Solutions of linear and nonlinear Partial Differential Equations with initial conditions and multivariate Faà di Bruno formula

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ABSTRACT

This paper presents a method for solving a wide class of linear and nonlinear Partial Differential Equations subject to certain initial conditions. The proposed method reduces the Partial Differential Equation and the given initial conditions into a set of equations which allows us to obtain directly and easily the solution of the initial-value problem. The particularity of the method is that it uses the multivariate Faà di Bruno formula due to Savits and Constantine [T.H. Savits, Some statistical applications of Faà di Bruno, *Journal of Multivariate Analysis*, **97** (10), 2131-2140, 2006; G.M. Constantine and T.H. Savits, A multivariate Faà di Bruno formula with applications, *Transactions of the American Mathematical Society*, **348** (2), 503-520, 1996].

Keywords: Partial Differential Equations, Linear and Nonlinear Problems, Initial Conditions, Multivariate Faà di Bruno Formula, Series and Polynomial Solutions.

1. Introduction

The purpose of this paper is to present a method for solving a large variety of linear and nonlinear Partial Differential Equations (PDEs) subject to certain initial conditions. The proposed method reduces the PDE and the given initial conditions into a set of equations which allows us to obtain directly and easily the solution of the initial-value problem. The particularity of the method presented in this paper is that it uses the multivariate Faà di Bruno (FdB) formula due to Savits and Constantine [41, 25].

In the 90's, Rach, Adomian and Meyers proposed a Modified Decomposition Method (MDM) ([7, Chapters 5 and 6], [37], [11], [13]) based on nonlinear transformation of series [9, 10]. The method presented in this work is inspired and based on the MDM.

The rest of the paper is organized as follows. The next section is devoted to preliminaries. In section 3, we recall the multivariate FdB formula due to Savits and Constantine. In section 4, the proposed method is presented. An example is solved in section 5 to illustrate the efficiency of the method. Finally, section 6 concludes this paper.

*Dedicated to the memory of Yves Cherruault, Emeritus Professor at the University Pierre and Marie Curie, Paris, France.

2. Preliminaries

We recall some notation that will be used in this paper (see [25, 41]).

Let $\mathbb{N}_0$ denotes the set of nonnegative integers, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$ and $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d$. Then define

$$|\boldsymbol{\nu}| = \sum_{i=1}^{d} \nu_i, \qquad \boldsymbol{\nu}! = \prod_{i=1}^{d} (\nu_i!), \qquad \mathbf{z}^{\boldsymbol{\nu}} = \prod_{i=1}^{d} z_i^{\nu_i},$$

$$D_{\mathbf{x}}^{\mathbf{0}} = \text{identity operator}, \quad D_{\mathbf{x}}^{\boldsymbol{\nu}} = \frac{\partial^{|\boldsymbol{\nu}|}}{\partial x_1^{\nu_1} \cdots \partial x_d^{\nu_d}} \quad \text{for } |\boldsymbol{\nu}| > 0$$

and for $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_d) \in \mathbb{N}_0^d$, $\boldsymbol{\ell} \leqslant \boldsymbol{\nu}$ if $\ell_i \leqslant \nu_i$ for $i = 1, \ldots, d$.

A function h is said to belong to $\mathcal{C}_{\boldsymbol{\nu}}(\mathbf{x}^0)$ if $D_{\mathbf{x}}^{\boldsymbol{\ell}} h$ exists and is continuous in a neighborhood of $\mathbf{x}^0$ for all $\boldsymbol{\ell} \leqslant \boldsymbol{\nu}$ and $h \in \mathcal{C}^n(\mathbf{x}^0)$ if $h \in \mathcal{C}_{\boldsymbol{\ell}}(\mathbf{x}^0)$ for all $|\boldsymbol{\ell}| \leqslant n$.

3. Multivariate Faà di Bruno formula

We now recall the multivariate Faà di Bruno formula given by Savits in [41].

Theorem 1 (T.H. Savits)**.** *(Multivariate Faà di Bruno Formula)*

Let $f(y_1, \ldots, y_m)$ and $\mathrm{g}^{(1)}(x_1, \ldots, x_d), \ldots, \mathrm{g}^{(m)}(x_1, \ldots, x_d)$ be real-valued functions and set

$$h(x_1, \ldots, x_d) = f[\mathrm{g}^{(1)}(x_1, \ldots, x_d), \ldots, \mathrm{g}^{(m)}(x_1, \ldots, x_d)]. \tag{3.1}$$

Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d)$ with $n = |\boldsymbol{\nu}| > 0$ and $\mathbf{x}^0$ be given. Assume $\mathrm{g}^{(1)}, \ldots, \mathrm{g}^{(m)} \in \mathcal{C}_{\boldsymbol{\nu}}(\mathbf{x}^0)$ and $f \in \mathcal{C}^n(\mathbf{y}^0)$, where $\mathbf{y}^0 = (\mathrm{g}^{(1)}(\mathbf{x}^0), \ldots, \mathrm{g}^{(m)}(\mathbf{x}^0))$. Let also set: $h_{\boldsymbol{\nu}}(\mathbf{x}) = D_{\mathbf{x}}^{\boldsymbol{\nu}} h(\mathbf{x}), f_{\boldsymbol{\lambda}}(\mathbf{y}) = D_{\mathbf{y}}^{\boldsymbol{\lambda}} f(\mathbf{y}), \mathrm{g}_{\boldsymbol{\mu}}^{(i)}(\mathbf{x}) = D_{\mathbf{x}}^{\boldsymbol{\mu}} \mathrm{g}^{(i)}(\mathbf{x})$ and $\mathbf{g}_{\boldsymbol{\mu}}(\mathbf{x}) = (\mathrm{g}_{\boldsymbol{\mu}}^{(1)}(\mathbf{x}), \ldots, \mathrm{g}_{\boldsymbol{\mu}}^{(m)}(\mathbf{x}))$. Under the above conditions $D_{\mathbf{x}}^{\boldsymbol{\nu}} h(\mathbf{x})$ exists in a neighborhood of $\mathbf{x}^0$ and can be explicitly expressed as below:

$$h_{\boldsymbol{\nu}}(\mathbf{x}) = \sum_{1 \leqslant |\boldsymbol{\lambda}| \leqslant |\boldsymbol{\nu}|} f_{\boldsymbol{\lambda}}[\mathbf{g}(\mathbf{x})] \sum_{p(\boldsymbol{\nu}, \boldsymbol{\lambda})} (\boldsymbol{\nu}!) \prod_{j=1}^{q} \frac{[\mathbf{g}_{\boldsymbol{\ell}_j}(\mathbf{x})]^{\mathbf{k}_j}}{(\mathbf{k}_j!)[\boldsymbol{\ell}_j!]^{|\mathbf{k}_j|}} \tag{3.2}$$

where

$$p(\boldsymbol{\nu}, \boldsymbol{\lambda}) = \left\{ (\mathbf{k}_1, \ldots, \mathbf{k}_q; \boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_q) : |\mathbf{k}_i| \geqslant 0, \ \sum_{i=1}^{q} \mathbf{k}_i = \boldsymbol{\lambda} \ \text{ and } \ \sum_{i=1}^{q} |\mathbf{k}_i| \, \boldsymbol{\ell}_i = \boldsymbol{\nu} \right\}.$$

In the above, $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_q$ is a complete listing of all vectors $\boldsymbol{\ell} \leqslant \boldsymbol{\nu}$ with $|\boldsymbol{\ell}| > 0$, $q = q(\boldsymbol{\nu}) = \left[\prod_{s=1}^{d} (\nu_s + 1) \right] - 1$, the vectors $\mathbf{k} \in \mathbb{N}_0^m$ and the vectors $\boldsymbol{\ell} \in \mathbb{N}_0^d$.

Remark 1. The form of Theorem 1 is stated somewhat different from that in Constantine and Savits [25], but it is equivalent.

Remark 2. It is to be noted that other approaches can be used for computing the multivariate Faà di Bruno Formula (see the references).

4. Presentation of the method

4.1. Description and analysis of the method

To show the basic principles and the main steps of the proposed method, we first treat the following problem:

$$(P)\quad \begin{cases} \dfrac{\partial^2 \mathrm{g}}{\partial t^2}(x,t) - \dfrac{\partial^2 \mathrm{g}}{\partial x^2}(x,t) + f[\mathrm{g}(x,t)] = E(x,t) & (4.1a) \\ \mathrm{g}(x,0) = \phi(x) & (4.1b) \\ \dfrac{\partial \mathrm{g}}{\partial t}(x,0) = \psi(x) & (4.1c) \end{cases}$$

where $E(x,t)$, $\phi(x)$ and $\psi(x)$ are known functions and $f[\mathrm{g}(x,t)]$ is a nonlinear function of $\mathrm{g}(x,t)$. In the above problem (P), we have: $m = 1$, $d = 2$, $x_1 = x$ and $x_2 = t$.

4.1.1. Method of solution

Let us assume that the function $\mathrm{g}(x,t)$ can be expanded in the form

$$\mathrm{g}(x,t) = \sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} a_{(\nu_1,\nu_2)}\, x^{\nu_1} t^{\nu_2} \tag{4.2}$$

where

$$a_{(\nu_1,\nu_2)} = \frac{1}{\nu_1!\nu_2!}\frac{\partial^{\nu_1+\nu_2}}{\partial x^{\nu_1}\partial t^{\nu_2}}\,\mathrm{g}(x,t)\bigg|_{x=0,t=0}$$

In order to find the solution of the problem (P) in the form (4.2), the unknown values of the coefficients $a_{(\nu_1,\nu_2)}$ have to be found.

4.1.1.1. Treatment of the PDE

It follows from (4.2) that

$$\frac{\partial^2 \mathrm{g}}{\partial x^2}(x,t) = \sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}(\nu_1+1)(\nu_1+2)\, a_{(\nu_1+2,\nu_2)}\, x^{\nu_1} t^{\nu_2} \tag{4.3}$$

$$\frac{\partial^2 \mathrm{g}}{\partial t^2}(x,t) = \sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}(\nu_2+1)(\nu_2+2)\, a_{(\nu_1,\nu_2+2)}\, x^{\nu_1} t^{\nu_2} \tag{4.4}$$

We assume that $h(x,t) = f[\mathrm{g}(x,t)]$ can be expanded in the form

$$f[\mathrm{g}(x,t)] = \sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} A_{(\nu_1,\nu_2)}\, x^{\nu_1} t^{\nu_2} \tag{4.5}$$

where

$$A_{(0,0)} = h(0,0) = f[\mathrm{g}(0,0)] = f[a_{(0,0)}]$$

and for $(\nu_1,\nu_2) \neq (0,0)$

$$\begin{aligned} A_{(\nu_1,\nu_2)} &= \frac{1}{\nu_1!\nu_2!}\, h_{(\nu_1,\nu_2)}(x,t)\bigg|_{x=0,t=0} \\ &= \frac{1}{\nu_1!\nu_2!}\frac{\partial^{\nu_1+\nu_2}}{\partial x^{\nu_1}\partial t^{\nu_2}}\, h(x,t)\bigg|_{x=0,t=0} \\ &= \frac{1}{\nu_1!\nu_2!}\frac{\partial^{\nu_1+\nu_2}}{\partial x^{\nu_1}\partial t^{\nu_2}}\, f[\mathrm{g}(x,t)]\bigg|_{x=0,t=0} \end{aligned}$$

In the above, $h_{(\nu_1,\nu_2)}(x,t) = \dfrac{\partial^{\nu_1+\nu_2}}{\partial x^{\nu_1}\partial t^{\nu_2}}\, f[\mathrm{g}(x,t)]$ is given by (3.2).

We assume that the function $E(x,t)$ can be expanded in the form

$$E(x,t) = \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} e_{(\nu_1,\nu_2)}\, x^{\nu_1} t^{\nu_2} \tag{4.6}$$

where

$$e_{(\nu_1,\nu_2)} = \frac{1}{\nu_1!\nu_2!} \frac{\partial^{\nu_1+\nu_2}}{\partial x^{\nu_1} \partial t^{\nu_2}}\, E(x,t)\Big|_{x=0,t=0}$$

Then, by substituting (4.3), (4.4), (4.5) and (4.6) in $(4.1a)$, we get

$$\sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} (\nu_2+1)(\nu_2+2)\, a_{(\nu_1,\nu_2+2)}\, x^{\nu_1} t^{\nu_2} - \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} (\nu_1+1)(\nu_1+2)\, a_{(\nu_1+2,\nu_2)}\, x^{\nu_1} t^{\nu_2}$$
$$+ \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} A_{(\nu_1,\nu_2)}\, x^{\nu_1} t^{\nu_2} = \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} e_{(\nu_1,\nu_2)}\, x^{\nu_1} t^{\nu_2}$$

and by equating like powers, we obtain

$$(\nu_2+1)(\nu_2+2)\, a_{(\nu_1,\nu_2+2)} - (\nu_1+1)(\nu_1+2)\, a_{(\nu_1+2,\nu_2)} + A_{(\nu_1,\nu_2)} = e_{(\nu_1,\nu_2)}, \quad \nu_1,\nu_2 = 0,1,2,\ldots \tag{4.7}$$

4.1.1.2. Treatment of the initial conditions

It follows from (4.2) that

$$\mathrm{g}(x,0) = \sum_{\nu_1=0}^{\infty} a_{(\nu_1,0)}\, x^{\nu_1} \tag{4.8}$$

$$\frac{\partial \mathrm{g}}{\partial t}(x,0) = \sum_{\nu_1=0}^{\infty} a_{(\nu_1,1)}\, x^{\nu_1} \tag{4.9}$$

We assume that the functions $\phi(x)$ and $\psi(x)$ can be expanded as

$$\phi(x) = \sum_{\nu_1=0}^{\infty} \phi_{\nu_1} x^{\nu_1} \tag{4.10}$$

$$\psi(x) = \sum_{\nu_1=0}^{\infty} \psi_{\nu_1} x^{\nu_1} \tag{4.11}$$

Substituting (4.8) and (4.10) into $(4.1b)$ leads to

$$\sum_{\nu_1=0}^{\infty} a_{(\nu_1,0)}\, x^{\nu_1} = \sum_{\nu_1=0}^{\infty} \phi_{\nu_1} x^{\nu_1}$$

consequently

$$a_{(\nu_1,0)} = \phi_{\nu_1}, \quad \nu_1 = 0,1,2,\ldots \tag{4.12}$$

Substituting (4.9) and (4.11) into $(4.1c)$ leads to

$$\sum_{\nu_1=0}^{\infty} a_{(\nu_1,1)}\, x^{\nu_1} = \sum_{\nu_1=0}^{\infty} \psi_{\nu_1} x^{\nu_1}$$

consequently

$$a_{(\nu_1,1)} = \psi_{\nu_1}, \quad \nu_1 = 0,1,2,\ldots \tag{4.13}$$

4.1.1.3. The problem solution

By grouping together (4.7), (4.12) and (4.13), we get

$$(A)\quad \left[\begin{array}{l} a_{(\nu_1,0)} = \phi_{\nu_1}, \quad \nu_1 = 0,1,2,\dots \\ \\ a_{(\nu_1,1)} = \psi_{\nu_1}, \quad \nu_1 = 0,1,2,\dots \\ \\ (\nu_2+1)(\nu_2+2)\, a_{(\nu_1,\nu_2+2)} - (\nu_1+1)(\nu_1+2)\, a_{(\nu_1+2,\nu_2)} + A_{(\nu_1,\nu_2)} = e_{(\nu_1,\nu_2)}, \quad \nu_1,\nu_2 = 0,1,2,\dots \end{array}\right.$$

(A) allows us to obtain the values of the coefficients $a_{(\nu_1,\nu_2)}$. Then, by substituting these values in (4.2), we obtain the exact solution of the problem (P). However, in practice, we use an approximation of the solution of the problem (P) in the form

$$\mathrm{g}(x,t) \approx \mathrm{g}_{\mathrm{M,N}}(x,t) = \sum_{\nu_1=0}^{M-1}\sum_{\nu_2=0}^{N-1} a_{(\nu_1,\nu_2)}\, x^{\nu_1} t^{\nu_2} \tag{4.14}$$

Remark 3. The accuracy of the approximation (4.14) can be improved by increasing the values of M and N.

5. Application

For a better understanding of the proposed method and to illustrate the solution procedure described in the preceding section, we have selected the following example which shows the simplicity and the efficiency of the proposed technique.

5.1. Example

In what follows, the method presented in section 4 will be used to solve the following problem taken from [44, 45].

Solve the following nonlinear Klein-Gordon equation:

$$(P_1)\quad \begin{cases} \dfrac{\partial^2 \mathrm{g}}{\partial t^2}(x,t) - \dfrac{\partial^2 \mathrm{g}}{\partial x^2}(x,t) + [\mathrm{g}(x,t)]^2 = x^2 t^2 & (5.1a) \\ \mathrm{g}(x,0) = 0 & (5.1b) \\ \dfrac{\partial \mathrm{g}}{\partial t}(x,0) = x & (5.1c) \end{cases}$$

5.1.1. Method of solution

Let

$$\mathrm{g}(x,t) = \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} a_{(\nu_1,\nu_2)}\, x^{\nu_1} t^{\nu_2} \tag{5.2}$$

5.1.1.1. Treatment of the PDE

It follows from (5.2) that

$$\frac{\partial^2 \mathrm{g}}{\partial x^2}(x,t) = \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} (\nu_1+1)(\nu_1+2)\, a_{(\nu_1+2,\nu_2)}\, x^{\nu_1} t^{\nu_2} \tag{5.3}$$

$$\frac{\partial^2 \mathrm{g}}{\partial t^2}(x,t) = \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} (\nu_2+1)(\nu_2+2)\, a_{(\nu_1,\nu_2+2)}\, x^{\nu_1} t^{\nu_2} \tag{5.4}$$

$$[\mathrm{g}(x,t)]^2 = \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} A_{(\nu_1,\nu_2)}\, x^{\nu_1} t^{\nu_2} \tag{5.5}$$

where

$$A_{(0,0)} = [a_{(0,0)}]^2$$

and for $(\nu_1,\nu_2) \neq (0,0)$

$$A_{(\nu_1,\nu_2)} = \frac{1}{\nu_1!\nu_2!} \frac{\partial^{\nu_1+\nu_2}}{\partial x^{\nu_1} \partial t^{\nu_2}} [\mathrm{g}(x,t)]^2 \Big|_{x=0,t=0}$$

In the above, $\dfrac{\partial^{\nu_1+\nu_2}}{\partial x^{\nu_1} \partial t^{\nu_2}} [\mathrm{g}(x,t)]^2$ is given by (3.2). Consequently, by using (3.2), we obtain

$$\begin{aligned} A_{(0,1)} &= 2a_{(0,0)}a_{(0,1)} \\ A_{(0,2)} &= 2a_{(0,0)}a_{(0,2)} + [a_{(0,1)}]^2 \\ &\vdots \\ A_{(1,0)} &= 2a_{(0,0)}a_{(1,0)} \\ A_{(1,1)} &= 2a_{(0,0)}a_{(1,1)} + 2a_{(0,1)}a_{(1,0)} \\ &\vdots \\ A_{(2,0)} &= 2a_{(0,0)}a_{(2,0)} + [a_{(1,0)}]^2 \end{aligned}$$

and so on.

Then, by substituting (5.3), (5.4) and (5.5) in (5.1a), we get

$$\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}(\nu_2+1)(\nu_2+2)\,a_{(\nu_1,\nu_2+2)}\,x^{\nu_1}t^{\nu_2}-\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}(\nu_1+1)(\nu_1+2)\,a_{(\nu_1+2,\nu_2)}\,x^{\nu_1}t^{\nu_2}$$
$$+\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}A_{(\nu_1,\nu_2)}\,x^{\nu_1}t^{\nu_2}=x^2t^2.$$

and by equating like powers, we obtain

$$(\nu_2+1)(\nu_2+2)\,a_{(\nu_1,\nu_2+2)}-(\nu_1+1)(\nu_1+2)\,a_{(\nu_1+2,\nu_2)}+A_{(\nu_1,\nu_2)}=\begin{cases}1 & \text{for } \nu_1=\nu_2=2,\\ 0 & \text{otherwise.}\end{cases} \tag{5.6}$$

5.1.1.2. Treatment of the initial conditions

It follows from (5.2) that

$$\mathrm{g}(x,0)=\sum_{\nu_1=0}^{\infty}a_{(\nu_1,0)}\,x^{\nu_1} \tag{5.7}$$
$$\frac{\partial \mathrm{g}}{\partial t}(x,0)=\sum_{\nu_1=0}^{\infty}a_{(\nu_1,1)}\,x^{\nu_1} \tag{5.8}$$

Substituting (5.7) into $(5.1b)$ leads to

$$\sum_{\nu_1=0}^{\infty}a_{(\nu_1,0)}\,x^{\nu_1}=0$$

consequently

$$a_{(\nu_1,0)}=0,\quad \nu_1=0,1,2,\ldots \tag{5.9}$$

Substituting (5.8) into $(5.1c)$ leads to

$$\sum_{\nu_1=0}^{\infty}a_{(\nu_1,1)}\,x^{\nu_1}=x$$

consequently

$$a_{(\nu_1,1)}=\begin{cases}1 & \text{for } \nu_1=1,\\ 0 & \text{otherwise.}\end{cases} \tag{5.10}$$

5.1.1.3. The problem solution

By grouping together (5.6), (5.9) and (5.10), we get

$$(B)\quad\left[\begin{array}{l} a_{(\nu_1,0)}=0,\quad \nu_1=0,1,2,\ldots \\ a_{(\nu_1,1)}=\begin{cases}1 & \text{for } \nu_1=1,\\ 0 & \text{otherwise.}\end{cases} \\ (\nu_2+1)(\nu_2+2)\,a_{(\nu_1,\nu_2+2)}-(\nu_1+1)(\nu_1+2)\,a_{(\nu_1+2,\nu_2)}+A_{(\nu_1,\nu_2)}=\begin{cases}1 & \text{for } \nu_1=\nu_2=2,\\ 0 & \text{otherwise.}\end{cases}\end{array}\right.$$

(B) allows us to obtain directly and easily the values of all the coefficients $a_{(\nu_1,\nu_2)}$ as

$$a_{(\nu_1,\nu_2)}=\begin{cases}1 & \text{for } (\nu_1,\nu_2)=(1,1),\\ 0 & \text{otherwise.}\end{cases}$$

Finally, by substituting the above values in (5.2), we obtain

$$\mathrm{g}(x,t)=xt \tag{5.11}$$

which is the exact solution of the problem (P_1).

6. Conclusion

In this paper, we have presented a method using the multivariate FdB formula due to Savits and Constantine for solving a large variety of linear and nonlinear PDEs subject to certain initial conditions. It has been shown that the proposed technique is simple, efficient and produces accurate results. Moreover, the method can also be used to solve PDEs with variable coefficients.

The author welcomes comments, suggestions and corrections.

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