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Abstract

This paper examines the cycling behavior of a deterministic and a stochastic version of the economic interpretation of the Lotka-Volterra model, the Goodwin model. We provide a characterization of orbits in the deterministic highly non-linear model. We then study the cycling behavior for a stochastic version, where a Brownian noise is introduced via an heterogeneous productivity factor. Sufficient conditions for existence of the system are provided. We prove that the system produces cycles around an equilibrium point in finite time for general volatility levels, using stochastic Lyapunov techniques for recurent domains. Numerical insights are provided.

Keywords: Lotka-Volterra model; Goodwin model; Brownian motion; Random perturbation; Business cycles; Stochastic Lyapunov techniques.

1 Introduction

The Lotka-Volterra equation is at the heart of population dynamics, but also possesses a famous economic interpretation. Introduced by Richard Goodwin [10] in 1967, the model in its modern form [6] reduces to the planar oscillator on a subset D of $\mathbb{R}_+$:

$$\left\{ \begin{array}{ll} dx_t & = x_t\left(\Phi(y_t) - \alpha\right) dt \\ dy_t & = y_t\left(\kappa(x_t) - \gamma\right) dt \end{array} \right. , \tag{1}$$

where x_t denotes the wage share of the working population and y_t the employment rate, α and γ are constant, and the following assumption is made on κ and Φ.

Assumption 1. *Consider system* (1).

(i) $\Phi \in \mathcal{C}^2([0,1))$*,* $\Phi'(y) > 0$*,* $\Phi''(y) \geq 0$ *for all* $y \in [0,1)$*,* $\Phi(0) < \alpha$ *and* $\lim_{y\to 1^-} \Phi(y) = +\infty$*.*

(ii) $\kappa \in \mathcal{C}^2(\mathbb{R}_+)$*,* $-\infty < \kappa'(x) < 0$ *for all* $x \in \mathbb{R}_+$*,* $\kappa(0) > \gamma$ *and* $\lim_{x\to+\infty} \kappa(x) = -\infty$*.*

Lemma 1 below asserts that Assumption 1 is sufficient to have $(x_t, y_t) \in D := \mathbb{R}_+^* \times (0,1)$ for any $t \geq 0$ if $(x_0, y_0) \in D$. This property preserves the above interpretation for x and y: the employment rate cannot exceed one for obvious reasons, but the wage share can, depending on the chosen economic assumptions, see [11]. This distinctive feature of the economic version (1) on its biological counterpart follows from a construction based on assumptions describing a closed capitalist economy. It can be done in three steps:

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(I) Assume a Leontief production function $P_t = \min(K_t/\nu; a_t y_t N_t)$ with full utilization of capital, i.e., $K_t/\nu = a_t y_t N_t$. Here, P_t is the yearly output, K_t the invested capital, $\nu > 0$ a capital-to-output ratio, $a_t := a_0 \exp(\alpha t)$ is the average productivity of workers and $N_t := N_0 \exp(\beta t)$ is the size of the labor class.

(II) The capital depreciates and receives investment, i.e., $dK_t/dt = (\kappa(x_t) - \delta)K_t$, where $\delta > 0$ is the depreciation rate and κ the investment function. Goodwin [10] originally invokes Say's law, i.e., $\kappa \ : \ x \in \mathbb{R}_+ \mapsto (1-x)/\nu$.

(III) Assume a reserve army effect for wage negotiation of the form $dw_t = \Phi(y_t)w_t dt$ where $w_t := a_t x_t$ represents the real wage of the total working population, and Φ is the Phillips curve.

Defining $\gamma := \alpha + \beta + \delta$ allows to retrieve (1) for $(x_t, y_t) := (w_t/a_t, K_t/(\nu a_t N_t))$. The class-struggle model (1) has been extensively studied because it allows to generate endogenous real business cycles affecting the production level P_t, e.g. [8, 9, 11, 15, 28, 29]. On this matter, Goodwin himself conceded that the model is "starkly schematized and hence quite unrealistic" [10]. It hardly connects with irregular observed trajectories, see [13, 22].

The objective of this paper is thus to study the following perturbed version of (1) by a standard Brownian motion $(W_t)_{t\geq 0}$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\begin{cases} dx_t &= x_t\left((\Phi(y_t) - \alpha + \sigma^2(y_t))dt + \sigma(y_t)dW_t\right) \\ dy_t &= y_t\left((\kappa(x_t) - \gamma + \sigma^2(y_t))dt + \sigma(y_t)dW_t\right) \end{cases} , \tag{2}$$

where σ is a positive function of y bounded by $\sigma_0 > 0$, and the filtration $\mathcal{F}_t$ is generated by paths of W. The form of σ is discussed in Remark 2 after. A stronger condition, Assumption 3, is assumed later on the behavior of σ to ensure that solutions of (2) remain in D. The example of Section 5 will also illustrate how such condition can hold. We modify the economic development (I), (II) and (III) by introducing the perturbation on one assumption, namely we assume that for $t \geq 0$,

$$da_t := a_t d\alpha_t = a_t\left(\alpha dt - \sigma(y_t)dW_t\right) \ , \quad a_0 \geq 0 \ , \tag{3}$$

instead of $da_t = a_t \alpha dt$. Using Itô formula with (3) in the previous reasoning retrieves (2). Productivity is one of the few exogenous parameters of the model, and one of those that were significantly invoked as influencial over business cycles, e.g. [7, 12]. Without arguing for the pertinence of that particular assumption, we simply suggest here that a standard continuous perturbation in this crucial parameter seems a good starting point.

To our knowledge, this is the first attempt to consider random noise in Goodwin interpretation of the famous prey-predator model. To stay in the spirit of the economic application, the present paper studies the cyclical behavior of the deterministic system (1) and the stochastic version (2). Namely, our contribution are as follows, developed in the present order:

- In Section 2, we fully characterize solutions of (1) and the period of their orbits. This generalizes standard results on Lotka-Volterra systems to bounded domains of existence.

- In Section 3, we provide existence conditions for regular solutions of (2). We use the entropy of (1) to estimate the deviation induced by (3). We provide a definition of stochastic orbits for (2). The proof that solutions of (2) draw stochastic orbits in finite time around a unique point is given in Section 4.

Our contribution has to be put in contrast with numerous studies of random perturbations of the Lotka-Volterra system. Apart from the obviously different origin of perturbations in the model, attention was mainly given to systems like (2) for its asymptotic behavior (e.g., [18, 21, 24]), regularity, persistence and extinction of species (e.g., [4, 20, 24, 25]), and the addition of regimes, jumps or delay (e.g., [2, 19, 31]). Here, we attempt to provide a relevant description of trajectories (x_t, y_t) and indirectly P_t, namely a cyclical behavior. This is done using stochastic Lyapunov techniques for recurrent domains as described in [17, 27]. By conveniently dividing the domain D, we obtain that almost every trajectory "cycles" around a point in finite time. The L_1-boundedness is out of reach with our method, but numerical simulations are presented in Section 5, not only to provide expectation of cycles, but also allow to conjecture a limit cycle phenomenon for the expectation of (x_t, y_t). It is somewhat unclear how Assumption 1 and late Assumption 3 on Φ, κ and σ, are relevant in these results. We show below thatthey are sufficient to obtain existence of regular solutions to (2). This actually emphasizes the role played by the entropy of the deterministic system in the well-posedness of the stochastic system and as a natural measure for perturbation.

2 Deterministic orbits

According to Assumption 1, there exists only one non-hyperbolic equilibrium point to (1) in D given by $(\hat{x}, \hat{y}) := (\kappa^{-1}(\gamma), \Phi^{-1}(\alpha))$. On the boundary of D, there exists also an additional equilibrium $(0, 0)$ which is eluded along the paper.

Definition 1. *Let V_1, V_2 and V be three functions defined by $V : (x, y) \in \mathbb{R}_+^* \times (0, 1) \mapsto V_1(x) + V_2(y)$ and*

$$V_1 \ : \ x \in \mathbb{R}_+^* \mapsto \int_{\hat{x}}^{x} \frac{\kappa(\hat{x}) - \kappa(s)}{s} ds \ , \quad V_2 \ : \ y \in (0,1) \mapsto \int_{\hat{y}}^{y} \frac{\Phi(s) - \Phi(\hat{y})}{s} ds \ .$$

Lemma 1. *Let $(x_0, y_0) \in D$. Let Assumption 1 hold. Then a solution (x_t, y_t) to (1) starting at (x_0, y_0) at $t = 0$ describes closed orbits given by the set of points $\{(x, y) \in D \ : \ V(x, y) = V(x_0, y_0)\}$, and $(x_t, y_t) \in D$ for all $t \geq 0$.*

Proof. It is well-known [11] that V is a Lyapunov function and a constant of motion for system (1): V_1 and V_2 take non-negative values with $V_1(\hat{x}) = V_2(\hat{y}) = 0$, and $dV/dt(x, y) = 0$. Additionally, under Assumption 1.(i)-(ii),

$$\lim_{x \uparrow +\infty} V_1(x) = \lim_{x \downarrow 0^+} V_1(x) = \lim_{y \uparrow 1^-} V_2(y) = \lim_{y \downarrow 0^+} V_2(y) = +\infty \ , \tag{4}$$

so that for any $(x_0, y_0) \in D$, $V(x_0, y_0) < +\infty$ and the solution stays in D.

□

The value of V characterizing an orbit, it is in bijection with its period. The following generalizes [14].

Theorem 1. *Let $(x_t, y_t)_{t \geq 0}$ be a solution to (1) satisfying Assumption 1, with $(x_0, x_0) \in D \backslash \{(\hat{x}, \hat{y})\}$. Let $V_0 := V(x_0, y_0)$, and $\underline{x} < \bar{x}$ the two solutions to equation $V_1(x) = V_0$. Define three functions F_1, F_2, G by*

$$F_1 \ : \ u \in \mathbb{R} \mapsto V_2(\Phi^{-1}(u^+ + \alpha)) \ , \quad F_2 \ : \ u \in \mathbb{R} \mapsto V_2(\Phi^{-1}(-u^- + \alpha)) \ , \quad G \ : \ z \in \mathbb{R} \mapsto V_0 - V_1(e^z) \ .$$

Then $(x_T, y_T) = (x_0, y_0)$ for T defined by

$$T(V_0) := \int_{log(\underline{x})}^{\log(\bar{x})} \frac{1}{F_1^{-1}(G(z))} - \frac{1}{F_2^{-1}(G(z))} dz \ . \tag{5}$$

Proof. Let $(x_0, y_0) \in D\backslash\{(\hat{x}, \hat{y})\}$, $V_0 = V(x_0, y_0) > 0$ and (x_t, y_t) a solution to (1) starting at (x_0, y_0). According to Lemma 1, $V_1(x_t) = V_0$ implies $V_2(y_t) = 0$. Then $\{x \in \mathbb{R}_+ \ : \ x_t = x \text{ for some } t \geq 0\} = [\underline{x}, \bar{x}] =: I$. Homogeneity of (1) allows to set $(x_0, y_0) = (\underline{x}, \hat{y})$ without loss of generality. Let $T_1 := \inf\{t \geq 0 \ : \ x_t = \bar{x}\}$. For $t \in [0, T_1]$, $(x_t, y_t) \in [\underline{x}, \bar{x}] \times [\hat{y}, \bar{y}]$, with $\bar{y}$ such that $V_2(\bar{y}) = V_0$. Let $z_t := \log(x_t)$ for $t \geq 0$. Then (1) rewrites $dz = (\Phi(y) - \alpha)dt$, $y = \Phi^{-1}(dz/dt + \alpha)$ and we get

$$\frac{d^2 z}{dt^2} = \Phi'(y)\frac{dy}{dt} = \Phi'\left(\Phi^{-1}\left(\frac{dz}{dt} + \alpha\right)\right)\Phi^{-1}\left(\frac{dz}{dt} + \alpha\right)[\kappa(e^z) - \gamma] \ .$$

Let $u := \Phi(y) - \alpha$ and define $\Psi := \Phi' \circ \Phi^{-1} \times \Phi^{-1}$, to rewrite again

$$\begin{cases} dz/dt &= u \\ du/dt &= \Psi\,(u + \alpha)\,[\kappa(e^z) - \gamma] \end{cases} . \tag{6}$$

Since $z_t \in [\underline{z}, \bar{z}] := [\log(\underline{x}), \log(\bar{x})]$ and $u_t \in [0, \Phi(\bar{y}) - \alpha]$ for $t \in [0, T_1]$, separation of variables in (6) provides two quantities F and G:

$$F(u) := \int_0^u \frac{s}{\Psi\,(s + \alpha)} ds = \int_{\underline{z}}^z [\kappa(e^s) - \gamma] ds =: G(z) \ . \tag{7}$$

The function F verifies $F(0) = G(\bar{z}) = 0$, is increasing on $[0, \Phi(\bar{y}) - \alpha]$ and decreasing on $[\Phi(\underline{y}) - \alpha, 0]$ with $\underline{y} < \hat{y}$ so that $V_2(\underline{y}) = 0$. Coming back to $y = \Phi^{-1}(u + \alpha)$ we get

$$F(u) = \int_0^u \frac{s}{\Phi'(\Phi^{-1}(s + \alpha))\Phi^{-1}(s + \alpha)} ds = \int_{\hat{y}}^{\Phi^{-1}(u+\alpha)} \frac{\Phi(s) - \Phi(\hat{y})}{s} ds = V_2(\Phi^{-1}(u + \alpha)) \ ,$$

implying that $F(u) \in [0, V(\underline{x}, \hat{y})]$ for $u \in [\Phi(\underline{y}) + \alpha, \Phi(\bar{y}) + \alpha]$. We can write $F = F_1 + F_2$ where F_1 and F_2 are the two restrictions of F on $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively. Notice that if $t \in [0, T_1]$, then $u_t := \Phi(y_t) + \alpha \in [0, \Phi(\bar{y} + \alpha)]$. Thus, $F_1(u_t)$ is a strictly increasing function of t taking its values in $[0, V(\underline{x}, \hat{y})]$. Getting back to $x = e^z$ for G, we have for $z \geq \underline{z}$

$$G(z) := \int_{\underline{x}}^{\hat{x} \wedge e^z} \frac{\kappa(s) - \gamma}{s} ds + \int_{\hat{x} \wedge e^z}^{e^z} \frac{\kappa(s) - \gamma}{s} ds = V_0 - V_1(e^z)$$

Since $\text{sign}(\kappa(x) - \gamma) = \text{sign}(\hat{x} - x)$ we have $\max_{z \in [\underline{z}, \bar{z}]} G(z) = G(\log(\hat{x})) = V_1(\underline{x}) = V(\underline{x}, \hat{y})$, while minimums are given by $G(\bar{z}) = G(\underline{z}) = 0$. This sums up with $G([\underline{z}, \bar{z}]) \subset [0, V_0]$, so we can write on this interval $F_1^{-1}(G(z)) = u = dz/dt$ which finally gives

$$T_1 = \int_{\underline{z}}^{\bar{z}} \frac{dz}{F_1^{-1}(G(z))} \ .$$

We apply the same method for the other half orbit, taking $(x_0, y_0) = (\bar{x}, \hat{y})$ and $T_2 := \inf\{t \geq 0 \ : \ x_t = \underline{x}\}$, to reach the other half of expression (5), i.e., $T(V_0) = T_1 + T_2$. □

Remark 1. A first order approximation of (1) at $(\hat{x}, \hat{y})$ provides a linear homogeneous system, which solution is trivially given by a linear combination of sines and cosines of $(-\hat{x}\Phi'(\hat{y})\hat{y}\kappa'(\hat{x})t)$. It follows that

$$\lim_{V_0 \to 0} T(V_0) = \frac{2\pi}{\sqrt{-\hat{x}\Phi'(\hat{y})\hat{y}\kappa'(\hat{x})}} > 0 \ .$$

3 Stochastic Goodwin model

We study a specific case of (2) where the deterministic part cancels at a unique point in D defined by $(\tilde{x}, \tilde{y}) := (\kappa^{-1}(\gamma - \sigma^2(\tilde{y})), \tilde{y})$ where $\tilde{y}$ comes from the following.

Assumption 2. *There is a unique $\tilde{y} \in (0,1)$ such that $\Phi(\tilde{y}) - \alpha + \sigma^2(\tilde{y}) = 0$*

For a stochastic differential equation to have a unique global solution for any given initial value, functions Φ and κ are generally required to satisfy linear growth and local Lipschitz conditions, see [17]. We can however consider the following Theorem of Khasminskii [17], which is a reformulation of Theorem 3.4, Theorem 3.5 and Corollary 3.1 of [17] to our context.

Theorem 2. *Consider the following stochastic differential equation for z taking values in $\mathbb{R}^2_+$:*

$$dz_t = \mu(z_t)dt + \sigma(z_t)dW_t \, . \tag{8}$$

Let $(D_n)_{n\geq 1}$ be an increasing sequence of open sets, and $(K_n)_{n\geq 1}$ a sequence of constants, verifying

(a) $\bar{D}_n \subset D$ for all $n \geq 1$,

(b) $\bigcup_n D_n = D$.

(c) For any $n \geq 1$, functions μ and σ are Lipschitz on D_n and verify $|\mu(z)|+|\sigma(z)| \leq K_n(1+|z|)$ for any $z \in D_n$.

Let $\varphi \in \mathcal{C}^{1,2,2}(\mathbb{R}_+ \times D)$ and $(K,k) \in \mathbb{R}^2_+$ be such that, denoting $\mathcal{L}^z$ the generator associated with (8),

(d) $\mathcal{L}^z\varphi(t, z_t) \leq K\varphi(t, z_t) + k$ on the set $\mathbb{R}_+ \times D$,

(e) $\lim_n \inf_{D\setminus D_n} \varphi(t,z) = +\infty$ for any $t \geq 0$.

Then for any $z \in D$, there exists a regular adapted solution to (8), unique up to null sets, with the Markov property and verifying $z_t \in D$ for all $t \geq 0$ almost surely.

To satisfy conditions (a) to (e), we study (2) under the additional sufficient growth conditions.

Assumption 3. *There exist two positive constants K, k such that*

(i) $\sigma^2(y)\Phi'(y) \leq KV_2(y) + k$ for all $y \in (0,1)$,

(ii) $-x\kappa'(x) - \kappa(x) \leq KV_1(x) + k$ for all $x \in \mathbb{R}^*_+$.

Remark 2. Assumption 3.(i) involves both Φ and σ to ensure that $y_t \in (0,1)$ for all $t \geq 0$ almost surely. Assumption 3.(ii) holds for polynomial growth of κ, suiting the classical conditions of existence on $\mathbb{R}_+$ for x_t. The dependence of σ could be generalized to x in full generality, implying a stronger condition than (ii). We refrain from doing this easy extension, emphasizing the unavoidable dependence in y.

For $\varphi \in \mathcal{C}^{1,2,2}(\mathbb{R}_+ \times D)$, we recall the diffusion operator associated with (2) by

$$\begin{aligned} \mathcal{L}\varphi(t,x,y) \;\; := \;\; & \Big[\tfrac{\partial\varphi}{\partial t} + \tfrac{\partial\varphi}{\partial x}x(\Phi(y) - \alpha + \sigma^2(y)) + \tfrac{\partial\varphi}{\partial y}y(\kappa(x) - \gamma + \sigma^2(y)) \\ & +\tfrac{\sigma^2(y)}{2}\left(\tfrac{\partial^2\varphi}{\partial x^2}x^2 + \tfrac{\partial^2\varphi}{\partial y^2}y^2 + 2\tfrac{\partial^2\varphi}{\partial x\partial y}xy\right)\Big](t,x,y). \end{aligned} \tag{9}$$

Theorem 3. *Let $(x_0, y_0) \in D$. Let Assumptions 1, 2 and 3 hold. Then there exists a solution $(x_t, y_t)_{t\geq 0}$ to (2) staying in D almost surely.*

Proof. Let us show that conditions (a) to (e) of Theorem 2 are fulfilled. Consider the sequence of sets $(D_n)_{n\geq 1}$ defined by $D_n = (1/(n+1), n) \times (1/(n+1), 1-1/(n+1))$. For any $n \geq 1$, D_n is open and $D_n \subset D_{n+1}$. (a) and (b) are satisfied with the limit $D = \mathbb{R}_+^* \times (0,1)$. According to Assumption 1, one can always find K_n big enough such that $\max\{|\Phi(y) - \alpha|; |\kappa(x) - \gamma|\} \leq K_n$ for any $(x, y) \in D_n$, and ensures the local Lipschitz condition (c).

Now consider V of Definition 1 which is $\mathcal{C}^{1,2,2}$ on D. Applying (9),

$$\begin{aligned}\mathcal{L}V(x,y) = &[\kappa(\hat{x}) - \kappa(x)]\,(\Phi(y) - \alpha + \sigma^2(y)) + [\Phi(y) - \Phi(\hat{y})]\,(\kappa(x) - \gamma + \sigma^2(y)) \\ &+ \left(\left[\kappa(x) - \kappa(\hat{x}) - x\kappa'(x)\right] + \left[\Phi(\hat{y}) - \Phi(y) + y\Phi'(y)\right]\right)\sigma^2(y)/2\,.\end{aligned}$$

Since $\alpha = \Phi(\hat{y})$ and $\gamma = \kappa(\hat{x})$,

$$\mathcal{L}V(x,y) = \left(\left[\kappa(\hat{x}) - \kappa(x) - x\kappa'(x)\right] + \left[\Phi(y) - \Phi(\hat{y}) + y\Phi'(y)\right]\right)\sigma^2(y)/2\,. \tag{10}$$

Assumption 3 implies $\mathcal{L}V(x,y) \leq \max(\sigma_0^2/2; 2K)V(x,y) + 2k$ for two positive constants K, k, checking condition (d). From Definition 1,

$$\inf_{x\in[0,+\infty)} V(x,y) = V_2(y) + \inf_{x\in[0,+\infty)} V_1(x) = V_2(y)$$

which implies that $\inf_{D\setminus D_n} V(x,y) \geq \inf\{V_2(y) \;:\; \max\{y, 1-y\} \leq 1/n\}$, the latter going to infinity with n, recall (4). Similarly, $\inf_{y\in(0,1)} V(x,y)$ goes to infinity as x goes to 0 or $+\infty$. Condition (e) is then satisfied, which allows to apply Theorem 2.

□

Remark 3. Notice that $\tilde{y} < \hat{y}$ and thus $\tilde{x} > \hat{x}$. Following Assumption 1, (10) at $(\tilde{x}, \tilde{y})$ provides $\mathcal{L}V(\tilde{x}, \tilde{y}) > 0$. It is straightforward that (2) has no equilibrium point in D, nor on its boundary $\{0\} \times (0,1) \cup \mathbb{R}_+^* \times \{0,1\}$. If the point $(0,0)$ cancels (2), we highlight that $\mathcal{L}V(0,0) < 0$ and by continuity, it holds on a small region $[0,\varepsilon)^2$. Recalling (4) implies that (x_t, y_t) will diverge from $(0,0)$ almost surely if $(x_0, y_0) \in D$.

A solution to (2) can be pictured as a trajectory continuously jumping from an orbit of (1) to another. Along this idea, V provides an estimate on trajectories, and can be related via Theorem 1 to the period T.

Theorem 4. *Let $(x_0, y_0) \in D$, $V_0 := V(x_0, y_0)$, and $(x_t, y_t)_{t\geq 0}$ be a regular solution to* (2)*. We first introduce a constant $0 \leq \rho \leq V_0$, the set $D(V_0, \rho) := \{(x,y) \in D \;:\; |V(x,y) - V_0| \leq \rho\} \subset D$ and the stopping time $\tau_\rho := \inf\{t > 0 \;:\; (x_t, y_t) \notin D(V_0, \rho)\}$. We then introduce two finite constants*

$$R(V_0,\rho) := \max_{D(V_0,\rho)} \left\{\sigma^2(y)\left(\kappa(\hat{x}) - \kappa(x) - x\kappa'(x) + y\Phi'(y) + \Phi(y) - \Phi(\hat{y})\right)\right\}$$

and

$$I(V_0,\rho) := \max_{D(V_0,\rho)} \left\{\sigma^2(y)\left(\kappa(\hat{x}) - \kappa(x) + \Phi(y) - \Phi(\hat{y})\right)^2\right\}\,.$$

Then for all $\mu > 0$

$$\mathbb{P}\left[\tau_\rho > \Theta(\rho,\mu)\right] \geq \left(1 - \frac{I(V_0,\rho)}{\mu^2}\right) \quad \textit{for} \quad \Theta(\rho,\mu) := \frac{2\left(\mu^2 + \mu\sqrt{\mu^2 + 2\rho R(V_0,\rho)} + \rho R(V_0,\rho)\right)}{(R(V_0,\rho)\sigma)^2}\,. \tag{11}$$

Proof. Fix $\mu > 0$. Now we define the $\mathcal{F}_{\tau_\rho}$-measurable set $A_\mu = \left\{\omega \in \Omega : \sup_{0<t\leq\tau_\rho} \left|M_t(\omega)\right| \leq \mu\right\}$ where $(M_t)_{t\geq 0}$ is a martingale defined by $M_t = 0$ for $t = 0$ and for $t > 0$ by

$$M_t = \frac{1}{\sqrt{t}} \int_0^t \sigma(y_s)\left(\kappa(\hat{x}) - \kappa(x_s) + \Phi(y_s) - \Phi(\hat{y})\right) dW(s) \ .$$

The process M is not right-continuous at $t = 0$ but still verifies $\mathbb{E}\left[M_t^2\right] \leq I(V_0, \rho)$ for all $0 < t \leq \tau_\rho$. The property holds by replacing M_t by its càdlàg representation. Doob's martingale inequality can then be applied: $\mathbb{P}\left[A_\mu\right] \geq 1 - I(V_0, \rho)/\mu^2$. At last, using Itô's formula, we have from (10):

$$\begin{aligned} |V(x_t, y_t) - V_0| \ &\leq \ \frac{1}{2}\int_0^t \sigma^2(y_s)\left|\kappa(\hat{x}) - \kappa(x_s) - x_s\kappa'(x_s) + y_s\Phi'(y_s) + \Phi(y_s) - \Phi(\hat{y})\right| ds \\ &\quad + \left|\int_0^t \sigma(y_s)\left(\kappa(\hat{x}) - \kappa(x_s) + \Phi(y_s) - \Phi(\hat{y})\right) dW_s\right| \end{aligned}$$

so that on $\{(t,\omega) \in \mathbb{R}_+ \times \Omega \ : \ (t,\omega) \in A_\mu \times [0, \tau_\rho(\omega)]\}$ $|V(x_t, y_t) - V_0| \leq \frac{1}{2}R(V_0, \rho)t + \mu\sqrt{t} =: S(t)$ almost surely. Also, $|e(t,\omega)| \leq \rho$ on that set. Put in another way, $\tau_\rho > S^{-1}(\rho) =: \Theta(\rho)$ on A_μ. According to Bayes rule,

$$\mathbb{P}\left[\tau_\rho > \Theta(\rho)\right] \geq \mathbb{P}\left[\tau_\rho > \Theta(\rho)\middle|A_\mu\right]\mathbb{P}\left[A_\mu\right] \geq \mathbb{P}\left[A_\mu\right] \geq \left(1 - \frac{I(V_0, \rho)}{\mu^2}\right)$$

□

We now introduce the main result of the paper. We provide the following tailor-made definition for the cycling behavior of (2).

Definition 2. *Let $(x^*, y^*) \in E \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in E\backslash\{(x^*, y^*)\}$. Let (x_t, y_t) be a stochastic process starting at (x_0, y_0) staying in E almost surely. We then introduce $(\rho_t)_{t\geq 0}$ the angle between $[x_t - x^*, y_t - y^*]^\top$ and $[x_0 - x^*, y_0 - y^*]^\top$. Let $S := \inf\{t > 0 \ : \ |\rho_t| \geq 2\pi$ or $(x_t, y_t) = (x^*, y^*)\}$ be a stopping time (a stochastic period). Then, the process (x_t, y_t) is said to orbit stochastically around (x^*, y^*) in E if $S < +\infty$ almost surely.*

Theorem 5. *Let $(x_0, y_0) \in D\backslash\{(\tilde{x}, \tilde{y})\}$ and (x_t, y_t) a solution to (2) starting at (x_0, y_0). Then (x_t, y_t) orbits stochastically around $(\tilde{x}, \tilde{y})$ in D.*

More precisely the system (2) produces clockwise orbits inside D. The angle ρ_t is only defined if $(x_t, y_t) \neq (\tilde{x}, \tilde{y})$. This can be ensured by either proving that $(x_t, y_t) \neq (\tilde{x}, \tilde{y})$ for all $t \geq 0$ almost surely, or by defining S as in Definition 2. See also Remark 4 The proof of Theorem 5 is removed to Section 4.

4 Proof of Theorem 5

4.1 Preliminary definitions and results

Recall that the probability space is given by $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration generated only by W is completed with null sets. Our proof, although unwieldy, allows us to describe precisely the possible trajectories of solutions of (2). it consists in defining subregions $(R_i)_i$ of the domain D, see Definition 4 below illustrated by Fig. 1, and prove that the process exits from them in finite time by the appropriate frontier. According to Theorem 3 any regular solution of (2) is a Markov process. We then repeatedly change the initial condition of the system, as equivalent of a time translation and use Definition 5 herefater. We obtain recurrence properties via Theorem 3.9 in [17]. Since it is repeatedly used hereafter, we provide here a version suited to our context.

Theorem 6. *Let $(x_t, y_t)_{t\geq 0}$ be a regular solution of (2) in D, starting at $(x_0, y_0) \in U$, for some $U \subset D$. Let $\varphi(t, x, y) \in \mathcal{C}^{1,2,2}(\mathbb{R}_+ \times U)$ verifying $\varphi(t, x, y) \geq 0$ for all $(t, x, y) \in U$ and $\mathcal{L}\varphi(s, x, y) \leq -\varphi(s)$ where $\varphi(s) \geq 0$ and $\lim_t \int_0^t \varphi(s)ds = +\infty$. Then (x_t, y_t) leaves the region U in finite time almost surely.*

Definition 3. *Let f be defined by $f : x \in \mathbb{R}_+ \mapsto f(x) := \Phi^{-1}(\alpha - \gamma + \kappa(x))$ as a concave decreasing function. For a solution (x_t, y_t) to (2), we define $\theta_t := y_t/x_t$ the finite variation process verifying $d\theta_t = \theta_t (\kappa(x) - \gamma + \alpha - \Phi(y))\, dt = \theta_t (\Phi(f(x)) - \Phi(y))\, dt$. Additionally, let $\tilde{\theta} := \tilde{y}/\tilde{x}$.*

Definition 4. *We define eight sets $(R_i)_{i=1,\dots,8}$ such that $\bigcap_{i=1}^8 R_i = (\tilde{y}, \tilde{x})$ and $\bigcup_{i=1}^8 R_i = D$, by*

$$\begin{cases} R_1 := \{(x, y) \in D \ : \ y \geq \tilde{y} \ \ and \ \ \theta_t \leq \tilde{\theta}\} \\ R_2 := \{(x, y) \in D \ : \ f(x) \leq y \leq \tilde{y}\} \\ R_3 := \{(x, y) \in D \ : \ y \leq f(x) \ \ and \ \ x \geq \tilde{x}\} \\ R_4 := \{(x, y) \in D \ : \ x \leq \tilde{x} \ \ and \ \ \theta \leq \tilde{\theta}\} \\ R_5 := \{(x, y) \in D \ : \ y \leq \tilde{y} \ \ and \ \ \theta_t \geq \tilde{\theta}\} \\ R_6 := \{(x, y) \in D \ : \ \tilde{y} \leq y \leq f(x)\} \\ R_7 := \{(x, y) \in D \ : \ y \geq f(x) \ \ and \ \ x_t \leq \tilde{x}\} \\ R_8 := \{(x, y) \in D \ : \ y \geq f(x) \ \ and \ \ x_t \leq \tilde{x}\}. \end{cases}$$

Definition 5. *Let (x_t, y_t) be a solution to (2) starting at $(x_0, y_0) = (x, y) \in D$. For any $i \in \{1, \dots, 8\}$, we define the stopping times $\tau_i(x, y) := \inf\{t \geq 0 \ : \ (x_t, y_t) \in R_i\}$.*

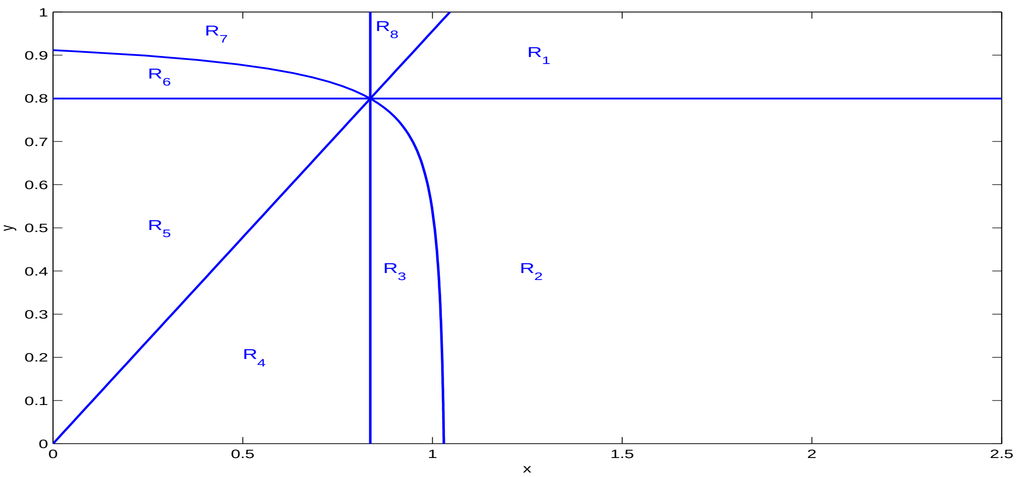

Figure 1: Covering of $D := \mathbb{R}_+^*) \times (0, 1)$ by $(R_i)_{i=1\dots8}$. Since $f(0) < 1$ and $\lim_{y\uparrow 1} \Phi(y) = +\infty$, the graph illustrates the general case.

Remark 4. It seems rather clear that the point $(\tilde{x}, \tilde{y})$ is not reached in finite time with a positive probability. In the following, the fact that $\mathcal{L}V(x, y) > \varepsilon$ for some small $\varepsilon > 0$ in a neighborhood of $(\tilde{x}, \tilde{y})$ implies that $(\tilde{x}, \tilde{y})$ is not a limit to almost every path of a solution to (2), recall Remark 3.

To ease the reading of the proof of Theorem 5 which follows from the following Propositions 1 to 10, we divide it in four quadrants around $(\tilde{x}, \tilde{y})$. We first prove that the process cycles, even in infinite time, for some particular starting points.

Proposition 1. *If* $(x_0, y_0) \in R_1$*, then* $\mathbb{P}[\tau_8(x_0, y_0) \leq \tau_7(x_0, y_0)] = 0$*. If* $(x_0, y_0) \in R_5$*, then* $\mathbb{P}[\tau_4(x_0, y_0) \leq \tau_3(x_0, y_0)] = 0$*.*

Proof. This is a direct consequence of the absence of Brownian motion in θ. Take $(x_0, y_0) \in R_1$. Then on $[0, \tau_3(x_0, y_0)]$, the process θ is non increasing almost surely, meaning that R_8 cannot be reached without first crossing region R_7. The other side is identical. □

Remark 5. Proposition 1 holds even if $\tau_i = +\infty$, for any i involved. It also implies that if $(x_0, y_0) \in R_1 \cup R_5$, then $\tau_i(x_0, y_0) \leq \tau_j(x_0, y_0)$ almost surely for $j \in \{mod(i + 1, 8)\}$.

4.2 Eastern quadrant

We ought to prove that for $(x_0, y_0) \in R_1$, the process reaches R_3 in finite time almost surely.

Proposition 2. *If* $(x_0, y_0) \in R_1$ *then* $\mathbb{P}[\tau_2(x_0, y_0) < +\infty] = 1$*.*

Proof. Let $\varphi : y \in [0, 1] \mapsto \sqrt{y}$. Then $\varphi(y) \geq \varphi(\tilde{y}) > 0$ for any y such that $(x, y) \in R_1$. Moreover,

$$\mathcal{L}\varphi(y) = \frac{\varphi(y)}{2}\left(\kappa(x) - \gamma + \frac{3}{4}\sigma^2(y)\right) \leq -\frac{\sigma^2(\tilde{y})\varphi(y)}{8} \leq -\frac{\sigma^2(\tilde{y})h(\tilde{y})}{8} < 0 \ .$$

Theorem 6 stipulates that (x_t, y_t) leaves R_1 in finite time almost surely which is only possible via R_2 according to Proposition 1. Reaching the boundary is prevented by Theorem 3.

□

Proposition 3. *If* $(x_0, y_0) \in R_2 \cup R_3$ *then* $\mathbb{P}[\tau_1(x_0, y_0) \wedge \tau_4(x_0, y_0) < +\infty] = 1$*.*

Proof. We follow the proof of Proposition 2 with $\varphi : x \in \mathbb{R}_+ \mapsto \sqrt{x}$.

□

Proposition 4. *If* $(x_0, y_0) \in R_1 \cup R_2$ *then* $\mathbb{P}[\tau_3(x_0, y_0) < +\infty] = 1$*.*

Proof. Step 1. Let $(\upsilon_n)_{n \geq 0}$ be a sequence of stopping times defined by $\upsilon_0 = 0$ and

$$\upsilon_n := \inf\{t \geq \upsilon_{n-1} \ : \ y_t = \tilde{y} \text{ or } (x_t, y_t) \in R_3\}, \quad n \geq 1.$$

By construction if $(x_{\upsilon_n}, y_{\upsilon_n}) \in R_3$ for some $n \geq 1$, then $\upsilon_k = \upsilon_n$ for all $k > n$. Following Propositions 1, 2 and 3, $\upsilon_n < +\infty$ for all $n \geq 1$ almost surely, and $\{\tau_3(x, y) = +\infty\} \subset \bigcap_{n \geq 1}\{y_{\upsilon_n} = \tilde{y}\}$. We prove in step 2 that this implies

$$\lim_{t \to \infty} \theta_t(\omega) = 0, \text{ for } \mathbb{P} - \text{a.e. } \omega \in \{\tau_3(x, y) = +\infty\} \ . \tag{12}$$

Providing that (12) holds we immediately get $\mathbb{P}[\tau_3(x_0, y_0) = +\infty] \leq \mathbb{P}[\lim_n x_{\upsilon_n} = +\infty] = 0$.

Step 2. If $\omega \in \{\tau_3(x, y) = +\infty\}$, then for all $n \geq 1$, $y_{\upsilon_n} = \tilde{y}$ and according to Proposition 3, (x_t, y_t) does not converge to the set $R_2 \cap R_3$. Since θ_t is a positive decreasing process for $(x_t, y_t) \in R_1 \cup R_2$, Doob's martingale convergence theorem implies that θ_t converges pathwise in $L^\infty([0, \tilde{\theta}))$. Assume now that θ_t does not converge to 0 with t on $E \subset \{\tau_3(x_0, y_0) = +\infty\}$. Then for any $\varepsilon > 0$, and for almost every $\omega \in E$

$$\lim_t \int_0^t \mathbb{1}_{\{\kappa(x_s(\omega)) - \gamma + \alpha - \Phi(y_s(\omega)) < -\varepsilon\}} ds = C_\varepsilon(\omega) < +\infty \ . \tag{13}$$

If the integral (13) explodes to $+\infty$ for some $\varepsilon > 0$ on some non null subset $F \subset E$, then for almost every $\omega \in F$,

$$\mathcal{L}\log\theta_t(\omega) = (\kappa(x_t(\omega)) - \gamma + \alpha - \Phi(y_t(\omega))) < -\varepsilon \mathbb{1}_{\{\kappa(x_t(\omega))-\gamma+\alpha-\Phi(y_t(\omega))<-\varepsilon\}}$$

and $\lim_{t\uparrow\infty}\log\theta_t(\omega) = -\infty$ for almost every $\omega \in F$, implying that θ_t converges to 0 on F, a contradiction with $F \subset E$, so that (13) holds on E. We then consider the random time $t_{\varepsilon,n}$, being the first time such that

$$\int_0^{t_{\varepsilon,n}} \mathbb{1}_{\{\kappa(x_s)-\gamma+\alpha-\Phi(y_s)<-\varepsilon\}} ds \geq C_\varepsilon - \frac{1}{n} \ , \tag{14}$$

and k_n the smallest index such that $\upsilon_{k_n} \geq t_{\epsilon,n}$. Note that $t_{\varepsilon,n}$ is not a $\mathbb{F}$-stopping time and k_n is not $\mathbb{F}$-adapted since they depend on C_ε which is $\mathcal{F}_\infty$-measurable. (14) implies that there exists a random time $s_n \in (\upsilon_{k_n}, \upsilon_{k_n} + 1/n)$ such that $-\varepsilon < \kappa(x_{s_n}) - \gamma + \alpha - \Phi(y_{s_n}) < 0$, otherwise we would have a contradiction of (13) on a subset of E:

$$\int_0^{\upsilon_{k_n}+1/n} \mathbb{1}_{\{\kappa(x_s)-\gamma+\alpha-\Phi(y_s)<-\varepsilon\}} ds \geq C_\varepsilon \ .$$

This implies that $\lim_n(s_n - \upsilon_{k_n})(\omega) = 0$ for almost every $\omega \in E$, and $(y_t)_{t\geq 0}$ being a continuous process

$$\lim_n y_{s_n}(\omega) = \tilde{y}, \qquad \text{for } \mathbb{P}-\text{a.e. } \omega \in D \subset \{\tau_3(x_0, y_0) = +\infty\} \ .$$

This is impossible for $\varepsilon > 0$ small enough since θ_t is strictly decreasing and thus E is a null set. (12) holds. $\square$

4.3 Southern quadrant

We show that starting from $R_2 \cap R_3$, (x_t, y_t) reaches R_5 in finite time almost surely.

Proposition 5. *If* $(x_0, y_0) \in R_2 \cap R_3$ *then* $\mathbb{P}[\tau_4(x_0, y_0) < +\infty] = 1$.

Proof. Step 1. We consider $\varphi_t := \varphi(x_t, y_t)$ with $\varphi : (x, y) \in D\backslash\{(x, y)\} \mapsto (y_t - \tilde{y})/(x_t - \tilde{x})$, and aim to prove that the process $F_t := F(\varphi(x_t, y_t))$ with $F : \varphi \in (-\pi/2, \pi/2) \mapsto \tan\left(\tan^{-1}(\varphi) + \tan^{-1}(c)\right)$ is a supermartingale on $R_1 \cup R_2 \cup R_3$, for $c \in (0, \tilde{\theta}^{-1})$. Notice that it is bounded in $R_1 \cup R_2 \cup R_3$. Applying Itô to φ first gives

$$d\varphi_t = \frac{dy_t}{x_t - \tilde{x}} - \frac{y_t - \tilde{y}}{(x_t - \tilde{x})^2} dx_t + \frac{\sigma^2(y_t)}{(x_t - \tilde{x})^2}\left[\frac{y_t - \tilde{y}}{x_t - \tilde{x}} x_t^2 - x_t y_t\right] dt \ .$$

Then, noticing that $F_t = (\varphi_t + c)(1 - \varphi_t c)$, we obtain

$$dF_t = \frac{1 + c^2}{(1 - \varphi_t c)^2}\left(d\varphi_t + \frac{c}{1 - \varphi_t c} d\langle\varphi\rangle_t\right) \ .$$

It is clear that $-(y - \tilde{y})(\Phi(y) - \alpha + \sigma^2(y)) \leq 0$ for all $y \in [0, 1)$. Now notice that for $(x, y) \in R_1$, we have $(x - \tilde{x})(\kappa(x) - \gamma + \sigma^2(y)) < 0$ so that

$$\begin{aligned}
\frac{(x - \tilde{x})^4}{\sigma^2(y)}\frac{(1 - \varphi c)^2}{1 + c^2}\mathcal{L}F &\leq (y - \tilde{y})(x - \tilde{x})x^2 - xy(x - \tilde{x})^2 + \frac{1}{\tilde{y}/\tilde{x} - \varphi}(\tilde{y}x - y\tilde{x})^2 \\
&= (x - \tilde{x})\left[x^2(y - \tilde{y}) - xy(x - \tilde{x}) + \tilde{x}(\tilde{y}x - y\tilde{x})\right] \\
&= (x - \tilde{x})^2 x\tilde{x}\left[\frac{y}{x} - \frac{\tilde{y}}{\tilde{x}}\right] < 0 \ .
\end{aligned}$$

Now on $R_2 \cup R_3$, $\hat{x} < \tilde{x}$ implies that $(\kappa(x) - \gamma) < 0$, so that

$$\begin{aligned}\frac{(x-\tilde{x})^4}{\sigma^2(y)}\frac{(1-\varphi c)^2}{1+c^2}\mathcal{L}F &\leq (y-\tilde{y})(x-\tilde{x})x^2 - y\tilde{x}(x-\tilde{x})^2 + \frac{1}{\tilde{y}/\tilde{x}-\varphi}(\tilde{y}x - y\tilde{x})^2 \\ &= (x-\tilde{x})\left[x^2(y-\tilde{y}) - y\tilde{x}(x-\tilde{x}) + \tilde{x}(\tilde{y}x - y\tilde{x})\right] \\ &= (x-\tilde{x})^2 x\,(y-\tilde{y}) < 0\ .\end{aligned}$$

Denoting $\tau_{1,4} := \tau_1(x_0,y_0) \wedge \tau_4(x_0,y_0)$, we conclude that $F_{t\wedge\tau_{1,4}}$ is a supermartingale for $t \geq 0$. Using optional sampling theorem, assisted by Proposition 3, $\tau_{1,4} < +\infty$ almost surely and

$$F_0 \geq \mathbb{E}\left[F_{\tau_{1,4}}\right] = \frac{1}{c}\mathbb{P}\left[\tau_4(x,y) < \tau_1(x,y)\right] + c\mathbb{P}\left[\tau_1(x_0,y_0) < \tau_4(x_0,y_0)\right]$$

Since $M := \max\{F(\varphi(x,y))\ :\ (x,y) \in R_2 \cap R_3\} < c$ then

$$\mathbb{P}\left[\tau_4(x_0,y_0) < \tau_1(x_0,y_0)\right] \geq \frac{c(c-M)}{c^2+1} > 0 \quad \forall (x_0,y_0) \in R_2 \cap R_3\ .$$

Step 2. According to Proposition 3, $\tau_{1,4} < +\infty$ almost surely for any $(x_0,y_0) \in R_2 \cap R_3$, and according to Proposition 4, $\tau_3(x_0,y_0) < +\infty$ $\mathbb{P}-$a.s. for all $(x_0,y_0) \in R_1$. Taking $(x_0,y_0) \in R_2 \cap R_3$, we define the sequence $(\tau_{1,4}^n, \tau_3^n)_{n\geq 0}$ with $\tau_3^0 = 0$ and

$$\left\{\begin{array}{ll}\tau_{1,4}^n := & \inf\{t \geq \tau_3^n\ :\ (x_t,y_t) \in R_1 \cup R_4\} \\ \tau_3^{n+1} := & \inf\{t \geq \tau_{1,4}^n\ :\ (x_t,y_t) \in (R_2 \cap R_3) \cup R_4\}\end{array}\right., \quad \text{for all } n \geq 1\ .$$

We then have $\{\tau_4(x_0,y_0) = +\infty\} \subset \cap_{n\geq 1}\{x_{\tau_{1,4}^n} > \tilde{x}\}$ for any $(x_0,y_0) \in R_2 \cap R_3$. The sequence $(\{x_{\tau_{1,4}^n} > \tilde{x}\})_{n\geq 1}$ is decreasing in the sense of inclusion, so that

$$\mathbb{P}\left[\tau_4(x,y) = +\infty\right] = \lim_n \mathbb{P}\left[x_{\tau_{1,4}^n} > \tilde{x}\right]\ . \tag{15}$$

Using Baye's rule,

$$\mathbb{P}\left[x_{\tau_{1,4}^n} > \tilde{x}\right] \leq \prod_{k=1}^n \mathbb{P}\left[x_{\tau_{1,4}^k} > \tilde{x} | x_{\tau_{1,4}^{k-1}} > \tilde{x}\right] \leq \prod_{k=1}^n \mathbb{P}\left[x_{\tau_{1,4}^k} > \tilde{x} | x_{\tau_3^k} > \tilde{x}\right]\ .$$

Using step 1 of the present proof and the Markov property of (x_t,y_t),

$$\mathbb{P}\left[x_{\tau_{1,4}^n} > \tilde{x}\right] \leq \prod_{k=1}^n \mathbb{P}\left[\tau_1(x_{\tau_3^k}, y_{\tau_3^k}) < \tau_4(x_{\tau_3^k}, y_{\tau_3^k})\right] \leq \prod_{k=1}^n \left(1 - \frac{c(c-M)}{c^2+1}\right)\ .$$

Plugging this inequality into (15) concludes the proof. ☐

Remark 6. Notice that by choosing c properly in the above proof, it is possible to be arbitrarily close to R_5 in finite time. The device is used later in Proposition 9.

Proposition 6. *If* $(x_0,y_0) \in R_3 \cap R_4$ *then* $\mathbb{P}\left[\tau_5(x_0,y_0) < +\infty\right] = 1$.

Proof. Step 1. We claim that $\tau_{2,5} := \tau_2(x_0,y_0) \wedge \tau_5(x_0,y_0) < +\infty$ almost surely. Consider the function $\varphi\ :\ (x,y) \in D \mapsto \sqrt{x_{t\wedge v_0}}$. The process $\varphi_t := \varphi(x_t,y_t)$ is a positive supermartingale on $R_2 \cup R_3 \cup R_4$:

$$\mathcal{L}\varphi(x,y) = \frac{\varphi(x,y)}{2}\left((\Phi(y) - \alpha + \frac{\sigma^2(y)}{2}\right) \leq -\frac{\sigma^2(y)\varphi(x,y)}{4}\ . \tag{16}$$

According to Doob's martingale convergence theorem, φ_t converges point-wise with t. Let $\varepsilon > 0$ and define $R_\varepsilon := \bigcup_{i=2}^4 R_i \cap \{x \geq \varepsilon\}$. Then $\varphi_t \geq \sqrt{\varepsilon}$ on R_ε, and similarly to Proposition 3, we use Theorem 6 to assert that (x_t, y_t) leaves R_ε in finite time almost surely. This being true for any $\varepsilon > 0$, $\lim_t \varphi_t(\omega) = 0$ for almost every $\omega \in \{\tau_{2,5}(\omega) = +\infty\}$. In R_5, this is only possible if $\lim_t y_t(\omega) = 0$ also, implying that $\lim_t(x_t(\omega), y_t(\omega)) = (0, 0)$ on this set. This being improbable, $\tau_{2,5} < +\infty$ almost surely.

Step 2. By denoting $\tau_4^0 = 0$, we then define the sequence $(\tau_{2,5}^n, \tau_4^n)_{n\geq 0}$ by

$$\begin{cases} \tau_4^n := & \inf\{t \geq \tau_{2,5}^{n-1} \ : \ x_t = \tilde{x} \text{ or } (x_t, y_t) \in R_5\} \\ \tau_{2,5}^n := & \inf\{t \geq \tau_4^n \ : \ (x_t, y_t) \in R_2 \cup R_5\} \end{cases} , \quad \text{for all } n \geq 1 .$$

If $(x_{\tau_{2,5}^0}, y_{\tau_{2,5}^0}) \in R_2$, then, according to Proposition 5, the process reaches back R_4 in finite time. Using step 1, we have that $\mathbb{P}[\tau_4^n < +\infty] = \mathbb{P}\left[\tau_{2,5}^n < +\infty\right] = 1$. By construction and Proposition 5, for $n \geq 1$

$$\{(x_{\tau_{2,5}^n}, y_{\tau_{2,5}^n}) \in R_2\} \subset \{x_{\tau_4^n} = \tilde{x}\} = \{(x_{\tau_{2,5}^{n-1}}, y_{\tau_{2,5}^{n-1}}) \in R_2\} = \{x_{\tau_{2,5}^{n-1}} > \tilde{x}\}. \tag{17}$$

Therefore, $\{\tau_5(x_0, y_0) = +\infty\} = \bigcap_{n\geq 0}\{(x_{\tau_{2,5}^n}, y_{\tau_{2,5}^n}) \in R_2\}$ and the sequence of sets

$$\left(\{(x_{\tau_{2,5}^n}, y_{\tau_{2,5}^n}) \in R_2\}\right)_{n\geq 0}$$

is decreasing in the sense of inclusion. Altogether we get

$$\mathbb{P}[\tau_5(x_0, y_0) = +\infty] = \lim_n \mathbb{P}\left[x_{\tau_{2,5}^n} > \tilde{x}\right] . \tag{18}$$

Now using Bayes formula and (17), we finally obtain for every $n \geq 1$

$$\mathbb{P}\left[x_{\tau_{2,5}^n} > \tilde{x}\right] \leq \prod_{k=1}^n \mathbb{P}\left[x_{\tau_{2,5}^k} > \tilde{x} | x_{\tau_{2,5}^{k-1}} > \tilde{x}\right] = \prod_{k=1}^n \mathbb{P}\left[x_{\tau_{2,5}^k} > \tilde{x} | x_{\tau_4^k} = \tilde{x}\right] \tag{19}$$

Putting (18) and (19) together, $\mathbb{P}[\tau_5(x_0, y_0) = +\infty] > 0$ implies that

$$\lim_n \mathbb{P}\left[x_{\tau_{2,5}^n} > \tilde{x} | x_{\tau_4^n} = \tilde{x}\right] = 1 . \tag{20}$$

Step 3. Let $\varphi \ : \ (t, x) \in \mathbb{R}_+^2 \mapsto \sqrt{x}\exp(\frac{1}{8}\sigma^2(\tilde{y})t)$. According to (16) the process $\varphi_t := \varphi(t, x_t)$ is a supermartingale on $[\tau_4^n, \tau_{2,5}^n]$. Fixing $t > 0$ and applying optional sampling theorem, we obtain

$$\mathbb{E}\left[\varphi(t \wedge \tau_{2,5}^n, x_{t\wedge\tau_{2,5}^n}) - \varphi(t \wedge \tau_4^n, x_{t\wedge\tau_4^n}) | x_{t\wedge\tau_4^n} = \tilde{x}\right] \leq 0 .$$

Since $\max(\tau_4^n, \tau_{2,5}^n) < +\infty$ almost surely, we apply Fatou's lemma and obtain

$$\mathbb{E}\left[\exp\left(\frac{1}{8}\sigma^2(\tilde{y})[\tau_{2,5}^n - \tau_4^n]\right)\sqrt{x_{\tau_{2,5}^n}}\left(\mathbb{1}_{\left\{x_{\tau_{2,5}^n} < \tilde{x}\right\}} + \mathbb{1}_{\left\{x_{\tau_{2,5}^n} > \tilde{x}\right\}}\right) | x_{\tau_4^n} = \tilde{x}\right] \leq \sqrt{\tilde{x}} . \tag{21}$$

Since $\sqrt{x_{\tau_{2,5}^n}}\mathbb{1}_{\left\{x_{\tau_{2,5}^n} < \tilde{x}\right\}} \geq 0$ and $\sqrt{x_{\tau_{2,5}^n}}\mathbb{1}_{\left\{x_{\tau_{2,5}^n} > \tilde{x}\right\}} \geq \sqrt{\tilde{x}}\mathbb{1}_{\left\{x_{\tau_{2,5}^n} > \tilde{x}\right\}}$ for all $n \geq 1$, (21) implies

$$\mathbb{E}\left[\exp\left(\frac{1}{8}\sigma^2(\tilde{y})[\tau_{2,5}^n - \tau_4^n]\right)\mathbb{1}_{\left\{x_{\tau_{2,5}^n} > \tilde{x}\right\}} | x_{\tau_4^n} = \tilde{x}\right] \leq 1 ,$$

leading to

$$\mathbb{E}\left[\left(\exp\left(\frac{1}{8}\sigma^2(\tilde{y})[\tau_{2,5}^n - \tau_4^n]\right) - 1\right)\mathbb{1}_{\{x(\tau_{2,5}^n)>\tilde{x}\}}\big|x_{\tau_4^n} = \tilde{x}\right] \leq 1 - \mathbb{P}\left[x_{\tau_{2,5}^n} > \tilde{x}|x_{\tau_4^n} = \tilde{x}\right] . \quad (22)$$

If $x_{\tau_4^n} = \tilde{x}$ then $y_{\tau_4^n} < \tilde{y}$ and by continuity $\{\tau_{2,5}^n > \tau_4^n\} \supset \{x_{\tau_4^n} = \tilde{x}\}$, implying

$$\exp\left(\frac{1}{8}\sigma^2(\tilde{y})[\tau_{2,5}^n(\omega) - \tau_4^n(\omega)]\right) > 1, \quad \text{for } \mathbb{P}-\text{a.e. } \omega \in \{x_{\tau_4^n} = \tilde{x}\} . \quad (23)$$

Let's assume that $\mathbb{P}[\tau_5(x_0, y_0) = +\infty] > 0$, so that (20) holds. According to (22), we get

$$0 \leq \mathbb{E}\left[\left(\exp\left(\frac{1}{8}\sigma^2(\tilde{y})[\tau_{2,5}^n - \tau_4^n]\right) - 1\right)\mathbb{1}_{\left\{x_{\tau_{2,5}^n}>\tilde{x}\right\}}\big|x_{\tau_4^n} = \tilde{x}\right] \to 0 \text{ as } n \to \infty .$$

Markov inequality then leads to the following convergence for any $\varepsilon > 0$:

$$\lim_n \mathbb{P}\left[\left(\exp\left(\frac{1}{8}\sigma^2(\tilde{y})[\tau_{2,5}^n - \tau_4^n]\right) - 1\right)\mathbb{1}_{\left\{x_{\tau_{2,5}^n}>\tilde{x}\right\}} > \varepsilon\big|x_{\tau_4^n} = \tilde{x}\right] = 0 .$$

Now Bayes rules with (23) provides

$$\begin{aligned}\mathbb{P}&\left[\left(\exp\left(\frac{1}{8}\sigma^2(\tilde{y})[\tau_{2,5}^n - \tau_4^n]\right) - 1\right) > \varepsilon\big|x_{\tau_{2,5}^n} > \tilde{x},\ x_{\tau_4^n} = \tilde{x}\right]\mathbb{P}\left[x_{\tau_{2,5}^n} > \tilde{x}|x_{\tau_4^n} = \tilde{x}\right] \\ &= \mathbb{P}\left[(\tau_{2,5}^n - \tau_4^n) > 8\ln(1+\varepsilon)/\sigma^2(\tilde{y})\big|x_{\tau_{2,5}^n} > \tilde{x},\ x_{\tau_4^n} = \tilde{x}\right]\mathbb{P}\left[x_{\tau_{2,5}^n} > \tilde{x}|x_{\tau_4^n} = \tilde{x}\right]\end{aligned}$$

which leads for any $\varepsilon > 0$ to $\mathbb{P}\left[(\tau_{2,5}^n - \tau_4^n) > \varepsilon|x_{\tau_{2,5}^n} > \tilde{x},\ x_{\tau_4^n} = \tilde{x}\right] \to 0$ as $n \to \infty$. From step 2, $\{\tau_5(x_0, y_0) = +\infty\} = \bigcap_{n\geq 0}\left(\{x_{\tau_4^n} = \tilde{x}\} \cap \{x_{\tau_{2,5}^n} > \tilde{x}\}\right)$. Therefore on this set, the continuous mapping theorem asserts that (x, y) at consecutive stopping times converge in probability. By continuity, this implies $\lim_n y_{\tau_4^n}(\omega) = \tilde{y}$ and $\lim_n x_{\tau_{2,5}^n}(\omega) = \tilde{x}$ for $\mathbb{P}-$a.e. $\omega \in \{\tau_5(x_0, y_0) = +\infty\}$. By the Markov property of (x_t, y_t), $\lim\limits_{t\to\infty}(x_t, y_t)(\omega) = (\tilde{x}, \tilde{y})$ for almost every $\omega \in \{\tau_5(x_0, y_0) = +\infty\}$. We conclude that $\mathbb{P}[\tau_5(x_0, y_0) = +\infty] = 0$. □

4.4 Western Quadrant

Proposition 7. *If* $(x_0, y_0) \in R_5 \cup R_6$ *then* $\mathbb{P}[\tau_7(x_0, y_0) < +\infty] = 1$.

Proof. Step 1. Consider $R_\varepsilon := R_5 \cup R_6 \cap \{x \leq \hat{x} - \varepsilon\}$ for arbitrarily fixed $\varepsilon > 0$. Assume that $(x_0, y_0) \in R_\varepsilon$. Denoting $\varphi_t := \varphi(x_t, y_t) \geq 0$ with $\varphi\ :\ (x, y) \in D \mapsto 1/y$ and recalling Definition 3, $\mathcal{L}\varphi(x, y) = -\varphi(x, y)(\kappa(x) - \gamma) < -(\kappa(\hat{x} - \varepsilon) - \gamma)/f(0) < 0$ for very $(x, y) \in R_\varepsilon$. Theorem 6 then states that (x_t, y_t) exits R_ε in finite time almost surely. Since that θ is non-decreasing on this set, and recalling Theorem 2, it is only possible via R_7 and $\mathbb{P}[\tau_7(x_0, y_0) < +\infty] = 1$. This holds for any $\varepsilon > 0$.

Step 2. Assume now that $(x_0, y_0) \in (R_5 \cup R_6)\backslash R_\varepsilon$. According to step 1, $\{\tau_7(x_0, y_0) = +\infty\} \subset \{x_t \geq \hat{x},\ \forall t \geq 0\}$ and thus $\{\tau_7(x_0, y_0) = +\infty\} \subset \{\theta_t \leq f(\hat{x})/\hat{x},\ \forall t \geq 0\}$. Because θ_t is non decreasing and according to Doob's martingale convergence theorem, θ_t converges to $\theta_0 \in L^\infty([\tilde{\theta}, f(\hat{\omega})/\hat{\omega}])$ on $\{\tau_7(x_0, y_0) = +\infty\}$. This implies that (x_t, y_t) converges with t to $R_6 \cap R_7$ on $\{\tau_7(x_0, y_0) = +\infty\}$. Since $\sigma(y) > \sigma(f(0))$, this convergence is improbable. □

4.5 Northern quadrant

Finally we prove that if $(x_0, y_0) \in R_7$, then the process reaches R_1 in finite time almost surely. One can notice that proofs are very similar to those of Subsections 4.2 and 4.3.

Proposition 8. *If* $(x_0, y_0) \in R_6 \cup R_7$ *then* $\mathbb{P}[\tau_5 \wedge \tau_8(x_0, y_0) < +\infty] = 1$.

Proof. Define the sequence of regions $\{B_n\}_{n\in\mathbb{N}}$ through $B_n = R_6 \cup R_7 \cap \{y < 1 - k/n\} \cap \{x > k/n\}$ where $k > 0$ is sufficiently small to have $(x_0, y_0) \in B_1$. Applying Itô to $\varphi : (x, y) \in D \mapsto \sqrt{\hat{x} - x_t}$ we find that for all $(x, y) \in B_n$

$$\begin{aligned}\mathcal{L}\varphi(x,y) = \qquad & -\frac{1}{2\varphi(x,y)}\left[x\left[\Phi(y) - \alpha + \sigma^2(y)\right] + \frac{1}{4}\frac{x^2}{\hat{x} - x}\sigma^2(y)\right] \leq -\frac{x^2\sigma^2(y)}{8(\hat{x} - x)^{3/2}} \\ & \leq -\frac{k^2\sigma^2(1 - k/n)}{8\sqrt{n}(n\hat{x} - k)^{3/2}} < 0\end{aligned}$$

while $\mathcal{L}\varphi(x, y) \leq 0$ in $R_6 \cup R_7$. Doob's supermartingale convergence theorem implies the existence of the pointwise limit $\varphi_\infty := \lim_t \varphi(x_{t\wedge\tau_{5,8}}, y_{t\wedge\tau_{5,8}})$ almost surely, where we use the notation $\tau_{5,8} := \tau_5(x_0, y_0) \wedge \tau_8(x_0, y_0)$. In addition, Theorem 6 guarantees that every set B_n is exited in finite time almost surely. Consequently if $\omega \in \{\tau_{5,8} = +\infty\}$, we have that either $\lim_t x_t(\omega) = 0$ or $\lim_t y_t(\omega) = 1$, a contradiction in either way. □

Proposition 9. *If* $(x_0, y_0) \in R_6 \cap R_7$ *then* $\mathbb{P}[\tau_8(x_0, y_0) < +\infty] = 1$.

Proof. The proof is identical to the one of Proposition 5, with small modifications. Here $\varphi_t := \varphi(x_t, y_t)$ with $\varphi : (x, y) \in D\backslash\{(\tilde{x}, \tilde{y})\} \mapsto (y - \tilde{y})/(x - \tilde{x})$ and $F : (x, y) \in D\backslash\{(\tilde{x}, \tilde{y})\} \mapsto \tan\left(\tan^{-1}(\varphi(x, y)) + \tan^{-1}(c)\right)$. The process $F_t := F(x_t, y_t)$ is a supermartingale on $R_5 \cup R_6 \cup R_7$ if we chose $c \in (0, (\tilde{\theta} + M/m)^{-1})$ where (m, M) are two positive constants given by $m := \min_{[\tilde{x},\hat{x}]\times[\hat{y},\tilde{y}]} x\hat{x}\sigma^2(y)$ and $M := \max_{[\tilde{x},\hat{x}]\times[\hat{y},\tilde{y}]} y(x - \hat{x})\left[\kappa(x) - \gamma\right] - x(y - \hat{y})\left[\Phi(y) - \alpha\right]$. The justification is the following. The domain $S_c := D\backslash\{\tilde{\theta} \leq \varphi(x, y) \leq 1/c\}$ contains the area of interest $R_5 \cup R_6 \cup R_7$. Using Proposition 5, we can prove that F_t is a supermartingale on $S_c\backslash[\hat{x}, \tilde{x}] \times [\tilde{y}, \hat{y}]$. On $[\hat{x}, \tilde{x}] \times [\tilde{y}, \hat{y}]$,

$$\begin{aligned}(x - \tilde{x})^2\frac{(1 - R_tc)^2}{1 + c^2}\mathcal{L}F_t &\leq y(x - \tilde{x})[\kappa(x) - \gamma] - x(y - \tilde{y})[\Phi(y) - \alpha] \\ &\quad + \sigma^2(y)\left[y(x - \tilde{x}) - x(y - \tilde{y}) + \tilde{x}\left(y - x(1/c - \tilde{x} + \tilde{x})\right)\right] \\ &\leq y(x - \tilde{x})[\kappa(x) - \gamma] - x(y - \tilde{y})[\Phi(y) - \alpha] - x\tilde{x}\sigma^2(y)(1/c - \tilde{\theta}) \\ &\leq M - m(1/c - \tilde{\theta}) \leq 0\,.\end{aligned}$$

We then reproduce step 2 of the proof of Proposition 5, using Propositions 7 and 8 instead of Propositions 2 and 3. □

Proposition 10. *If* $(x_0, y_0) \in R_7 \cap R_8$ *then* $\mathbb{P}[\tau_1(x_0, y_0) < +\infty] = 1$.

Proof. We follow Proposition 6 with the minor following modifications.

1 We consider $\tau_{1,6} := \tau_1(x_0, y_0) \wedge \tau_6(x_0, y_0)$ the exit time of $R_7 \cup R_8$. The process $\varphi_t := \varphi(x_t, y_t)$ with $\varphi : (x, y) \in D \mapsto x_t^{-2}$ verifies

$$\mathcal{L}h_t = -2h_t\left(\Phi(y_t) - \alpha + \frac{3}{2}\sigma^2(y_t)\right) < -\varepsilon h_t < -\varepsilon h(\tilde{\theta}) < 0$$

for some $\varepsilon > 0$. Indeed $\Phi(y) - \alpha + \sigma^2(y) \geq 0$ and is null only if $y = \tilde{y}$, whereas $\sigma^2(y) = 0$ only if $y = 1$. Applying Theorem 6 to $R_7 \cup R_8$, $\tau_{1,6} < +\infty$ almost surely.

Step 2. If $(x_{\tau_{1,6}}, y_{\tau_{1,6}}) \in R_6$, then the process reaches R_8 in finite time almost surely according to Proposition 9. We define the sequence $(\tau_{1,6}^n, \tau_8^n)_{n\geq 0}$ with $\tau_8^0 := 0$ and

$$\left\{ \begin{array}{ll} \tau_{1,6}^n := & \inf\{t \geq \tau_8^n \ : \ (x_t, y_t) \in R_6 \cup R_1\} \\ \tau_8^{n+1} := & \inf\{t > \tau_{1,6}^n \ : \ (x_t, y_t) \in (R_7 \cap R_8) \cup R_1\} \end{array} \right. , \quad \text{for all } n \geq 0 \ .$$

Proceeding as in step 2 Proposition 6, we obtain that $\mathbb{P}\left[\tau_1(x_0, y_0) = +\infty\right] > 0$ implies that

$$\lim_n \mathbb{P}\left[x_{\tau_{1,6}^n} < \tilde{x} | x_{\tau_8^n} = \tilde{x}\right] = 1 \ . \tag{24}$$

Step 3. Define $m := \inf\{2(\Phi(y) - \alpha) + 3\sigma^2(y) \ : \ y \in [\tilde{y}, 1)\}$, which is strictly positive according to step 1. Consider the new process $\varphi_t := \varphi(x_t, y_t)$ with $\varphi \ : \ (x, y) \in D \mapsto \exp(-mt)x_t^2$. It is a positive submartingale on $[0, \tau_{1,6}^0]$, and similarly to step 3 of Proposition 6, we can obtain

$$\begin{aligned} \tilde{x}^2 \leq \quad & \mathbb{E}\left[x^2_{\tau_{1,6}^n} e^{-m(\tau_{1,6}^n - \tau_8^n)} | x_{\tau_8^n} = \tilde{x}\right] \leq \tilde{x}^2 \mathbb{E}\left[e^{-m(\tau_{1,6}^n - \tau_8^n)} \mathbb{1}_{\left\{x_{\tau_{1,6}^n} < \tilde{x}\right\}} | x_{\tau_8^n} = \tilde{x}\right] \\ & + \tilde{\theta}^{-2} \mathbb{E}\left[e^{-m(\tau_{1,6}^n - \tau_8^n)} \mathbb{1}_{\left\{x_{\tau_{1,6}^n} \geq \tilde{x}\right\}} | x_{\tau_8^n} = \tilde{x}\right] \ . \end{aligned}$$

Assuming (24), we have

$$0 \leq \mathbb{E}\left[(1 - e^{-m(\tau_{1,6}^n - \tau_8^n)}) \mathbb{1}_{\left\{x_{\tau_{1,6}^n} < \tilde{x}\right\}} | x_{\tau_8^n} = \tilde{x}\right] \leq (1/\tilde{\lambda} - 1)(1 - \mathbb{P}\left[x_{\tau_{1,6}^n} < \tilde{x} | x_{\tau_8^n} = \tilde{x}\right]) \xrightarrow{n} 0 \ .$$

We then proceed exactly as in step 3 of Proposition 6 to finish the proof. $\square$

5 Example

In this section we assume that investment follows Say's law and Philips curve is provided by [11, 15].

Assumption 4. *We let $\kappa \ : \ x \in \mathbb{R}_+ \mapsto (1 - x)/\nu$ and $\Phi \ : \ y \in [0, 1) \mapsto \Phi(y) := \frac{\phi_1}{(1-y)^2} + \phi_0$.*

Assumption 1 holds under Assumption 4. The unique non-hyperbolic equlibrium point in $D = \mathbb{R}_+^* \times (0, 1)$ is given by $(\hat{x}, \hat{y}) = \left(1 - \nu\gamma, 1 - \sqrt{\phi_1/(\alpha - \phi_0)}\right)$. Functions of Definition 1 are given by $V(x, y) = V_1(x) + V_2(y)$ with

$$\begin{array}{rcl} V_1(x) & = & \frac{1}{\nu}\left(x - \hat{x}\left(\log\left(\frac{x}{\hat{x}}\right) + 1\right)\right) \ , \\ V_2(y) & = & \phi_1\left(\log\left(\frac{1-\hat{y}}{1-y}\right) + \left(\frac{\hat{y}}{1-\hat{y}}\right)\log\left(\frac{\hat{y}}{y}\right) + \frac{1}{y-y^2} - \frac{1}{\hat{y}-\hat{y}^2}\right) \ . \end{array} \tag{25}$$

Although period T given by Theorem 1 is not explicit here, numerical computations allow to approximate it with a linear function of V_0, see first part of Fig. 2. Following Remark 1, T does not converge to 0 with solutions of (1) concentrating to $(\hat{x}, \hat{y})$. A local phase portrait with values of T is provided in second part of Fig. 2.

Assumption 5. *Let $\sigma \ : \ y \in [0, 1] \mapsto \sigma_0(1 - y)$ with $\sigma_0 > 0$.*

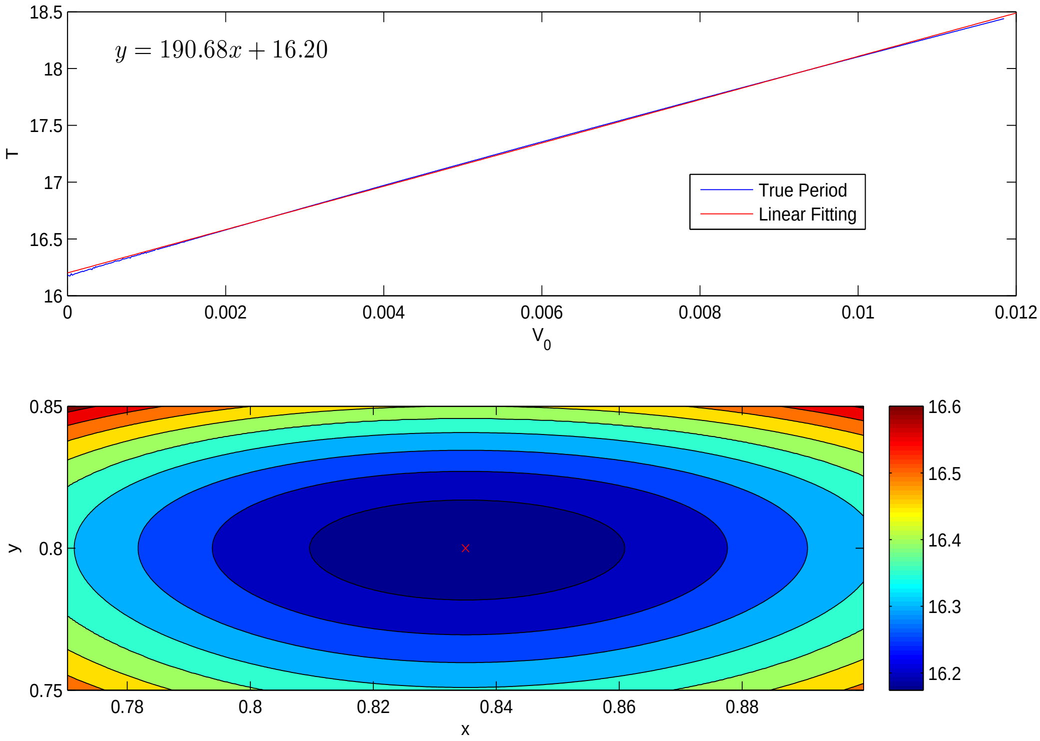

Figure 2: Up: values of T as a function of V. Down: Contour lines with values of T in a subset of D. Parameters set at $(\alpha, \gamma, \nu, \phi_0, \phi_1) = (0.025, 0.055, 3, 0.040064, 0.000064)$. Equilibrium point at $(\hat{x}, \hat{y}) = (0.8350, 0.80)$.

If we implicitly assume that the perturbation of the average growth rate α of the productivity is due to the flow of workers coming in and out of the fraction y_t employed at time t, Assumption 5 conveniently expresses that this perturbation decreases with the employment rate since higher employment implies lower perturbation on the constant average rate α_t. Other models can of course be considered.

Assumption 5 together with Assumption 4, and comparing with (25), satisfy Assumption 3. Indeed for all $y \in (0, 1)$,

$$\sigma^2(y)\Phi'(y) = \frac{2\sigma_0^2\phi_1}{(1-y)} \leq 2\sigma_0^2\left(V_2(y) - \phi_1\left(\frac{1}{1-\hat{y}}\left(\hat{y}\log(\hat{y}) - \frac{1}{\hat{y}}\right) + \log(1-\hat{y})\right)\right) \tag{26}$$

and along with the sub-linearity of the log function,

$$-\kappa(x) - x\kappa'(x) = \frac{2x-1}{\nu} \leq \frac{2}{1-\hat{x}}\left(V_1(x) + \hat{x} - \hat{x}\log(\hat{x})\right) \ . \tag{27}$$

In line with Assumptions 1 and 3 the vertical asymptote at $y = 1$ implies that $\sigma^2(y)\Phi(y) \leq K_0V_2(y) + k_0$ for some $K_0, k_0 \in \mathbb{R}_+^2$. Under Assumption 3 and following (26) and (27), $K_0 = 0$ and $k_0 = \sigma_0^2(\phi_1 + \phi_0)$.

Assumption 5 also implies that $(1-\tilde{y})^2$ is the root of a quadratic polynomial $\sigma_0^2(1-\tilde{y})^4 - (\alpha + \phi_0)(1-\tilde{y})^2 + \phi_1 = 0$. The latter shall have a unique root in $(0, 1)$ to satisfy Assumption 2. The following example of condition is sufficient.

Assumption 6. *We assume* $\phi_1 \leq (\alpha + \phi_0)/2$ *and* $\sigma_0 \leq \max\{(\alpha + \phi_0)/(2\sqrt{\phi_1}), (\alpha + \phi_0 - \phi_1)\}$.

We are now able to claim the existence of $K, k \in \mathbb{R}_+^2$ such that $|R(V_0, \rho)| \leq K(V_0 + \rho) + k$, where R is defined in Proposition 4. A direct application provides

$$R(V_0, \rho) := \max_{D(V_0,\rho)} \left\{ \sigma_0^2 (1-y)^2 \left(\frac{2x - \hat{x}}{\nu} + \frac{2\phi_1 y}{(1-y)^3} + \frac{\phi_1}{(1-y)^2} - \frac{\phi_1}{(1-\hat{y})^2} \right) \right\}.$$

Using (27), this estimate becomes $|R(V_0, \rho)| \leq K(V_0 + \rho) + k$ where $K := (2\sigma_0^2)/(1 - \hat{x})$ and k is an explicitly calculable constant. Following the same procedure with (26), $I(V_0, \rho) \leq K(V_0 + \rho)^2 + k'$ with the same K and $k' \neq k$. Now choosing $\mu = (\rho - (K(V_0 + \rho) + k)\theta/2)/\sqrt{\theta}$ for some $\theta \geq 0$, so that $\Theta(\rho) = \theta$, Proposition 4 provides

$$\mathbb{P}\left[\tau_\rho > \theta\right] \geq \left(1 - \frac{(K(V_0 + \rho)^2 + k')\theta}{\left(\frac{1}{2}(K(V_0 + \rho) + k)\theta - \rho\right)^2} \right).$$

Theorem 5 is a straightly observable phenomenon with simulations, see Fig. 3. Under the assumptions of this section, the system has been simulated using XPPAUT with a fourth order Runge-Kutta scheme for the deterministic part, and an Euler scheme for the Brownian part. Fig. 3 illustrates the effect of the volatility level σ_0 on trajectories of the system, as for the economic quantity $P_t := a_t y_t N_t$.

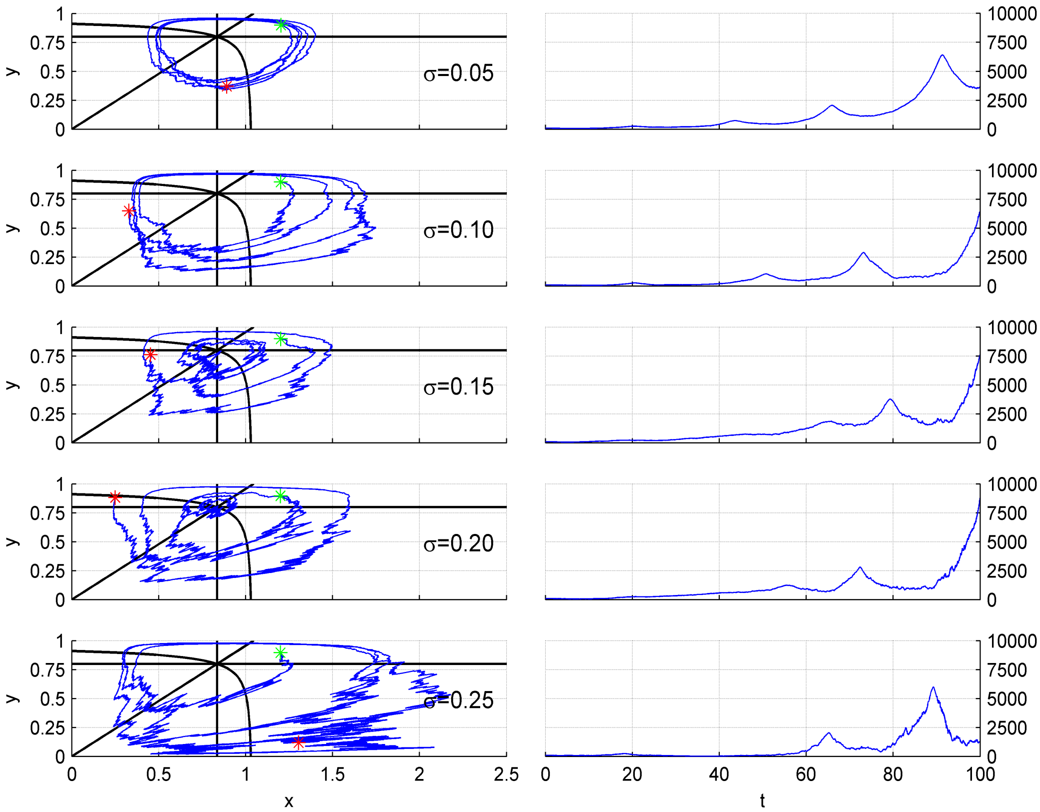

Figure 3: Left column : phase diagram (x, y) of subsample paths of trajectories for (2) with different values of volatility σ_0, starting from the green star and stopping at the red start. Right column: evolution of output P_t over time for the subsample path.

Apart from specific subregions of D as R_1 or R_5 where Corollary 3.2 in [17] can provide an estimate for the expectation of the exit time, a bound for the expected period $\mathbb{E}[S]$ seems out of reach. Numerical simulations have nevertheless always provide reasonnable finite periods of stochastic orbits of (2). We thus expect that $\mathbb{E}[S]$ is finite for a wide range of values of $(x_0, y_0) \in D$. Let us start with $(x_0, y_0) \in R_1 \cap R_8$ and reformulate S of Definition 2 as the time the process crosses the line $y = \hat{\theta}x$ for the second time. This is equivalent to take $(x_0, y_0) \in R_4 \cap R_5$. Resorting to numerical methods, we have simulated the system 2000 times for 100 different starting points in $R_1 \cap R_8$ and recorded the position at the time when this line is crossed the second time, that is the positions after a full loop. Fig. 4 contains such examination for an array of values of σ_0. The expected time $\mathbb{E}[S]$ to complete a full-loop is also illustrated. As observed, there seems to be a stable attractive fixed point to $y_0 \mapsto \mathbb{E}[y_S]$ for sufficiently large values of σ_0. If the starting point is picked too close to $(\hat{x}, \hat{y})$, the expected crossing value after one loop is further away from it. On the other hand, if the one starts extremely far away from $(\hat{x}, \hat{y})$, say with $y_0 < 0.25$, then the expected value after on loop is higher. This implies that after many loops, the expectation converges, and so does $\mathbb{E}[S]$ with the number of loops around $(\tilde{x}, \tilde{y})$. Assuming that $\mathbb{E}[S] < +\infty$ for enough initial points, Theorem 6 can be used with V at points $(0, 0)$ and $(\tilde{x}, \tilde{y})$ to prove the following conjecture.

Conjecture 1. *Consider the function $\mathcal{S} : y \in (0, \tilde{y}) \mapsto \mathbb{E}[y_S] \in (0, \tilde{y})$ such that (x_t, y_t) is a solution to (2) with $(x_0, y_0) = (y/\tilde{\theta}, y)$, and S is the finite stopping time defined by Theorem 5. Then $\mathcal{S}$ has at least one fixed point in $(0, \tilde{y})$.*

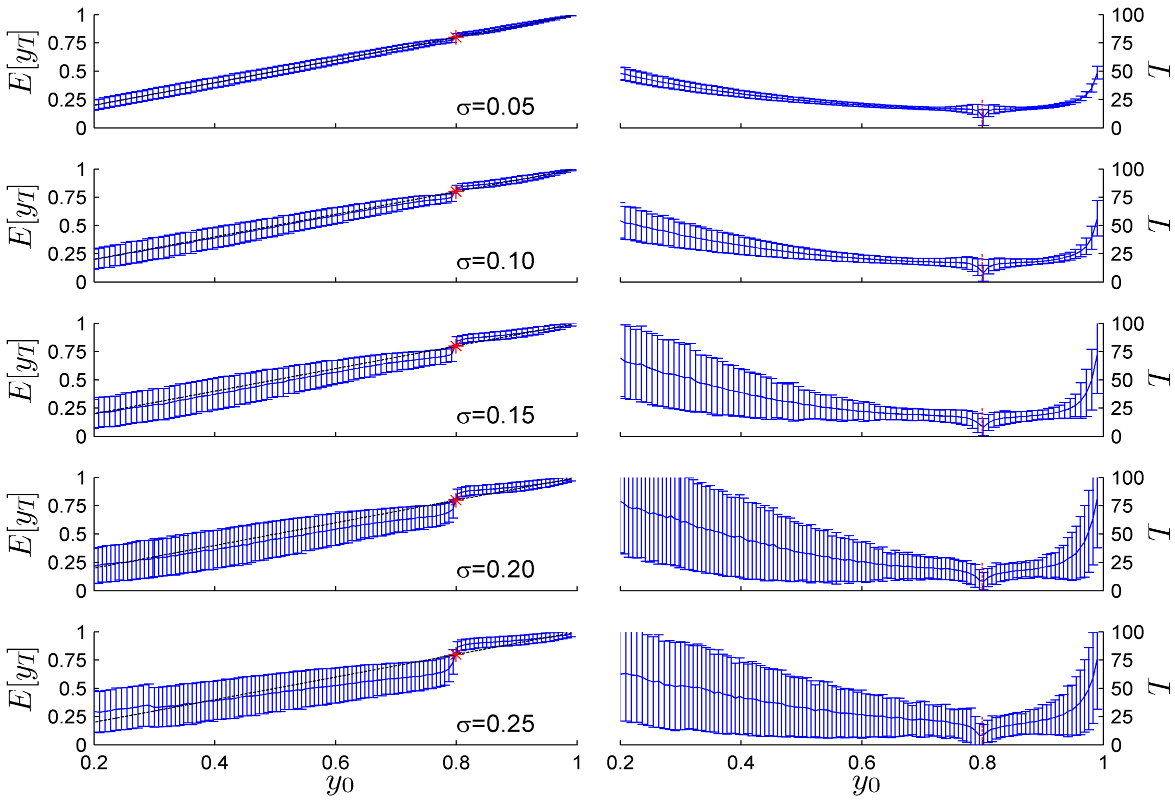

Figure 4: Expected values of employment y after one full loop y_T (left), and expected elapsed time T (right). Computation performed in MATLAB, with 2000 simulations for every value single one of the 100 initial values taken along the line $y = \hat{\theta}x$.

6 Concluding remarks

This contribution attempts to draw the attention of dynamical system analysis onto macroeconomic models. Before looking into complex models of finance and crises, e.g. [5, 11, 15], we focus here on a Brownian perturbation added into a non-linear version of the Lotka-Volterra system used in Economics, the Goodwin model. To begin with, we recall the usual results for the deterministic planar oscillator: we provide the constant Entropy function and describe the period of the closed orbits drawned by the system. We then provide sufficient conditions for the stochastically perturbed system to stay in the meaningful domain D which is a a bounded subset of $\mathbb{R}^2_+$ for the y-component. The entropy function is actually of great use for the last result, additionally to prior estimates on variations of the system.

We finally prove what seems a fundamental and staightforward property of the system, namely that a solution (x_t, y_t) rotates with perturbations around a unique point $(\tilde{x}, \tilde{y})$. The definition of stochastic orbits provided here conventienly suits the intuition of how the deterministic concept can be extended. However it has clearly not the ambition to be a definitive concept and further investigations might confirm its usefulness or its precarity. The proof exploits the concept of reccurent domains in an intensive manner.

We expect that economists seek interest in (2), as other perturbed macroeconomic systems (e.g. [16, 23]), for the possibility to adjust the model to observed past data (e.g. [1] and [13, 22]) and find a possible synthetic explanation for perturbations of business cycles (see [7, 12]).

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