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1. Introduction

In classical continuum mechanics, the motion of a material medium relative to a rectangular Cartesian system of spatial coordinates z^i, $i=1,2,3$, is represented by a set of three single-valued functions $\overset{K}{Z}(z^i, T)$, $K=1,2,3$, of the coordinates z^i and the time T. The functions $\overset{K}{Z}$ representing a real continuous motion are subject to the condition

$$\mathfrak{D} \equiv \det \partial_i \overset{K}{Z} \neq 0, \qquad \partial_i \equiv \frac{\partial}{\partial z^i}. \tag{1.1}$$

Every set of three values $Z^K = \overset{K}{Z}(z^i, T)$ for the functions $\overset{K}{Z}$ serves as names for the *material points* $\boldsymbol{P}$ of the continuous medium*. Every set of values for the four *space-time coordinates* (z^i, T) serves as names for an *event* $\boldsymbol{E}$.

* We shall use the convention that a set of functions like $\overset{K}{Z}(z^i, T)$ will be written with the labeling index set over the kernel letter. When we wish to regard the values of such a set of functions as coordinates we write Z^K. Thus, for example, the partial derivatives of the functions $\overset{K}{Z}$ are again functions $\partial_i \overset{K}{Z}(z^i, T)$ of the coordinates z^i, T. However, from the condition (1.1), it follows that we can solve for the z^i as functions of the Z^K and T. We write the solution in the form $z^i = \overset{i}{z}(Z^K, T)$. The gradients $\partial_i Z^K$ can be regarded as functions of the Z^K and T, in which case we denote these functions by $\partial_i Z^K(Z^K, T) \equiv \partial_i \overset{K}{Z}(\overset{i}{z}(Z^K, T), T)$. This convention avoids excessive use of new symbols and gives a precise meaning to the two different quantities $\partial_i \overset{K}{Z}$ and $\partial_i Z^K$.

The functions $\overset{K}{Z}$ representing the motion specify the material point Z^K which experiences the event (z^i, T). We restrict attention in what follows to local properties of motions. We assume that the functions $\overset{K}{Z}$ are *analytic* at an event $\underset{0}{\boldsymbol{E}}$ with coordinates $(\underset{0}{z}^i, \underset{0}{T})$. Thus we assume the existence of a power series

$$\overset{K}{Z}(z^i,T)=\overset{K}{Z}(\underset{0}{z}^i,\underset{0}{T})+\overset{K}{Z}_i(z^i-\underset{0}{z}^i)+\overset{K}{Z}_4(T-\underset{0}{T})+\tfrac{1}{2}\overset{K}{Z}_{ij}(z^i-\underset{0}{z}^i)(z^j-\underset{0}{z}^j)+\cdots,\ \overset{K}{Z}_i\neq 0, \quad (1.2)$$

convergent at every event $\boldsymbol{E}$ in some neighborhood $N(\underset{0}{\boldsymbol{E}})$ of the event $\underset{0}{\boldsymbol{E}}$. In all that follows, by "analytic" we shall mean analytic at the event $\underset{0}{\boldsymbol{E}}$ and all functions of the coordinates that appear will be assumed analytic at $\underset{0}{\boldsymbol{E}}$. From (1.1) follows the existence of a unique inverse

$$z^i = \overset{i}{z}(Z^K, T). \quad (1.3)$$

The functions $\overset{i}{z}$ will be analytic at the point with coordinate values $\underset{0}{Z}^K=\overset{K}{Z}(\underset{0}{z}^i, \underset{0}{T})$, $T=\underset{0}{T}$.

Let $A^i_j(T)$ be any set of nine analytic functions of T such that

$$\delta^{kl}A^i_k A^j_l = \delta^{ij}. \quad (1.4)$$

$A^i_j(T)$ is an orthogonal matrix. Let $d^i(T)$ be any three analytic functions of the time T. Then in terms of a given motion $\overset{K}{Z}(z^i, T)$ we define the *class of all motions* $\overset{K}{Z}'(z^i, T)$ *which differ from* $\overset{K}{Z}(z^i, T)$ *by a rigid motion* as follows:

$$\overset{K}{Z}'(z^i, T) = \overset{K}{Z}(A^i_j z^j + d^i,\ T + \text{constant}). \quad (1.5)$$

Conversely, any two motions $(\overset{K}{Z}', \overset{K}{Z})$ between which there exists a relation of the form (1.5) are said to differ from each other by a rigid motion.

An important problem in classical continuum mechanics is the construction of constitutive equations for the stress, internal energy, and heat flux. In general, these constitutive equations depend on a motion and are required to transform in a definite way when a motion is replaced by any motion differing from it by a rigid motion. For example, the theory of finite deformations of elastic media is based on constitutive equations for the stress tensor t^{ij} having the general form [*12, 8, 11*]

$$t^{ij}(z^k, T) = \overset{ij}{T}\big(\overset{K}{Z}(z^k, T),\ \partial_m \overset{M}{Z}(z^k, T)\big), \quad (1.6)$$

where, for definiteness, we may assume that the functions $\overset{ij}{T}$ are analytic and single valued in all 12 arguments $\overset{K}{Z}$ and $\partial_i \overset{K}{Z}$. The functional form of $\overset{ij}{T}$ depends on the elastic properties of the material medium. However, for *all* elastic materials the $\overset{ij}{T}$ are required to satisfy the invariance condition

$$\overset{ij}{T}(\overset{K}{Z}', \partial_m \overset{M}{Z}') = A^i_k(T)\, A^j_l(T)\, \overset{kl}{T}(\overset{K}{Z}, \partial_m \overset{M}{Z}), \quad (1.7)$$

where $(\overset{K}{Z}{}', \overset{K}{Z})$ is any pair of motions differing from each other by a rigid motion. Using known methods of invariant theory one can prove that any set of functions $\overset{ij}{T}$ satisfying (1.7) is reducible to the form

$$\overset{ij}{T} = \frac{\partial \overset{i}{z}}{\partial Z^K} \frac{\partial \overset{j}{z}}{\partial Z^L} \overset{KL}{P} (\overset{M}{Z}, \overset{PQ}{C}, \mathfrak{D}), \qquad \overset{KL}{C} \equiv \delta^{ij} \partial_i \overset{K}{Z} \partial_j \overset{L}{Z}. \tag{1.8}$$

In all that precedes, we have regarded a motion as being relative to a fixed rectangular Cartesian system of spatial coordinates z^i. However, the invariant-theoretic problems of continuum mechanics involving the class of all motions differing from each other by a rigid motion can be attacked from another point of view which is more convenient for our present purposes. Consider the group of coordinate transformations

$$\begin{aligned} z^{i'} &= A^{i'}_j (T) z^j + d^{i'} (T), \\ T' &= T + \text{constant}, \end{aligned} \tag{1.9}$$

where $A^{i'}_j (T)$ is an orthogonal matrix. Let $\overset{K}{Z}(z^i, T)$ represent a motion relative to the frame (z^i, T). Let $\overset{K}{Z}{}'(z^{i'}, T')$ be the scalar transform of $\overset{K}{Z}(z^i, T)$. Thus

$$\overset{K}{Z}{}'(z^{i'}, T') = \overset{K}{Z}(z^i, T) \tag{1.10}$$

according to the definition of a scalar in tensor calculus. The functions $\overset{K}{Z}{}'(z^{i'}, T')$ define a motion relative to the frame $(z^{i'}, T')$. The transformation law (1.7) implies that

$$t^{i'j'}(z^{m'}, T') = A^{i'}_k(T) A^{j'}_l(T) t^{kl}(z^m, T). \tag{1.11}$$

If we define a set of 16 quantites $\tau^{\mu\nu}$, $\mu, \nu = 1, 2, 3, 4$ by setting

$$\tau^{\mu\nu} = \begin{bmatrix} t^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \tag{1.12}$$

we see that the law of transformation (1.11) for t^{ij} is precisely the law of transformation implied by transforming the quantities $\tau^{\mu\nu}$ as a 4-dimensional tensor under the group of coordinate transformations (1.9) in four variables, where the time T is regarded as the 4th coordinate. Thus, transforming the functions $\overset{K}{Z}$ representing a motion as a set of three scalars under the group (1.9), we see that the fundamental assumption (1.11) of continuum mechanics assumes a simple and familiar form in terms of a 4-dimensional tensor law of transformation under the group (1.9).

In presenting these familiar ideas we have attempted to indicate the important role in mechanics of the theory of invariants of a motion under a group of coordinate transformations in a 4-dimensional space. In this work we shall introduce three such groups of transformations: the Euclidean group, the Galilean group, and the Lorentzian group. Each of these groups of coordinate transformations on 4 real variables defines a type of geometric space. In each of these spaces, we shall define a motion of a continuous medium. We then define Euclidean kinematics, Galilean kinematics, and Lorentzian kinematics as the theory of the invariants of a motion in these 4-dimensional spaces. We have attempted to design a formalism which treats these three kinematical theories on a parallel basis. This has been done so that a comparison of classical kinematical concepts, definitions, and theorems with their relativistic counterparts is made easier. Another advantage of this symmetrical treatment is that we can borrow ideas from the more familiar classical invariant theory of a motion and transfer them by analogy to the relativistic case. The Euclidean and Galilean groups are firmly interlocked with classical mechanics; the Lorentzian group, with electromagnetic theory and relativistic mechanics. However, kinematics being the science of motion in itself, independent of the natural laws presumed to govern the motion, we have considered here only briefly the application of kinematical results to these more restricted theories.

2. Euclidean, Galilean, and Lorentzian space-time

So as to fix the meaning we attach to the words *space, geometry, field,* and *invariant,* allow us to describe briefly how they shall be used.

An n-dimensional geometric space $\mathcal{S}$ is a set of points $\boldsymbol{p}$ such that to every point $\boldsymbol{p}$ there corresponds a subset of points $\mathfrak{N}(\boldsymbol{p})$ containing $\boldsymbol{p}$ which can be placed into one to one correspondence with all the ordered sets of n real numbers $x^\mu = (x^1, x^2, \ldots, x^n)$ lying in some interval $\overset{0}{x}{}^\mu - h^\mu \leqq x^\mu \leqq \overset{0}{x}{}^\mu + h^\mu$, $h^\mu > 0$ and such that $\boldsymbol{p}$ corresponds to $\overset{0}{x}{}^\mu$, *together with a group* $\mathfrak{G}$ *of allowable coordinate transformations* $x^{\mu'} = \overset{\mu'}{x}(x^\mu)$, $x^\mu = \overset{\mu}{x}(x^{\mu'})$. The class of all coordinate systems related by elements of the group $\mathfrak{G}$ is called the class of *allowable coordinate systems* for the points $\mathfrak{N}(\boldsymbol{p})$ in the neighborhood of $\boldsymbol{p}$. The characterization of a space $\mathcal{S}$ may involve also a set of functions $\Phi_1, \Phi_2, \ldots, \Phi_N$ of the coordinates x^μ having an assigned transformation law under the group $\mathfrak{G}$. By choosing different groups $\mathfrak{G}$ and different sets of functions $\boldsymbol{\Phi}$ and different transformation laws for the set $\boldsymbol{\Phi}$, we obtain various examples of geometric spaces. Thus ordinary 3-dimensional Euclidean metric space corresponds to letting $\mathfrak{G}$ be the group of orthogonal transformations, where with this choice of $\mathfrak{G}$, the set of functions $\boldsymbol{\Phi}$ is empty. However, we can also let $\mathfrak{G}$ be the group of general analytic transformations $\mathfrak{G}_A$ provided we append a Euclidean metric tensor field $g_{ij}(x^k)$. The two spaces so defined are regarded as equivalent. Curved Riemannian spaces, affinely connected spaces, conformal spaces, *etc.*, correspond to various other choices for the group $\mathfrak{G}$ and the functions $\boldsymbol{\Phi}$ together with their transformation law [*3, 10*]. The foregoing example of Euclidean metric space shows that different choices of the pair of objects $(\mathfrak{G}, \boldsymbol{\Phi})$ may serve to define spaces which are regarded as equivalent.

By a *field* $\boldsymbol{\Phi}$ in a space $\mathcal{S}$ with group $\mathfrak{G}$ we shall mean a set of functions of the coordinates $\Phi_\Omega(x^\mu)$, $\Omega = 1, 2, \ldots, N$ having an assigned law of transformation under the group $\mathfrak{G}$. By "an assigned law of transformation" we mean a rule such that when the representation of the field $\boldsymbol{\Phi}$ by functions $\Phi_\Omega(x^\mu)$ in any one allowable coordinate system x^μ is given, the representation $\Phi_{\Omega'}(x^{\mu'})$ of the field in any other allowable coordinate system is uniquely determined by the functions $\Phi_\Omega(x^\mu)$ and the coordinate transformation relating the $x^{\mu'}$ and x^μ.

A *tensor field* in a space $\mathcal{S}$ with group $\mathfrak{G}$ is a set of functions of the coordinates having a transformation law under $\mathfrak{G}$ of the general form

$$\begin{aligned} \Phi^{\mu'\nu'\cdots}_{\lambda'\cdots\cdots}(x^{\sigma'}) &= |(x'\,x)|^{-w}\,(\mathrm{sgn}\,(x'\,x))^{y}\,\frac{\partial x^{\mu'}}{\partial x^{\mu}}\,\frac{\partial x^{\nu'}}{\partial x^{\nu}}\cdots\frac{\partial \overset{\lambda}{x}}{\partial x^{\lambda'}}\cdots\Phi^{\mu\nu\cdots}_{\lambda\cdots\cdots}(\overset{\sigma}{x}), \\ (x'\,x) &\equiv \det\frac{\partial x^{\mu'}}{\partial x^{\nu}}. \end{aligned} \tag{2.1}$$

If $y=0$, $\mathbf{\Phi}$ is called a *tensor field of weight* w. If $y=1$, $\mathbf{\Phi}$ is called an *axial tensor field of weight* w. If $w=y=0$, $\mathbf{\Phi}$ is called an *absolute tensor* field. The number of superscripts and subscripts on a tensor field determine its *contravariant* and *covariant rank*, respectively. The rank of a tensor field is the sum of its contravariant and covariant ranks. Tensors of rank zero are called *scalars*, and tensors of rank one are called *vectors*.

An *affine connection* is a field $\mathbf{\Gamma}$ having a law of transformation of the form [*10*]

$$\Gamma^{\mu'}_{\lambda'\nu'} = \frac{\partial^2 x^{\mu}}{\partial x^{\lambda'}\,\partial x^{\nu'}}\,\frac{\partial x^{\mu'}}{\partial x^{\mu}} + \frac{\partial x^{\mu'}}{\partial x^{\mu}}\,\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\,\frac{\partial x^{\nu}}{\partial x^{\nu'}}\,\Gamma^{\mu}_{\lambda\nu}. \tag{2.2}$$

We shall have occasion to consider only *symmetric* affine connections: $\Gamma^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\nu\mu}$. If $g_{\mu\nu}$ is any symmetric non-singular absolute tensor field, the *Christoffel symbols* based on $g_{\mu\nu}$ are defined by

$$\left\{{\varrho \atop \mu\nu}\right\}_g \equiv \tfrac{1}{2}\, g^{\varrho\lambda}(\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu}), \tag{2.3}$$

where $g^{\varrho\lambda}$ is the inverse of $g_{\mu\nu}$, $g^{\varrho\lambda} g_{\lambda\mu}=\delta^{\varrho}_{\mu}$. Christoffel symbols have the transformation law (2.2) of an affine connection. The *Riemann curvature tensor* based on an affine connection $\mathbf{\Gamma}$ is defined by*

$$R_{\lambda\varrho\nu}{}^{\mu}(\mathbf{\Gamma}) \equiv 2\,\partial_{[\lambda}\Gamma^{\mu}_{\varrho]\nu} + 2\Gamma^{\mu}_{\tau[\lambda}\Gamma^{\tau}_{\varrho]\nu}. \tag{2.4}$$

If $R_{\lambda\varrho\nu}{}^{\mu}(\mathbf{\Gamma})=0$, the affine connection $\mathbf{\Gamma}$ is said to be *flat* or *integrable*. We denote the Riemann curvature tensor based on the Christoffel symbols $\left\{{\varrho \atop \mu\nu}\right\}_g$ by $R_{\lambda\varrho\nu}{}^{\mu}(\boldsymbol{g})$. If $R_{\lambda\varrho\nu}{}^{\mu}(\boldsymbol{g})=0$, and if $g_{\mu\nu}$ is positive definite, the field $g_{\mu\nu}$ is called a *Euclidean metric tensor*.

The covariant derivative of a tensor field based on an affine connection $\mathbf{\Gamma}$ is the tensor field defined by

$$\underset{\Gamma}{\nabla}_{\mu}\Phi^{\lambda\cdots}_{\nu\cdots} \equiv \partial_\mu \Phi^{\lambda\cdots}_{\nu\cdots} + \Gamma^{\lambda}_{\mu\varrho}\,\Phi^{\varrho\cdots}_{\nu\cdots}\cdots - \Gamma^{\varrho}_{\mu\nu}\,\Phi^{\lambda\cdots}_{\varrho\cdots}\cdots - w\,\Gamma^{\varrho}_{\varrho\mu}\,\Phi^{\lambda\cdots}_{\nu\cdots}. \tag{2.5}$$

If the components of the affine connection are Christoffel symbols based on a tensor $g_{\sigma\pi}$, then we write $\underset{g}{\nabla}_{\mu}$ for the operator of covariant differentiation.

An *invariant property* of a field $\mathbf{\Phi}$ in a spaces $\mathcal{S}$ with group $\mathfrak{G}$ is a property possessed in common by each of its representations $\Phi_{\Omega}(x^{\mu})$. The *invariants* of

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* Square brackets enclosing a set of indices denote the alternating sum over all permutations of the enclosed indices divided by $k!$, where k is the number of such indices. Round brackets denote the sum over all permutations divided by $k!$. Thus, for example, $a_{[ij]}\equiv\frac{1}{2}(a_{ij}-a_{ji})$, and $a_{(ij)}\equiv\frac{1}{2}(a_{ij}+a_{ji})$.

a field consist in all of its invariant properties and in other fields which can be defined in terms of it. The *joint invariants* of a set of fields $\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots$ consist in all of the invariant properties of the fields held singly and jointly. *Differential invariants* or *joint differential invariants* of a set of fields are joint invariant algebraic relations between the components of the fields and their partial derivatives of all orders with respect to the coordinates.

The *geometry* of a set of fields in a space $\mathcal{S}$ with group $\mathfrak{G}$ is the theory of the joint invariants of the fields under the group $\mathfrak{G}$.

Consider now the three 4-dimensional geometric spaces $\mathcal{S}_{\mathrm{E}}$, $\mathcal{S}_{\mathrm{G}}$, and $\mathcal{S}_{\mathrm{L}}$ defined by the following groups of allowable coordinate transformations on four real variables z^μ, $\mu = 1, 2, 3, 4$*.

I. *Euclidean space-time* $\mathcal{S}_{\mathrm{E}}$ *and the group* $\mathfrak{G}_{\mathrm{E}}$ *of Euclidean transformations:*

$$\begin{aligned} z^{i'} &= A^{i'}_j(z^4)\, z^j + d^{i'}(z^4), \\ z^{4'} &= z^4 + \text{constant}, \end{aligned} \tag{2.6}$$

where $A^{i'}_j$ and $d^{i'}$ are analytic functions of z^4 and $A^{i'}_j$ is an orthogonal matrix.

II. *Galilean space-time* $\mathcal{S}_{\mathrm{G}}$ *and the group* $\mathfrak{G}_{\mathrm{G}}$ *of Galilean transformations:*

$$\begin{aligned} z^{i'} &= A^{i'}_j\, z^j + u^{i'}\, z^4 + \text{constant}, \\ z^{4'} &= z^4 + \text{constant}, \end{aligned} \tag{2.7}$$

where $A^{i'}$ is a constant orthogonal matrix and the $u^{i'}$ are constants.

III. *Lorentzian space-time* $\mathcal{S}_{\mathrm{L}}$ *and the group* $\mathfrak{G}_{\mathrm{L}}$ *of Lorentz transformations:*

$$z^{\mu'} = L^{\mu'}_\nu\, z^\nu, \tag{2.8}$$

where $L^{\mu'}_\nu$ is a constant matrix satisfying the equations

$$\eta^{\mu\nu}\, L^{\lambda'}_\mu\, L^{\tau'}_\nu - \eta^{\lambda'\tau'}, \tag{2.9}$$

$$\eta^{\mu\nu} = \eta^{\mu'\nu'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

A *motion of a material medium* $\boldsymbol{M}$ *in Euclidean or Galilean space-time* is defined by any three absolute scalar fields $\overset{K}{Z}(z^\mu)$, $K = 1, 2, 3$ such that the matrix $\overset{KL}{C^{-1}}(z^\mu)$ defined by

$$\overset{KL}{C^{-1}} \equiv \delta^{ij}\, \partial_i \overset{K}{Z}\, \partial_j \overset{L}{Z} \tag{2.10}$$

is *positive definite*.

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* From this point on, Greek lower case indices will always range over the four values 1, 2, 3, 4. Greek upper case indices will be reserved for a variable range depending on the context. Latin lower and upper case indices will always range over the three values 1, 2, 3. The summation convention applies to all types of indices.

A motion of a material medium $\mathbf{M}$ *in Lorentzian space-time* is defined by any three absolute scalar fields $\overset{K}{Z}(z^\mu)$ such that the matrix $\overset{KL}{X^{-1}}(z^\mu)$ defined by

$$\overset{KL}{X^{-1}} \equiv \eta^{\mu\nu}\, \partial_\mu \overset{K}{Z}\, \partial_\nu \overset{L}{Z} \tag{2.11}$$

is *positive definite, i.e.*, $\overset{KL}{X^{-1}} V_K V_L > 0$ for all $V_K \not\equiv 0$*.

The geometry of a motion in $\mathscr{S}_\mathrm{E}$, $\mathscr{S}_\mathrm{G}$, and $\mathscr{S}_\mathrm{L}$ will be called *Euclidean kinematics, Galilean kinematics,* and *Lorentzian kinematics,* respectively.

3. Klein's principle and general coordinates in Euclidean, Galilean, and Lorentzian space-time

The use of curvilinear coordinates for 3-dimensional Euclidean space is familiar in mechanics. Many authors in continuum mechanics, especially in finite elasticity theory [*11*], use curvilinear and *deforming* spatial coordinates x^i in Galilean space-time. This type of coordinate system is related to an inertial rectangular Cartesian coordinate system z^i by a general analytic transformation of the form

$$\begin{aligned} x^i &= \overset{i}{x}(z^i, z^4), \\ x^{4'} &= z^4 + \text{constant}. \end{aligned} \tag{3.1}$$

Unless the transformation (3.1) is a Galilean transformation, the spatial coordinate system x^i is said to be non-inertial or curvilinear, or both non-inertial and curvilinear. Non-inertial spatial coordinate systems in classical mechanics are further classified as rigid, deforming, accelerated, rotating, *etc.* Though the use of non-inertial curvilinear spatial coordinates is accepted practice in classical mechanics, the fourth coordinate (time) is rarely transformed more generally than in a Galilean transformation. Thus there has arisen a large body of literature [*5, 7*] concerned with the invariants of a motion under the more general group of transformations (3.1). Now the Euclidean and Galilean groups are subgroups of the more general transformations (3.1); however, the Lorentz group is not a subgroup of (3.1) since the fourth coordinate in a Lorentz transformation is transformed more generally than in $(3.1)_2$. The utility of introducing a more general class of coordinates than the z^μ in space-time once accepted, there seems little motivation for giving undue special attention to the group (3.1) in this work, which attempts a uniform treatment of Euclidean, Galilean, and Lorentzian kinematics. What we shall do is to develop a formalism for kinematics in terms of invariants under the *group* $\mathfrak{G}_\mathrm{A}$ *of unrestricted analytic transformations on all four coordinates of events.* General coordinates in space-time will be denoted by x^μ and a typical element of the group $\mathfrak{G}_\mathrm{A}$ is written in the form

$$x^{\mu'} = \overset{\mu'}{x}(x^\mu), \qquad x^\mu = \overset{\mu}{x}(x^{\mu'}). \tag{3.2}$$

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* In § 7 we introduce a group of transformations $Z^{K'} = \overset{K'}{H}(Z^K)$ of the material coordinates Z^K. At the appropriate point in the discussion, it will be shown how a motion in any of these spaces determines an inverse relation $z^\mu = \overset{\mu}{z}(Z^K, \tau)$ between the space-time coordinate z^μ, the material coordinates Z^K and a suitable fourth variable τ. When the scalar fields $\overset{KL}{C^{-1}}$ and $\overset{KL}{X^{-1}}$ are considered as functions of the Z^K and τ, we shall write $(C^{-1})^{KL}$, $(X^{-1})^{KL}$ consistent with the fact that these quantities, so regarded, transform as tensors under transformations of the material coordinates.

The groups $\mathfrak{G}_E$, $\mathfrak{G}_G$, and $\mathfrak{G}_L$ are all subgroups of $\mathfrak{G}_A$. Once such a formalism for the three kinematical systems has been developed, the classical problem of introducing moving and deforming coordinates is of course solved since (3.1) is a subgroup of (3.2). That is, any invariant of a motion under (3.2) is automatically an invariant under the subgroup (3.1) of these more general transformations. The concepts needed to construct such a formalism for kinematics are embodied in KLEIN's *principle* [*10*, p. 65]:

If in any space with group $\mathfrak{G}_1$ the subgroup $\mathfrak{G}_2$ is introduced, consisting in all transformations which leave a figure (field) $\boldsymbol{\Phi}_1$ invariant, then the geometry of a figure $\boldsymbol{\Phi}_2$ with respect to $\mathfrak{G}_2$ is identical with the geometry of the set of figures $(\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2)$ with respect to $\mathfrak{G}_1$.

Let us illustrate the application of KLEIN's principle that we intend to make by the following familiar example. Suppose we have given a tensor field $f^{i\,\dots}_{j\,\dots}$ in ordinary 3-dimensional Euclidean metric space where the group $\mathfrak{G}_2$ is the orthogonal group, *i.e.*, $f^{i\,\dots}_{j\,\dots}$ is a Cartesian tensor. Let $\mathfrak{G}_1$ be the group of general analytic coordinate transformations in 3-dimensions. $\mathfrak{G}_2$ is a subgroup of $\mathfrak{G}_1$. Let $g_{ij}(x^k)$ be an absolute symmetric positive definite tensor field under $\mathfrak{G}_1$ such that its Riemann curvature tensor vanishes. Then in the space with group $\mathfrak{G}_1$ we know [*6*, § 10] that there exist preferred coordinate systems z^i such that $g_{ij}(z^k) = \delta_{ij}$. Furthermore, any such pair of coordinate systems are related to each other by an orthogonal transformation. Thus the group $\mathfrak{G}_2$ can be defined as the subgroup of $\mathfrak{G}_1$ which leaves the *canonical form* δ_{ij} of the Euclidean metric field g_{ij} invariant. Let $\varphi^{i\,\dots}_{j\,\dots}(x^k)$ be any field in the space with group $\mathfrak{G}_1$ having any law of transformation under $\mathfrak{G}_1$ such that

$$\varphi^{i\,\dots}_{j\,\dots}(z^k) = f^{i\,\dots}_{j\,\dots}(z^k) \tag{3.3}$$

in every preferred coordinate system in which $g_{ij} = \delta_{ij}$. According to KLEIN's principle, the theory of the invariants of the field $f^{i\,\dots}_{j\,\dots}$ under the group $\mathfrak{G}_2$ is identical with the theory of the joint invariants of the fields $(\varphi^{i\,\dots}_{j\,\dots}, g_{km})$ under the group $\mathfrak{G}_1$ of general analytic transformations or under any group containing $\mathfrak{G}_2$ as a subgroup.

Consider the group $\mathfrak{G}_A$ of general analytic transformations (3.2) of the four coordinates of events. Our objective is to define three sets of fields $\{\boldsymbol{\Phi}\}_\Delta$, $\Delta = \mathrm{E}$, G, L, having an assigned law of transformation under $\mathfrak{G}_A$ such that (1) there exists a subclass of preferred coordinates z^μ in which the fields $\{\boldsymbol{\Phi}\}_\Delta$ assume certain canonical forms and (2) the subgroups $\mathfrak{G}_E$, $\mathfrak{G}_G$, and $\mathfrak{G}_L$ consist in all the transformations of $\mathfrak{G}_A$ which leave invariant the canonical forms of the sets of fields $\{\boldsymbol{\Phi}\}_E$, $\{\boldsymbol{\Phi}\}_G$, and $\{\boldsymbol{\Phi}\}_L$, respectively. Once we determine such a set of fields we invoke KLEIN's principle and give new but equivalent definitions of Euclidean, Galilean, and Lorentzian kinematics. That is, these three theories can then be defined as the theories of the joint invariants of the combined sets of fields $\overset{K}{Z}(x^\mu)$, $\{\boldsymbol{\Phi}\}_\Delta(x^\mu)$ under the group $\mathfrak{G}_A$.

Case I. Euclidean space-time. Let $t_\mu(x^\nu) \not\equiv 0$ be an absolute covariant vector field under $\mathfrak{G}_A$ such that

$$\partial_{[\mu} t_{\nu]} = 0. \tag{3.4}$$

Let $g^{\mu\nu}(x^\pi)$ be a symmetric contravariant singular absolute tensor field under $\mathfrak{G}_A$ such that

$$g^{\mu\nu} t_\nu = 0, \qquad g^{\mu\nu} v_\mu v_\nu > 0, \tag{3.5}$$

for all $v_\mu \neq 0$ and not parallel to t_ν. The condition (3.4) is necessary and sufficient for the existence of a scalar field $t(x^\mu)$ such that

$$t_\mu = \partial_\mu t. \tag{3.6}$$

Moreover, the field $t(x^\mu)$ is uniquely determined by (3.6) to within an additive constant. Let $\overset{i}{y}(x^\mu)$, $i = 1, 2, 3$ be any three analytic functions of the coordinates such that*

$$\Theta = \varepsilon^{\mu\nu\varrho\tau} \partial_\mu \overset{1}{y} \partial_\nu \overset{2}{y} \partial_\varrho \overset{3}{y} \partial_\tau t \neq 0. \tag{3.7}$$

From (3.7) it follows that we can solve for the coordinates x^μ as functions of the variables y^i and $T = t(x^\mu)$

$$x^\mu = \overset{\mu}{x}(y^i, T). \tag{3.8}$$

Consider the functions $g^{ij}(y^k, T)$ defined by

$$g^{ij}(y^k, T) \equiv g^{\mu\nu}(\overset{\pi}{x}) \partial_\mu y^i \partial_\nu y^j. \tag{3.9}$$

We assume that the Riemann curvature tensor based on the positive definite symmetric $g^{ij}(y^k, T)$ vanishes for all values of y^i and T corresponding to the events in $N(\underset{0}{\boldsymbol{E}})$,

$$R_{jkl}{}^{i}(\boldsymbol{g}) = 0. \tag{3.10}$$

These conditions are necessary and sufficient that we be able to choose functions $y^{i'} = \overset{i'}{y}(y^i, T)$ such that the $g^{i'j'}$ defined with respect to the $y^{i'}$ have values $g^{i'j'} = \delta^{i'j'}$. Thus there exists a coordinate transformation

$$\begin{aligned} z^{i'} &= \overset{i'}{z}(x^\mu), \\ z^{4'} &= t(x^\mu) + \text{constant}, \end{aligned} \tag{3.11}$$

such that in the coordinate system $z^{\mu'}$ the fields $g^{\mu'\nu'}(z^{\pi'})$ and $t_{\mu'}(z^{\pi'})$ have the *canonical form*

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad t_\mu = (0, 0, 0, 1). \tag{3.12}$$

Applying the assumed tensor law of transformation to these canonical forms we then see that the Euclidean group of transformations (2.6) can be defined as the subgroup of $\mathfrak{G}_A$ which leaves these canonical forms invariant.

Thus *Euclidean kinematics is the theory of the joint invariants under the group $\mathfrak{G}_A$ of the set of fields*

$$\overset{K}{Z}(x^\mu), \qquad g^{\mu\nu}(x^\pi), \qquad t_\mu(x^\pi), \tag{3.13}$$

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* $\varepsilon^{\mu\nu\varrho\tau}$ and $\varepsilon_{\mu\nu\varrho\tau}$ are the completely antisymmetric axial tensor fields of weights $+1$ and -1, respectively, whose components are $+1$, -1, or 0 in every coordinate system and $\varepsilon^{1234} = \varepsilon_{1234} = +1$.

where the condition (2.10) invariant under $\mathfrak{G}_E$ is replaced by the condition

$$\overset{KL}{C^{-1}} V_K V_L > 0, \qquad V_K \not\equiv 0, \qquad \overset{KL}{C^{-1}} \equiv g^{\mu\nu} \partial_\mu \overset{K}{Z} \partial_\nu \overset{L}{Z}. \tag{3.14}$$

A coordinate system z^μ in Euclidean space-time such that (3.12) holds will be called a *Euclidean frame.* We shall call $g^{\mu\nu}$ the *space metric,* and we shall call t_μ the *covariant space normal.*

Case II. Galilean space-time. Let $\Gamma^\lambda_{\mu\nu}(x^\pi)$ be an affine connection under $\mathfrak{G}_A$. We assume that $\mathbf{\Gamma}$ is a flat or integrable connection.

$$R_{\dot{\lambda}\dot{\varrho}\nu}{}^{;\mu}(\mathbf{\Gamma}) = 0. \tag{3.15}$$

Let $g^{\mu\nu}$ and t_μ be tensors under $\mathfrak{G}_A$ having the same properties assigned to these fields as in Case I above, but which in addition satisfy the conditions

$$\begin{aligned} \underset{\Gamma}{\nabla}_\lambda g^{\mu\nu} &= \partial_\lambda g^{\mu\nu} + \Gamma^\mu_{\lambda\varrho} g^{\varrho\nu} + \Gamma^\nu_{\lambda\varrho} g^{\mu\varrho} = 0, \\ \underset{\Gamma}{\nabla}_\nu t_\mu &= \partial_\nu t_\mu - \Gamma^\lambda_{\mu\nu} t_\lambda = 0, \end{aligned} \tag{3.16}$$

jointly with the connection $\Gamma^\lambda_{\mu\nu}$. That is, the covariant derivatives of $g^{\mu\nu}$ and t_μ based on the connection $\Gamma^\lambda_{\mu\nu}$ vanish identically.

From (3.15) follows the existence of preferred coordinate systems in which all of the components of the connection vanish [*1*, § 29]. Any two such systems are related by a *linear* transformation. From (3.16) it follows that, in any of the coordinate systems in which the connection vanishes, the components of $g^{\mu\nu}$ and t_μ are constants. Set $z^4 = t(x^\mu)$. This will be a linear transformation leaving the connection zero, and t_μ will assume its canonical form

$$t_\mu = (0, 0, 0, 1). \tag{3.17}$$

From (3.5) it then follows that $g^{\mu\nu}$ is reduced to the form

$$g^{\mu\nu} = \begin{bmatrix} g^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.18}$$

where g^{ij} is a constant symmetric positive definite matrix. Thus by a further linear transformation of the first three coordinates not involving z^4, preserving the condition (3.17), and the vanishing of the connection, we can reduce $g^{\mu\nu}$ to its canonical form

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.19}$$

It is then an easy matter to verify that the Galilean group (2.7) is the subgroup of $\mathfrak{G}_A$ which leaves invariant all three canonical forms (3.17), (3.18), and $\Gamma^\lambda_{\mu\nu} = 0$.

Thus *Galilean kinematics is the theory of the joint invariants under $\mathfrak{G}_A$ of the set of fields*

$$\overset{K}{Z}(x^\mu), \qquad g^{\mu\nu}(x^\pi), \qquad t_\mu(x^\pi), \qquad \Gamma^\lambda_{\mu\nu}(x^\pi), \tag{3.20}$$

where the $\overset{K}{Z}$ satisfy the invariant condition (3.14). The preferred coordinate systems z^μ in Galilean space-time in which we have (3.17), (3.19), and $\Gamma^\varrho_{\mu\nu} = 0$ will be called *Galilean frames*, and $\Gamma^\mu_{\nu\lambda}$ will be called the *Galilean connection*.

Case III. Lorentzian space-time. Let $\gamma^{\mu\nu}(x^\pi)$ be a non-singular symmetric tensor field with signature 2 whose Riemann curvature tensor vanishes. These are necessary and sufficient conditions for the existence of preferred coordinates z^μ such that [6, § 27]

$$\gamma^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{3.21}$$

The Lorentz transformations (2.8) consists in the group of all transformations of $\mathfrak{G}_A$ which leave the canonical form (3.21) for $\gamma^{\mu\nu}$ invariant.

Thus *Lorentzian kinematics is the theory of the joint invariants under* $\mathfrak{G}_A$ *of the set of fields*

$$\overset{K}{Z}(x^\mu), \qquad \gamma^{\mu\nu}(x^\mu), \tag{3.22}$$

where the condition (2.11) invariant under $\dot{G}_L$ is replaced by the condition

$$\overset{KL}{X^{-1}} V_K V_L > 0, \qquad V_K \not\equiv 0, \qquad \overset{KL}{X^{-1}} \equiv \gamma^{\mu\nu} \partial_\mu \overset{K}{Z} \, \partial_\nu \overset{L}{Z}, \tag{3.23}$$

invariant under $\mathfrak{G}_A$. The preferred coordinates z^μ in Lorentzian space-time in which we have (3.21) will be called *Lorentz frames*. The tensor field $\gamma^{\mu\nu}$ and its inverse $\gamma_{\mu\nu}$ will be called the *Lorentz metric*.

Thus we have succeeded in formulating all three kinematical systems as theories of the joint invariants of a motion and a suitable set of fields under a common group of coordinate transformations $\mathfrak{G}_A$. Quantities transforming as a tensor under $\mathfrak{G}_A$ will be called *world tensors*. An affine connection under $\mathfrak{G}_A$ will be called a *world connection*. The Galilean connection is a world connection, $g^{\mu\nu}$, t_μ, and $\gamma^{\mu\nu}$ are world tensors, and the $\overset{K}{Z}$ are world scalars.

4. Euclidean and Galilean kinematics

The results we present in this section and the following one are not intended to represent an exhaustive, systematic study of the invariants of a motion in $\mathscr{S}_E$, $\mathscr{S}_G$, and $\mathscr{S}_L$. Rather, the remarks and equations in these sections are intended merely to illustrate the world invariant formalism and to show the ease with which familiar results, often obtained otherwise by cumbersome methods, follow easily and elegantly.

Since the Galilean group is a subgroup of the Euclidean group and a motion is defined in just the same way in both spaces, it is clear that any Euclidean invariant of a motion is also a Galilean invariant. Comparing the lists of fields (3.13) and (3.20) we see that, in the world invariant formalism, this means that any joint invariant of the fields (3.13) is also a joint invariant of the fields (3.20). Thus it is appropriate that we discuss these two kinematical theories simultaneously. Any invariant of a motion in $\mathscr{S}_E$ or $\mathscr{S}_G$ which does not depend on the Galilean connection is a Euclidean invariant of the motion. However, if an invariant depends explicitly on the Galilean connection, then this invariant will be a Galilean invariant of the motion but not a Euclidean invariant of the motion.

Euclidean kinematics as we have defined it here is equivalent to an invariant theory proposed by DEFRISE [20]. We have announced the problem somewhat differently, but the two theories are in fact the same. The paper by DEFRISE contains a number of geometrical results pertaining to a motion in Euclidean space-time. Some of these will be included in the discussion here. The references contain other sources of related material.

Consider the axial world scalar of weight 1 defined by

$$\mathfrak{D} \equiv \frac{1}{3!} \varepsilon^{\mu\nu\varrho\tau} \partial_\mu \overset{K}{Z} \partial_\nu \overset{L}{Z} \partial_\varrho \overset{M}{Z} \partial_\tau t \, \varepsilon_{KLM} \tag{4.1}$$

and the axial world vector of weight 1 defined by

$$\mathfrak{v}^\mu = \frac{1}{3!} \varepsilon^{\nu\varrho\tau\mu} \partial_\nu \overset{K}{Z} \partial_\varrho \overset{L}{Z} \partial_\tau \overset{M}{Z} \varepsilon_{KLM}. \tag{4.2}$$

In a Euclidean frame, $\mathfrak{D} = \det \partial_i \overset{K}{Z} = \pm \sqrt{\det \overset{KL}{C^{-1}}} \neq 0$. Since the law of transformation for $\mathfrak{D}$ is $\mathfrak{D}' = (x'x)^{-1}\mathfrak{D}$ and $(x'x)$ is never zero, $\mathfrak{D} \neq 0$ in any coordinate system. Thus we can define the absolute world contravariant vector field

$$v^\mu \equiv \frac{\mathfrak{v}^\mu}{\mathfrak{D}} \tag{4.3}$$

called the *world velocity vector of the motion.* The form which any world tensor or other type of world invariant takes in every Euclidean, Galilean, or Lorentz frame, depending on the context, will be called its *canonical form.* The canonical form of the world velocity vector v^μ is

$$v^\mu = (v^i, 1). \tag{4.4}$$

Since $\mathfrak{D}$ is never zero, we can always solve for any system of general coordinates x^μ in terms of the material coordinates $Z^K = \overset{K}{Z}$ and the *time* $T = t(x^\mu)$. Thus we always have relations of the form

$$x^\mu = \overset{\mu}{x}(Z^K, T), \tag{4.5}$$

where the functions $\overset{\mu}{x}$ are single-valued and analytic. In terms of the $\overset{\mu}{x}$ the world velocity vector v^μ is given by

$$v^\mu = \frac{\partial x^\mu}{\partial T} = \overset{\mu}{v}(\overset{K}{Z}, t), \qquad \overset{\mu}{v}(Z^K, T) \equiv \frac{\partial \overset{\mu}{x}}{\partial T}. \tag{4.6}$$

The result (4.5) serves to promote the geometric interpretation of a motion in terms of a congruence of lines in space-time which are nowhere tangent to the surfaces $t(x^\mu) = T = \text{constant}$. Such a surface is called *an instantaneous space.* The material coordinates Z^K serve as names for the lines of the congruence and T is an admissible parameter whose value is never stationary as one moves along a line of the congruence. A line of the congruence (4.5) is called the *world line* of the corresponding material point Z^K. Each surface $t(x^\mu) = T$ is an ordinary 3-dimensional Euclidean metric space imbedded in 4-dimensional Euclidean or Galilean space-time. One can introduce a general system of parameters or *instantaneous space coordinates*

$$y^i = \overset{i}{y}(x^\mu) \tag{4.7}$$

on each of the one parameter family of surfaces $t(x^\mu) = T$. We can also arrange matters so that the x^μ are given in terms of the y^i and T by functions

$$x^\mu = \overset{\mu}{x}(y^i, T) \tag{4.8}$$

analytic in all four variables y^i and T. The material coordinates Z^K of a material medium constitute one such set of instantaneous space coordinates. The induced surface metric $g^{ij}(y^k, T)$ defined by

$$g^{ij}(y^k, T) \equiv g^{\mu\nu}(\overset{\pi}{x})\, \partial_\mu y^i\, \partial_\nu y^j \tag{4.9}$$

is always Euclidean. Let the functions $(C^{-1})^{KL}(Z^M, T)$ be defined by (*cf.* the remarks in the footnote, page 186)

$$(C^{-1})^{KL}(Z^M, T) \equiv \overset{KL}{C^{-1}}\left(\overset{\mu}{x}(Z^M, T)\right), \tag{4.10}$$

where $\overset{KL}{C^{-1}}$ is the set of scalar functions defined in (3.14) and the $\overset{\mu}{x}$ are the functions occurring in (4.5). If we choose material coordinates Z^K for the instantaneous space coordinates y^i, the components of the instantaneous space metric $g^{KL}(Z^M, T)$ are obviously given by

$$g^{KL}(Z^M, T) = (C^{-1})^{KL}(Z^M, T). \tag{4.11}$$

Thus for any motion in Euclidean space-time or Galilean space-time we have

$$R_{\dot{K}\dot{L}\dot{M}}{}^N(C_{PQ}) = 0. \tag{4.12}$$

The quantities $(C^{-1})^{KL}$ or the inverse C_{KL}, $C_{KL}(C^{-1})^{LM} = \delta_K^M$, are called *material measures of deformation* [8, p. 140].

A motion in Euclidean or Galilean space-time is called *rigid* if and only if the functions $(C^{-1})^{KL}$ are independent of the time T, *i.e.*,

$$(\dot{C^{-1}})^{KL} = \frac{\partial (C^{-1})^{KL}}{\partial T} = v^\mu\, \partial_\mu \overset{KL}{C^{-1}} = 0. \tag{4.13}$$

Let $g_{KL}(Z^M, T)$ denote the inverse of g^{KL}. The distance between two neighboring material points Z^K and $Z^K + dZ^K$ at time T is given by

$$dS^2 = g_{KL}\, dZ^K\, dZ^L = C_{KL}\, dZ^K\, dZ^L. \tag{4.14}$$

Thus a motion is rigid if and only if $d\dot{S} = 0$ for every pair of neighboring material points.

The *Lie derivative* [10, p. 106] of a tensor field $\varphi^{\mu\dots}_{\nu\dots}$ with respect to an absolute contravariant vector field v^μ is a tensor field of the same type as $\varphi^{\mu\dots}_{\nu\dots}$ defined by

$$\underset{v}{\pounds}\, \varphi^{\mu\dots}_{\nu\dots} \equiv v^\lambda\, \partial_\lambda \varphi^{\mu\dots}_{\nu\dots} - \varphi^{\lambda\dots}_{\nu\dots}\, \partial_\lambda v^\mu - \cdots + \varphi^{\mu\dots}_{\lambda\dots}\, \partial_\nu v^\lambda + \cdots + w\, \partial_\lambda v^\lambda\, \varphi^{\mu\dots}_{\nu\dots}. \tag{4.15}$$

Consider then the contravariant absolute symmetric world tensor $\Delta^{\mu\nu}$ defined by

$$\begin{aligned} \Delta^{\mu\nu} &\equiv -\tfrac{1}{2}\, \underset{v}{\pounds}\, g^{\mu\nu} \\ &\equiv -\tfrac{1}{2}\left(v^\lambda\, \partial_\lambda g^{\mu\nu} - g^{\lambda\nu}\, \partial_\lambda v^\mu - g^{\mu\lambda}\, \partial_\lambda v^\nu\right), \end{aligned} \tag{4.16}$$

where v^μ is the world velocity vector of the motion. The tensor $\Delta^{\mu\nu}$ has the canonical form

$$\Delta^{\mu\nu} = \begin{bmatrix} d^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \qquad d^{ij} \doteq \tfrac{1}{2}\left(\partial_i v^j + \partial_j v^i\right). \tag{4.17}$$

The quantities d^{ij} are the familiar Cartesian components of the *rate of deformation tensor* [8, p. 150]. Since $\Delta^{\mu\nu}$ has the canonical form (4.17), we shall call it the *world rate of deformation tensor*. Since $\Delta^{\mu\nu}$ is defined independently of the Galilean connection, it is a Euclidean invariant as well as a Galilean invariant of a motion*.

The world scalar invariant equation

$$v^\mu \partial_\mu \overset{KL}{C^{-1}} = -2\Delta^{\mu\nu} \partial_\mu \overset{K}{Z} \partial_\nu \overset{L}{Z} \tag{4.18}$$

can be easily verified by referring all quantities to a Euclidean frame. Thus a sufficient condition for a motion to be rigid is that $\Delta^{\mu\nu}=0$. This condition is also necessary, for on referring all quantities to a Euclidean frame we get

$$v^\mu \partial_\mu \overset{KL}{C^{-1}} = 0 = -2d^{ij} \partial_i \overset{K}{Z} \partial_j \overset{L}{Z}, \quad \det \partial_i \overset{K}{Z} \neq 0, \tag{4.19}$$

from which it follows that $d^{ij}=0$. Thus every component of $\Delta^{\mu\nu}$ vanishes in a Euclidean frame if the motion is rigid. Since $\Delta^{\mu\nu}$ transforms as a tensor under general transformations of the coordinates, its components will vanish in every system of coordinates if the motion is rigid. Thus *the vanishing of the world rate of deformation tensor $\Delta^{\mu\nu}$ is a necessary and sufficient condition that a motion in $\mathcal{S}_{\mathrm{E}}$ or $\mathcal{S}_{\mathrm{G}}$ be rigid.*

The field $\mathfrak{g}$ defined by

$$\mathfrak{g} = \frac{1}{3!} \varepsilon_{\mu\varrho\lambda\tau} \varepsilon_{\nu\omega\zeta\varphi} g^{\varrho\omega} g^{\lambda\zeta} g^{\tau\varphi} v^\mu v^\nu \tag{4.20}$$

is a world scalar of weight -2 having the constant value $\mathfrak{g}=1$ in every Euclidean frame. The familiar absolute scalar invariants of the rate of deformation tensor d^{ij} are given in world invariant form by the formulae

$$\Theta_1 = \frac{\mathfrak{g}^{-1}}{2} \varepsilon_{\mu\varrho\lambda\tau} \varepsilon_{\nu\omega\zeta\varphi} g^{\varrho\omega} g^{\lambda\zeta} \Delta^{\mu\nu} v^\tau v^\varphi, \tag{4.21}$$

$$\Theta_2 = \frac{\mathfrak{g}^{-1}}{2!} \varepsilon_{\mu\varrho\lambda\tau} \varepsilon_{\nu\omega\zeta\varphi} g^{\varrho\omega} \Delta^{\lambda\zeta} \Delta^{\mu\nu} v^\tau v^\varphi, \tag{4.22}$$

$$\Theta_3 = \frac{\mathfrak{g}^{-1}}{3!} \varepsilon_{\mu\varrho\lambda\tau} \varepsilon_{\nu\omega\zeta\varphi} \Delta^{\varrho\omega} \Delta^{\lambda\zeta} \Delta^{\mu\nu} v^\tau v^\varphi. \tag{4.23}$$

The canonical form of these absolute scalars is

$$\Theta_1 = \operatorname{trace} d^{ij}, \qquad \Theta_2 = \text{sum of the principal minors of } d^{ij}, \qquad \Theta_3 = \det d^{ij}. \tag{4.24}$$

As an illustration of how the theory may be applied to problems in mechanics, consider the world tensor $\tau^{\mu\nu}$ defined by

$$\tau^{\mu\nu} = \lambda \Theta_1 g^{\mu\nu} + 2\mu \Delta^{\mu\nu} - p g^{\mu\nu}. \tag{4.25}$$

The canonical form of this tensor is

$$\tau^{\mu\nu} = \begin{bmatrix} t^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \quad t^{ij} = \lambda d^{kk} \delta^{ij} + 2\mu d^{ij} - p \delta^{ij}. \tag{4.26}$$

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* In pure geometry (see, *e.g.*, [10, p. 346]), a motion in a space characterized by a set of fields $\varphi^{\mu\cdots}_{\nu\cdots}$ is a vector field w^π satisfying one or more equations of the type $\underset{w}{\pounds} \varphi^{\mu\cdots}_{\nu\cdots} = 0$. Thus we are using the word motion in quite a different sense here. However, much of the theory of motions in the sense of pure geometry can be applied to the study of motions of continuous media.

The quantities t^{ij} transform according to required law (1.11) under the group $\mathfrak{G}_E$ relating the Euclidean frames. Equations $(4.26)_2$ are the familiar constitutive equations for the stress tensor of a classical linear viscous fluid, where λ and μ are the viscosities and p is the pressure [8, p. 126]. The pressure is assumed to be some function of the world scalars $\overset{KL}{C^{-1}}$ and the temperature. From the point of view of world invariant kinematics, the stress $\tau^{\mu\nu}$ is a world tensor differential invariant of a motion in Euclidean space-time satisfying the invariant equations $\tau^{\mu\nu}=\tau^{\nu\mu}$, $\tau^{\mu\nu}t_\nu=0$.

The Lie derivative of the world rate of deformation tensor $\Delta^{\mu\nu}$ is again a tensor given by

$$\overset{*}{\Delta}{}^{\mu\nu}\equiv \underset{v}{\pounds}\,\Delta^{\mu\nu}=v^\lambda\,\partial_\lambda\Delta^{\mu\nu}-\Delta^{\lambda\nu}\,\partial_\lambda v^\mu-\Delta^{\mu\lambda}\,\partial_\lambda v^\nu. \tag{4.27}$$

The canonical form of $\overset{*}{\Delta}$ is

$$\overset{*}{\Delta}{}^{\mu\nu}=\begin{bmatrix}\overset{*}{d}{}^{ij} & 0\\ 0 & 0\end{bmatrix},\qquad \overset{*}{d}{}^{ij}=\frac{\partial d^{ij}}{\partial z^4}+\partial_l d^{ij}\,v^l-d^{lj}\,\partial_l v^i-d^{jl}\,\partial_l v^i. \tag{4.28}$$

The quantities $\overset{*}{d}{}^{ij}$ transform as a 3-dimensional tensor under $\mathfrak{G}_E$. We see that the above process may be continued indefinitely to obtain an infinite sequence of world tensors $\Delta^{\mu\nu}, \overset{*}{\Delta}{}^{\mu\nu}, \ldots, \overset{(n)}{\Delta}{}^{\mu\nu}, \ldots$. All of these fields have a canonical form similar to (4.28) involving a sequence of 3-dimensional tensor fields $d^{ij}, \overset{*}{d}{}^{ij}, \ldots, \overset{(n)}{d}{}^{ij}, \ldots$. Such a sequence of differential invariants of a motion under the group $\mathfrak{G}_E$ has been considered by ERICKSEN & RIVLIN [*12*] in the formulation of constitutive equations for the stress in a visco-elastic material. If $\tau^{\mu\nu}$ is *any* tensor invariant of a motion in Euclidean space-time such that it has the canonical form $(4.26)_1$, then t^{ij} is an admissible stress tensor defined in terms of the motion and satisfying the transformation law (1.11) under the group $\mathfrak{G}_E$. Thus we see how the present formalism can be put to use in the problem of formulating admissible consititutive relations for the stress tensor in classical continuum mechanics. The Lie derivative of $\tau^{\mu\nu}$ with respect to the world velocity v^μ has the canonical form

$$\overset{*}{\tau}{}^{\mu\nu}=\begin{bmatrix}\overset{*}{t}{}^{ij} & 0\\ 0 & 0\end{bmatrix},\qquad \overset{*}{t}{}^{ij}=\frac{\partial t^{ij}}{\partial z^4}+\partial_l t^{ij}\,v^l-t^{lj}\,\partial_l v^i-t^{jl}\,\partial_l v^i. \tag{4.29}$$

The quantities $\overset{*}{t}{}^{ij}$ transform as a tensor under $\mathfrak{G}_E$ and have been used in formulating the constitutive equations of a class of materials called hypo-elastic [*16, 21*]. The tensor $\overset{*}{t}{}^{ij}$ is called the *convected time flux* of the stress tensor. The proof of its invariance under $\mathfrak{G}_E$ has been discussed from numerous points of view [*7, 8, 13, 14, 21*].

In terms of the world tensors $g^{\mu\nu}$ and v^μ we can define the non-singular symmetric contravariant world tensor $p^{\mu\nu}$ given by

$$p^{\mu\nu}\equiv g^{\mu\nu}-v^\mu v^\nu. \tag{4.30}$$

Let $p_{\mu\nu}$ denote the inverse of $p^{\mu\nu}$. The canonical form of $p_{\mu\nu}$ is

$$p_{\mu\nu}=\begin{bmatrix}\delta_{ij} & -v^i\\ -v^j & -1+v^k v^k\end{bmatrix},\qquad \det p_{\mu\nu}=-1. \tag{4.31}$$

Let $\left\{ {\varrho \atop \mu\nu} \right\}_p$ denote the Christoffel symbols based on the tensor $p_{\mu\nu}$. Except for special motions, these Christoffel symbols define a world connection in Galilean space-time which is distinct from the Galilean connection. The Riemannian curvature tensor based on the $\left\{ {\varrho \atop \mu\nu} \right\}_p$ does not vanish except for special motions. The Galilean connection is independent of a motion; the Christoffel symbols based on $p_{\mu\nu}$ are a type of differential invariant of a motion in $\mathscr{S}_E$ or $\mathscr{S}_G$. Before proceeding, we list some invariant algebraic relations satisfied by $p_{\mu\nu}$:

$$p^{\mu\nu} t_\nu = -v^\mu, \qquad p_{\mu\nu} v^\nu = -t_\mu, \qquad p_{\mu\varrho} p_{\nu\lambda} g^{\varrho\lambda} = p_{\mu\nu} + t_\mu t_\nu. \tag{4.32}$$

The canonical form of the Christoffel symbols is

$$\left\{ {i \atop jk} \right\}_p = v^i d^{jk}, \quad \left\{ {i \atop j4} \right\}_p = -\omega^{ij} - v^i v^s d^{sj}, \quad \left\{ {i \atop 44} \right\}_p = -\frac{\partial v^i}{\partial z^4} - v^s d^{si} - v^s \omega^{si} + v^i v^s v^r d^{sr},$$
$$\left\{ {4 \atop 4i} \right\}_p = -v^s d^{is}, \quad \left\{ {4 \atop 44} \right\}_p = v^r v^s d^{rs}, \quad \left\{ {4 \atop ij} \right\}_p = d^{ij}, \tag{4.33}$$

where $\omega^{ij} \equiv \frac{1}{2}(\partial_i v^j - \partial_j v^i)$.

Since adding any world tensor $S^\varrho_{\mu\nu} = S^\varrho_{\nu\mu}$ to the Christoffel symbols yields another set of quantities transforming as the components of a symmetric world connection [*1*, p. 6], we see that one can construct an infinity of world connections all of which are part of the Euclidean and Galilean geometry of a motion. All that is needed is to be able to define a tensor of rank 3, $S^\varrho_{\mu\nu}$, in terms of the motion, $g^{\mu\nu}$ and t_μ. The tensor $p_{\mu\nu} v^\varrho$ is one such admissible tensor, there being an infinite number of others. Without further motivation derived from physical applications or intuition, there seems little to recommend an intensive study of this variety of differential invariants of a motion. DEFRISE [*20*] has based the determination of a world connection in Euclidean space-time upon the intuitive notion that the world lines of the particles of the medium shall be a system of "parallel straight lines" in the 4-dimensional sense, plus other ideas based on the parallel transport of tensor fields. Having determined such a connection, DEFRISE proceeds to study in detail various other invariants of a motion that can be defined with his connection, such as its Riemann tensor. The equations used by DEFRISE to determine a world connection $\Omega^\varrho_{\mu\nu}$ (hereafter referred to as the Defrise connection) are equivalent to the following:

$$\underset{\Omega}{\nabla}_\nu v^\mu = 0, \qquad \underset{\Omega}{\nabla}_\lambda g^{\mu\nu} = 2\Delta^{\mu\nu} t_\lambda, \qquad \underset{\Omega}{\nabla}_\nu t_\mu = 0, \tag{4.34}$$

where the operator $\underset{\Omega}{\nabla}_\mu$ denotes covariant differentiation based on the Defrise connection $\Omega^\varrho_{\mu\nu}$. These equations have a unique solution for all 64 components of $\Omega^\varrho_{\mu\nu}$ in terms of the fields $g^{\mu\nu}$, t_λ, v^μ and their derivatives. The canonical form of the Defrise connection is

$$\Omega^i_{jk} = 0, \qquad \Omega^4_{\mu\nu} = 0, \qquad \Omega^i_{4j} = -\partial_j v^i, \qquad \Omega^i_{44} = -\frac{\partial v^i}{\partial z^4} + v^j \partial_j v^i. \tag{4.35}$$

The difference between the Defrise connection and the Christoffel symbols based on $p_{\mu\nu}$ is a world tensor given by

$$S^\varrho_{\mu\nu} = \left\{ {\varrho \atop \mu\nu} \right\}_p - \Omega^\varrho_{\mu\nu} = p_{\mu\lambda} p_{\nu\zeta} \Delta^{\lambda\zeta} v^\varrho + p_{\mu\lambda} \Delta^{\lambda\varrho} t_\nu + p_{\nu\lambda} \Delta^{\lambda\varrho} t_\mu. \tag{4.36}$$

The canonical form of $S^{\varrho}_{\mu\nu}$ is

$$S^i_{jk} = v^i d^{jk}, \quad S^i_{j4} = -v^i v^s d^{js}, \quad S^4_{4i} = -v^s s^{is}, \quad S^4_{ij} = d^{ij}, \quad S^i_{44} = v^i v^s v^r d^{sr} - 2v^s d^{is}, \quad S^4_{44} = v^r v^s d^{rs}. \tag{4.37}$$

As a variant of DEFRISE's procedure, we can solve the equations

$$\underset{\Psi}{\nabla_\lambda} g^{\mu\nu} = 0, \quad v^\nu \underset{\Psi}{\nabla_\nu} v^\mu = 0, \quad \underset{\Psi}{\nabla_\nu} t_\mu = 0, \quad \underset{\Psi}{\nabla_\lambda} v^{[\mu} g^{\nu]\lambda} = 0 \tag{4.38}$$

for the components of yet another symmetric world connection $\Psi^{\varrho}_{\mu\nu}$. According to $(4.38)_2$, the world lines of the material points undergoing the motion will be paths of the connection $\Psi^{\varrho}_{\mu\nu}$ [*1*, § 22]. The canonical form of this connection is

$$\Psi^i_{jk} = 0, \quad \Psi^4_{\mu\nu} = 0, \quad \Psi^i_{4j} = \omega^{ij}, \quad \Psi^i_{44} = (\omega^{si} - d^{si})\, v^s - \frac{\partial v^i}{\partial z^4}, \tag{4.39}$$

and the difference between this connection and the Defrise connection is the world tensor $U^{\varrho}_{\mu\nu}$ given by

$$U^{\varrho}_{\mu\nu} \equiv \Psi^{\varrho}_{\mu\nu} - \Omega^{\varrho}_{\mu\nu} = p_{\mu\lambda} \Delta^{\lambda\varrho} t_\nu + p_{\nu\lambda} \Delta^{\lambda\varrho} t_\mu. \tag{4.40}$$

The formulae (4.36) and (4.40) are convenient to have when one considers a question of the type: What is the invariant significance of the absolute derivative of the velocity field with respect to the Christoffel symbols $\{{}^{\varrho}_{\mu\nu}\}_p$? The answer follows simply from the result (4.36). Let $\underset{p}{\nabla_\mu}$ denote covariant differentiation based on the $\{{}^{\varrho}_{\lambda\mu}\}_p$. We then have

$$\underset{p}{\nabla_\nu} v^\mu = \partial_\nu v^\mu + \{{}^{\mu}_{\nu\varrho}\}_p v^\varrho = \partial_\nu v^\mu + \Omega^\mu_{\nu\varrho} v^\varrho + S^\mu_{\nu\varrho}. \tag{4.41}$$

The first two terms on the right hand side of (4.41) cancel each other by $(4.34)_1$. Substituting from (4.36) for the $S^\mu_{\nu\varrho}$ yields

$$\underset{p}{\nabla_\nu} v^\mu = p_{\nu\lambda} \Delta^{\mu\lambda}. \tag{4.42}$$

If we multiply this equation by $p^{\nu\varrho}$ and sum, we get

$$v^{\varrho\mu} = p^{\nu\varrho} \underset{p}{\nabla_\nu} v^\mu = \Delta^{\varrho\mu}. \tag{4.43}$$

Since $\Delta^{\mu\nu}$ satisfies the invariant equation $\Delta^{\mu\nu} t_\nu = 0$, we have from (4.42)

$$v^\nu \underset{p}{\nabla_\nu} v^\mu = v^\nu p_{\nu\lambda} \Delta^{\mu\lambda} = -t_\lambda \Delta^{\mu\lambda} = 0. \tag{4.44}$$

Thus *the world lines of the motion are paths of the connection* $\{{}^{\varrho}_{\mu\nu}\}_p$.

Let $\underset{\Gamma}{\nabla_\mu}$ denote covariant differentiation based on the Galilean connection $\Gamma^{\varrho}_{\mu\nu}$. The *world acceleration vector* of a motion in Galilean space-time is defined by

$$\alpha^\mu \equiv v^\nu \underset{\Gamma}{\nabla_\nu} v^\mu. \tag{4.45}$$

The canonical form of the world acceleration vector is

$$\alpha^\mu = (a^i, 0), \qquad a^i = \frac{\partial v^i}{\partial z^4} + v^j \partial_j v^i. \tag{4.46}$$

The transformation law relating the components a^i and $a^{i'}$ in two Galilean frames is

$$a^{i'} = A^{i'}_j a^j, \tag{4.47}$$

where $A_j^{i'}$ is a constant orthogonal matrix. Note that the world acceleration vector of a motion in Euclidean space-time is undefined.

Consider the world tensor $W^{\mu\nu}$ defined by

$$W^{\mu\nu} \equiv g^{\mu\lambda} \underset{\Gamma}{\nabla}_\lambda v^\nu. \tag{4.48}$$

The canonical form of this tensor is

$$W^{\mu\nu} = \begin{bmatrix} w^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \qquad w^{ij} = \partial_i v^j. \tag{4.49}$$

The symmetric and antisymmetric parts of $W^{\mu\nu}$ are world tensors

$$\Delta^{\mu\nu} = W^{(\mu\nu)}, \qquad \Omega^{\mu\nu} = W^{[\mu\nu]}, \qquad W^{\mu\nu} = \Delta^{\mu\nu} + \Omega^{\mu\nu}, \tag{4.50}$$

where the canonical form of $\Omega^{\mu\nu}$ is

$$\Omega^{\mu\nu} = \begin{bmatrix} \omega^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \omega^{ij} = \tfrac{1}{2}(\partial_i v^j - \partial_j v^i). \tag{4.51}$$

Since ω^{ij} is the classical measure of vorticity, we call $\Omega^{\mu\nu}$ the *world vorticity tensor*. Now the world rate of deformation tensor $\Delta^{\mu\nu}$ was defined in (4.16) independently of the Galilean connection; whereas, from (4.50) it is not immediately apparent that the symmetric part of $W^{\mu\nu}$ can be expressed solely in terms of the fields v^μ, $g^{\mu\nu}$ and the derivatives of these fields. That this is the case follows from $(3.16)_1$.

$$W^{(\mu\nu)} = g^{\lambda(\mu} \partial_\lambda v^{\nu)} + g^{\lambda(\mu} \Gamma^{\nu)}_{\lambda\varrho} v^\varrho, \tag{4.52}$$

and from $(3.16)_1$ we get

$$2 g^{\lambda(\mu} \Gamma^{\nu)}_{\lambda\varrho} = - \partial_\varrho g^{\mu\nu}. \tag{4.53}$$

Substituting this last result into (4.52) we verify the identity $W^{(\mu\nu)} \equiv -\frac{1}{2} \underset{v}{\pounds}\, g^{\mu\nu}$. The components of the world vorticity tensor $\Omega^{\mu\nu}$ cannot in this same way be expressed solely in terms of the fields v^μ, $g^{\nu\pi}$, t_σ and their derivatives That is, vorticity is not a Euclidean invariant of a motion. But this is clear from an intuitive point of view since vorticity measures a rate of rotation and rotation does not have an invariant significance under the Euclidean group, but does have an invariant significance under the smaller Galilean group.

5. Lorentzian kinematics

The study of invariants of a motion in Lorentzian space-time has important applications in relativistic mechanics and electromagnetic theory. The symmetrical treatment of all four coordinates for events has found greater usage and favor in relativity theory than in classical mechanics. However, as we have seen, the formulation of classical invariant-theoretic problems in terms of a 4-dimensional geometry is easily accomplished and has some formal manipulative advantages. There appears still to exist some misunderstanding in applied work concerning the introduction of general coordinates in space-time and the physical implications of such a process. The equations of classical mechanics, as well as all of the purely kinematical considerations we have given here, can be phrased and presented in terms of world invariants of a suitable set of fields in space-time.

This does not alter the physical hypotheses or interpretations constituting the theory any more or less than the familiar use of curvilinear coordinates for positions in space. McVittie [5] has considered certain of the Galilean invariants of a motion as limiting cases of Lorentz invariants with infinitesimal velocities. Here we have preferred an independent development of each of these theories of motion treating each as an exact science. One of the best sources for ideas and results in Lorentzian kinematics is Møller's book on relativity theory [9]. Following is a brief discussion of a few Lorentz invariants of a motion whose importance, relative to others we might consider, is suggested by applications in relativistic theories of elastic bodies and fluids.

Consider the world axial vector field of weight 1 defined by

$$\mathfrak{v}^{\mu} \equiv \frac{1}{3!}\, \varepsilon^{\varrho\nu\lambda\mu}\, \partial_{\varrho}\overset{K}{Z}\, \partial_{\nu}\overset{L}{Z}\, \partial_{\lambda}\overset{M}{Z}\, \varepsilon_{KLM}. \tag{5.1}$$

The world scalar of weight 2 given by

$$\boldsymbol{\theta} \equiv \gamma_{\mu\nu}\, \mathfrak{v}^{\mu}\, \mathfrak{v}^{\nu} \tag{5.2}$$

is always *negative*. This follows on substituting the definition of $\mathfrak{v}^{\mu}$ into (5.2) and deriving the identity

$$\boldsymbol{\theta} = \det \gamma_{\mu\nu} \det \overset{KL}{X^{-1}} < 0. \tag{5.3}$$

The inequality holds since the determinant of the Lorentz metric is always negative and the determinant of $\overset{KL}{X^{-1}}$ is by assumption always positive. The absolute world vector defined by

$$w^{\mu} \equiv \frac{\mathfrak{v}^{\mu}}{\sqrt{-\boldsymbol{\theta}}}, \qquad \gamma_{\mu\nu}\, w^{\mu}\, w^{\nu} = -1, \tag{5.4}$$

is called the ***relativistic world velocity vector of the motion.***

Let the motion be referred to an arbitrary Lorentz frame. We then have

$$\begin{aligned} \overset{KL}{X^{-1}} &= \delta^{ij}\, \partial_i\overset{K}{Z}\, \partial_j\overset{L}{Z} - \partial_4\overset{K}{Z}\, \partial_4\overset{L}{Z}, \\ \delta^{ij}\, \partial_i\overset{K}{Z}\, \partial_j\overset{L}{Z} &= \overset{KL}{X^{-1}} + \partial_4\overset{K}{Z}\, \partial_4\overset{L}{Z}. \end{aligned} \tag{5.5}$$

Taking the determinant of both sides of this last equation yields

$$\mathfrak{D}^2 = (\det \partial_i \overset{K}{Z})^2 = \det \overset{KL}{X^{-1}} \left(1 + \underset{KL}{X}\, \partial_4\overset{K}{Z}\, \partial_4\overset{L}{Z}\right) > 0, \tag{5.6}$$

where $\underset{KL}{X}$ is the inverse of the matrix $\overset{KL}{X^{-1}}$. Since $\overset{KL}{X^{-1}}$ is positive definite, so is its inverse, and from (5.6) we see that $\mathfrak{D}^2$ is never zero for any motion of a material medium in Lorentzian space-time. This means that we can always solve for the first three Lorentz coordinates z^i of a Lorentz frame in terms of the Z^K and z^4

$$z^i = \overset{i}{z}(Z^K, z^4). \tag{5.7}$$

The canonical m of the axial vector $\mathfrak{v}^{\mu}$ is

$$\mathfrak{v}^{\mu} = \mathfrak{D}\left(\frac{\partial \overset{i}{z}}{\partial z^4}, 1\right) = \mathfrak{D}(v^i, 1), \tag{5.8}$$

where v^i is the "classical velocity" of a material point relative to the Lorentz frame z^μ, where we think of z^4 as classical time. This is not a Lorentz invariant notion since z^4 and T have different transformation laws. The canonical form of (5.2) is

$$\boldsymbol{\theta} = \mathfrak{D}^2 (v^i v^i - 1) < 0, \tag{5.9}$$

where the inequality follows from (5.3). Thus we see that the hypothesis that $\overset{KL}{X^{-1}}$ is positive definite for the motion of a material medium in Lorentzian space-time leads to the familiar result that the "classical velocity" of the motion relative to *any* Lorentz frame is always less than 1, where this upper limit for the speed of any motion is identified with the speed of light in the chosen system of units.

The components of the relativistic world velocity vector w^μ in a Lorentz frame have the values

$$w^\mu = \left(\frac{v^i}{\sqrt{1-v^2}}, \frac{1}{\sqrt{1-v^2}} \right), \qquad v^2 \equiv v^i v^i. \tag{5.10}$$

In relativistic mechanics, the fourth component of w^μ in a Lorentz frame less 1 is called the *kinetic energy per unit of mass* [*4*].

An affine connection in $\mathscr{S}_{\mathrm{L}}$ is determined by the Christoffel symbols based on the Lorentz metric. All of these symbols vanish in a Lorentz frame. Let $\underset{\gamma}{\nabla}_\mu$ denote covariant differentiation based on the Christoffel symbols of $\gamma_{\mu\nu}$. The *world velocity gradients* are defined by $\underset{\gamma}{\nabla}_\mu w^\nu$. The relativistic counterparts of the world rate of deformation tensor and the world vorticity tensor of Euclidean and Galilean kinematics can be defined as follows:

$$\underset{\mathrm{L}}{\Delta}^{\mu\nu} \equiv \gamma^{\lambda(\mu} \underset{\gamma}{\nabla}_\lambda w^{\nu)}, \qquad \underset{\mathrm{L}}{\Omega}^{\mu\nu} \equiv \gamma^{\lambda[\mu} \underset{\gamma}{\nabla}_\lambda w^{\nu]}, \tag{5.11}$$

where we shall have the identity

$$\underset{\mathrm{L}}{\Delta}^{\mu\nu} \equiv -\tfrac{1}{2} \underset{w}{\pounds} \gamma^{\mu\nu}, \tag{5.12}$$

analogous to the classical case (4.16), (4.50).

In the case of Euclidean or Galilean kinematics, a rigid motion can be defined by either of the conditions $v^\mu \partial_\mu \overset{KL}{C^{-1}} = 0$, or $\Delta^{\mu\nu} = 0$. In the case of a motion in Lorentzian space-time, however, the two analogous conditions

$$w^\mu \partial_\mu \overset{KL}{X^{-1}} = 0, \qquad \underset{\mathrm{L}}{\Delta}^{\mu\nu} = 0 \tag{5.13}$$

are *not* equivalent. There exist motions for which we have $(5.13)_1$ and do not have $(5.13)_2$. In fact, if w^μ satisfies $(5.13)_2$, then it is a *translation* [*10*, p. 349]. Its components in a Lorentz frame will have the form (5.10), where the v^i will be constants. The motion

$$\overset{K}{Z}(z^i, z^4) = A^K_i(z^4)\, z^i, \qquad A^K_i A^L_i = \delta^{KL}, \tag{5.14}$$

representing a rotation about the origin of the Lorentz frame z^i is an admissible motion in $\mathscr{S}_{\mathrm{L}}$ for all values of z^i and z^4 such that

$$\overset{KL}{X^{-1}} = \delta^{KL} - \dot{A}^K_i \dot{A}^L_j z^i z^j \tag{5.15}$$

is positive definite. This will be true for all sufficiently small values of z^i. If the time derivatives of the A^K_i are constants, then the first of the conditions (5.13) is satisfied. The motion (5.14) is a rigid motion in Euclidean or Galilean space-time. Thus it is reasonable to define a rigid motion in Lorentzian space-time by the first of the conditions (5.13). We are aware that there has been considerable debate as to what a useful and appropriate definition of a rigid motion in relativistic kinematics should be. We see that the conditions $(5.13)_2$ would be too restrictive since they rule out all but the uniform translations. Pursuing the analogy with Euclidean kinematics, consider the functions $(X^{-1})^{KL}(Z^K, z^4)$ defined by

$$(X^{-1})^{KL}(Z^K, z^4) \equiv \overset{KL}{X^{-1}}\left(z^i(Z^K, z^4), z^4\right), \tag{5.16}$$

where the functions $\overset{i}{z}$ are those occurring in (5.7). Thus, treating z^4 as a parameter, we can construct the Riemann curvature tensor $R_{\dot{L}\dot{M}\dot{N}}{}^K(X_{KL})$, $X_{KL}(X^{-1})^{LM} = \delta^M_K$. MØLLER [9] calls a motion in $\mathscr{S}_\mathrm{L}$ satisfying the conditions

$$R_{\dot{L}\dot{M}\dot{N}}{}^K(X_{PQ}) = 0 \tag{5.17}$$

a *Euclidean motion*. The rigid motion (5.14) is *not* a Euclidean motion. Equation (5.17) is to be compared with its classical counterpart (4.12), which holds for *any* motion in $\mathscr{S}_\mathrm{E}$ or $\mathscr{S}_\mathrm{G}$. Moreover, (4.12) is a Euclidean and Galilean invariant property of any motion in $\mathscr{S}_\mathrm{E}$ or $\mathscr{S}_\mathrm{G}$; whereas, if (5.17) is satisfied by a motion relative to one Lorentz frame, it need not be satisfied by the same motion referred to another Lorentz frame. That is, since z^4 is not an invariant parameter in (5.17) under Lorentz transformations, this condition is not Lorentz invariant.

In special relativity theory, the equations representing conservation of energy, conservation of momentum, conservation of angular momentum, and the equivalence of momentum and energy flux take the form [4]

$$\underset{\gamma}{\nabla_\nu} P^{\mu\nu} = 0, \qquad P^{\mu\nu} = P^{\nu\mu}, \tag{5.18}$$

where $P^{\mu\nu}$ is the stress-energy-momentum tensor. It is customary to write $P^{\mu\nu}$ in the form

$$P^{\mu\nu} = \tau^{\mu\nu} - \varrho\, w^\mu w^\nu, \tag{5.19}$$

where $\tau^{\mu\nu}$ embodies the relativistic counterparts of the classical stress, internal energy, and heat flux. In classical mechanics, it is customary to provide constitutive equations for all of these quantities in terms of the motion, the temperature, the electromagnetic field, *etc.*, and to require their invariance under various groups of transformations such as the rigid motions. But in relativity theory, owing to the open question of a proper definition of a rigid motion which is not a uniform translation, a clear and concise statement of the class of admissible constitutive equations for $\tau^{\mu\nu}$ depending on a motion cannot, to our knowledge, be found in the literature. *The relativistic counterpart of the fundamental assumption* (1.11) *of classical continuum mechanics has not been enunciated.* It would seem that a sound relativistic generalization of, say, classical elasticity theory rests on questions of this nature.

6. Convected coordinates in Euclidean and Galilean space-time

Given a motion in Galilean or Euclidean space-time, there are, in addition to the Galilean and Euclidean frames, other classes of preferred coordinates defined in a natural way in terms of the motion [12, 14]. One such class of coordinate systems are the convected coordinates belonging to a motion [11, 7]. A convected coordinate system of a motion in $\mathcal{S}_E$ or $\mathcal{S}_G$ can be defined as any coordinate system x^Ω (we use upper case Greek indices to indicate convected coordinates) in which the world velocity vector v^μ and the covariant space normal t_μ have the *convected form*

$$v^\Omega = (0,0,0,1), \qquad t_\Omega = (0,0,0,1). \tag{6.1}$$

The existence of such coordinate systems is easy to perceive. Let $\overset{K}{X}(Z^L)$ be any three functionally independent functions of the Z^L. If we transform the components v^μ and t_μ in an arbitrary system of coordinates x^μ to the coordinate system x^Ω determined by

$$\begin{aligned} x^K &= \overset{K}{X}(\overset{L}{Z}(x^\mu)), \\ x^4 &= t(x^\mu) = T, \end{aligned} \tag{6.2}$$

we conclude immediately that the coordinate system (x^K, T) is a convected coordinate system. Any two convected coordinate systems x^Ω and $x^{\Omega'}$ are related by a transformation having the general form

$$\begin{aligned} x^{K'} &= \overset{K'}{x}(x^K), \\ x^{4'} &= x^4 + \text{constant}. \end{aligned} \tag{6.3}$$

From $(3.5)_1$ it follows immediately that in a convected frame

$$g^{\Omega 4} = g^{4\Omega} = 0, \tag{6.4}$$

and from the definition of the scalars $\overset{KL}{C^{-1}}$ we have

$$\overset{KL}{C^{-1}} = g^{MN} \frac{\partial \overset{K}{Z}}{\partial x^M} \frac{\partial \overset{L}{Z}}{\partial x^N}. \tag{6.5}$$

If we consider the special case in which $\overset{K}{X} = \overset{K}{Z}$, we have the simpler relation

$$\overset{KL}{C^{-1}} = g^{KL}. \tag{6.6}$$

Thus *the non-vanishing components of $g^{\mu\nu}$ in a convected frame $x^K = Z^K$ of a motion are equal respectively to the six material measures of deformation* $\overset{KL}{C^{-1}}$. Some workers in elasticity theory [11] who employ convected coordinates almost exclusively refer to the quantities C_{KL} as the metric tensor. The result (6.6) provides the principal motivation for the use of such terminology. However, it must be realized that the equality (6.6) holds only in a restricted class of coordinate systems.

Consider next the rate of deformation tensor $\Delta^{\mu\nu}$, whose components in a general frame are given by (4.16). Substituting the convected form of the velocity vector into this formula, we conclude that

$$\Delta^{\Omega 4} = \Delta^{4\Omega} = 0, \qquad \Delta^{KL} = -\frac{1}{2}\frac{\partial g^{KL}}{\partial T}. \tag{6.7}$$

More generally, we have for the series of world tensors $\overset{*}{\Delta}{}^{\mu\nu}, \ldots, \overset{(n)}{\Delta}{}^{\mu\nu}, \ldots$ [*cf.* (4.27)]

$$\overset{(n)}{\Delta}{}^{\Omega 4} = \overset{(n)}{\Delta}{}^{4\Omega} = 0, \qquad \overset{(n)}{\Delta}{}^{KL} = -\frac{1}{2}\frac{\partial^n g^{KL}}{\partial T^n}. \tag{6.8}$$

Since the covariant derivatives of $g^{\mu\nu}$ and t_μ with respect to the Galilean connection vanish in an arbitrary coordinate system, we have

$$\partial_\Theta t_\Omega - \Gamma^{\Psi}_{\Theta\Omega} t_\Psi = 0, \tag{6.9}$$

$$\partial_\Psi g^{\Theta\Omega} + \Gamma^{\Omega}_{\Psi\Delta} g^{\Delta\Theta} + \Gamma^{\Theta}_{\Psi\Delta} g^{\Omega\Delta} = 0. \tag{6.10}$$

From (6.9) and $(6.1)_2$ it follows that, in a convected frame,

$$\Gamma^{4}_{\Omega\Theta} = 0. \tag{6.11}$$

From (6.10) we obtain the equations

$$\Gamma^{K}_{LM} = \left\{{K \atop LM}\right\}_g, \qquad \Gamma^{(K}_{4M} g^{L)M} = -\frac{1}{2}\frac{\partial g^{KL}}{\partial T} = \Delta^{KL}, \tag{6.12}$$

where the Γ^{K}_{LM} are the Christoffel symbols based on g_{KL}, $g_{KL} g^{LM} = \delta^{M}_{K}$.

The components of the world vorticity tensor $\Omega^{\Phi\Psi}$ in a convected frame are given by

$$\Omega^{\Theta 4} = -\Omega^{4\Theta} = 0, \qquad \Omega^{KL} = g^{M[K}\Gamma^{L]}_{M4}. \tag{6.13}$$

The world acceleration vector has components given by

$$a^4 = 0, \qquad a^K = \Gamma^{K}_{44}. \tag{6.14}$$

The formulae of this section are useful for the interpretation of any system of deforming and accelerated coordinates in Euclidean or Galilean space-time not necessarily associated with the motion of a material medium. That is, consider the class of all coordinate systems in $\mathscr{S}_E$ or $\mathscr{S}_G$ in which we have $t_\mu = (0, 0, 0, 1)$. Equation (6.7) then gives an interpretation of the time dependence of the non-vanishing components of the space metric $g^{\mu\nu}$ in one of these general types of coordinate systems. $-\frac{1}{2}\frac{\partial g^{KL}}{\partial T}$ is a measure of the rate of deformation of the coordinate system. Similarly, the formulae (6.12), (6.13), and (6.14) provide an interpretation of the non-vanishing components of the Galilean connection in such a system of coordinates.

7. The geometry of the space of material points and material symmetry

In continuum mechanics we assign a geometry to the 3-dimensional space $\mathscr{S}_M$ with coordinates Z^K by introducing a group $\mathfrak{G}_M$ of *allowable material coordinate transformations*. A typical element of this group has the form

$$Z^{K'} = A^{K'}_{L} Z^L + D^{K'}, \tag{7.1}$$

where, in all of the applications with which we are familiar, it is sufficiently general to assume that $\mathfrak{G}_M$ is some subgroup of the 3-dimensional orthogonal group. Thus the matrix of coefficients in (7.1) is an *orthogonal matrix*. By demanding invariance of constitutive equations under $\mathfrak{G}_M$ we obtain further restrictions on the form of these equations. The group $\mathfrak{G}_M$ determines the *material* *symmetry* of the medium. Let us see how this relation between the group of material coordinate transformations $\mathfrak{G}_M$ and the idea of material symmetry arises. It is customary to require that, for some instant T_0, the functions $\overset{K}{Z}(z^i, T)$ representing a motion relative to a Euclidean or Galilean frame (z^i, T) reduce to the form

$$z^i = \delta^i_K \overset{K}{Z}(z^i, T_0) = \delta^i_K Z^K. \tag{7.2}$$

That is, the material coordinates Z^K at the instant T_0 coincide with the Cartesian coordinates z^i. Now if the material points are all identical and arranged in space in a uniform homogeneous array, the intuitive notion is that we cannot in this way ascribe unique names to each material point, but that the names (material coordinates Z^K) are determined only to within an arbitrary orthogonal transformation. This is the intuitive picture of an *isotropic homogeneous material medium*. Thus the appropriate group $\mathfrak{G}_M$ for a material with this symmetry is the *complete orthogonal group*. If some of the material points have different properties than others, such as in a crystal, and if they are arranged in space in some non-uniform or inhomogeneous array at the instant T_0, the class of equivalent (allowable) material coordinate systems Z^K will be smaller than the corresponding class for isotropic homogeneous materials. Thus the group $\mathfrak{G}_M$ in the general case will be some proper subgroup of the orthogonal group. The continuum theory of elastic homogeneous crystals is based on constitutive equations for the stress which are invariant under a group of material coordinate transformations $\mathfrak{G}_M$, where the set of matrices $A^{K'}_L$ in (7.1) constitute the elements of one of the 32 crystallographic subgroups of the orthogonal group characteristic of the *point* symmetry of the crystal. In the theory of finite elastic deformations of crystals and in the classical linear theory of elastic crystals the constitutive equations are assumed invariant to arbitrary translations D^K in (7.1). This last assumption represents an approximation to a more detailed description of the symmetry of a crystal in which one would require invariance only to a group of discrete translations D^K. However, the present formalism does not rule out the possibility of an accurate and detailed continuum description of such "microscopic" structure or symmetry. Other choices for the group $\mathfrak{G}_M$ describe materials having transverse isotropic symmetry, orthotropic symmetry, *etc.*, [*19*].

Let us illustrate the way in which the invariance of constitutive equations under the group $\mathfrak{G}_M$ restricts their general form by considering the case of the stress tensor in finite elasticity theory. In § 1 we remarked that if the stress tensor t^{ij} in an elastic body of any symmetry whatever were invariant under the group of rigid motions, then it must reduce to the form

$$t^{ij} = \frac{\partial \overset{i}{x}}{\partial Z^K} \frac{\partial \overset{j}{x}}{\partial Z^L} \overset{KL}{P}\left(\overset{M}{Z}, \overset{PQ}{C^{-1}}, \mathfrak{D}\right). \tag{7.2 a}$$

If we now demand that t^{ij} transform as an absolute scalar under the group $\mathfrak{G}_M$ of all orthogonal transformations, whereby we assume that the material is isotropic and homogeneous, then it is known that the functions $\overset{KL}{P}$ are expressible in the special form [*12*]

$$\overset{KL}{P} = \mathcal{F}_0 \delta^{KL} + \mathcal{F}_1 \overset{KL}{C^{-1}} + \mathcal{F}_2 \overset{KL}{C^{-2}}. \tag{7.3}$$

where the $\mathscr{F}$'s are functions of the scalar invariants of the matrix $\overset{KL}{C^{-1}}$.

$$\mathscr{F}_\Omega = \mathscr{F}_\Omega(\mathrm{I}, \mathrm{II}, \mathrm{III}), \qquad \Omega = 0, 1, 2, \tag{7.4}$$
$$\mathrm{I} \equiv \operatorname{trace} \overset{KL}{C^{-1}}, \ \mathrm{II} \equiv \text{sum of the principal minors of } \overset{KL}{C^{-1}}, \ \mathrm{III} \equiv \det \overset{KL}{C^{-1}} \equiv \mathfrak{D}^2.$$

The constitutive equations for the stress in a material with less symmetry than an isotropic homogeneous medium will involve functions $\overset{KL}{P}$ of $\overset{K}{Z}$, $\overset{KL}{C^{-1}}$, and $\mathfrak{D}$ more complicated than (7.3).

In presenting these few remarks and examples concerning the relation between material symmetry and the invariant-theoretic problems encountered in the formulation of constitutive relations in continuum mechanics, we have attempted to make clear that the problem is one of invariance under at least two distinct groups of transformations: 1) invariance to rigid motions, 2) invariance under a group of material coordinate transformations. A third group not considered here is the group of *unit* transformations. It is important that these three demands for invariance not be confused and that each of them be satisfied. A confusion of this sort is the apparent source of difficulty in some recent attacks on the foundations of classical elasticity theory [*18*, *et al.*]. As we have seen, these invariances are not equivalent.

In the 3-dimensional space $\mathscr{S}_{\mathrm{M}}$ with group $\mathfrak{G}_{\mathrm{M}}$ we can introduce general coordinates X^K and appeal once again to KLEIN's principle so as to obtain an equivalent statement of the invariant theoretic problem. Let $\mathfrak{G}_{\mathrm{MA}}$ denote the group of unrestricted analytic transformations on the material coordinates X^K with typical element*

$$X^{K'} = \overset{K'}{Y}(X^K), \qquad X^K = \overset{K}{Y}(X^{K'}). \tag{7.5}$$

Let $\boldsymbol{H}_\Omega(X^K)$, $\Omega = 1, 2, \ldots, N$ be a set of fields having an assigned law of transformation under $\mathfrak{G}_{\mathrm{MA}}$ such that the fields $\boldsymbol{H}_\Omega$ possess certain canonical forms $\boldsymbol{H}_\Omega(Z^K)$ in a subclass Z^K of the general coordinate systems X^K and such that the group $\mathfrak{G}_{\mathrm{M}}$ consists in all the transformations of $\mathfrak{G}_{\mathrm{MA}}$ which leave these canonical forms invariant. By KLEIN's principle, the geometry of a field or set of fields such as $C_{KL}(Z^M, T)$ with respect to the group $\mathfrak{G}_{\mathrm{M}}$ is equivalent to the geometry of the fields $C_{KL}(X^M, T)$ together with the fields $\boldsymbol{H}_\Omega(X^K)$ under the group $\mathfrak{G}_{\mathrm{MA}}$.

Since we assume that $\mathfrak{G}_{\mathrm{M}}$ is some subgroup of the orthogonal group, we can always choose for one of the fields $\boldsymbol{H}_\Omega$ a symmetric positive definite Euclidean metric tensor $G_{KL}(X^M)$ and identify the frames Z^K with some subclass of the coordinate systems in which we have $G_{KL}(Z^M) = \delta_{KL}$. If $\mathfrak{G}_{\mathrm{M}}$ is the complete orthogonal group, G_{KL} is the only field in the set $\boldsymbol{H}_\Omega$ since the orthogonal group consists in all the transformations which leave its canonical form δ_{KL} invariant. The procedure of introducing general coordinates in the space $\mathscr{S}_{\mathrm{M}}$ of an isotropic homogeneous material can be illustrated by writing down the form of (7.3) which is invariant under $\mathfrak{G}_{\mathrm{MA}}$. It is as follows [*8*]:

$$P^{KL}(X^M, T) = \mathscr{F}_0 G^{KL}(X^M) + \mathscr{F}_1 (C^{-1})^{KL}(X^M, T) + \mathscr{F}_2 (C^{-2})^{KL}, \tag{7.6}$$

* Here we must use a different symbol for the functions $\overset{K}{Y}$ representing a material coordinate transformation so as not to confuse these functions of 3 variables with the motion $\overset{K}{X}(x^\mu)$ represented by functions of 4 variables x^μ.

where we put $(C^{-1})^K_L = G_{LM}(C^{-1})^{KM}$, $(C^{-2})^{KL} = G^{KM}(C^{-1})^L_N (C^{-1})^N_M$. The $\mathcal{F}$'s are functions of I, II, and III, now given by

$$\mathrm{I} = (C^{-1})^K_K, \quad \mathrm{II} = \frac{1}{2!}\,\delta^{KL}_{PR}(C^{-1})^P_K (C^{-1})^R_L, \quad \mathrm{III} = \frac{1}{3!}\,\delta^{KLM}_{PRS}(C^{-1})^P_K (C^{-1})^R_L (C^{-1})^S_M. \qquad (7.7)$$

8. Two-point tensor fields and world invariant kinematics

We have seen that in continuum mechanics we are interested in functions of a motion such as the stress tensor in elasticity theory which are invariant under two groups of transformations. The first of these groups involves transformations of the coordinates of points (events) in a 4-dimensional space and the second of these groups involves transformations of the coordinates of points (material points) in a 3-dimensional space. *Multiple point fields* are familiar objects in pure geometry and have been used to advantage in continuum mechanics [*2*, *19*, *17*]. The concept of a 2-point field is an easy generalization of the concept of a 1-point field set forth in § 2. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two spaces, not necessarily of the same dimension, with groups $\mathfrak{G}_1$ and $\mathfrak{G}_2$, respectively. Let X^Ω, $\Omega = 1, 2, \ldots, N$ denote coordinates for the points in $\mathcal{S}_1$ and let x^μ, $\mu = 1, 2, \ldots, n$ denote coordinates for points in $\mathcal{S}_2$. A *2-point field* is a set of functions $F_\Delta(X^\Omega, x^\mu)$ of the coordinates of a point in $\mathcal{S}_1$ and of a point in $\mathcal{S}_2$. Thus the components F_Δ of a 2-point field will depend in general on $n+N$ independent variables. A 2-point field has a law of transformation for its components under independent transformations of the coordinates of either point. The law of transformation may be any law consistent with the property that a representation $F_\Delta(X^\Omega, x^\mu)$ in any system of reference (X^Ω, x^μ) determines a unique representation $F_\Delta(X^{\Omega'}, x^{\mu'})$ in every other allowable system of reference $(X^{\Omega'}, x^{\mu'})$. An *invariant property* of a 2-point field is a property held in common by all of its representations. Joint invariants, differential invariants, *etc.*, are defined as in the case of 1-point fields.

A *2-point tensor field* is a set of functions $T^{\mu\ldots\Omega\ldots}_{\nu\ldots\Theta\ldots}(X^\Omega, x^\mu)$ with a transformation law of the general form

$$\begin{aligned} T^{\mu'\ldots\Omega'\ldots}_{\nu'\ldots\Theta'\ldots} = {} & |(x'\,x)|^{-w}\,|(X'\,X)|^{-W}\,(\mathrm{sgn}\,(x'\,x))^y\,(\mathrm{sgn}\,(X'\,X))^Y \times \\ & \times \frac{\partial x^{\mu'}}{\partial x^\mu}\cdots\frac{\partial X^{\Omega'}}{\partial X^\Omega}\cdots\frac{\partial x^\nu}{\partial x^{\nu'}}\cdots\frac{\partial X^\Theta}{\partial X^{\Theta'}}\,T^{\mu\ldots\Omega\ldots}_{\nu\ldots\Theta\ldots}. \end{aligned} \qquad (8.1)$$

An *absolute 2-point tensor field* is one for which $w = W = y = Y = 0$. Names of 2-point fields with other values of the exponents w, W, y, and Y are assigned on a basis similar to the case of 1-point fields. 1-point tensor fields are special cases of 2-point tensor fields which are constant scalars with respect to transformations of one of the points. Thus, in all of the work preceding this section, the 1-point fields in space-time can be regarded as special cases of 2-point fields.

Let $\mathcal{S}_1$ be the 3-dimensional space $\mathcal{S}_{\mathrm{M}}$ of material points with group $\mathfrak{G}_{\mathrm{MA}}$ and let $\mathcal{S}_2$ be one of the 4-dimensional spaces $\mathcal{S}_{\mathrm{E}}$, $\mathcal{S}_{\mathrm{G}}$, or $\mathcal{S}_{\mathrm{L}}$ with group $\mathfrak{G}_{\mathrm{A}}$. The invariant theoretic problems of classical continuum mechanics can be phrased in terms of 2-point fields $F_\Delta(X^K, x^\mu)$. We call such a 2-point field a *world-material* *field*. We have already met with an example of such a world-material field $(C^{-1})^{KL}(X^M, T)$ that we defined by setting

$$(C^{-1})^{KL}(X^M, T) = \overset{KL}{C^{-1}}\big(\overset{\mu}{x}(X^M, T)\big). \tag{8.2}$$

One readily verifies that $(C^{-1})^{KL}$ is a world-material tensor field having a transformation law of the general form (8.1). The dependence of this field on the space-time coordinates x^μ is somewhat special since it depends on the x^μ only in the combination $T = t(x^\mu)$. Now it is possible also to regard $(C^{-1})^{KL}$ as a 1-point field in space-time, $\overset{KL}{C^{-1}}(x^\mu)$. But these quantities do not have the convenient tensor law of transformation under the group $\mathfrak{G}_{\mathrm{MA}}$. Rather, the $\overset{KL}{C^{-1}}(x^\mu)$ have the odd transformation law

$$\overset{K'L'}{C^{-1}}(x^\mu) = \overset{K'}{\underset{K}{B}}(x^\mu)\, \overset{L'}{\underset{L}{B}}(x^\mu)\, \overset{KL}{C^{-1}}(x^\mu), \tag{8.3}$$

where the functions $\overset{K'}{\underset{K}{B}}$ are determined by

$$\overset{K'}{\underset{K}{B}}(x^\mu) = \frac{\partial \overset{K'}{Y}}{\partial X^K}\big(\overset{K}{X}(x^\mu)\big), \tag{8.4}$$

and the $\overset{K}{X}(x^\mu)$ are the functions representing the motion relative to the general coordinate system x^μ. Thus the transformation law of the $\overset{KL}{C^{-1}}(x^\mu)$ under the group $\mathfrak{G}_{\mathrm{MA}}$ depends on the motion $\overset{K}{X}(x^\mu)$ and cannot be written down without it. Of course, we used the motion to define the functions $(C^{-1})^{KL}(X^K, t)$, but they have a more convenient tensor law of transformation under $\mathfrak{G}_{\mathrm{MA}}$.

The concept of a 2-point field and a pair of spaces $\mathscr{S}_{\mathrm{E}}$, $\mathscr{S}_{\mathrm{G}}$, or $\mathscr{S}_{\mathrm{L}}$ and $\mathscr{S}_{\mathrm{M}}$ each with its own geometry leads naturally to the interpretation of a motion $X^K = \overset{K}{X}(x^\mu)$ as a mapping between the points of space-time and the material points of $\mathscr{S}_{\mathrm{M}}$. The mapping is one-to-one only in the direction $x^\mu \to X^K$. In the other direction $X^K \to x^\mu$, a single material point is mapped onto a 1-dimensional set of events, the world line of X^K.

The geometry of world-material fields is enriched still further by adding the *connectors* or *shifters* $g^\mu_K(X^M, x^\pi)$, $g^K_\mu(X^M, x^\pi)$ to the list of 1-point fields $g^{\mu\nu}$, t_π, $\Gamma^\varrho_{\sigma\pi}$, $\gamma^{\lambda\xi}$, $H^{KL\ldots}(X^M), \ldots, G_{KL}(X^M)$ which characterize the geometry of each of our two spaces separately. The connectors are 2-point world-material absolute tensor fields providing a linkage or connection between the two spaces. Quantities similar to the connectors we now introduce have been considered previously in [*19*, *17*]. For our purposes here, we shall define the components of the connectors in a general system of coordinates (X^K, x^μ) as follows: Let there be a class of *preferred* Euclidean, Galilean, or Lorentz frames z^μ_0 and rectangular Cartesian material frames Z^K_0 such that the spatial frames z^i_0 and the material frames Z^K_0 are "at rest" relative to each other and whose axes coincide. With respect to such a system of reference (Z^K_0, z^μ_0) we assume that the connectors have the *joint canonical* form

$$g^\mu_K = (\delta^i_K, 0), \qquad g^K_\mu = (\delta^K_i, 0). \tag{8.5}$$

The components of the connectors in a general system of reference are then given by

$$g^{\mu}_{K}(X^M, x^\pi) = \delta^i_L \frac{\partial Z^L_0}{\partial X^K} \frac{\partial x^\mu}{\partial z^i_0}, \qquad g^K_\mu(X^M, x^\pi) = \delta^L_i \frac{\partial X^K}{\partial Z^L_0} \frac{\partial z^i_0}{\partial x^\mu}. \tag{8.6}$$

Stated more simply and directly, we assume that the connectors are absolute 2-point tensor fields such that by suitable choice of a Euclidean, Galilean, or Lorentz frame and a Cartesian material frame they are reducible to the joint canonical form (8.5). A pair of coordinate systems (Z^K_0, z^μ_0) in which we have (8.5) is called a *common frame.* If the components of the connectors corresponding to $\mathscr{S}_{\mathrm{E}}$ and $\mathscr{S}_{\mathrm{G}}$ are referred to an arbitrary Cartesian material frame Z^K and an arbitrary Euclidean or Galilean frame z^μ they will have the *canonical* form

$$\begin{aligned} g^{\mu}_{K} &= (S^i_K, 0), & g^K_\mu &= (S^K_i, V^K), \\ V^K &= -S^K_i V^i, & S^i_K S^K_j &= \delta^i_j, \end{aligned} \tag{8.7}$$

where V^i is the *relative velocity* of the origin of the spatial frame z^i and the material frame Z^K, and S^i_K is an orthogonal matrix representing the relative orientation of the two systems of axes Z^K and z^i. In Euclidean space-time, the S^K_i and V^i will be general functions of z^4, while in Galilean space-time they will be constants. Let V^μ be a world vector in $\mathscr{S}_{\mathrm{E}}$ or $\mathscr{S}_{\mathrm{G}}$ with the canonical form

$$V^\mu = (V^i, 1). \tag{8.8}$$

The Euclidean, Galilean and Lorentzian connectors satisfy the invariant relations

$$\begin{aligned} g^{\mu}_{K} g^L_\mu &= \delta^L_K, & g^{\mu}_{K} g^K_\nu &= \delta^\mu_\nu - V^\mu t_\nu, \\ g^K_\mu g^L_\nu g^{\mu\nu} &= G^{KL}, & G^{KL} g^{\mu}_{K} g^{\nu}_{L} &= g^{\mu\nu}, \\ g^{\mu}_{K} g^{\nu}_{L} \gamma_{\mu\nu} &= G_{KL}, & g^K_\mu g^L_\nu \gamma^{\mu\nu} &= G^{KL}. \end{aligned} \tag{8.9}$$

All of these relations can be verified by referring all quantities to the common frame. Since they are tensor equations holding in one frame, they will hold in a general system of coordinates.

As an illustration of the kind of world invariants of a motion in $\mathscr{S}_{\mathrm{E}}$ or $\mathscr{S}_{\mathrm{G}}$ that one can construct with the connectors, we consider the problem of defining a world tensor measure of the finite rotation of a motion relative to the common frame. The considerations given here are natural generalizations of those given in [*17*, § 4] to the case of arbitrary moving and deforming coordinate systems in space-time.

Consider the world tensor defined by

$$c_{\mu\nu} \equiv G_{KL}(\overset{M}{X}(x^\pi))\, \partial_\mu X^K \partial_\nu X^L. \tag{8.10}$$

The canonical form of this tensor is

$$c_{\mu\nu} = \begin{bmatrix} c_{ij} & -c_{ik} v^k \\ -c_{jk} v^k & c_{kl} v^k v^l \end{bmatrix}, \qquad c_{ij} \equiv \delta_{KL}\, \partial_i Z^K \partial_j Z^L. \tag{8.11}$$

The quantities c_{ij} are the *spatial measures of finite deformation* introduced into elasticity theory by Cauchy and Green [*8*]. It follows immediately from the canonical form (8.11) that $\det c_{\mu\nu}=0$ so that $c_{\mu\nu}$ is a singular tensor. Its null eigenvector is the world velocity vector v^μ. Thus in defining a world spatial measure of finite deformation it proves more convenient to use the tensor $c^{*\mu\nu}$ defined by

$$c^{*\mu\nu} \equiv G^{KL}\, \partial_K x^\mu\, \partial_L x^\nu, \tag{8.12}$$

where $\overset{\mu}{x}(X^K, T)$ are the functions (4.5). The canonical form of $c^{*\mu\nu}$ is

$$c^{*\mu\nu} = \begin{bmatrix} (c^{-1})^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \tag{8.13}$$

where the $(c^{-1})^{ij}$ are the components of the inverse of c_{ij}. $c^{*\mu\nu}$ has the null eigenvector t_μ and $(c^{-1})^{ij}$ is positive definite. Consider the eigenvalue equation

$$c^{*\mu\nu} \underset{\Omega}{n_\nu} = \underset{\Omega}{c}^{-1} g^{\mu\nu} \underset{\Omega}{n_\nu}, \qquad \Omega = 1, 2, 3. \tag{8.14}$$

This equation has 3 solutions $\left(\underset{\Omega}{n_\nu}, \underset{\Omega}{c}^{-1}\right)$ with positive $\underset{\Omega}{c}$ and vectors $\underset{\Omega}{n_\nu}$ satisfying the two conditions

$$g^{\mu\nu} \underset{\Omega}{n_\mu} \underset{\Theta}{n_\nu} = \delta_{\Omega\Theta}, \qquad v^\mu \underset{\Omega}{n_\mu} = 0. \tag{8.15}$$

The scalars $\left(\underset{\Omega}{c}^{\frac{1}{2}} - 1\right)$ are called the *principal extension ratios*. The canonical form of the vectors $\underset{\Omega}{n_\mu}$ is

$$\underset{\Omega}{n_\mu} = \left(\underset{\Omega}{n_i}, -v^i \underset{\Omega}{n_i}\right), \tag{8.16}$$

where the unit vectors $\underset{\Omega}{n_i}$ determine the *principal axes of strain in the deformed body* [8, 17]. All of these results follow from the canonical form of $c^{*\mu\nu}$, and we have simply placed them in world invariant form.

Now consider the world-material vector fields $\underset{\Omega}{N}(X^K, T)$ satisfying the eigenvalue equation

$$(C^{-1})^{KL} \underset{\Omega}{N_L} = \underset{\Omega}{C}^{-1} G^{KL} \underset{\Omega}{N_L}. \tag{8.17}$$

Since $(C^{-1})^{KL}$ is positive definite, the eigenvalues $\underset{\Omega}{C}^{-1}$ are all positive, and there exist three linearly independent eigenvectors $\underset{\Omega}{N_L}$ satisfying

$$G^{KL} \underset{\Omega}{N_K} \underset{\Theta}{N_L} = \delta_{\Omega\Theta}. \tag{8.18}$$

Consider next the world-material vector fields $\underset{\Omega}{n_K}(X^M, T)$ obtained from the fields $\underset{\Omega}{n_\mu}(x^\pi)$ according to the rule

$$\underset{\Omega}{n_K}(X^M, T) \equiv g^\mu_K(X^M, \overset{\pi}{x}) \underset{\Omega}{n_\mu}(\overset{\pi}{x}). \tag{8.19}$$

Since we have the identity $(8.9)_4$, it follows from (8.15) and (8.19) that

$$G^{KL} \underset{\Omega}{n_K} \underset{\Theta}{n_L} = \delta_{\Omega\Theta}. \tag{8.20}$$

Thus $\underset{\Omega}{N}_K$ and $\underset{\Omega}{n}_K$ are two sets of orthogonal unit vectors at (X^K, T). Therefore, there exists a unique matrix $R^K_L(X^M, T)$ satisfying

$$\underset{\Omega}{n}_L = R^K_L \underset{\Omega}{N}_K, \qquad G^{MN} R^K_M R^L_M = G^{KL}, \qquad R^K_L = \sum_{\Omega} \underset{\Omega}{n}_L \underset{\Omega}{N}_N G^{KN}. \tag{8.21}$$

When the material system of coordinates is rectangular Cartesian, R^K_L is an orthogonal matrix. The sense in which this matrix is a measure of the finite rotation of a motion and the relation in which it stands to the classical measure of infinitesimal rotation has been explained in [*19, 17*]. From the canonical forms of $c^{*\mu\nu}$ and $(C^{-1})^{KL}$ we see that c_{ij} and $(C^{-1})^{KL}$ have equal eigenvalues. Since these eigenvalues are absolute scalars under general transformations of the coordinates (X^K, x^μ), we shall have in general

$$\underset{\Omega}{c}(\overset{\mu}{x}) = \underset{\Omega}{C^{-1}}, \tag{8.22}$$

provided we order the two sets of eigenvalues appropriately. With this result we can show that the vector fields $\underset{\Omega}{n}_\mu$ given by

$$\underset{\Omega}{n}_\mu = \partial_\mu X^K \underset{\Omega}{N}_K(\overset{K}{X}, t) \sqrt{\underset{\Omega}{C}} \tag{8.23}$$

satisfy all of the equations (8.14) and (8.15). If the eigenvalues of C_{KL} are distinct so are the eigenvalues of $c^{*\mu\nu}$ and the eigenvectors $\underset{\Omega}{N}_K$ and $\underset{\Omega}{n}_\mu$ are uniquely determined. This is not true if two or more of the eigenvalues are equal. However, in the case of distinct eigenvalues, the motion determines a unique matrix R^K_L provided we order each set of eigenvectors in some definite way such as that corresponding to the equalities (8.22), (8.23).

Multiplying (8.23) by $g^\mu_M \underset{\Omega}{N}_L$ and summing on μ and Ω we get

$$\begin{aligned} G_{MK} R^K_L &= \sum_{\Omega} \underset{\Omega}{n}_\mu \underset{\Omega}{N}_L g^\mu_M = \sum_{\Omega} \underset{\Omega}{n}_M \underset{\Omega}{N}_L = \partial_\mu X^K g^\mu_M \sum_{\Omega} \sqrt{\underset{\Omega}{C}} \underset{\Omega}{N}_K \underset{\Omega}{N}_L, \\ R_{ML} &= \partial_\mu X^K g^\mu_M (C^{\frac{1}{2}})_{KL}. \end{aligned} \tag{8.24}$$

Multiplying this last equation through by $(C^{-\frac{1}{2}})^{LN} g^M_\nu$ and using $(8.9)_2$ we get finally

$$\partial_\nu X^K = t_\nu V^\mu \partial_\mu X^K + (C^{-\frac{1}{2}})^{KL} R_{ML} g^M_\nu. \tag{8.25}$$

The canonical form of this world-material tensor equation is

$$\partial_i X^K = (C^{-\frac{1}{2}})^{KL} R_{ML} S^M_i, \tag{8.26}$$

$$\frac{\partial X^K}{\partial T} = (C^{-\frac{1}{2}})^{KL} R_{ML} V^M + V^i \partial_i X^K + \frac{\partial X^K}{\partial T}. \tag{8.27}$$

Equation (8.26) corresponds to the result (4.19) of [*17*]. The last equation is an identity satisfied as a consequence of (8.26) and $(8.7)_2$. Equation (8.26) is the familiar decomposition of a deformation $\partial_i X^K$ into a pure stretching without rotation followed by a rigid rotation. In Euclidean space-time, the orthogonal matrix S^M_i will in general depend on the time. As we have said, it represents the time dependent relation between the material coordinate axes and the moving, rotating axes of the Euclidean frame.

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