The VOCO multibody software in the context of real time simulation

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Abstract

The railway multibody software developed for more than 20 years by INRETS (now IFSTTAR) under the name VOCO®, has been dedicated from the origin to highly non linear elements, such as the dry friction dampers of freight bogies and the wheel rail contact based on measured profiles. A second important step has been the discovery of a particular method in order to simulate on sinuous tracks. In the aim of industrial applications, the specification has always been to reach the goal of real time. Although it is not possible in all the cases, the recent non-hertzian contact development is allowing real-time simulation to be achieved.

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Keywords: Algorithms, Multibody Systems, Stiff differential equations, Stability analysis, Rail-wheel interaction, Curving performance

1 - Origins of the VOCO software

Before developing a railway vehicle dynamics software, the future INRETS team has been involved from 1983 in experimental research in order to simulate the stability behaviour of a scaled freight bogie on a large test bench. At that time, computer models were linear, have been used successfully to design the bogies of the TGV project in France, but the non linear freight bogie Y25 was not adequately modelled by this computing approach because of the dry friction elements.

SNCF and RATP asked INRETS to concentrate on this bogie, experimentally or by simulation. Experiments were done in 1984 and were successful [HELIOT 84] [CHOLLET 92]. The numerical modelling step started in 1985.

A few months later, the first program dealing with the 6 dry friction dampers of a 8 DOF planar model (lateral and yaw) was set up. Despite the simplifying assumptions, the critical speeds were of sufficient accuracy, compared to the scaled measurements and to the SNCF on-line tests. This success led the steering committee of the project to confirm the INRETS team in the direction of computer modelling.

2 - The dry friction element

The simulation of a dry friction element was at this time a difficult point. INRETS researchers found the more general way to simulate the Coulomb friction with a stiff – but defined – serial stiffness. This class of solution is called "regulated" and is implanted in several other multibody packages. In accordance with the multibody formulation used here, this stiffness was of the order of the real structure elastic stiffness.

The algorithm is particular and has never been published. An intermediate point, I (figure 1) is used to calculate, from the spring deformation AI, the corresponding force $F<\mu F_o$, μ being the friction coefficient, and to switch to the sliding branch when Coulomb's limit is reached : AI=constant, F=μFo.

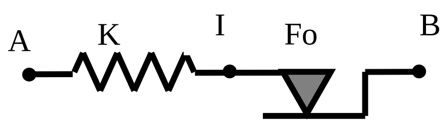

Figure 1 – Dry friction element used in the first VOCO version for the LENOIR system

In one branch, I is associated to B, in the other to A, with the same point speed used to give the direction of the force. This sort of model, called "regulated", was compatible with the solver, an Euler-Cromer symplectic integrator:

$$v(t+\Delta t) = v(t)+a(t)\Delta t \quad (1)$$

$$x(t+\Delta t) = x(t)+v(t+\Delta t)\Delta t$$

Real time simulations were done even with this stiff serial stiffness with the TEKTRONIX microcomputer available before 1990.

In the case of the Y25 side bearers, the model was more complex due to a serial free play in the real element. The standard dry friction 1 DOF connection of VOCO is offering this option from this time (figure 2).

Figure 2 – Dry friction element with free play, used to simulate the side bearers

Later, with the 6 DOF/body models, the connection library was enriched by a dry friction plane element loaded with a variable force F_o coming from any other connection.

With the help of these stable algorithms, despite a great number of dry friction elements, the simulation of freight wagons is not an obstacle to real time simulations.

3 - Simulations in curves at small angles, the Transformed Space

Around 1990, the software was limited to the simulation of critical speed tests on a straight line, corresponding to the scaled tests performed on the test bench of Grenoble.

For simplification and execution speed, the calculation of displacements was limited to "small angles" between bodies and relatively to the track:

$$\sin(\alpha) = \alpha, \quad \cos(\alpha) = 1 \qquad (2)$$

The cooperation with RATP, a metro operator, led naturally to explore the beginning of a curve with this assumption. First it seemed reasonable to use the straight line vehicle model until the rotation in this curve was 0.1 radian. Then came a discovery which extended the way the models can be used.

The first simulated curve was a circle, but it seemed simpler to describe it as a parabola:

$$y* = \frac{1}{2R} s^2 \qquad (3)$$

To check this method, a simulation has been done during a long time along this curve, and the researcher found that the vehicle was still running in this parabola in a steady-state equilibrium, kilometres from the beginning. The situation is that in both reference frames, the vehicle is accelerated to a ratio of:

$$\ddot{y} = \ddot{T}_h = \dot{s}^2/R \qquad (4)$$

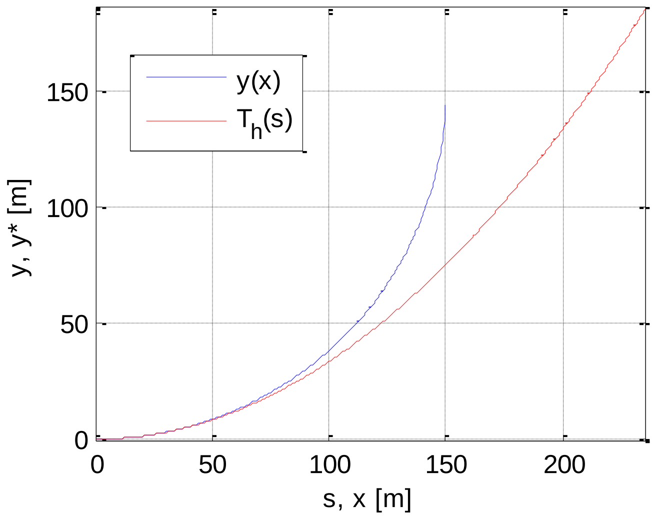

Figure 3 – Example of a R=150 m circle y(x) and its approximation as a parabola T_h(s)

A first report with this new method was published in 1990 [AYASSE 90]. This curve simulation method needed a more complete theoretical analysis. Ayasse and Maupu described the set of simplification assumptions and their range validity in a report in French [AYASSE 03]. Mark Hoffmann, after a cooperation with INRETS, used it and explained the method and the auxiliary functions T_h and T_v in English in its PhD Thesis (chapter 2, [HOFFMANN 06]).

The curve description has been particularly adapted to the curvilinear reference used in the theoretical description and in the measurement of tracks: like other events along the track, the versine measurement is made in function of the curvilinear abscissa.

This method is applied for 25 years in the different variants of VOCO softwares, including VOCODYM, and VOCOLINK, a SIMULINK variant, up to the present version of VOCO which is in FORTRAN. It gives access to the simulation of sinuous tracks with the basic model and the simplified "small angles" dynamic equations, without the need (however it is possible) to separate the long and short wavelength events. This is certainly a reason, today, for the particularly fast performances of the code. It is not excluded that some industrial codes could use a similar method.

4 - Modelling with a Multibody formulation

Another reason that may lead to the real time performances of VOCO is in the compact way the parameters are stored.

4.1 - Multibody formulation in compact matrices

The first version of the software was a single model of the Y25 bogie, followed by a generic model with 2 bogies and 4 wheelsets, where it was not possible to modify the general architecture but it was possible to control the primary and secondary suspension parameters.

Around 1993, with the translation of the software from HP BASIC to FORTRAN, AYASSE proposed a multibody architecture dedicated to the railway vehicle. This method has never been published outside the French notice [SEBES 04]. It is based on the use of compact matrix with several dimensions, with a strong hierarchy between the different dimensions.

The body reference is done by a 3 index numbering, I, J, K

- I is the number of the carbody
- I,J is the number of the bogie linked to the Ith carbody
- I,J,K is the wheelset number, linked to the Jth bogie of the Ith carbody

The body properties, the connection parameters, the position of the connection and the connection properties are stored in different matrix. There are presently 14 different types of connections classified in two families: linear and non linear. There can be 17 parameters for a connection, from the x,y,z relative position from the first body, to the different parameters of a dry friction element.

From this architecture, it was possible to create generic functions aiming to manipulate bodies and connection, in order to:

- duplicate bogies and carbodies with, in general, longitudinal translations,
- make use of symmetry properties, in order to limit the number of manipulations when creating the connections;

From this time, a VOCO vehicle model is a list of ASCII numbers defining the type of elements and parameters, arranged in loops corresponding to the matrix elements. For several years, the symmetries and duplications were made by FORTRAN subroutines, while it is now completely made under a MATLAB graphic interface completed with a linear modal analysis.

When simulating a full train, the performances are of course slowing down with the number of bodies. Table 1 presents a comparison of a full ALSTOM DUPLEX TGV (26 wheelsets) with a simple model of the single locomotive (4 wheelsets). The influence is practically proportional to the number of wheelsets.

Table 1 – Simulations of a 40" run with different model size and contact type

< **40" real time :**	TGV Duplex, 10 cars	TGV duplex, locomotive	*Loco time x 26/4*
Multi-Hz	**38"**	**6.4"**	*42"*
Semi-Hz	99"	**16"**	*104"*
Semi-Hz + rail roll	154"	**24"**	*156"*
Semi-Hz +rr +plast.	168" (56")	**26"**	*169"*

In the example of Table 1, the compared vehicles are running 1 kilometre during 40 seconds at 25 m/s, on a sinuous track with measured defects. Contact details are written in the output file only for the wheelset 1; plasticity is taken into account in the last case with σ_{MAX} = 400 MPa. Time step: dt=2.5 10^{-4} second. Output is sampled at 50 Hz.

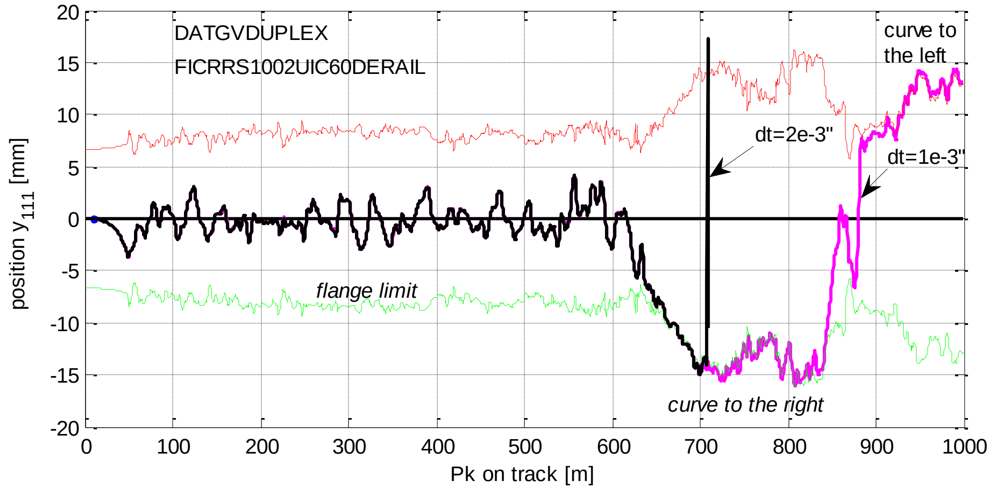

Figure 4 – Research of the divergence in a simulation on track with irregularities (Red and green lines : flange limits; black and magenta lines : trajectories with two different time steps)

In the case of Table 1 and figure 4, with the full TGV train model, the simplest contact model lead to real time simulation, and with the most complex contact model, the time step can be increased to 10^{-3} without numerical stability problems, leading to accelerate the slowest simulation of Table 1 from 168" to 56", not far from real time[1]. On the figure 4 is shown a divergence of the wheelset lateral displacement when flanging in the first curve, when the time integration step is increased from 1 (magenta line) to 2 milliseconds (500 Hz, black line). This straightforward divergence is seen as an advantage of the integration method.

Time output, contact output and real time simulations

The typical time step with a "safety margin" is 2.5 10^{-4} second (5 kHz). At this frequency, the huge amount of calculated parameters is not necessary in general, even for animations in simulators. VOCO is delivering the DOF states (position and speed) of the bodies at a frequency chosen by the user. In Table 1 by example, the sampling is 50 Hz, it is currently 200 Hz for statistics on accelerations. Of course, this does have an influence on the rapidity of the execution. Some other examples are given in Table 2.

A second point that can affect the output speed is the user demand on the contact details. These details are not necessary for simulator animations but for contact studies.

In the reference case of the Manchester benchmark, see Table 2, the detailed output is written only for the left wheel of the first wheelset. For the two wheels, the

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[1] Nota : in this article, calculations are done with a Dell Precision T3400 under Windows Vista 64 bits with a 3 years old 2.83 MHz single core CPU unit (Windows performance index 5.0); better results can be obtained with recent machines or other operating systems.

real time performance goes down to 1.99, and for all the 8 wheels of this vehicle, to 1.86. In the same time, the file where are stored these data growth from 17.7 to 20 Mo which is approximately the same variation ratio.

In the case of a full TGV train, which is necessary because the vehicles are interconnected, it is preferable to limit the detailed results to one or two wheelsets.

4.2 - Initial equilibrium

VOCO's vehicle models were initially limited to plane models with constant vertical loads on elements; after 1993, with the use of 6 DOF models and unilateral contacts, it became necessary to define the initial loads on elements, leading to the creation of an equilibration step as the last operation necessary before storing the model, and before using it in simulations.

This led to separate the connection types in two parts:

- the linear connections, used for the equilibrium, typically the primary and secondary [GIMENEZ 91] suspension stiffness;
- the non linear elements, generally defined around the equilibrium position. It is the case for bump stops with free play, and dry friction elements.

The equilibrium step is done by running a full non linear integration, with a variation of the gravity acceleration g between 0 and 9.81 m/s^2 in a few seconds, with an exponential saturation. Viscous dampers are replaced by critical damping values. The kinetic energy is surveyed during the equilibrium and the process is stopped if a static equilibrium is found. The equilibrium forces or initial displacements are stored in a matrix and the whole model is stored for a future use.

When speaking about real time simulations, one must have in mind that the modelling and equilibrium steps can be time consuming.

5 - Real time wheel rail contact models

The wheel rail contact is certainly the most important and particular part of a multi-body software dedicated to railway vehicles.

In the first version of VOCO, before 1990, the contact was based on the simple model of a cone wearing on a string, using a simple "equivalent conicity" as the main parameter governing the wheelset oscillation wavelength.

As the first version was simplified to 2 DOF, yaw and lateral, the mean loads on one wheel was a constant, the corresponding contact ellipse was determined by the theory of Hertz, and the corresponding Kalker's coefficients were read from tables.

Flange contact was defined by a separate spring and by the track free play.

5.1 - Multi-elliptic model: the Contact Angle Function

Around 1989, G. SAUVAGE went from SNCF to INRETS and proposed a way to calculate a multi-elliptic contact in a complete 6 DOF bogie model taking into account the real shape of wheels and rails, leading eventually to contact jumps between different parts of the rail and the wheel [SAUVAGE 90].

However at this time, this wheelset model was very far from real time; it was decided to make different simplified versions in order to simulate more quickly with the microcomputers which were available at this time.

- A variant has been to pre-compute an equivalent contact ellipse; this solution has been implemented later in the LUCIFER software from SNCF, giving the VOCODYM software used in SNCF and in the ALSTOM group. This solution, presented during the 1991 IAVSD symposium [PASCAL 91], has inspired a lot of other softwares, including SIMPACK.

- A second variant, the Contact Angle Function method (CAF method), due to AYASSE has been to linearize the parameters in order to store them in compact tables, allowing the presence of several ellipses during the simulation. [AYASSE 02]. [AYASSE 06]. On figure 5 is shown the linearization of the so called CAF method, another table not presented here is describing the rail radius, which is the basis of theoretical rail profiles description.

This last CAF method led to real time simulations at this time, because the multibody has only 2 DOF per solid. It has been slightly improved between 1992 and 2000. The profiles can be non symmetric, the number of linear zones is not limited, and increased when the user wants to be close to measured profiles. There is no limit to the number of ellipses (even more than the 3 on top of figure 6). However there were some limitations, only convex profiles were possible for the rails.

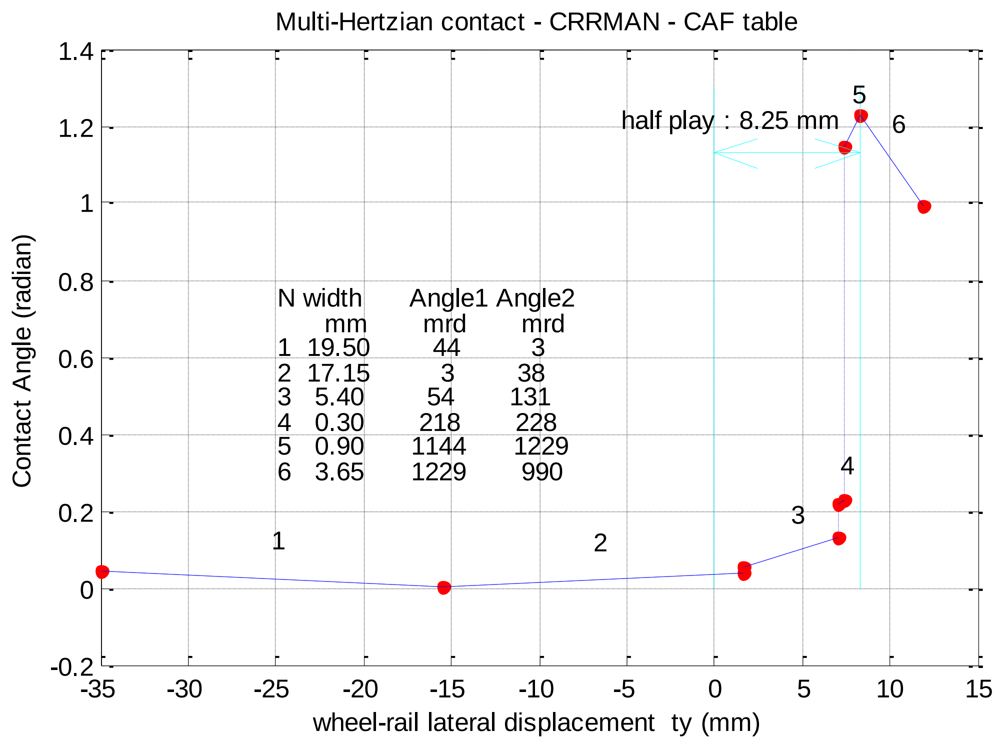

Figure 5 – Linearization of the Contact Angle Function, Manchester benchmark case [IWNICKI 99], left wheel: 6 zones

However the evolutions of the railway operators needs in research, led to more accurate and time consuming solutions. Around 2000, it appears that the use of a fine wheel rail contact model has a particular interest in rail fatigue studies [DANG VAN 09]. Multi-elliptic contacts have the disadvantage of frequent overlaps between the ellipses, giving a poor definition of the contact pressure. Another limit of the CAF multi-elliptic approach was the interpolation of the contact jumps in the case of rail variations like in switches. A solution was found, leading to create artificial jumps and to multiply the ellipses [AYASSE 02] but it becomes rather complex to analyse despite good stability.

5.2 - Semi-elliptic method: the STRIPES model

Ayasse and Chollet [AYASSE 05] proposed to improve the approach, in a way which was close to Kik and Piotrowski's proposition used in ADAMS [KIK 1996], with an algorithm named STRIPES. A comparison of these different approaches has been made in [PIOTROWSKI 05].

The particularity of the STRIPES algorithm for the normal problem is to take into account curvatures that are variable along all the longitudinal strips defining the contact. Another particularity is an adequate correction of the curvatures in order to be compatible with Hertz' theory for the ratio a/b of slender ellipses, in Hertzian cases. This correction is completed by a smoothing step, optimised by comparison to finite element models of non Hertzian cases [QUOST 06].

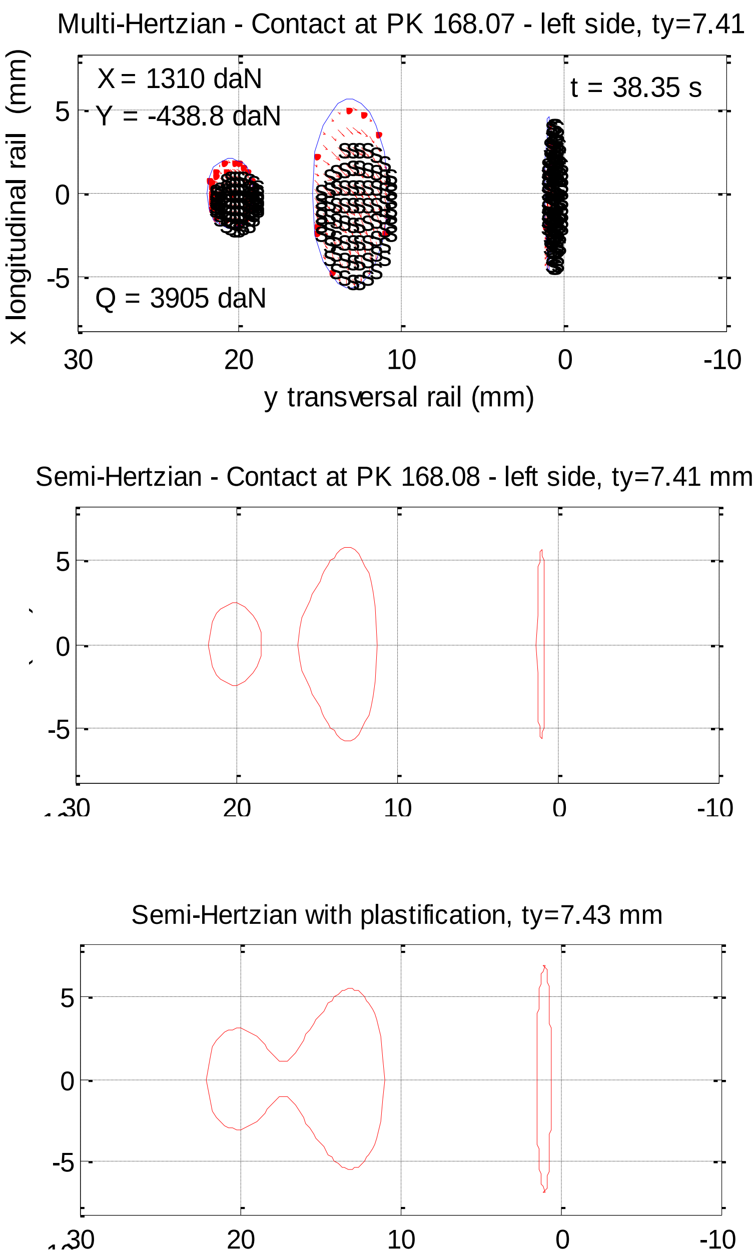

Figure 6 – Different contact models, shapes when flanging, Manchester benchmark [IWNICKI 99]

Concerning the tangential problem, the classical application of FASTSIM algorithm on a big surface matrix (top of figure 6: applied to the ellipses, saturation indicators 'S' on the elements) could give a risk of slowing down the calculation under real time. An adaptation led however to reasonable performances. The saturation algorithm is based on a single loop on the strips, associated to a saturation formula on each strip, taking into account the spin, similarly to the SHEN HEDRICK ELKINS approach. There are of course differences (example on figure 7), which must be appreciated relative to the uncertainty on the friction coefficient.

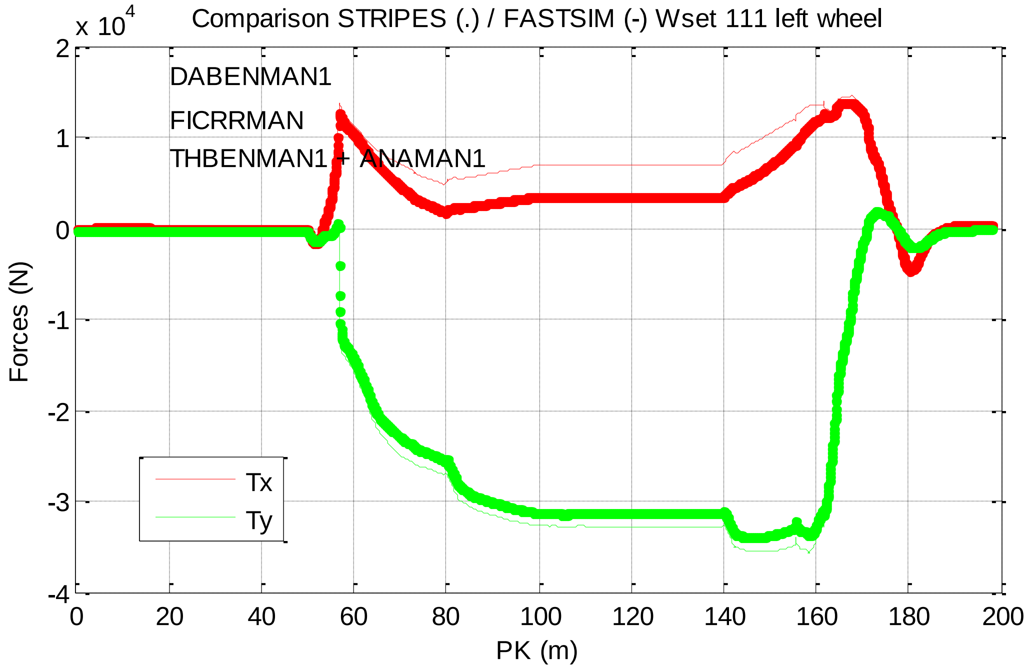

Figure 7 – Tangent forces calculations, Manchester benchmark - Difference between VOCO's internal algorithm (semi-Hertzian elastic) and FASTSIM

5.3 - Multi-elliptic method based on Semi-elliptic data tables

As the STRIPES algorithm was easily managing varying rail profiles, like in switches [SEBES 05] in comparison to the Multi-elliptic traditional method based on CAF, a simplified version was considered, in order to keep this advantage, but with a faster performance.

The principle is to determine a main strip and one or more secondary strips, instead of around 20 or 30 typical strips of a full semi-Hertzian method. It is simply acting proportionally to the number of strips. As far as the dynamic behaviour is concerned, there is in general not so much difference, but the gain in computation speed is significant. This multi-elliptic method is the most adequate for real time simulations with VOCO.

5.4 - On some other functionalities influencing real time simulations

Simulation at low speed: improvement of the algorithm

In some cases, despite the vehicle speed being low, contact models can have stabilisation problems at the contact level. This issue is due to the presence in the lateral creep of the ratio y'/Vx (Vx=x'). The contact algorithm of VOCO has been improved around 2007 to facilitate the low speed simulations. An example of the performance at a speed of 44 mm/s is given in Table 2 on the first line. As the vehicle speed is very low, the internal time step can be increased, improving the simulation speed performance. Under VOCO, a vehicle can stop and go back.

Taking into account plasticity

The last version of STRIPES improved the stress description by taking plasticity into account [SEBES 12]. Even with this sophistication, whose effect on the contact surface can be seen on figure 6, the reference simulation can be done in real time (cf Table 2). On a simulation where there are no major load variations, the plasticity computation cost could be +10 to +35 %.

5.5 - Solving contact in the time loop

STRIPES and its multi-elliptic variant are based on a "virtual interpenetration" of wheel and rail profiles [AYASSE & CHOLLET 05]. It is similar to the penalty method used in finite elements where the contact force is the product of a contact stiffness with the interpenetration.

Using this contact stiffness in the context of railway dynamics where abrupt changes of contact status are likely to occur would lead to time steps that are too small. In order to keep a time step of the order of magnitude of 0.1 ms, a serial stiffness is introduced between the wheelset centre of gravity and the contact zone. This is very similar to what has been done in order to handle dry friction dampers (see section 2). Even with this serial stiffness, some numerical instability may occur. In that case a smaller time step may be activated in the wheel/rail contact subroutine. This smaller time step is only activated if the jerk is too high, so it is dependent on the dynamics. Examples of the lowest time step value are given in Table 2. This special treatment is particularly efficient in term of real time, in case of curve negotiations with wheel flanging.

6 - Today's performances

Optimizing the time step is a way of improving the computing speed; however, it is depending on the vehicle model, and depending on the operating conditions.

The real time performance of a simulation is dependent on many parameters. Table 2 presents a comparison between the simulation speed and time step at the limit of the integration algorithm stability, in different cases based on the Manchester benchmark case 1 (line 3 of the table). For comparisons, the Manchester benchmark is an interesting reference because the vehicles and tracks are completely defined and published.

Table 2 – Semi-Hertzian elastic model – Different simulations based on the Manchester benchmark case 1, at different speeds

Manchester benchmark variants	Speed Vx	Nominal time step	Minimal time step	Output sampling	Real time ratio
1	0.044 m/s	1.5e-3 s	99e-6 s	2 Hz	9.84
2	0.44 m/s	1.5e-3 s	0.11e-6 s	20 Hz	6.6
3	4.4 m/s	1.5e-3 s	0.17e-6 s	200 Hz	2.05
4	11 m/s	1.e-3 s	1.6e-6 s	500 Hz	0.98
5	11 m/s	1.e-3 s	1.6e-6 s	50 Hz	4.15

The "real time ratio" is the ratio between the real time and the simulation time. In the example of Table 2, for the lines 1 to 4, the output sampling is done at a constant spatial step of 22 mm along the track. On line 4, flanging at 11 m/s has more dynamic consequences and with this example, a numerical divergence is observed if the nominal time step is greater than one millisecond.

The 5th line example, the same than 4th line but sampled at 50 Hz, shows the output file writing impact.

Conclusion

By keeping simple expressions originally dedicated to straight line conditions, with the help of some particular dispositions, the VOCO software is in the range of achieving real time simulation rates, despite being mainly designed to implement advanced contact models in the past years. This is mainly due to the stability of the algorithms developed for this code, like for dry friction and wheel flanging, leading to a reasonable reduction of the integration time step when a discontinuity is encountered.

In order to control the maintenance of the infrastructure, the rail industry is requesting more and more sophisticated simulations, consequently the recent developments of VOCO are directed to the multi-body modelling of the railway track and not to the use in real time applications. This possibility is however present.

The use of a fast tool is always an advantage for developments. In the future use of multibody softwares in virtual homologations, parameter variation studies are a reason to promote the fastest solutions.

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