A unied framework for vehicle routing problems with stochastic travel and service times

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1 Introduction

Vehicle routing problems (VRPs) are concerned with the design of efficient routes that deliver goods and services from (to) central depots to (from) customer locations, satisfying specific business constraints. In the last 50 years, a vast amount of research has been devoted to solve different VRP variants. Nonetheless, most of the solution methods for VRPs are based on the premise that problem parameters such as travel times and customer demands are known in advance. However, in a practical setting, more often than not the problem parameters are uncertain and neglecting their stochastic nature may result in poor routing decisions. In this research, we tackle a family of problems that have received relatively little attention in the literature: the vehicle routing problems with stochastic travel and service times. The main contributions of this research are twofold: proposing a unified framework based on queueing theory to model stochastic travel and service times and illustrating how this framework can be embedded into a routing optimization engine to solve different problem variants.

2 Modeling stochastic travel and service times

The family of vehicle routing problems with stochastic travel and service times can be defined on a complete and undirected graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{0, \ldots, n\}$ is the vertex set and $\mathcal{E} = \{e = (v_i, v_j) : v_i, v_j \in \mathcal{V}, v_i \neq v_j\}$ is the edge set. Vertices $v_i \in \mathcal{V} \backslash \{0\}$ represent the customers and vertex 0 represents the depot. An edge weight $\tilde{t}_e$ is associated with edge $e = (v_i, v_j) = (v_j, v_i) \in \mathcal{E}$, and it represents the random travel time along edge e. Each customer v_i has a random service time $\tilde{s}_{v_i}$, a known demand d_{v_i} for a given product, and a time window $[t^-_{v_i}, t^+_{v_i}]$, where $t^-_{v_i}$ and $t^+_{v_i}$ are the earliest and latest times when the customer must be served. Both travel and service times are assumed to follow known distributions. Customers are served by an unlimited fleet of homogeneous vehicles located at the depot. Each vehicle has a maximum capacity Q and a maximum route duration T. The objective is to design a route set $\mathcal{R}$ of minimum total expected duration $E\left[\tilde{T}(\mathcal{R})\right] = \sum_{r \in \mathcal{R}} E\left[\tilde{T}_r\right]$, where $\tilde{T}(\mathcal{R})$ is the total (random) duration of the route set, $\tilde{T}_r$ is the (random) duration of route r, and $E(\cdot)$ denotes the expected value. Each route $r \in \mathcal{R}$ is a sequence of vertices $r = \big(0, v_{(1)}, \ldots, v_{(i)}, \ldots, v_{(n_r)}, 0\big)$, where $v_{(i)} \in \mathcal{V} \setminus \{0\}$ is the i-th node in the sequence, n_r is the number of customers serviced by the route, and $e = (v_{(i)}, v_{(i+1)}) \in \mathcal{E}$ (with $v_{(0)} = v_{(n_r+1)} = 0$). Aside from the classical capacity constraint, each route $r \in \mathcal{R}$ satisfies a set of constraints $\mathcal{C}$ that involve the route's duration, such as not exceeding the time limit T or violating the customer time windows. The aforementioned problem definition encompasses a number of problems such as the classical VRP with stochastic travel and service times (Laporte et al., 1992), the VRP with stochastic travel times (Van Woensel et al., 2003), and the VRP with stochastic travel times and time windows (Taş et al., 2013), among others.

To tackle the VRP with stochastic travel and service times, the literature reports several methods based on the expected values of the random variables that model travel and service times. However, working with expected values could lead to partial information regarding the performance of the route (e.g., expected route duration). In order to find more refined information such as the distribution of the route duration, it is necessary to compute the convolution of the distribution functions that represent the travel times along the arcs and the service times at the vertices. While the derivation of this convolution is straightforward in a few special cases, such as when all the travel and service times follow a normal distribution, it is in general a complex task. We propose an approach based on the family of Phase-type (PH) distributions and queueing theory to model the stochastic travel and service times and derive their convolution.

The family of PH distributions is dense on the set of continuous density functions with support on $[0,\infty)$, meaning that there exists a PH distribution arbitrarily close to any positive continuous distribution. The expected value, probability density function, and moments of a PH distribution can be found in closed form. This family of distributions also possesses a number of closure properties: the convolution of PH distributions is again a PH distribution, where the initial vector and generator matrix are easily obtained from the original parameters of the distributions; the stationary waiting time distribution in an M/PH/1 queue is PH; and the convex mixture of PH distributions is again a PH distribution.

Uninterrupted traffic flow on a road or a highway can be studied using queueing theory (Heidemann, 1996; Vandaele et al., 2000), since the relevant characteristics of a given road (arc) can be mapped to parameters of a queueing model, allowing sensitivity analysis and assessment of what-if scenarios. Parameters such as the maximum road density, the vehicle flow rate, and the free-flow speed can be employed to construct a queueing system for a segment of the road with a given length (a fraction of the length of an arc). Then, using queueing theory, we can compute performance measures such as the expected sojourn time in the road segment.

We propose to model each segment of the road as an M/M/c queue (Poisson arrivals and exponential service times with c parallel servers). In such a model, the sojourn time in the road segment consists of an exponentially distributed service phase and with a certain probability an exponentially distributed wait phase. Hence, the distribution of time across each arc can be computed using the distribution of time in each segment. This type of distribution is called a Coxian-C2 distribution, which is a class of the family of PH distributions. The service time of each customer can also be closely approximated by a PH distribution, using appropriate fitting algorithms. Therefore, the distribution of time spent traveling across a sequence of arcs (i.e., a particular route) whose travel times follow PH distributions, and visiting the customers on the vertices whose service times are approximated by PH distributions, is the convolution of a set of PH distributions which is also PH. The parameters of the resulting PH convolution can be found from the parameters of travel and service time distributions.

3 Embedding the queueing model into a routing engine

The proposed queueing model can be seen as a black box $\texttt{PH}(r)$ that takes as input a route r and computes, based on the travel time distribution of all arcs traversed by the route and the service time distribution of each customer visited by the route, the PH distribution of the total duration of r (i.e., $\tilde{T}_r \triangleq \sum_{e \in r} \tilde{t}_e + \sum_{v_i \in r} \tilde{s}_{v_i}$), namely, PH_r. Besides the customary total expected duration, PH_r provides several performance metrics in connection to the total duration of a route. For instance, it is possible to calculate the probability that the route does not exceed time limit T or the probability of arriving at a customer v_i within the time window $[t^-_{v_i}, t^+_{v_i}]$. By using $\texttt{PH}(r)$ as a *route evaluator*, our approach is able to tackle different stochastic VRP variants.

Our routing (search) engine is based on the multi-space sampling heuristic (MSH) by Mendoza and Villegas (2012). Like the original MSH, our heuristic follows a two-phase solution strategy. In the first phase, it samples multiple solution representation spaces; while in the second phase, it uses the sampled elements to build a solution to the problem on hand. The approach operates as follows. At each iteration t, the algorithm selects a *sampling heuristic* from a set $\mathcal{H}$ of randomized traveling salesman problem (TSP) heuristics and uses it to build a giant tour p^t visiting all customers. Then, the algorithm makes a call to a *splitting procedure* to retrieve a tuple $\langle \Omega^t, s^t \rangle$, where Ω^t is the set of all feasible routes that can be obtained from p^t without altering the order of the customers, and $s^t \in \mathcal{S}$ is the best solution that can be built using routes from Ω^t. To verify the satisfaction of constraints in $\mathcal{C}$ and to compute the expected duration $E[\tilde{T}_r]$ of a route r that is candidate to join Ω^t the splitting procedure invokes `PH`(r) to obtain the PH distribution of the route duration (i.e., travel plus service times). The routes in Ω^t join a set of *sampled routes* Ω, while s^t is used to update an upper bound $f(s^*)$ on the objective function of the final solution. In the assembly phase, the heuristic solves a set partitioning formulation of the underlying problem over Ω, using $f(s^*)$ as an upper bound.

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