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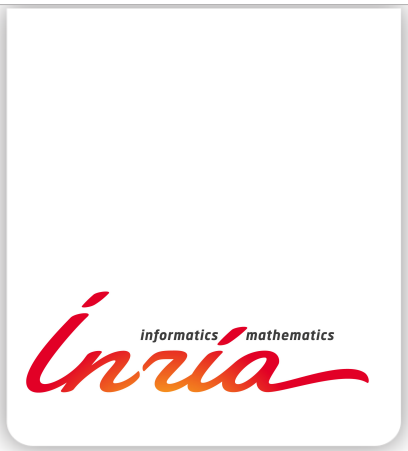

Fast polynomial evaluation and composition

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Abstract: The library *fast_polynomial* for Sage compiles multivariate polynomials for subsequent fast evaluation. Several evaluation schemes are handled, such as Hörner, divide and conquer and new ones can be added easily. Notably, a new scheme is introduced that improves the classical divide and conquer scheme when the number of terms is not a pure power of two. Natively, the library handles polynomials over gmp big integers, boost intervals, python numeric types. And any type that supports addition and multiplication can extend the library thanks to the template design. Finally, the code is parallelized for the divide and conquer schemes, and memory allocation is localized and optimized for the different evaluation schemes. This extended abstract presents the concepts behind the *fast_polynomial* library. The sage package can be downloaded at: `http://trac.sagemath.org/sage_trac/ticket/13358`. In Section 1, we present the notion of evaluation tree and function scheme that unifies and extends state of the art algorithms for polynomial evaluation, such as the Hörner scheme [Mul06] or divide and conquer algorithms [Mul06, Est60, BK75, BZ11]. Section 2 reviews the different optimisations implemented in the library (multi-threads, template, fast exponentiation), that allows the library to compete with state-of-the art implementations. Finally, Section 3 shows experimental results.

Key-words: computer algebra, polynomial evaluation, parallel implementation

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Évaluation et composition rapide de polynômes

Résumé : La bibliothèque *fast_ polynomial* permet de compiler des polynômes multivariés sage pour les évaluer ensuite rapidement en Sage. Plusieurs schémas d'évaluation sont disponibles, comme Hörner, scindage binaire, ou d'autres qui peuvent être ajoutés facilement. La bibliothèque peut manipuler des polynômes à coefficients de type entier long gmp, interval boost, objet python. Par un système de template, elle est facilement extensible à d'autres types qui peuvent être ajoutés et multipliés entre eux. Enfin l'évaluation peut être parallélisée et l'espace mêmoire est optimisé pour chaque schéma d'évaluation. Ce rapport présente les concepts sous-jacents à la bibliothèque *fast_ polynomial*. Le paquet Sage est téléchargeable à l'adresse `http://trac.sagemath.org/sage_trac/ticket/13358`.

Mots-clés : calcul formel, évaluation polynomiale, implantation parallèle

1 Polynomial preprocessing

Given a polynomial with integer, floating points, or even polynomial coefficients, there is several way to evaluate it. Some are better suited than others for specific data type. An evaluation tree specifies how the polynomial will be evaluated.

Definition 1.1. *An* evaluation tree $\mathcal{T}_p$ *associated to a polynomial* p *is an acyclic graph with a root node* R*. Each node* N *corresponds to a monomial of* p *and has* 2 *labels, denoted by* $c(N)$*, the* coefficient *associated to* N*, and* $d(N)$*, the* partial degree *associated to* N*. The result of an* evaluation tree *on* x *is defined recursively:*

$$\mathcal{T}(x) = \begin{cases} c(R)x^{d(R)} & \text{if } R \text{ is the only node of } \mathcal{T}. \\ (c(R) + \sum_i \mathcal{S}_i(x))x^{d(R)} & \text{otherwise, where } \mathcal{S}_i \text{ are the children tree of } R. \end{cases}$$

Each node of an evaluation tree is naturally associated with a term of the input polynomial. However, the *partial degree* of a node N is not the degree of monomial associated to N. The degree of the monomial associated to N is rather the sum of the partial degrees of its ancestors.

If we order the terms of p in a decreasing lexicographical ordering, we induce naturally an ordering on the nodes of $\mathcal{T}_p$. This ordering is also a topological ordering of $\mathcal{T}_p$ and will be denoted subsequently by $<_t$. The first node is the bigger for $<_t$ and will have index 0. The last node is the root of the tree and will have index n. In particular, all the children of a node of index i have an index lower than i.

1.1 Function scheme

A way to define an evaluation scheme for univariate polynomials is to use a *function scheme*.

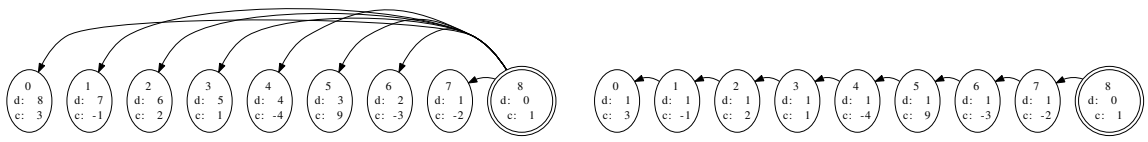
Definition 1.2. *Let* $f : \mathbb{N} \to \mathbb{N}$ *be a function such that* $0 < f(k) \leq k$ *for all* $k \geq 1$*. Let* p *be a univariate polynomial of degree* n*. We define recursively the evaluation tree* $\mathcal{T}_p^f$ *associated to the* function scheme f*.*

If p *has one term, then* $\mathcal{T}_p^f$ *is reduced to one node of coefficient and degree those of the term in* p*. Otherwise,* p *can be written uniquely* $p(x) = a(x)x^{f(n)} + b(x)$*. The evaluation tree* $\mathcal{T}_p^f$ *is obtained by adding the tree* $\mathcal{T}_a^f$ *as a child of the root of the tree* $\mathcal{T}_b^f$*.*

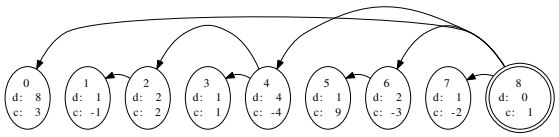
Most classical schemes such as Hörner [Mul06] or Estrin (divide and conquer [Mul06, Est60, BK75, BZ11]) schemes can be described with simple function schemes:

Direct: $D(k) = k$ Hörner: $H(k) = 1$ Estrin: $E(k) = 2^{\lfloor \log k \rfloor}$

Example 1.3. *Let* p *be the polynomial* $3x^8 - x^7 + 2x^6 + x^5 - 4x^4 + 9x^3 - 3x^2 - 2x + 1$*. Then the following trees are all evaluation trees of* p*, with different evaluation scheme.*



Direct scheme Hörner scheme



Estrin scheme

Remark 1.4. *For multivariate polynomials, the function scheme can be applied recursively to each variable.*

Remark 1.5. *Function schemes can be defined and used in* fast_polynomial *library, as documented in the module* method*. It is thus possible to combine easily different schemes. For example, let f be the function $f(k) = 2^{\lfloor \log k \rfloor}$ if $k > 10$ and $f(k) = 1$ otherwise. The corresponding evaluation tree is a divide and conquer scheme for the upper part and a Hörner scheme for the sub polynomials of degree less than* 10*.*

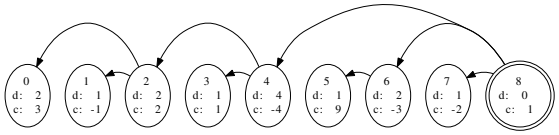
1.2 A new balanced divide and conquer scheme

The Estrin scheme is a divide and conquer algorithm well suited to evaluate polynomials on elements whose size increases linearly with each multiplications ([BK75, BZ11]). These elements include multiple precision integers or univariate polynomials. However, the computation time of evaluating $\mathcal{T}_p^E$ reaches thresholds when the number of terms of p is a pure power of 2 (see Figure 1 in Section 3).

We introduce in this library a new evaluation scheme that avoids the time penalty of the classical divide and conquer. It is defined by the *balanced* function scheme.

$$\text{Balanced: } B(k) = \lfloor \frac{k}{2} \rfloor$$

Example 1.6. *[continued] The balanced divide and conquer evaluation trees contains lower partial degrees in this example.*



Balanced divide and conquer scheme

1.3 Lazy height

We associate to each node of the tree a *lazy height*, that will determine the number of temporary variables required during the evaluation. In particular, the *lazy height* must be kept as low as possible. Classically, the height of a node is always greater then the height of its children. In our case, the *lazy height* of a node is greater than the *lazy height* of its children only if it has two or more children. In particular, this ensures us that for any tree, the maximal *lazy height* is at most logarithmic in the number of nodes.

Definition 1.7. *Let N be a tree node. The* lazy height *of N, denoted $lh(N)$, is defined recursively. Let $C_1, ..., C_k$ be the child nodes of N such that $c_1 >_t \cdots >_t c_k$.*

$$lh(n) = \begin{cases} 0 & \text{if } N \text{ has } 0 \text{ or } 1 \text{ child.} \\ \max\limits_{2 \le i \le k} (lh(C_i)) + 1 & \text{otherwise.} \end{cases}$$

Example 1.8. *Consider again the polynomial $p = 3x^8 - x^7 + 2x^6 + x^5 - 4x^4 + 9x^3 - 3x^2 - 2x + 1$. In the case of Hörner scheme, the maximal* lazy height *of the associated evaluation tree is 0, whereas its classical height is 8. The* lazy height *associated to the Direct scheme is 1. And we can check that the Estrin scheme and the Balanced scheme have both maximal* lazy heights *1.*

2 Evaluation

2.1 Coefficients walk

Once the tree data structure has been computed, the evaluation can be done efficiently. If p is a univariate polynomial of degree n, we can use the following pseudo-code.

```
for i from 0 <= i < n:
    N = nodes[i]
    c, d, h = N.coefficient, N.partial_degree, N.lheight
    p = (m[h] + c)*x^d
    m[h] = 0
    if   i == n: return p
    elif i <  n: m[N.parent.lheight] += p
```

If the values x^d have been precomputed (see next Section), each step costs one multiplication and one addition. The mutable variables are p and $m[0], \ldots, m[L]$, where L is the lazy height of the root node. Their number is at most $O(\log n)$.

2.2 Powers computation

The powers x^d appearing in the evaluation loop can be computed several times for the same d. In order to optimize the evaluation, these powers can be precomputed using fast exponentiation methods.

Assume that p is a dense univariate polynomial of degree n. Table 1 shows that the balanced scheme, as well as the Estrin scheme, require at most a logarithmic number of different powers to compute.

Direct	Hörner	Estrin	Balanced
$1, \ldots, n$	1	2^k	$\lfloor \frac{n}{2^k} \rfloor, \lfloor \frac{n}{2^k} \rfloor + 1$
		$0 \le k \le \log n$	$0 \le k \le \log n$

Table 1: Degrees appearing in the evaluation tree of a dense polynomial of degree n.

2.3 Template system and multi-thread

The code is written with templates, and is specialized for different *C/C++* object. This allows the library to compete with state-of-the art ad hoc implementations, and to be easily extended with new numeric types (see *interfaces/README* in the package).

Moreover, the evaluation tree can be evaluated with multiple threads in parallel. The parallelization mechanism is implemented with openMP directives.

3 Benchmarks

The Figure 1 shows the performance of the balanced scheme implemented in *fast_polynomial* for the evaluation over multi precision integers. We see in particular that the balanced scheme doesn't suffer the staircase effect shown by the classical divide and conquer algorithms for pure powers of 2. The results suggest also that an implementation of the balanced scheme directly in Flint could improve the polynomial composition and evaluation over big integers in some cases.

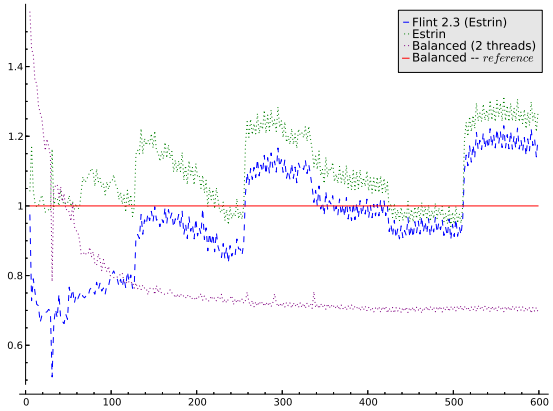


Figure 1: Comparison of the Balanced scheme with the Estrin scheme, the Balanced scheme with 2 threads, and the state of the art Flint library. The abscisse represents the degree of the polynomial p, the bitsize of its coefficients, and the bitsize of the integer on which it is evaluated. The ordinate represents the computation time for the different methods divided by the computation time for the Balanced scheme.

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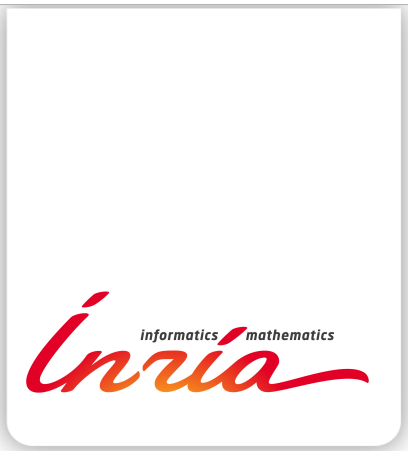

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