Randomness Zoo
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Definition 1 (MLR: Martin-Löf Randomness (Definition 3.2.1 in [ML66])).

1. A Martin-Löf test, a ML-test for short, is a uniformly c.e. sequence \((G_m)_{m \in \mathbb{N}}\) of open sets such that \(\forall n \mu(G_m) \leq 2^{-m}\).

2. A set \(R\) fails the test if \(R \in \bigcap_{m \in \mathbb{N}} G_m\) otherwise \(R\) passes the test.

3. \(R\) is Martin-Löf Random if \(R\) passes each ML-test.

1 Weaker than MLR

Definition 2 (Scan rule function). For a partial function \(f : 2^{\omega} \to \{\text{scan}, \text{select}\} \times \mathbb{N}\) we denote \(n : 2^{\omega} \to \mathbb{N}\) and \(\delta : 2^{\omega} \to \{\text{scan}, \text{select}\}\).

And we say that \(f\) is a scan rule if for all \(\sigma, \rho \in 2^{\omega}\) such that \(\sigma \prec \rho\) we have \(n(\sigma) \neq n(\rho)\).

The sequence of string observed by \(f\) on \(Z \in \text{cs}\) is defined by (if \(f\) is well defined on this points):

\[
V_f^A(0) = A(n(\varepsilon))
\]
\[
V_f^A(k + 1) = V_f^A(k).A(n(V_f^A(k)))
\]

and bits selected by \(f\) are:

\[
T_f^A(0) = \varepsilon
\]
\[
T_f^A(n + 1) = \begin{cases} 
T_f^A(n) & \text{if } \delta(V_f^A(n)) = \text{scan} \\
T_f^A(n).V_f^A(n + 1) & \text{if } \delta(V_f^A(n)) = \text{select}
\end{cases}
\]

and we say that \(f\) is well defined on \(Z\) if \(V_f^A(k)\) is well define for all \(k\) and \((T_f^A(n))\) converge to an infinite string and we denote this infinite string by \(T_f^A\).

Definition 3 (Martingale). A martingale is a function \(M : 2^{\omega} \to \mathbb{R}^+ \cup 0\) such that for all \(\sigma \in 2^{\omega}\)

\[
M(\sigma) = \frac{M(\sigma0) + M(\sigma1)}{2}
\]

We say that \(M\) succeed on \(Z \in 2^\omega\) if

\[
\limsup_{n \to \infty} M(Z \upharpoonright n) = \infty
\]

Definition 4 (KLR: KolmogorovLoveland randomness (Definition in [Kol63] and [Lov66])). \(R \in 2^\omega\) is KL-random if for any partial computable scan rule function \(f\) and any partial computable martingale \(M\) such that \(f\) is well defined on \(R\) the martingale martingale \(M\) does not succeed on \(T_f^A\).
Definition 5 (KLStoch: KolmogorovLoveland stochasticity (Definition in [Lov66])). Let \( \#0 : 2^{<\omega} \rightarrow \mathbb{N} \) the function giving the number of “0” in a string.

A sequence \( R \) is KolmogorovLoveland stochastic if for all partial computable scan rule \( f \) such that \( f \) is well defined on \( \mathbb{Z} \) we have:

\[
\lim_{n \to \infty} \frac{\#0\left(T_f^A \upharpoonright n\right)}{n} = \frac{1}{2}
\]

Definition 6 (MWCStoch: Mises-Wald-Church stochasticity (Definition in [vM19] and [Chu40])). A sequence \( R \) is Mises-Wald-Church stochastic if for all partial computable monotonic scan rule function \( f \) is well defined on \( \mathbb{Z} \) we have:

\[
\lim_{n \to \infty} \frac{\#0\left(T_f^A \upharpoonright n\right)}{n} = \frac{1}{2}
\]

Definition 7 (ChStoch: Church stochasticity (Definition in [vM19] and [Chu40])). A sequence \( R \) is Church stochastic if for all total computable monotonic scan rule function \( f \) we have:

\[
\lim_{n \to \infty} \frac{\#0\left(R_f(0)R_f(1)\ldots R_f(n)\right)}{n} = \frac{1}{2}
\]

Definition 8 (PInjR: partial injective randomness (Definition in [MN06])). A sequence \( R \) is partial injective random if for any total computable injective function \( g : \mathbb{N} \rightarrow \mathbb{N} \) and any partial computable martingale \( M \) this martingale is defined and does not succeed on the sequence \( R_f(1)R_f(2)\ldots R_f(n)R_f(n+1)\ldots \).

Definition 9 (InjR: injective randomness (Definition in [?])). A sequence \( R \) is injective random if for any total computable injective function \( g : \mathbb{N} \rightarrow \mathbb{N} \) and any total computable martingale \( M \) this martingale does not succeed on the sequence \( R_f(1)R_f(2)\ldots R_f(n)R_f(n+1)\ldots \).

Definition 10 (PermR: partial permutation randomness (Definition in [BHKM09])). A sequence \( R \) is partial permutation random if for any total computable bijective function \( g : \mathbb{N} \rightarrow \mathbb{N} \) and any partial computable martingale \( M \) this martingale is defined and does not succeed on the sequence \( R_f(1)R_f(2)\ldots R_f(n)R_f(n+1)\ldots \).

Definition 11 (PCR: partial computable randomness (Definition in [AS97])). A sequence \( R \) is partial computable random if for all partial computable martingale \( M \) if \( M(R \upharpoonright n) \) is define for all \( n \) and \( M \) does not succeed on \( R \).
Definition 12 (CR: computable randomness (Definition in [Sch71])). A sequence $R$ is computable random if for all total computable martingale $\mathcal{M}$ this martingale succeed on $R$.

Definition 13 (SR: Schnorr randomness (Definition in [Sch71])). A Schnorr test is a uniformly c.e. sequence $(G_m)_{m \in \mathbb{N}}$ of open sets such that $\forall n \mu(G_m) = 2^{-m}$.

$R$ is Schnorr random if for any Schnorr test $(G_m)_{m \in \mathbb{N}}$ $R \notin \bigcap_{m \in \mathbb{N}} G_m$.

Definition 14 (FBoundR: finitely bounded randomness (Definition in [BDN12])). $R$ is finitely bounded random if $R$ passes any Martin-Löf test $(U_n)$ such that for every $n$, $\#U_n < \infty$ (with $\#U_n$ the number of sting enumerated in $U_n$).

Definition 15 (CBoundR: computably bounded randomness (Definition in [BDN12])). A Martin-Löf test $(U_n)$ is computably bounded if there is some total computable function $f$ such that $\#U_n \leq f(n)$ for every $n$.

$R$ is computably bounded random if $R$ passes every computably bounded Martin-Löf test.

Definition 16 (WR: weakly randomness (Definition in [Kur81])). $R$ is weakly random if $R \in U$ for every $\Sigma^0_1$ set $U \subseteq 2^\omega$ of measure 1.

Definition 17 (PolyR: polynomial randomness (Definition in [Wan96])). A sequence $R$ is polynomially random if any martingale computable in polynomial time $\mathcal{M}$ do not succeed on $R$.

Definition 18 ($\dim^s_{\text{comp}}R$: computable $s$-randomness (Definition in [Lut03] and [May02])). A computable $s$-test is a uniformly computable sequence $(G_m)_{m \in \mathbb{N}}$ of computable open sets such that for all $n$

\[ \sum_{x \in G_m} 2^{-s|x|} \leq 2^{-n}. \]

$R$ is computably $s$-random if for all $s' < s$ and computable $s$-tests $(G_m)$ we have:

$R \notin \bigcap_{m \in \mathbb{N}} G_m$

Definition 19 (Cdim$s^s$R: constructive $s$-randomness (Definition in [Lut03] and [May02])). A constructive $s$-test is a uniformly computable sequence $(G_m)_{m \in \mathbb{N}}$ of computable enumerable open sets such that for all $n$

\[ \sum_{x \in G_m} 2^{-s|x|} \leq 2^{-n}. \]
$R$ is computably $s$-random if for all $s' < s$ and computable $s$-tests $(G_m)$ we have:

$$R \notin \bigcap_{m \in \mathbb{N}} G_m$$

2 Stronger than MLR

**Definition 20 (DiffR: difference randomness (Definition in [FN])).** A difference test is given by a sequence $(V_m)_{m \in \mathbb{N}}$ of uniformly c.e. sets and a $\Pi_1^0$ set $P$ such that $\mu(P \cap V_m) \leq 2^{-m}$ for every $m$.

A sequence $R$ is difference random if for any difference test $(V_m)_{m \in \mathbb{N}}, P$ we have

$$R \not\in P \bigcap_{m \in \mathbb{N}} (\cap_{m \in \mathbb{N}} V_m).$$

**Definition 21 (BalancedR: balanced randomness (Definition in [FHM+10])).** A balanced test is a sequence $(V_m)_{m \in \mathbb{N}}$ of c.e. sets such that $V_i = W_{f(i)}$ for some $2^n$-c.e. function $f$ and $\mu(V_m) \leq 2^{-m}$ for every $m$.

A sequence $R$ is balanced random if $R \not\in \cap_{m \in \mathbb{N}} V_m$.

**Definition 22 (WDemR: weak Demuth randomness (Definition in [Dem82])).** A Demuth test is a sequence $(V_m)_{m \in \mathbb{N}}$ of c.e. sets such that $V_i = W_{f(i)}$ for some $\omega$-c.e. function $f$ and $\mu(V_m) \leq 2^{-m}$ for every $m$.

A sequence $R$ is weak Demuth random if for any Demuth test $(V_m)$ we have $R \not\in V_m$ for almost all $m$.

**Definition 23 (DemR: Demuth randomness (Definition in [Dem82])).** A Demuth test is a sequence $(V_m)_{m \in \mathbb{N}}$ of c.e. sets such that $V_i = W_{f(i)}$ for some $\omega$-c.e. function $f$ and $\mu(V_m) \leq 2^{-m}$ for every $i$.

A sequence $R$ is Demuth random if for any Demuth test $(V_m)$ we have $R \not\in V_m$ for almost all $m$.

**Definition 24 (W2R: weak 2-randomness (Definition in [Kau91])).** A sequence $R$ is weak 2-random if $R \not\in U$ for every $\Pi^0_2$ set $U \subset 2^\omega$ of measure 0.

**Definition 25 (LimitR: limit randomness (Definition in [KN11])).** A limit test is a sequence $(V_m)_{m \in \mathbb{N}}$ of c.e. sets such that $V_i = W_{f(i)}$ for some $\Delta^0_2$-computable function $f$ and $\mu(V_m) \leq 2^{-m}$ for every $m$.

A sequence $R$ is limit random if for any limit test $(V_m)$ we have $R \not\in V_m$ for almost all $m$.

**Definition 26 (\Delta^1_1R: \Delta^1_1 randomness (Definition in [ML70])).** $R$ is $\Delta^1_1$-random if $R$ avoids each null $\Delta^1_1$-class.
Definition 27 ($\Pi^1_1$-MLR: $\Pi^1_1$-Martin-Löf Randomness (Definition in [HN07])).

A $\Pi^1_1$-Martin-Löf test is a sequence $(G_m)_{m \in \mathbb{N}}$ of open sets such that $\forall n \mu(G_m) \leq 2^{-m}$ and the relation $\{\langle m, \sigma \rangle | [\sigma] \subseteq G_m\}$ is $\Pi^1_1$

$R$ is $\Pi^1_1$-Martin-Löf Random if $R$ passes each $\Pi^1_1$-ML-test.

Definition 28 ($\Pi^1_1$R: $\Pi^1_1$-Randomness (Definition in [HN07])). $R$ is $\Pi^1_1$-random if $R$ avoids each null $\Pi^1_1$-class.
References


