]
see e.g. Section D.2 of [5] for the construction of $X_{a_1,h}^\ast$. Then, the proof is complete. □

In [2], using a wavelet series expansion, under our assumptions of Proposition 3.9, the authors obtained a continuous modification $X_{a_1,h}^\ast$ satisfying a.s. for all $\eta > 0$,
\[
\sup_{t,t' \in [a,b], t \neq t'} \frac{|X_{a_1,h}^\ast(t) - X_{a_1,h}^\ast(t')|}{|t-t'|(H_1-1/\alpha_1) (|\log |t-t'|| + 1)^{2/\alpha_1+\eta}} < +\infty.
\]
Since $1/2 < 2/\alpha_1$, our result is sharper. Moreover it is quasi-optimal since, for $\eta > 0$, one can find $h$ such that a.s.
\[
\sup_{t,t' \in [a,b], t \neq t'} \frac{|X_{a_1,h}^\ast(t) - X_{a_1,h}^\ast(t')|}{|t-t'|(H_1-1/\alpha_1) (|\log |t-t'|| + 1)^{-\eta}} = +\infty,
\]
by Theorem 6.1 of [2]. Let us also quote that following our method based on [5], one may obtain an upper bound for the global modulus of continuity of linear fractional stable sheets, which is sharper than the one given in [3].

A Proof of Proposition 3.2

Let us consider $K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A] \subset (1, 2) \times (1/2, 1) \times \mathbb{R}$ such that $1/\alpha_1 < H_1 \leq H_2 < 1$. Let us note that it is enough to prove Proposition 3.2 for $A$ large enough. Then, in this proof, we assume, without loss of generality that $A > e$ (so that $\log \xi > 1$ for $\xi > A$).

For all $x = (\alpha, H, t) \in K$, we set
\[ \beta(x) = H - 1/\alpha \in (0, 1) \]
and remark that $\beta(x) \in [\beta_1, \beta_2] \subset (0, 1)$ with
\[ \beta_1 = H_1 - 1/\alpha_1 \text{ and } \beta_2 = H_2 - 1/\alpha_2. \]
Moreover, for all $x = (\alpha, H, t) \in K$ and all $\xi \in \mathbb{R}$, let us note that
\[ f_+(\alpha, H, t, \xi) = g(\beta(x), t, \xi) \]
with $g$ defined on $(0, 1) \times \mathbb{R} \times \mathbb{R}$ by
\[ g(\beta, t, \xi) := (t - \xi)^\beta_+ - (-\xi)^\beta_+. \]
Let us now consider $x = (\alpha, H, t) \in K$ and $x' = (\alpha', H', t') \in K$. Then, by (3.12),
\[ V_{m,n}(x) - V_{m,n}(x') = \left( g(\beta(x), t, \xi_n)m(\xi_n)^{-1/\alpha} - g(\beta(x'), t', \xi_n)m(\xi_n)^{-1/\alpha'} \right). \]
Proposition 3.2 follows from the following lemma, which proof is given at the end of this section.
Lemma A.1 Let \(0 < \beta_1 \leq \beta_2 < 1\) and \(A > e\).

1. There exists a finite positive constant \(c_1(A, \beta_1, \beta_2)\) such that for all \(\beta, \beta' \in [\beta_1, \beta_2],\) all \(t, t' \in [-A, A]\) and all \(\xi \in \mathbb{R},\)

\[
|g(\beta, t, \xi) - g(\beta', t', \xi)| \leq c_1(A, \beta_1, \beta_2)\left(|t - t'|^{\beta_1} + |\beta - \beta'|\right)h_{A,1}(\xi, \beta_2)
\]

with

\[
h_{A,1}(\xi, c) = 1_{|\xi| \leq 2A} + |\xi|^{c-1} \log |\xi| 1_{|\xi| > 2A}.
\]

2. Moreover, there exists a finite positive constant \(c_2(A, \beta_1)\) such that for all \(\beta \in [\beta_1, \beta_2]\) and \(t \in [-A, A],\)

\[
|g(\beta, t, \xi)| \leq c_2(A, \beta_1)h_{A,2}(\xi, \beta_2)
\]

with

\[
h_{A,2}(\xi, c) = 1_{|\xi| \leq 2A} + |\xi|^{c-1} 1_{|\xi| > 2A}.
\]

Setting for almost every \(\xi \in \mathbb{R}\)

\[
\begin{cases}
F_1(x, x', \xi) := |g(\beta(x), t, \xi) - g(\beta(x'), t', \xi)|m(\xi)^{-1/\alpha}, \\
F_2(x, x', \xi) := |g(\beta(x'), t', \xi)||m(\xi)^{-1/\alpha} - m(\xi)^{-1/\alpha'}|
\end{cases} 
\]

we then have

\[
|V_{m,1}(x) - V_{m,1}(x')| \leq F_1(x, x', \xi_1) + F_2(x, x', \xi_1).
\]

Before we apply Lemma A.1 to bound \(F_1\) and \(F_2,\) let us remark that for all \(\xi \in \mathbb{R},\)

\[
h_{A,2}(\xi, \beta_2) \leq h_{A,1}(\xi, \beta_2) \leq c_3(A, \beta_2)\left(1_{|\xi| \leq e} + |\xi|^{\beta_2} - \log |\xi| 1_{|\xi| > e}\right) \quad (A.17)
\]

with \(c_3(A, \beta_2)\) a finite positive constant, which does not depend on \(\xi.\) Then, combining this remark with Lemma A.1, for almost every \(\xi \in \mathbb{R},\)

\[
F_1(x, x', \xi) \leq c_1(A, \beta_1, \beta_2)c_3(A, \beta_2)\left(|t - t'|^{\beta_1} + |\beta(x) - \beta(x')|\right)h_{m,K}(\xi)
\]

with \(h_{m,K}\) defined by Equation (3.14). Since \(\alpha_1 > 1,\) by definition of the function \(\beta,\) it follows that for almost every \(\xi \in \mathbb{R},\)

\[
F_1(x, x', \xi) \leq c_1(A, \beta_1, \beta_2)c_3(A, \beta_2)\tau(x - x')h_{m,K}(\xi),
\]

with \(\tau(x - x') = |t - t'|^{\beta_1} + |H - H'| + |\alpha - \alpha'|.\)

Moreover, applying Assertion 2 of Lemma A.1, Equation (A.17) and the mean value theorem, for almost every \(\xi \in \mathbb{R},\)

\[
F_2(x, x', \xi) \leq c_2(A, \beta_1)c_3(A, \beta_2)|\alpha - \alpha'|h_{m,K}(\xi).
\]
In view of the previous computations, we have: almost surely,

\[ |V_{m,1}(x) - V_{m,1}(x')| \leq c_{3,1}(K)\tau(x - x')h_{m,K}(\xi_1) \]

with \( c_{3,1}(K) := c_3(A, \beta_2)(c_1(A, \beta_1, \beta_2) + c_2(A, \beta_1)) \). This concludes the proof of Proposition 3.2.

We conclude this section by the proof of Lemma A.1.

**Proof.** [Proof of Lemma A.1] Let \( 0 < \beta_1 < \beta_2 < 1 \) and \( A > e \). Let \( \beta, \beta' \in [\beta_1, \beta_2] \subset (0, 1) \) and \( t, t' \in [-A, A] \). Let us write for all \( \xi \in \mathbb{R} \),

\[ |g(\beta, t, \xi) - g(\beta', t', \xi)| \leq g_1(\beta', t, t', \xi) + g_2(\beta, \beta', t, \xi) \]

with

\[
\left\{ \begin{array}{ll}
g_1(\beta', t, t', \xi) & := |g(\beta', t', \xi) - g(\beta', t, \xi)| \\
g_2(\beta, \beta', t, \xi) & := |g(\beta', t, \xi) - g(\beta, t, \xi)|. 
\end{array} \right.
\]

**Step 1: Control of** \( g_1 \). Let us note that if \( t = t' \), \( g_1(\beta', t, t', \xi) = 0 \) for all \( \xi \in \mathbb{R} \). Then, in this step, we assume now, without loss of generality that \( t < t' \). This implies that

\[ g_1(\beta', t, t', \xi) = \begin{cases} 
0 & \text{if } \xi \geq t' \\
(t' - \xi)^{\beta'} & \text{if } t \leq \xi < t' \\
(t' - \xi)^{\beta'} - (t - \xi)^{\beta'} & \text{if } \xi < t. 
\end{cases} \]

Let \( \xi \in \mathbb{R} \) with \( |\xi| > 2A \). If \( \xi < 0 \) it follows that \( \xi < t < t' \). Since \( \beta' > 0 \), applying the mean value theorem,

\[ g_1(\beta', t, t', \xi) \leq \beta' |t - t'| |c_{\xi, t, t'} - \xi|^{\beta' - 1} \]

with \( c_{\xi, t, t'} \in (t, t') \subset [-A, A] \). Moreover, since \( |\xi| > 2A \)

\[ |c_{\xi, t, t'} - \xi| \geq |\xi| - |c_{\xi, t, t'}| \geq |\xi| - A \geq |\xi|/2 \]

and then

\[ g_1(\beta', t, t', \xi) \leq 2^{1-\beta'} |t - t'| |\xi|^{\beta' - 1} \]

since \( \beta' \in (0, 1) \). Therefore, for \( |\xi| > 2A \),

\[ g_1(\beta', t, t', \xi) \leq 4A |t - t'|^{\beta_1} |\xi|^{\beta_2 - 1} \quad (A.18) \]

since \( |t - t'| \leq 2A, \beta' \in [\beta_1, \beta_2] \subset (0, 1) \) and \( 2A > 1 \).

Now let \( \xi \in \mathbb{R} \) with \( |\xi| \leq 2A \). Since \( 0 < \beta' < 1 \), we have

\[ |a^{\beta'} - b^{\beta'}| \leq |a - b|^{\beta'} \]
for all \( a, b \geq 0 \). By definition of \( g \), it follows that
\[
g_1(\beta', t', t, \xi) \leq \left| (t' - \xi)_+ - (t - \xi)_+ \right|^\beta' \leq |t' - t|^\beta' \leq 2A|t' - t|^\beta_1
\]
since \(-A \leq t < t' \leq A\), \( 0 < \beta_1 \leq \beta' < 1 \) and \( A > 1 \). From this last inequality and Equation (A.18), we deduce that for all \( \xi \in \mathbb{R} \),
\[
g_1(\beta', t', t, \xi) \leq 4A|t - t'|^\beta_1 h_{A,2}(\xi, \beta_2) \tag{A.19}
\]
with \( h_{A,2}(\xi, \beta_2) = 1_{|\xi| \leq 2A} + |\xi|^\beta_2 - 1_{|\xi| > 2A} \).

**Step 2: Control of \( g_2 \).** Let us recall that for all \( \xi \in \mathbb{R} \),
\[
g_2(\beta, \beta', t, \xi) = \left| (t - \xi)_+^{\beta'} - (t - \xi)^{\beta} + (-\xi)^{\beta} - (-\xi)^{\beta'} \right|.
\]
Then, applying the mean value theorem, for all \( \xi \in \mathbb{R} \),
\[
g_2(\beta, \beta', t, \xi) \leq |\beta - \beta'| \sup_{\beta_1 \leq c \leq \beta_2} \left( (t - \xi)_+^c \log(t - \xi) - (-\xi)_+^c \log(-\xi) \right)
\]
where for \( c > 0 \),
\[
(x)_+^c \log(x)_+ = \begin{cases} x^c \log x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}
\]
Let us first consider \( \xi \in [-2A, 2A] \). Then, \((-\xi)_+ \in [0, 2A] \) and \((t - \xi)_+ \in [0, 3A] \) since \( t \in [-A, A] \). Therefore,
\[
g_2(\beta, \beta', t, \xi) \leq \tilde{c}_1(A, \beta_1, \beta_2) |\beta - \beta'| \tag{A.20}
\]
with
\[
\tilde{c}_1(A, \beta_1, \beta_2) = 2 \max_{\beta_1 \leq \epsilon \leq \beta_2} \max_{0 < u \leq 3A} u^\epsilon |\log u| = 2 \max \left( \frac{1}{e^{\beta_1}}, (3A)^{\beta_2} \log(3A) \right) < +\infty.
\]
Let us now assume that \( \xi < -2A \). Then, \( \xi < t \) and
\[
g_2(\beta, \beta', t, \xi) \leq |\beta - \beta'| \sup_{\beta_1 \leq c \leq \beta_2} \left( (t - \xi)_+^c \log(t - \xi) - (-\xi)_+^c \log(-\xi) \right)
\]
with \( t - \xi > 0 \) and \(-\xi > 0\). Let us remark that \(-\xi \in (-\xi/2, -3\xi/2) \) since \(-\xi > 0\) and that
\[
-\xi/2 < -A - \xi \leq t - \xi \leq A - \xi < -3\xi/2
\]
since \( t \in [-A, A] \) and \( \xi < -2A \). Then, for each \( c \in [\beta_1, \beta_2] \subset (0, 1) \), by the mean value theorem,
\[
|(t - \xi)_+^c \log(t - \xi) - (-\xi)_+^c \log(-\xi)| \leq |u_{t, \xi, c}|^{c-1}(c|\log u_{t, \xi, c}| + 1)
\]
with $u_{t,\xi,c} \in (-\xi/2, -3\xi/2)$. Since $u_{t,\xi,c} \in (-\xi/2, -3\xi/2)$ and $-\xi/2 > A > e$, we get

$$|(t - \xi)^c \log(t - \xi) - (-\xi)^c \log(-\xi)| \leq 4|\xi|^{\beta_2-1} \log |\xi|$$

for all $c \in [\beta_1, \beta_2] \subset (0, 1)$. Hence, for $\xi < -2A$,

$$g_2(\beta, \beta', t, \xi) \leq 4|\beta - \beta'||\xi|^{\beta_2-1} \log |\xi|.$$ 

Note that this last inequality still holds for $\xi > 2A$ since in this case, $g_2(\beta, \beta', t, \xi) = 0$.

Then, we have proved that for all $\xi \in \mathbb{R}$,

$$g_2(\beta, \beta', t, \xi) \leq \tilde{c}_2(A, \beta_1, \beta_2)|\beta - \beta'| h_{A,1}(\xi, \beta_2)$$  \hspace{1cm} (A.21)

with $\tilde{c}_2(A, \beta_1, \beta_2) = \max(\tilde{c}_1(A, \beta_1, \beta_2), 4)$ and

$$h_{A,1}(\xi, \beta_2) = 1_{|\xi| \leq 2A} + |\xi|^{\beta_2-1} \log |\xi| 1_{|\xi| > 2A}.$$ 

**Step 3: Proof of Assertion 1.** It follows from Equations (A.19) and (A.21) choosing $c_1(A, \beta_1, \beta_2) = \tilde{c}_2(A, \beta_1, \beta_2) + 4A \in (0, +\infty)$ and using the fact that $h_{A,2}(\xi, \beta_2) \leq h_{A,1}(\xi, \beta_2)$ since $A > e$.

**Step 4: Proof of Assertion 2.** Let us remark that

$$g(\beta', t', \xi) = g(\beta', t', \xi) - g(\beta', 0, \xi)$$

since $g(\beta', 0, \xi) = (-\xi)^{\beta'_+} - (-\xi)^{\beta'_+} = 0$. Hence, applying Equation (A.19) with $t = 0$ and $\beta' = \beta$,

$$|g(\beta', t', \xi)| \leq 4A|t'|^{\beta_1} h_{A,2}(\xi, \beta_2) \leq 4A^{\beta_1+1} h_{A,2}(\xi, \beta_2),$$

which concludes the proof. \hfill \square

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