HAL open science

Energy decay for Maxwell's equations with Ohm's law in partially cubic domains

Kim Dang Phung

To cite this version:

Kim Dang Phung. Energy decay for Maxwell's equations with Ohm's law in partially cubic domains. Communications on Pure and Applied Analysis, 2013, 12 (5), pp.2229-2266. 10.3934/cpaa.2013.12.2229 . hal-00845761

HAL Id: hal-00845761

https://hal.science/hal-00845761

Submitted on 17 Jul 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Energy decay for Maxwell's equations with Ohm's law in partially cubic domains

Kim Dang Phung
Université d'Orléans, Laboratoire MAPMO, CNRS UMR 7349,
Fédération Denis Poisson, FR CNRS 2964,
Bâtiment de Mathématiques
B.P. 6759, 45067 Orléans Cedex 2, France.
E-mail: kim_dang_phung@yahoo.fr

Abstract .- We prove a polynomial energy decay for the Maxwell's equations with Ohm's law in partially cubic domains with trapped rays. We extend the results of polynomial decay for the scalar damped wave equation in partially rectangular or cubic domain. Our approach have some similitude with the construction of reflected gaussian beams.

Keywords .- Maxwell's equations; decay estimates; trapped ray.

1 Introduction and main result

The problems dealing with Maxwell's equations with nonzero conductivity are not only theoretical interesting but also very important in many industrial applications (see e.g. [3], [7], [8], [14]). This paper is concerned with the energy decay of Maxwell's equations with Ohm's law in a bounded cylinder $\Omega \subset \mathbb{R}^3$ with trapped rays. Precisely, let $\rho > 0$ and D be an open simply connected bounded set in $\mathbb{R}^2$ with C^2 boundary ∂D. Consider $\Omega = D \times (-\rho, \rho)$ with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ where $\Gamma_0 = \overline{D} \times \{-\rho, \rho\}$ and $\Gamma_1 = \partial D \times (-\rho, \rho)$. The domain Ω is occupied by an electromagnetic medium of constant electric permittivity ε_o and constant magnetic permeability μ_o. For the sake of simplicity, we assume from now that $\varepsilon_o \mu_o = 1$.

Let E and H denote the electric and magnetic fields respectively. Define the energy by

$$\mathcal{E}(t) = \frac{1}{2} \int_\Omega \left(\varepsilon_o \left| E(x,t) \right|^2 + \mu_o \left| H(x,t) \right|^2 \right) dx \ . \tag{1.1}$$

The Maxwell's equations with Ohm's law are described by

$$\left\{ \begin{array}{rl} \varepsilon_o \partial_t E - \operatorname{curl} H + \sigma E = 0 & \text{in } \Omega \times [0,+\infty) \\ \mu_o \partial_t H + \operatorname{curl} E = 0 & \text{in } \Omega \times [0,+\infty) \\ \operatorname{div}(\mu_o H) = 0 & \text{in } \Omega \times [0,+\infty) \\ E \times \nu = H \cdot \nu = 0 & \text{on } \partial\Omega \times [0,+\infty) \\ (E,H)(\cdot,0) = (E_o, H_o) & \text{in } \Omega \ . \end{array} \right. \tag{1.2}$$

Here, (E_o, H_o) is the initial data in the energy space $L^2(\Omega)^6$ and ν denotes the outward unit normal vector to $\partial\Omega$. The conductivity σ is such that $\sigma \in L^\infty(\Omega)$ and $\sigma \geq 0$. It is well-known that when the conductivity is identically null, then the above system is conservative and when σ is bounded from below by a positive constant, then an exponential energy decay rate holds for the Maxwell's equations with Ohm's law in the energy space. The situation becomes more delicate if the conductivity is locally active, i.e., when the following constraint holds

$$\sigma = 1_{|\omega}\, a \text{ where } \omega \subsetneq \Omega, a \in L^{\infty}(\Omega) \text{ and } a \geq \textit{constant} > 0 \ .$$

Here and hereafter, we denote by $1_{|\cdot}$ the characteristic function of a set in the place where $\cdot$ stays. From the works of Bardos, Lebeau and Rauch [2] for the scalar wave operator, there is a geometric condition on the location of ω which implies an exponential decay rate for the Maxwell's equations with Ohm's law in the energy space. In our paper, ω is a non-empty connected open subset ω of Ω as follows. Consider

$$\omega = \left\{ x' \in D; \inf_{y' \in \partial D} |x' - y'| < r_o \right\} \times (-\rho, \rho) \text{ for some small } r_o \in (0, \rho/2) \ .$$

Such case of ω does not satisfy the geometric condition of the works of Bardos, Lebeau and Rauch [2] and we do not hope an exponential decay rate for the the Maxwell's equations with Ohm's law in the energy space.

Our geometry presents parallel trapped rays and can be compared to the one in [12] or in [4],[10] for the two dimensional case. It generalizes the cube (see [8]) and therefore explicit and analytical results are harder to obtain. Our main result is as follows.

Theorem 1 .- *If* $\sigma = 1_{|\omega}\, a$ where $a \in L^{\infty}(\Omega)$ and $a \geq \textit{constant} > 0$, *then there exist* $c > 0$ *and* $\gamma > 0$ *such that for any* $t \geq 0$

$$\mathcal{E}(t) \leq \frac{c}{t^{\gamma}} \left(\mathcal{E}(0) + \left\| (\operatorname{curl} E_o, \operatorname{curl} H_o) \right\|^2_{L^2(\Omega)^6} \right)$$

for every solution of the system (1.2) of Maxwell's equations with Ohm's law with initial data $(E_o, H_o) \in L^2(\Omega)^6$ *such that*

$$\left\{ \begin{array}{rl} (\operatorname{curl} E_o, \operatorname{curl} H_o) \in L^2(\Omega)^6 & , \\ \operatorname{div} H_o = 0 & \textit{in } \Omega \\ E_o \times \nu = H_o \cdot \nu = 0 & \textit{on } \partial\Omega \\ \operatorname{div}\left(1_{|\Omega \backslash \omega}\, E_o\right) = 0 & \textit{in } \Omega \backslash \omega \ . \end{array} \right.$$

In literature, the exponential energy decay rate of a linear dissipative system can be deduced from an observability estimate with (1.1). Precisely, in order to get an exponential decay rate in the energy space we should have the following observability inequality

$$\exists C, T_c > 0 \quad \forall \zeta \geq 0 \quad \int_{\Omega} |(E, H)(\cdot, \zeta)|^2 \, dx \leq C \int_{\zeta}^{\zeta + T_c} \int_{\Omega} \sigma |E|^2 \, dx dt$$

or simply, in virtue of a semigroup property,

$$\exists C, T_c > 0 \quad \int_{\Omega} |(E_o, H_o)|^2 \, dx \leq C \int_0^{T_c} \int_{\Omega} \sigma |E|^2 \, dx dt$$

for any initial data (E_o, H_o) in the energy space $L^2(\Omega)^6$. We can also look for establishing the above observability inequality for any initial data in the energy space intersecting suitable invariant subspaces. Such estimate is established in [11] under the geometric control condition of Bardos, Lebeau and Rauch [2] for the scalar wave operator on (Ω, ω) and when the conductivity has the property that $\sigma(x) \geq \textit{constant} > 0$ for all $x \in \omega$ and $\sigma(x) = 0$ for all $x \in \Omega \backslash \overline{\omega}$. Now, our main result gives a polynomial energy decay with regular initial data when the conductivity is only active on a neighborhood of the lateral boundary. Our proof is based on a new kind of observation inequality (see (4.1) below) which can also be seen as an interpolation estimate. It relies with the construction of a particular solution for the operator $i\partial_s + h\left(\Delta - \partial_t^2\right)$ inspired by the gaussian beam techniques (see [13] and [15]).

The plan of the paper is as follows. In the next section, we recall the known results about the Maxwell's equations with Ohm's law that will be used. Section 3 contains the proof of our main result, while Section 4 is concerned with the interpolation estimate. In Section 5, we present an interpolation estimate with weight functions, while Section 6 includes its proof. Finally, two appendix are added dealing with inequalities involving Fourier analysis and the Poisson summation formula.

2 The Maxwell's equations with Ohm's law

We begin to recall some well-known results concerning the Maxwell's equations with Ohm's law: well-posedness, energy identity, standard orthogonal decomposition and asymptotic behaviour in time of the energy of the electromagnetic field.

2.1 Well-posedness of the problem

Let us introduce the spaces

$$\mathcal{V} = L^2(\Omega)^3 \times \left\{ G \in L^2(\Omega)^3 ;\ \operatorname{div} G = 0,\ G \cdot \nu_{|\partial\Omega} = 0 \right\} , \tag{2.1.1}$$

$$\begin{aligned} \mathcal{W} = \Big\{ (F,G) \in L^2(\Omega)^6 ;\ \operatorname{curl} F \in L^2(\Omega)^3 ,\ F \times \nu_{|\partial\Omega} = 0, \\ \operatorname{div} G = 0,\ G \cdot \nu_{|\partial\Omega} = 0,\ \operatorname{curl} G \in L^2(\Omega)^3 \Big\} . \end{aligned} \tag{2.1.2}$$

It is well-known that if $(E_o, H_o) \in \mathcal{V}$, there is a unique weak solution $(E,H) \in C^0([0,+\infty), \mathcal{V})$. Further, if $(E_o, H_o) \in \mathcal{W}$, there is a unique strong solution $(E,H) \in C^0([0,+\infty), \mathcal{W}) \cap C^1([0,+\infty), \mathcal{V})$. We can easily check that the energy $\mathcal{E}$, defined by (1.1), is a continuous positive non-increasing real function on $[0,+\infty)$. Further for any initial data $(E_o, H_o) \in \mathcal{W}$ and any $t_2 > t_1 \geq 0$,

$$\mathcal{E}(t_2) - \mathcal{E}(t_1) + \int_{t_1}^{t_2} \int_\Omega \sigma(x) \left|E(x,t)\right|^2 dxdt = 0 \tag{2.1.3}$$

and

$$\mathcal{E}_1(t_2) - \mathcal{E}_1(t_1) + \int_{t_1}^{t_2} \int_\Omega \sigma(x) \left|\partial_t E(x,t)\right|^2 dxdt = 0 , \tag{2.1.4}$$

where

$$\mathcal{E}_1(t) = \frac{1}{2} \int_\Omega \left(\varepsilon_o \left|\partial_t E(x,t)\right|^2 + \mu_o \left|\partial_t H(x,t)\right|^2 \right) dx . \tag{2.1.5}$$

2.2 Orthogonal decomposition

Both E and $\mu_o H$ can be described, by means of the scalar and vector potentials p and A with the Coulomb gauge, in a unique way as follows.

Proposition 2.1 -. *For any $(E_o, H_o) \in \mathcal{W}$, there is a unique $(p, A) \in C^1\left([0,+\infty), H_0^1(\Omega)\right) \times C^2\left([0,+\infty), H^1(\Omega)^3\right)$ such that the solution (E, H) of the Maxwell's equations with Ohm's law (1.2) satisfies*

$$\left\{ \begin{array}{l} E = -\nabla p - \partial_t A \\ \mu_o H = \operatorname{curl} A \end{array} \right. \tag{2.2.1}$$

$$\left\{ \begin{array}{rl} \varepsilon_o \mu_o \partial_t^2 A + \operatorname{curl}\operatorname{curl} A = \mu_o\left(-\varepsilon_o \partial_t \nabla p + \sigma E\right) & \text{in } \Omega \times [0,+\infty) \\ \operatorname{div} A = 0 & \text{in } \Omega \times [0,+\infty) \\ A \times \nu = 0 & \text{on } \partial\Omega \times [0,+\infty) \end{array} \right. \tag{2.2.2}$$

and we have the following relations

$$\|E\|^2_{L^2(\Omega)^3} = \|\nabla p\|^2_{L^2(\Omega)^3} + \|\partial_t A\|^2_{L^2(\Omega)^3} \ , \tag{2.2.3}$$

$$\|\varepsilon_o \partial_t \nabla p\|_{L^2(\Omega)^3} \leq \|\sigma E\|_{L^2(\Omega)^3} \ , \tag{2.2.4}$$

$$\exists c > 0 \qquad \|A\|^2_{L^2(\Omega)^3} \leq c \|\operatorname{curl} A\|^2_{L^2(\Omega)^3} \ . \tag{2.2.5}$$

Further, since $\operatorname{curl} H \in L^2(\Omega)^3$, $\operatorname{curl}\operatorname{curl} A \in L^2(\Omega)^3$ *and* $\operatorname{div} A \in H_0^1(\Omega)$.

The proof is essentially given in [11, p. 121] from a Hodge decomposition and is omitted here. Now, the vector field A has the nice property of free divergence and satisfies a second order vector wave equation with homogeneous boundary condition $A \times \nu = \operatorname{div} A = 0$ and with a second member in $C^1\left([0,+\infty), L^2(\Omega)^3\right)$ bounded by $2\mu_o \|\sigma E\|_{L^2(\Omega)^3}$.

2.3 Invariant subspaces, asymptotic behavior and exponential energy decay

Let ω_+ be a non-empty connected open subset of Ω with a Lipschitz boundary $\partial\omega_+$. In this subsection we suppose that

$$\sigma = 1_{|\omega_+} a \text{ where } a \in L^\infty(\Omega) \text{ and } a \geq constant > 0 \ .$$

Denote $\omega_- = \Omega \backslash \overline{\omega_+}$ and suppose that its boundary $\partial\omega_-$ is Lipschitz and has no more than two connected components γ_1, γ_2.

We recall that the range of the curl, $\operatorname{curl} H^1(\omega_-)^3$, is closed in $L^2(\omega_-)^3$ (see [6, p. 257] or [5, p. 54]) and

$$\operatorname{curl} H^1(\omega_-)^3 = \left\{ U \in L^2(\omega_-)^3 ; \ \operatorname{div} U = 0 \text{ in } \omega_-, \ \int_{\gamma_i} U \cdot \nu = 0 \quad \text{for } i \in \{1,2\} \right\} \ . \tag{2.3.1}$$

Its orthogonal space for the $\left(L^2(\omega_-)\right)^3$ norm is

$$\left(\operatorname{curl} H^1(\omega_-)^3\right)^\perp = \left\{ V \in L^2(\omega_-)^3 ; \ \operatorname{curl} V = 0 \text{ in } \omega_-, \ V \times \nu = 0 \text{ on } \partial\omega_- \right\} \ . \tag{2.3.2}$$

Let us introduce $\mathcal{S}_\sigma = \left(\operatorname{curl} H^1(\omega_-)^3 \cap L^2(\Omega)^3\right) \times L^2(\Omega)^3$. The space $\mathcal{W} \cap \mathcal{S}_\sigma$ is stable for the system of Maxwell's equations with Ohm's law, which can be seen by multiplying by $V \in$

$\left(\operatorname{curl} H^1(\omega_-)^3\right)^\perp$ the equation $\varepsilon_o \partial_t E - \operatorname{curl} H + \sigma E = 0$. Then, we can add the following well-posedness result. If $(E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_\sigma$, there is a unique solution $(E, H) \in C^0([0,+\infty), \mathcal{W} \cap \mathcal{S}_\sigma) \cap C^1([0,+\infty), \mathcal{V} \cap \mathcal{S}_\sigma)$.

It has been proved (see [11, p. 124]) that if ω_- is a non-empty connected open set then $\lim_{t\to+\infty} \mathcal{E}(t) = 0$ for any initial data $(E_o, H_o) \in \mathcal{V} \cap \mathcal{S}_\sigma$. Further, the following result (see [11, p. 124]) plays a key role.

Proposition 2.2 -. *If ω_- is a non-empty connected open set, then there exists $c > 0$ such that for all initial data $(E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_\sigma$ of the system (1.2) of Maxwell's equations with Ohm's law, we have*

$$\forall t \geq 0 \quad \mathcal{E}(t) \leq c\mathcal{E}_1(t) \ . \tag{2.3.3}$$

Remark 2.3 -. The estimate (2.3.3) is still true under the assumption $\partial\omega_+ \cap \partial\Omega \neq \emptyset$ and without adding the hypothesis saying that ω_- is a connected set. Indeed, the proof given in [11, p. 127] can be divided into two steps. In the first step, we begin to establish the existence of $c > 0$ such that

$$\|\nabla p\|^2_{L^2(\omega_+)^3} + \|\partial_t A\|^2_{L^2(\Omega)^3} + \|H\|^2_{L^2(\Omega)^3} \leq c\left(\mathcal{E}_1(t) + \sqrt{\mathcal{E}(t)}\sqrt{\mathcal{E}_1(t)}\right) \tag{2.3.4}$$

for any $(E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_\sigma$. Here, we used a standard compactness-uniqueness argument for H, (2.2.5) of Proposition 2.1 for $\partial_t A$, and for ∇p from the fact that $\sigma(x) \geq \mathit{constant} > 0$ for all $x \in \omega_+$ and (2.1.5). Till now, we did not utilize that ω_- is a connected set. The second step (see [11, p. 128]) did consist to prove that

$$\|\nabla p\|^2_{L^2(\omega_-)^3} \leq c\left(\|\nabla p\|^2_{L^2(\omega_+)^3} + \|\partial_t A\|^2_{L^2(\Omega)^3}\right) \ . \tag{2.3.5}$$

Finally, we concluded by virtue of (2.2.3) of Proposition 2.1. This last estimate becomes easier to obtain under the assumption $\partial\omega_+ \cap \partial\Omega \neq \emptyset$ and without adding the hypothesis saying that ω_- is a connected set. Indeed, since $(E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_\sigma$ and $-\Delta p = \operatorname{div} E$, $p \in H^1_0(\Omega)$ solves the following elliptic system

$$\begin{cases} \Delta p = 0 \quad \text{in } \omega_- \\ p_{|\partial\omega_-} \in H^{1/2}(\partial\omega_-) \\ p = 0 \quad \text{on } \partial\omega_+ \cap \partial\Omega \ . \end{cases} \tag{2.3.6}$$

Thus, by the elliptic regularity, the trace theorem and the Poincaré inequality, we have the following estimate

$$\|\nabla p\|_{L^2(\omega_-)^3} \leq c_1 \|p\|_{H^{1/2}(\partial\omega_-)} \leq c_2 \|p\|_{H^{1/2}(\partial\omega_+)} \leq c_3 \|\nabla p\|_{L^2(\omega_+)^3} \tag{2.3.7}$$

for suitable constants $c_1, c_2, c_3 > 0$. Hence, combining (2.3.4) and (2.3.7) with (2.2.5), (2.3.3) follows if $\partial\omega_+ \cap \partial\Omega \neq \emptyset$ and without adding the hypothesis saying that ω_- is a connected set.

The exponential energy decay rate for the Maxwell's equations with Ohm's law in the energy space is as follows.

Proposition 2.4 -. *Assume that*

(i) $\sigma = 1_{|\omega_+} a$ where $a \in L^\infty(\Omega)$ and $a \geq \mathit{constant} > 0$;

(ii) $\partial\omega_+ \cap \partial\Omega \neq \emptyset$ or ω_- is a non-empty connected open set;

(iii) at any point in ω_- and any direction, the generalized ray of the scalar wave operator $\partial_t^2 - \Delta$ is uniquely defined.

Let ϑ be a subset of Ω such that any generalized ray of the scalar wave operator $\partial_t^2 - \Delta$ meets $\overline{\vartheta}$. If $\overline{\vartheta} \cap \Omega \subset \omega_+$, then there exist $c > 0$ and $\beta > 0$ such that for any $t \geq 0$, we have

$$\mathcal{E}(t) \leq ce^{-\beta t}\mathcal{E}(0) \ , \tag{2.3.8}$$

for every solution of the system (1.2) of Maxwell's equations with Ohm's law with initial data $(E_o, H_o) \in \mathcal{V} \cap \mathcal{S}_\sigma$.

The proof of Proposition 2.4 is done in [11, p. 129] when ω_- is a non-empty connected open set. Here, we only recall the key points of the proof. From the geometric control condition, the following estimate holds without using the fact that ω_- is a non-empty connected open set.

$$\exists C, T_c > 0 \quad \forall \zeta \geq 0 \quad \mathcal{E}_1(\zeta) \leq C \int_{\zeta}^{T_c+\zeta} \int_{\Omega} \left(\sigma(x) \left| \partial_t E(x,t) \right|^2 + \sigma(x) \left| E(x,t) \right|^2 \right) dxdt \tag{2.3.9}$$

for any initial data $(E_o, H_o) \in \mathcal{W} \cap \mathcal{S}_\sigma$. Next by (2.3.3), we deduced that

$$\exists C, T_c > 0 \quad \forall \zeta \geq 0 \quad \mathcal{E}(\zeta) + \mathcal{E}_1(\zeta) \leq C \int_{\zeta}^{T_c+\zeta} \int_{\Omega} \left(\sigma(x) \left| \partial_t E(x,t) \right|^2 + \sigma(x) \left| E(x,t) \right|^2 \right) dxdt \ . \tag{2.3.10}$$

Finally, we concluded by virtue of a semigroup property with (2.1.3) and (2.1.4). Notice that no assumption $\operatorname{div}(E(\cdot,\zeta)) = 0$ in Ω is set up. This is important because the free divergence of the electric field is not preserved by the Maxwell's equations with Ohm's law. Now, thanks to Remark 2.3, the proof works as well when $\partial\omega_+ \cap \partial\Omega \neq \emptyset$ and without adding the hypothesis saying that ω_- is a connected set

3 Proposition 3.1 and proof of Theorem 1

Let us consider the solution U of the following system

$$\left\{ \begin{array}{rl} \partial_t^2 U + \operatorname{curl}\operatorname{curl} U = 0 & \text{in } \Omega \times \mathbb{R} \\ \operatorname{div} U = 0 & \text{in } \Omega \times \mathbb{R} \\ U \times \nu = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ \left(U(\cdot,0), \partial_t U(\cdot,0) \right) = \left(U^0, U^1 \right) & \text{in } \Omega \ , \\ \left(U^0, U^1 \right) \in \mathcal{X} & , \\ \left(U^1, \operatorname{curl}\operatorname{curl} U^0 \right) \in \mathcal{X} & , \end{array} \right. \tag{3.1}$$

where

$$\mathcal{X} = \left\{ (F,G) \in L^2(\Omega)^6 ; \ \operatorname{curl} F \in L^2(\Omega)^3 , F \times \nu_{|\partial\Omega} = 0, \operatorname{div} F = 0, \operatorname{div} G = 0 \right\} \ . \tag{3.2}$$

It is well-known that the above system is well-posed with a unique solution U so that $(U(\cdot,t), \partial_t U(\cdot,t))$ and $\left(\partial_t U(\cdot,t), \partial_t^2 U(\cdot,t)\right)$ belong to $\mathcal{X}$ for any $t \in \mathbb{R}$. Let us define the following two conservations of energies.

$$\mathcal{G}(U,0) = \mathcal{G}(U,t) \equiv \int_{\Omega} \left(\left| \partial_t U(x,t) \right|^2 + \left| \operatorname{curl} U(x,t) \right|^2 \right) dx \ , \tag{3.3}$$

$$\mathcal{G}(\partial_t U,0) = \mathcal{G}(\partial_t U,t) \equiv \int_{\Omega} \left(\left| \operatorname{curl}\operatorname{curl} U(x,t) \right|^2 + \left| \operatorname{curl} \partial_t U(x,t) \right|^2 \right) dx \ . \tag{3.4}$$

Further, for such solution U, the following two inequalities hold by standard compactness-uniqueness argument and classical embedding (see [1] and [5, p. 50]).

$$\mathcal{G}(U,t) \leq c\mathcal{G}(\partial_t U,t) \ , \tag{3.5}$$

$$\left\| U(\cdot,t) \right\|^2_{H^1(\Omega)^3} \leq c \left\| \operatorname{curl} U(\cdot,t) \right\|^2_{L^2(\Omega)^3} \ , \tag{3.6}$$

for some $c > 0$ and any $t \in \mathbb{R}$.

Let $\vartheta_{r_o}(\partial D)$ be an r_o-neighborhood of the boundary of D, that is

$$\vartheta_{r_o}(\partial D) = \left\{ x' \in D ; \inf_{y' \in \partial D} |x' - y'| < r_o \right\} \text{ for some small } r_o \in (0, \rho/2) \ . \tag{3.7}$$

Recall that $\sigma = 1_{|\omega} a$ where

$$\omega = \vartheta_{r_o}(\partial D) \times (-\rho, \rho) \ . \tag{3.8}$$

Consider

$$\omega_o = (D \backslash \vartheta_{r_o}(\partial D)) \times (\rho - 2r_o, \rho - r_o) \ . \tag{3.9}$$

Proposition 3.1 -. *For any $T_o > 0$, there exist $c, \gamma > 0$ such that for any $h > 0$, the solution U of (3.1) satisfies*

$$\int_0^{T_o} \int_{\omega_o} |\partial_t U(x,t)|^2 \, dxdt \leq c \frac{1}{h^\gamma} \int_{-c\frac{1}{h^\gamma}}^{c\frac{1}{h^\gamma}} \int_\omega \left(|\partial_t U(x,t)|^2 + |U(x,t)|^2 \right) dxdt + h\mathcal{G}(\partial_t U, 0) \ . \tag{3.10}$$

We shall leave the proof of Proposition 3.1 till later (see Section 4). Now we turn to prove Theorem 1. Remark that $(E,H)(\cdot,0) \in \mathcal{W} \cap \mathcal{S}_\sigma$ and therefore from the previous section for any $t \geq 0$, $(E,H)(\cdot,t) \in \mathcal{W} \cap \mathcal{S}_\sigma$.

Step 1 .- Denote (E_1, H_1) the electromagnetic field of the following Maxwell's equations with Ohm's law

$$\left\{ \begin{array}{rl} \varepsilon_o \partial_t E_1 - \operatorname{curl} H_1 + \left(\sigma + 1_{|\omega_o}\right) E_1 = 0 & \text{in } \Omega \times [0, +\infty) \\ \mu_o \partial_t H_1 + \operatorname{curl} E_1 = 0 & \text{in } \Omega \times [0, +\infty) \\ \operatorname{div}(\mu_o H_1) = 0 & \text{in } \Omega \times [0, +\infty) \\ E_1 \times \nu = H_1 \cdot \nu = 0 & \text{on } \partial\Omega \times [0, +\infty) \\ (E_1, H_1)(\cdot, T_h) = (E, H)(\cdot, T_h) & \text{in } \Omega \ . \end{array} \right. \tag{3.11}$$

Notice that $(E,H)(\cdot,T_h) \in \mathcal{W} \cap \mathcal{S}_\sigma \subset \left(\mathcal{V} \cap \mathcal{S}_{\sigma + 1_{|\omega_o}}\right)$. Therefore by Proposition 2.4, there exist $c, \beta > 0$ (independent of T_h) such that for any $t \geq 0$ we have

$$\int_\Omega \left(\varepsilon_o |E_1(x, t+T_h)|^2 + \mu_o |H_1(x, t+T_h)|^2 \right) dx \leq c e^{-\beta t} \mathcal{E}(T_h) \ . \tag{3.12}$$

On the other hand, let $(E_2, H_2) = (E_1, H_1) - (E, H)$. Then it solves

$$\left\{ \begin{array}{rl} \varepsilon_o \partial_t E_2 - \operatorname{curl} H_2 + \left(\sigma + 1_{|\omega_o}\right) E_2 = -1_{|\omega_o} E & \text{in } \Omega \times [0, +\infty) \\ \mu_o \partial_t H_2 + \operatorname{curl} E_2 = 0 & \text{in } \Omega \times [0, +\infty) \\ \operatorname{div}(\mu_o H_2) = 0 & \text{in } \Omega \times [0, +\infty) \\ E_2 \times \nu = H_2 \cdot \nu = 0 & \text{on } \partial\Omega \times [0, +\infty) \\ (E_2(\cdot, T_h), H_2(\cdot, T_h)) = (0, 0) & \text{in } \Omega \ , \end{array} \right. \tag{3.13}$$

and by a standard energy method, we get that for any $t \geq 0$

$$\int_\Omega \left(\varepsilon_o |E_2(x, t+T_h)|^2 + \mu_o |H_2(x, t+T_h)|^2 \right) dx \leq \frac{t}{\varepsilon_o} \int_{T_h}^{t+T_h} \int_{\omega_o} |E(x.s)|^2 \, dxds \ . \tag{3.14}$$

Now we are able to bound the quantity $\mathcal{E}(T_h) = \mathcal{E}(t+T_h) + \int_{T_h}^{t+T_h} \int_\Omega \sigma(x) |E(x,s)|^2 \, dxds$ as follows. By using (1.1), (2.1.3), (3.12) and (3.14), we deduce that

$$\mathcal{E}(T_h) \leq 2ce^{-\beta t} \mathcal{E}(T_h) + \frac{2t}{\varepsilon_o} \int_{T_h}^{t+T_h} \int_{\omega_o} |E(x.s)|^2 \, dxds + \int_{T_h}^{t+T_h} \int_\Omega \sigma(x) |E(x,s)|^2 \, dxds \tag{3.15}$$

which implies by taking t large enough, the existence of constants $C, T_c > 1$ such that

$$\mathcal{E}(T_h) \leq C \int_{T_h}^{T_c + T_h} \left(\int_\Omega \sigma(x) |E(x,t)|^2 \, dx + \int_{\omega_o} |E(x,t)|^2 \, dx \right) dt \ . \tag{3.16}$$

Step 2 .- Recall the existence of the vector potential A from Proposition 2.1 and let U be the solution of

$$\left\{ \begin{array}{rl} \partial_t^2 U + \operatorname{curl}\operatorname{curl} U = 0 & \text{in } \Omega \times \mathbb{R} \\ \operatorname{div} U = 0 & \text{in } \Omega \times \mathbb{R} \\ U \times \nu = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ (U, \partial_t U)(\cdot, 0) = (A, \partial_t A)(\cdot, 0) & \text{in } \Omega \ , \end{array} \right. \tag{3.17}$$

then by a standard energy method, for any $T_1 > 0$

$$\int_0^{T_1}\int_\Omega \left(\left|\partial_t (U-A)\right|^2 + \left|\mathrm{curl}\,(U-A)\right|^2 \right) dxdt \leq T_1^2 \int_0^{T_1}\int_\Omega \mu_o \left|-\varepsilon_o \partial_t \nabla p + \sigma E\right|^2 dxdt \tag{3.18}$$

which implies from (2.2.4) of Proposition 2.1 that

$$\int_0^{T_1}\int_\Omega \left(\left|\partial_t (U-A)\right|^2 + \left|\mathrm{curl}\,(U-A)\right|^2 \right) dxdt \leq 4\mu_o T_1^2 \int_0^{T_1}\int_\Omega \left|\sigma E\right|^2 dxdt \ . \tag{3.19}$$

Now we are able to bound the quantity $\mathcal{E}(0) = \mathcal{E}(T_h) + \int_0^{T_h}\int_\Omega \sigma(x)\left|E(x,s)\right|^2 dxds$ as follows. Since $E = -\nabla p + \partial_t (U-A) - \partial_t U$, we deduce by (2.1.3) and (3.16) that

$$\begin{aligned}
\mathcal{E}(0) &\leq C \int_{T_h}^{T_c+T_h} \left(\int_\Omega \sigma \left|E\right|^2 dx + \int_{\omega_o} \left|-\nabla p + \partial_t (U-A) - \partial_t U\right|^2 dx \right) dt + \int_0^{T_h}\int_\Omega \sigma \left|E\right|^2 dxdt \\
&\leq C\left(1+T_h^2\right) \int_0^{T_c+T_h}\int_\Omega \left(\sigma \left|E\right|^2 + \left|\nabla p\right|^2 \right) dxdt + C\int_{T_h}^{T_c+T_h}\int_{\omega_o} \left|\partial_t U\right|^2 dxdt \ ,
\end{aligned} \tag{3.20}$$

where in the last line, we used (3.19). Here and hereafter, C will be used to denote a generic constant, not necessarily the same in any two places.

Step 3 .- Now we fix $T_h = c\frac{1}{h^\gamma}$ where c and γ are given by Proposition 3.1. Taking $T_o = T_c$ in Proposition 3.1, we obtain that for any $h > 0$,

$$\int_0^{T_c}\int_{\omega_o} \left|\partial_t U\right|^2 dxdt \leq T_h \int_{-T_h}^{T_h}\int_\omega \left(\left|\partial_t U\right|^2 + \left|U\right|^2 \right) dxdt + h\mathcal{G}\left(\partial_t U, 0\right) \ . \tag{3.21}$$

By a translation in time and (3.4), the above inequality implies that

$$\int_{T_h}^{T_c+T_h}\int_{\omega_o} \left|\partial_t U\right|^2 dxdt \leq T_h \int_0^{2T_h}\int_\omega \left(\left|\partial_t U\right|^2 + \left|U\right|^2 \right) dxdt + h\mathcal{G}\left(\partial_t U, 0\right) \ . \tag{3.22}$$

But from (3.19),

$$\begin{aligned}
\int_0^{2T_h}\int_\omega \left|\partial_t U\right|^2 dxdt &= \int_0^{2T_h}\int_\omega \left|-E - \nabla p + \partial_t (U-A)\right|^2 dxdt \\
&\leq CT_h^2 \int_0^{2T_h}\int_\Omega \left(\sigma\left|E\right|^2 + \left|\nabla p\right|^2 \right) dxdt
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
\int_0^{2T_h}\int_\omega \left|U\right|^2 dxdt &\leq 2\int_0^{2T_h}\int_\Omega \left|U-A\right|^2 dxdt + 2\int_0^{2T_h}\int_\omega \left|A\right|^2 dxdt \\
&\leq C\int_0^{2T_h}\int_\Omega \left|\mathrm{curl}\,(U-A)\right|^2 dxdt + 2\int_0^{2T_h}\int_\omega \left|A\right|^2 dxdt \\
&\leq CT_h^2\int_0^{2T_h}\int_\Omega \sigma\left|E\right|^2 dxdt + 2\int_0^{2T_h}\int_\omega \left|A\right|^2 dxdt \ .
\end{aligned} \tag{3.24}$$

Therefore, plugging (3.23) and (3.24) into (3.22), we get

$$\int_{T_h}^{T_c+T_h}\int_{\omega_o} \left|\partial_t U\right|^2 dxdt \leq CT_h^3 \int_0^{2T_h} \left(\int_\Omega \left(\sigma\left|E\right|^2 + \left|\nabla p\right|^2 \right) dx + \int_\omega \left|A\right|^2 dx \right) dt + h\mathcal{G}\left(\partial_t U, 0\right) \ . \tag{3.25}$$

Finally, combining (3.20) and (3.25), we have

$$\mathcal{E}(0) \leq CT_h^3 \int_0^{2T_h} \left(\int_\Omega \left(\sigma\left|E\right|^2 + \left|\nabla p\right|^2 \right) dx + \int_\omega \left|A\right|^2 dx \right) dt + h\mathcal{G}\left(\partial_t U, 0\right) \ . \tag{3.26}$$

In conclusion, we proved that there exist $c,\gamma>0$ such that for any $h>0$, the solution (E,H) of (1.2) with initial data in $\mathcal{W}\cap\mathcal{S}_\sigma$ satisfies

$$\mathcal{E}(0)\leq c\frac{1}{h^\gamma}\int_0^{c\frac{1}{h^\gamma}}\left(\int_\Omega\left(\sigma\left|E\right|^2+\left|\nabla p\right|^2\right)dx+\int_\omega\left|A\right|^2dx\right)dt+h\mathcal{G}\left(\partial_tU,0\right)\ . \tag{3.27}$$

Step 4 .- By formula (2.1.3) and since $\mathcal{G}\left(\partial_tU,mc\frac{1}{h^\gamma}\right)=\mathcal{G}\left(\partial_tU,0\right)$ for any m, this last inequality becomes

$$\begin{array}{l} N\mathcal{E}(0)-\sum\limits_{m=0,..,N-1}\int_0^{mc\frac{1}{h^\gamma}}\int_\Omega\sigma\left|E\right|^2dxdt \\ =\sum\limits_{m=0,..,N-1}\mathcal{E}\left(mc\frac{1}{h^\gamma}\right) \\ \leq c\frac{1}{h^\gamma}\sum\limits_{m=0,..,N-1}\int_{mc\frac{1}{h^\gamma}}^{(m+1)c\frac{1}{h^\gamma}}\left(\int_\Omega\left(\sigma\left|E\right|^2+\left|\nabla p\right|^2\right)dx+\int_\omega\left|A\right|^2dx\right)dt+Nh\mathcal{G}\left(\partial_tU,0\right)\ , \end{array} \tag{3.28}$$

for any $N>1$. We choose $N\in\left(c\frac{1}{h^\gamma},1+c\frac{1}{h^\gamma}\right]$. Therefore, there exist $c,\gamma>0$ such that for any $h>0$,

$$\mathcal{E}(0)\leq c\int_0^{c\left(\frac{1}{h}\right)^\gamma}\left(\int_\Omega\left(\sigma\left|E\right|^2+\left|\nabla p\right|^2\right)dx+\int_\omega\left|A\right|^2dx\right)dt+h\mathcal{G}\left(\partial_tU,0\right)\ . \tag{3.29}$$

On the other hand, since $(U,\partial_tU)(\cdot,0)=(A,\partial_tA)(\cdot,0)$,

$$\begin{array}{ll} \mathcal{G}\left(\partial_tU,0\right) & =\left\|\mathrm{curl}\,E(\cdot,0)\right\|^2_{L^2(\Omega)^3}+\left\|\mu_o\,\mathrm{curl}\,H(\cdot,0)\right\|^2_{L^2(\Omega)^3} \\ & =\mu_o^2\left\|\partial_tH(\cdot,0)\right\|^2_{L^2(\Omega)^3}+\left\|\left(\partial_tE+\mu_o\sigma E\right)(\cdot,0)\right\|^2_{L^2(\Omega)^3} \\ & \leq c\left(\mathcal{E}_1(0)+\mathcal{E}(0)\right)\leq c\left\|(E_o,H_o)\right\|^2_{D(\mathcal{M})} \end{array} \tag{3.30}$$

where $\mathcal{M}$ is the m-accretive operator in $\mathcal{V}\cap\mathcal{S}_\sigma$ with domain $D(\mathcal{M})=\mathcal{W}\cap\mathcal{S}_\sigma$, defined as follows.

$$\begin{array}{l} \left\|(F,G)\right\|^2_{\mathcal{V}}=\varepsilon_o\left\|F\right\|^2_{L^2(\Omega)^3}+\mu_o\left\|G\right\|^2_{L^2(\Omega)^3}\ , \\ \mathcal{M}=\begin{pmatrix} \frac{1}{\varepsilon_o}\sigma & -\frac{1}{\varepsilon_o}\mathrm{curl} \\ \frac{1}{\mu_o}\mathrm{curl} & 0 \end{pmatrix}\ . \end{array} \tag{3.31}$$

Therefore, combining (3.29) and (3.30), we get the existence of constants $c,\gamma>0$ such that for any $h>0$,

$$\mathcal{E}(0)\leq c\int_0^{c\left(\frac{1}{h}\right)^\gamma}\left(\int_\Omega\left(\sigma\left|E\right|^2+\left|\nabla p\right|^2\right)dx+\int_\omega\left|A\right|^2dx\right)dt+ch\left\|(E_o,H_o)\right\|^2_{D(\mathcal{M})}\ . \tag{3.32}$$

Similarly, we get the existence of constants $c,\gamma>0$ such that for any $\zeta\geq0$ and $h>0$,

$$\mathcal{E}(\zeta)\leq c\int_\zeta^{\zeta+c\left(\frac{1}{h}\right)^\gamma}\left(\int_\Omega\left(\sigma\left|E\right|^2+\left|\nabla p\right|^2\right)dx+\int_\omega\left|A\right|^2dx\right)dt+h\left\|(E_o,H_o)\right\|^2_{D(\mathcal{M})}\ . \tag{3.33}$$

Step 5 .- Denote $\left(\mathcal{T}(t)\right)_{t\geq0}$ the unique semigroup of contractions generated by $-\mathcal{M}$. First, suppose that $(E_o,H_o)\in D\left(\mathcal{M}^3\right)$ and let us define the functional of energy

$$\mathcal{E}_2(t)=\frac{1}{2}\int_\Omega\left(\varepsilon_o\left|\partial_t^2E(x,t)\right|^2+\mu_o\left|\partial_t^2H(x,t)\right|^2\right)dx \tag{3.34}$$

which satisfies

$$\mathcal{E}_2(t_2)-\mathcal{E}_2(t_1)+\int_{t_1}^{t_2}\int_\Omega\sigma(x)\left|\partial_t^2E(x,t)\right|^2dxdt=0\ . \tag{3.35}$$

Let $X_o = -\mathcal{M}^2 (E_o, H_o)$, then $(\mathcal{T}(t))_{t\geq 0} X_o = \left(\partial_t^2 E, \partial_t^2 H\right)$, $\|\mathcal{T}(\zeta) X_o\|_{\mathcal{V}}^2 = 2\mathcal{E}_2(\zeta)$ and $\|X_o\|_{D(\mathcal{M})}^2 \leq c\,\|(E_o, H_o)\|_{D(\mathcal{M}^3)}^2$. Further, by uniqueness of the orthogonal decomposition in (2.1.1) of Proposition 2.1, (3.33) implies that for any $(E_o, H_o) \in D\left(\mathcal{M}^3\right)$

$$\mathcal{E}_2(\zeta) \leq c \int_\zeta^{\zeta + c\left(\frac{1}{h}\right)^\gamma} \left(\int_\Omega \left(\sigma \left|\partial_t^2 E\right|^2 + \left|\partial_t^2 \nabla p\right|^2 \right) dx + \int_\omega \left|\partial_t^2 A\right|^2 dx \right) dt + ch \,\|(E_o, H_o)\|_{D(\mathcal{M}^3)}^2 \ . \tag{3.36}$$

Since by Proposition 2.2, $\mathcal{E}(\zeta) \leq c\mathcal{E}_1(\zeta)$ and in a similar way $\mathcal{E}_1(\zeta) \leq c\mathcal{E}_2(\zeta)$ for some $c > 0$, taking account of the first line of (2.2.1) and (2.2.4), (3.36) becomes

$$\mathcal{E}(\zeta) + \mathcal{E}_1(\zeta) + \mathcal{E}_2(\zeta) \leq c \int_\zeta^{\zeta + c\left(\frac{1}{h}\right)^\gamma} \int_\Omega \left(\sigma |E|^2 + \sigma |\partial_t E|^2 + \sigma \left|\partial_t^2 E\right|^2 \right) dx dt + h \,\|(E_o, H_o)\|_{D(\mathcal{M}^3)}^2 \ . \tag{3.37}$$

Step 6 .- Denote

$$\mathcal{H}(\zeta) = \frac{\mathcal{E}_2(\zeta) + \mathcal{E}_1(\zeta) + \mathcal{E}(\zeta)}{\|(E_o, H_o)\|_{D(\mathcal{M}^3)}^2} \ . \tag{3.38}$$

By (2.1.3), (2.1.4) and (3.35), the function $\mathcal{H}$ is a continuous positive decreasing real function on $[0, +\infty)$, bounded by one. Taking $h = \frac{1}{2}\mathcal{H}(\zeta)$ in (3.37), we get the existence of constants $c, \gamma > 0$ such that for any $\zeta \geq 0$,

$$\mathcal{E}(\zeta) + \mathcal{E}_1(\zeta) + \mathcal{E}_2(\zeta) \leq c \int_\zeta^{\zeta + c\left(\frac{1}{\mathcal{H}(\zeta)}\right)^\gamma} \int_\Omega \left(\sigma |E|^2 + \sigma |\partial_t E|^2 + \sigma \left|\partial_t^2 E\right|^2 \right) dx dt \ , \tag{3.39}$$

that is, using (2.1.3), (2.1.4) and (3.35),

$$\mathcal{H}(\zeta) \leq c \left(\mathcal{H}(\zeta) - \mathcal{H}\left(\left(\frac{c}{\mathcal{H}(\zeta)} \right)^\gamma + \zeta \right) \right) \quad \forall \zeta \geq 0 \ . \tag{3.40}$$

From [12, p. 122, Lemma B], we deduce that there exist $C, \gamma > 0$ such that for any $t > 0$

$$\mathcal{E}(t) + \mathcal{E}_1(t) + \mathcal{E}_2(\zeta) \leq \frac{C}{t^\gamma} \|(E_o, H_o)\|_{D(\mathcal{M}^3)}^2 \ , \tag{3.41}$$

that is

$$\|\mathcal{T}(t) Y_o\|_{D(\mathcal{M}^2)}^2 \leq \frac{C}{t^\gamma} \|Y_o\|_{D(\mathcal{M}^3)}^2 \quad \forall Y_o \in D\left(\mathcal{M}^3\right) \ . \tag{3.42}$$

Since $\mathcal{M}$ is an m-accretive operator in $\mathcal{V} \cap \mathcal{S}_\sigma$ with dense domain, one can restrict it to $D\left(\mathcal{M}^2\right)$ in a way that its restriction operator is m-accretive. Thus the following two properties holds.

$$\forall Z_o \in D\left(\mathcal{M}^2\right) \quad \exists ! Y_o \in D\left(\mathcal{M}^3\right) \quad Y_o + \mathcal{M} Y_o = Z_o \ ; \tag{3.43}$$

$$\|Y_o\|_{D(\mathcal{M}^2)} \leq \|Y_o + \mathcal{M} Y_o\|_{D(\mathcal{M}^2)} \quad \forall Y_o \in D\left(\mathcal{M}^3\right) \ . \tag{3.44}$$

Consequently,

$$\begin{aligned}
\|\mathcal{T}(t) Z_o\|_{D(\mathcal{M})}^2 &= \|\mathcal{T}(t)(Y_o + \mathcal{M} Y_o)\|_{D(\mathcal{M})}^2 \quad \text{by (3.43)} \\
&\leq C \,\|\mathcal{T}(t) Y_o\|_{D(\mathcal{M}^2)}^2 \\
&\leq \tfrac{C}{t^\gamma} \|Y_o\|_{D(\mathcal{M}^3)}^2 \quad \text{by (3.42)} \\
&\leq \tfrac{C}{t^\gamma} \left(\|Y_o\|_{D(\mathcal{M}^2)}^2 + \|Y_o + \mathcal{M} Y_o\|_{D(\mathcal{M}^2)}^2 \right) \\
&\leq \tfrac{C}{t^\gamma} \|Y_o + \mathcal{M} Y_o\|_{D(\mathcal{M}^2)}^2 \quad \text{by (3.44)} \\
&\leq \tfrac{C}{t^\gamma} \|Z_o\|_{D(\mathcal{M}^2)}^2 \quad \text{by (3.43)}
\end{aligned} \tag{3.45}$$

for suitable positive constant $C > 0$ which is not necessarily the same in any two places.

Now, suppose that $(E_o, H_o) \in D(\mathcal{M})$. Since $\mathcal{M}$ is an m-accretive operator in $\mathcal{V} \cap \mathcal{S}_\sigma$ with dense domain, one can restrict it to $D(\mathcal{M})$ in a way that its restriction operator is m-accretive. Thus the following two properties holds.

$$\forall (E_o, H_o) \in D(\mathcal{M}) \quad \exists! Z_o \in D\left(\mathcal{M}^2\right) \quad Z_o + \mathcal{M} Z_o = (E_o, H_o) \ ; \tag{3.46}$$

$$\|Z_o\|_{D(\mathcal{M})} \leq \|Z_o + \mathcal{M} Z_o\|_{D(\mathcal{M})} \quad \forall Z_o \in D\left(\mathcal{M}^2\right) \ . \tag{3.47}$$

We conclude that

$$\begin{aligned} 2\mathcal{E}(t) = \|\mathcal{T}(t)(E_o, H_o)\|_{\mathcal{V}}^2 &= \|\mathcal{T}(t)(Z_o + \mathcal{M} Z_o)\|_{\mathcal{V}}^2 \quad \text{by } (3.46) \\ &\leq C \|\mathcal{T}(t) Z_o\|_{D(\mathcal{M})}^2 \\ &\leq \tfrac{C}{t^\gamma} \|Z_o\|_{D(\mathcal{M}^2)}^2 \quad \text{by } (3.45) \\ &\leq \tfrac{C}{t^\gamma} \left(\|Z_o\|_{D(\mathcal{M})}^2 + \|Z_o + \mathcal{M} Z_o\|_{D(\mathcal{M})}^2 \right) \\ &\leq \tfrac{C}{t^\gamma} \|Z_o + \mathcal{M} Z_o\|_{D(\mathcal{M})}^2 \quad \text{by } (3.47) \\ &\leq \tfrac{C}{t^\gamma} \|(E_o, H_o)\|_{D(\mathcal{M})}^2 \quad \text{by } (3.46) \\ &\leq \tfrac{C}{t^\gamma} \left(\mathcal{E}(0) + \mathcal{E}_1(0)\right) \end{aligned} \tag{3.48}$$

for suitable positive constant $C > 0$ which is not necessarily the same in any two places.

4 Proposition 4.1 and proof of propositon 3.1

Recall that $\omega_o = (D \backslash \vartheta_{r_o}(\partial D)) \times (\rho - 2r_o, \rho - r_o)$. In this Section, we establish an interpolation estimate in L^2 norm in order to prove Proposition 3.1.

Proposition 4.1 -. *There exist $c, \gamma > 0$ such that for any $T_o > 0$, and $h \in (0, 1]$, the solution U of (3.1) satisfies*

$$e^{-cT_o^2} \int_0^{T_o} \int_{\omega_o} |U(x,t)|^2 \, dxdt \leq c \frac{1}{h^\gamma} \int_{-c\frac{1}{h^\gamma}}^{c\frac{1}{h^\gamma}} \int_\omega \left(|\partial_t U(x,t)|^2 + |U(x,t)|^2 \right) dxdt + ch\mathcal{G}(U, 0) \ . \tag{4.1}$$

We shall leave the proof of Proposition 4.1 till later (see Section 5). Now we turn to prove Proposition 3.1 as follows.

Let $\psi \in C_0^\infty(0, 5T_o/3)$ be such that $\psi = 1$ in $(T_o/3, 4T_o/3)$. Then,

$$\begin{aligned} &\int_{T_o/3}^{4T_o/3} \int_{\omega_o} |\partial_t U(x,t)|^2 \, dxdt \\ &\leq \int_0^{5T_o/3} \int_{\omega_o} |\psi(t) \partial_t U(x,t)|^2 \, dxdt \\ &\leq \int_0^{5T_o/3} \int_{\omega_o} \left| 2\psi(t) \psi'(t) \partial_t U(x,t) + \psi^2(t) \partial_t^2 U(x,t) \right| |U(x,t)| \, dxdt \\ &\leq \tfrac{C}{\sqrt{h}} \int_0^{5T_o/3} \int_{\omega_o} |U(x,t)|^2 \, dxdt + \sqrt{h} \mathcal{G}(\partial_t U, 0) \quad \forall h > 0 \\ &\leq \tfrac{C}{\sqrt{h}} \left[c \tfrac{1}{h^\gamma} \int_{-c\frac{1}{h^\gamma}}^{c\frac{1}{h^\gamma}} \int_\omega \left(|\partial_t U(x,t)|^2 + |U(x,t)|^2 \right) dxdt + ch\mathcal{G}(U, 0) \right] + \sqrt{h} \mathcal{G}(\partial_t U, 0) \end{aligned} \tag{4.2}$$

where in the last line, we used Proposition 4.1. Therefore, it implies thanks to (3.5) that for any $T_o > 0$, there exist $c, \gamma > 0$ such that for any $h \in (0, 1]$,

$$\int_{T_o/3}^{4T_o/3} \int_{\omega_o} |\partial_t U(x,t)|^2 \, dxdt \leq c \frac{1}{h^\gamma} \int_{-c\frac{1}{h^\gamma}}^{c\frac{1}{h^\gamma}} \int_\omega \left(|\partial_t U(x,t)|^2 + |U(x,t)|^2 \right) dxdt + ch\mathcal{G}(\partial_t U, 0) \ . \tag{4.3}$$

By a translation in time and (3.3), we obtain (3.10) for any $h \in (0,1]$. Since

$$\int_0^{T_o} \int_{\omega_o} \left|\partial_t U(x,t)\right|^2 dxdt \leq Ch\mathcal{G}\left(\partial_t U, 0\right) \quad \forall h > 1 \ , \tag{4.4}$$

we get the desired estimate of Proposition 3.1.

5 Proposition 5.1 and proof of Proposition 4.1

Notice that $C_0^\infty\left(B\left(x_o, r_o/2\right)\right) \subset C_0^\infty(\Omega)$ for any $x_o \in \overline{\omega_o}$, where $B(x_o, r)$ denotes the ball of center x_o and radius r.

Let $\ell \in C^\infty\left(\mathbb{R}^3\right)$ be such that $0 \leq \ell(x) \leq 1$, $\ell = 1$ in $\mathbb{R}^3 \backslash \omega$, $\ell(x) \geq \ell_o > 0$ for any $x \in \overline{\omega_o}$, $\ell = \partial_\nu \ell = 0$ on Γ_1 and both $\nabla\ell$ and $\Delta\ell$ have support in ω.

The proof of Proposition 4.1 comes from the following result.

Proposition 5.1 -. *There exist $c, \delta, \eta > 0$ such that for any $x_o \in \overline{\omega_o}$ and $h \in (0,1]$, $\lambda \geq 1$, the solution U of (3.1) satisfies*

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}} \chi(x)\, e^{-\frac{1}{2}\left(\frac{1}{h}|x-x_o|^2+t^2\right)} \ell(x) \left|U(x,t)\right|^2 dxdt \\
&\leq c\frac{1}{\sqrt{\lambda}}\mathcal{G}(U,0) + ce^{-\frac{1}{ch}\frac{\lambda^\delta}{h^\eta}}\mathcal{G}(U,0) + c\frac{\lambda^\delta}{h^\eta}\left\|(U,\partial_t U)\right\|^2_{L^2\left(\omega\times\left(-c\frac{\lambda^\delta}{h^\eta}, c\frac{\lambda^\delta}{h^\eta}\right)\right)^6} \ ,
\end{aligned}$$

where $\chi \in C_0^\infty\left(B\left(x_o, r_o/2\right)\right)$, $0 \leq \chi \leq 1$.

We shall leave the proof of Proposition 5.1 till later (see Section 6). Now we turn to prove Proposition 4.1.

Let $h \in (0, h_o]$ where $h_o = \min\left(1, (r_o/8)^2\right)$. We begin by covering $\overline{\omega_o}$ with a finite collection of balls $B\left(x_o^i, 2\sqrt{h}\right)$ for $i \in I$ with $x_o^i \in \overline{\omega_o}$ and where I is a countable set such that the number of elements of I is $\frac{c_o}{h\sqrt{h}}$ for some constant $c_o > 0$ independent of h. Then, for each x_o^i, we introduce $\chi_{x_o^i} \in C_0^\infty\left(B\left(x_o^i, r_o/2\right)\right) \subset C_0^\infty(\Omega)$ be such that $0 \leq \chi_{x_o^i} \leq 1$ and $\chi_{x_o^i} = 1$ on $B\left(x_o^i, r_o/4\right) \supset B\left(x_o^i, 2\sqrt{h}\right)$. Consequently, for any $T_o > 0$,

$$\begin{aligned}
e^{-\frac{1}{2}T_o^2}\int_0^{T_o}\int_{\omega_o} \left|U(x,t)\right|^2 dxdt &\leq \frac{1}{\ell_o}\int_0^{T_o}\int_{\omega_o} e^{-\frac{1}{2}t^2}\ell(x)\left|U(x,t)\right|^2 dxdt \\
&\leq \frac{1}{\ell_o}e^2\sum_{i\in I}\int_0^{T_o}\int_{B\left(x_o^i, 2\sqrt{h}\right)} \chi_{x_o^i}(x)\, e^{-\frac{1}{2}\left(\frac{1}{h}\left|x-x_o^i\right|^2+t^2\right)}\ell(x)\left|U(x,t)\right|^2 dxdt \\
&\leq \frac{1}{\ell_o}e^2\sum_{i\in I}\int_{\Omega\times\mathbb{R}} \chi_{x_o^i}(x)\, e^{-\frac{1}{2}\left(\frac{1}{h}\left|x-x_o^i\right|^2+t^2\right)}\ell(x)\left|U(x,t)\right|^2 dxdt \ .
\end{aligned} \tag{5.1}$$

By virtue of Proposition 5.1, $\int_{\Omega\times\mathbb{R}} \chi_{x_o^i}(x)\, e^{-\frac{1}{2}\left(\frac{1}{h}\left|x-x_o^i\right|^2+t^2\right)}\ell(x)\left|U(x,t)\right|^2 dxdt$ is bounded independently of x_o^i and it implies that for some constants $c, \delta, \eta > 0$,

$$\begin{aligned}
&e^{-\frac{1}{2}T_o^2}\int_0^{T_o}\int_{\omega_o} \left|U(x,t)\right|^2 dxdt \\
&\leq c\frac{1}{h\sqrt{h}}\left(\frac{1}{\sqrt{\lambda}}\mathcal{G}(U,0) + ce^{-\frac{1}{ch}\frac{\lambda^\delta}{h^\eta}}\mathcal{G}(U,0) + c\frac{\lambda^\delta}{h^\eta}\left\|(U,\partial_t U)\right\|^2_{L^2\left(\omega\times\left(-c\frac{\lambda^\delta}{h^\eta}, c\frac{\lambda^\delta}{h^\eta}\right)\right)^6}\right) \ .
\end{aligned} \tag{5.2}$$

Finally, we choose $\lambda \geq 1$ such that $\lambda = \left(\frac{h_o}{h}\right)^5$ in order that $\frac{1}{h\sqrt{h}}\frac{1}{\sqrt{\lambda}} \leq Ch$ for some $C > 0$. Then there exist $c, \gamma > 0$ such that for any $h \in (0, h_o]$,

$$e^{-\frac{1}{2}T_o^2} \int_0^{T_o} \int_{\omega_o} |U(x,t)|^2 \, dxdt \leq c\frac{1}{h^\gamma} \int_{-c\frac{1}{h^\gamma}}^{c\frac{1}{h^\gamma}} \int_\omega \left(|\partial_t U(x,t)|^2 + |U(x,t)|^2 \right) dxdt + ch\mathcal{G}(U,0) \ . \tag{5.3}$$

Since $e^{-\frac{1}{2}T_o^2} \int_0^{T_o} \int_{\omega_o} |U|^2 \, dxdt \leq Ch\mathcal{G}(U,0)$ for any $h \geq h_o$, the proof of Proposition 4.1 is complete.

6 Proof of Proposition 5.1

Let $h \in (0,1]$, $x_o \in \overline{\omega_o}$, $\chi \in C_0^\infty(B(x_o, r_o/2))$ be such that $0 \leq \chi \leq 1$. Let us introduce

$$a_o(x,t) = e^{-\frac{1}{4}\left(\frac{1}{h}|x-x_o|^2 + t^2\right)} \text{ and } \varphi(x,t) = \chi(x)\, a_o(x,t) \ . \tag{6.1}$$

By the Fourier inversion formula,

$$\begin{aligned} & \int_{\Omega\times\mathbb{R}} \chi(x)\, e^{-\frac{1}{4}\left(\frac{1}{h}|x-x_o|^2 + t^2\right)} \ell(x) |U(x,t)|^2 \, dxdt \\ = & \int_{\Omega\times\mathbb{R}} \varphi(x,t)\, a_o(x,t)\, \ell(x) |U(x,t)|^2 \, dxdt \\ = & \int_{\Omega\times\mathbb{R}} \left(\frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{i(x\xi + t\tau)} \widehat{\varphi U}(\xi,\tau)\, d\xi d\tau \right) \cdot a_o(x,t)\, \ell(x)\, U(x,t)\, dxdt \\ = & \int_{\Omega\times\mathbb{R}} \left(\frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|<\lambda} e^{i(x\xi + t\tau)} \widehat{\varphi U}(\xi,\tau)\, d\xi d\tau \right) \cdot a_o(x,t)\, \ell(x)\, U(x,t)\, dxdt \\ & + \int_{\Omega\times\mathbb{R}} \left(\frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|\geq\lambda} e^{i(x\xi + t\tau)} \widehat{\varphi U}(\xi,\tau)\, d\xi d\tau \right) \cdot a_o(x,t)\, \ell(x)\, U(x,t)\, dxdt \end{aligned} \tag{6.2}$$

for any $\lambda \geq 1$. Here we recall that

$$\widehat{F}(\xi,\tau) = \int_{\mathbb{R}^4} e^{-i(x\xi + t\tau)} F(x,t)\, dxdt \quad \text{and} \quad F(x,t) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{i(x\xi + t\tau)} \widehat{F}(\xi,\tau)\, d\xi d\tau \tag{6.3}$$

when F and $\widehat{F}$ belong to $L^1\left(\mathbb{R}^4\right)^3$. On the other hand, from (A1) of Appendix A, (3.6), (3.3) and the fact that each component of U solves the wave equation, we have that for any U solution of (3.1),

$$\begin{aligned} & \left| \int_{\Omega\times\mathbb{R}} \left(\frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|\geq\lambda} e^{i(x\xi + t\tau)} \widehat{\varphi U}(\xi,\tau)\, d\xi d\tau \right) \cdot a_o(x,t)\, \ell(x)\, U(x,t)\, dxdt \right| \\ \leq & \, c\frac{1}{\sqrt{\lambda}} \mathcal{G}(U,0) \ . \end{aligned} \tag{6.4}$$

It remains to study the following quantity

$$\int_{\Omega\times\mathbb{R}} \left(\frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|<\lambda} e^{i(x\xi + t\tau)} \widehat{\varphi U}(\xi,\tau)\, d\xi d\tau \right) \cdot a_o(x,t)\, \ell(x)\, U(x,t)\, dxdt \ . \tag{6.5}$$

We claim that there exists $c, \delta, \eta > 0$ such that for any $x_o \in \overline{\omega_o}$ and $h \in (0,1]$, $\lambda \geq 1$, we have

$$\begin{aligned} & \left| \int_{\Omega\times\mathbb{R}} \left(\frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{|\tau|<\lambda} e^{i(x\xi + t\tau)} \widehat{\varphi U}(\xi,\tau)\, d\xi d\tau \right) \cdot a_o(x,t)\, \ell(x)\, U(x,t)\, dxdt \right| \\ \leq & \, c\frac{1}{\sqrt{\lambda}} \mathcal{G}(U,0) + ce^{-\frac{1}{ch}\frac{\lambda^\delta}{h^\eta}} \mathcal{G}(U,0) + c\frac{\lambda^\delta}{h^\eta} \left\| (U, \partial_t U) \right\|^2_{L^2\left(\omega \times \left(-c\frac{\lambda^\delta}{h^\eta}, c\frac{\lambda^\delta}{h^\eta}\right)\right)^6} \ , \end{aligned} \tag{6.6}$$

which implies Proposition 5.1 using (6.2), (6.4).

The proof of our claim is divided into many subsections. In the next subsection, we shall introduce suitable sequences of Fourier integral operators. First, we add a new variable $s \in [0,L]$. Next, we construct a particular solution of the equation (6.1.10) below for $(x,t,s) \in \mathbb{R}^4 \times [0,L]$ with good properties on Γ_0.

6.1 Fourier integral operators

Let $\varphi \in C_0^\infty(\Omega)$ and $f = f(x,t) \in L^\infty\left(\mathbb{R}; L^2\left(\mathbb{R}^3\right)\right)$ be such that $\widehat{\varphi f} \in L^1\left(\mathbb{R}^4\right)$. Let $h \in (0,1]$, $L \geq 1$, $\lambda \geq 1$ and $(x_o, \xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z}+1)$. Denote $x = (x_1, x_2, x_3)$ and $x_o = (x_{o1}, x_{o2}, x_{o3})$. First, let us introduce for any $(x,t,s) \in \mathbb{R}^4 \times [0,L]$ and $n \in \mathbb{Z}$,

$$\begin{aligned}
&\left(\mathcal{A}(x_o, \xi_{o3}, n) f\right)(x,t,s) \\
&= \tfrac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x_1\xi_1 + x_2\xi_2 + t\tau)} e^{i\left[(-1)^n x_3 + 2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho\right]\xi_3} e^{-i\left(|\xi|^2 - \tau^2\right)hs} \widehat{\varphi f}(\xi,\tau) \\
&\quad a\left(x_1 - x_{o1} - 2\xi_1 hs, x_2 - x_{o2} - 2\xi_2 hs, (-1)^n x_3 + 2n\tfrac{\xi_{o3}}{|\xi_{o3}|}\rho - x_{o3} - 2\xi_3 hs, t + 2\tau hs, s\right) d\xi d\tau
\end{aligned} \tag{6.1.1}$$

where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^2 \times [\xi_{o3}-1, \xi_{o3}+1]$,

$$a(x,t,s) = \left(\frac{1}{(is+1)^{3/2}} e^{-\frac{1}{4h}\frac{|x|^2}{is+1}}\right)\left(\frac{1}{\sqrt{-ihs+1}} e^{-\frac{1}{4}\frac{t^2}{-ihs+1}}\right) . \tag{6.1.2}$$

Next, let us introduce for any $(x,t,s) \in \mathbb{R}^4 \times [0,L]$,

$$\left(\mathbb{A}(x_o, \xi_{o3}) f\right)(x,t,s) = \sum_{n=-2Q}^{2P+1} (-1)^n \left(\mathcal{A}(x_o, \xi_{o3}, n) f\right)(x,t,s) \ , \tag{6.1.3}$$

$$\left(\mathbb{B}(x_o, \xi_{o3}) f\right)(x,t,s) = \sum_{n=-2Q}^{2P+1} \left(\mathcal{A}(x_o, \xi_{o3}, n) f\right)(x,t,s) \ , \tag{6.1.4}$$

where $(P,Q) \in \mathbb{N}^2$ is the first couple of integer numbers satisfying

$$\left\{\begin{array}{l} P \geq \frac{1}{4\rho}\left(\sqrt{(|\xi_{o3}|+2)(L^2+1)} + 2(|\xi_{o3}|+1)L\right) \ , \\ Q \geq \frac{1}{4\rho}\left(\sqrt{(|\xi_{o3}|+2)(L^2+1)} + 2\rho - r_o\right) \ . \end{array}\right. \tag{6.1.5}$$

We check after a lengthy but straightforward calculation that for any $(x,t,s) \in \mathbb{R}^4 \times [0,L]$,

$$\left\{\begin{array}{l} \left(i\partial_s + h\left(\Delta - \partial_t^2\right)\right)\left(\mathbb{A}(x_o, \xi_{o3}) f\right)(x,t,s) = 0 \ , \\ \left(i\partial_s + h\left(\Delta - \partial_t^2\right)\right)\left(\mathbb{B}(x_o, \xi_{o3}) f\right)(x,t,s) = 0 \ , \end{array}\right. \tag{6.1.6}$$

and that for any $(x_1, x_2, t, s) \in \mathbb{R}^3 \times [0,L]$,

$$\left\{\begin{array}{rl} \left(\mathbb{A}(x_o, \xi_{o3}) f\right)\left(x_1, x_2, \frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right) & = 0 \ , \\ \left(\mathbb{A}(x_o, \xi_{o3}) f\right)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right) & = \left(\mathcal{A}(x_o, \xi_{o3}, -2Q) f\right)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right) \\ & \quad + \left(\mathcal{A}(x_o, \xi_{o3}, 2P+1) f\right)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right) \ , \end{array}\right. \tag{6.1.7}$$

$$\left\{\begin{array}{rl} \partial_{x_3}\left(\mathbb{B}(x_o, \xi_{o3}) f\right)\left(x_1, x_2, \frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right) & = 0 \ , \\ \partial_{x_3}\left(\mathbb{B}(x_o, \xi_{o3}) f\right)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right) & = \partial_{x_3}\left(\mathcal{A}(x_o, \xi_{o3}, -2Q) f\right)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right) \\ & \quad - \partial_{x_3}\left(\mathcal{A}(x_o, \xi_{o3}, 2P+1) f\right)\left(x_1, x_2, -\frac{\xi_{o3}}{|\xi_{o3}|}\rho, t, s\right) \ . \end{array}\right. \tag{6.1.8}$$

Let $f_j = f_j(x,t) \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^3))$ be such that $\widehat{\varphi f_j} \in L^1(\mathbb{R}^4)$ for any $j \in \{1,2,3\}$. Let us introduce

$$F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \quad \text{and} \quad \mathbb{V}(x_o, \xi_{o3}) F = \begin{pmatrix} \mathbb{A}(x_o,\xi_{o3}) f_1 \\ \mathbb{A}(x_o,\xi_{o3}) f_2 \\ \mathbb{B}(x_o,\xi_{o3}) f_3 \end{pmatrix} \tag{6.1.9}$$

then

$$\left(i\partial_s + h\left(\Delta - \partial_t^2\right)\right)\left(\mathbb{V}(x_o,\xi_{o3}) F\right)(x,t,s) = 0 \quad \forall (x,t,s) \in \mathbb{R}^4 \times [0,L] \ . \tag{6.1.10}$$

On another hand, let U be the solution of (3.1). Denote

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{then} \quad \begin{cases} \forall j \in \{1,2,3\} \quad \partial_t^2 u_j - \Delta u_j = 0 & \text{in } \Omega \times \mathbb{R} \\ u_1 = u_2 = 0 & \text{on } \Gamma_0 \times \mathbb{R} \\ \partial_{x_3} u_3 = 0 & \text{on } \Gamma_0 \times \mathbb{R} \end{cases} \tag{6.1.11}$$

because $\operatorname{div} U = 0$ and $U \times \nu = 0$. Further, by (3.6) and (3.3),

$$\exists c > 0 \qquad \left\| u_j(\cdot,t) \right\|_{H^1(\Omega)}^2 + \left\| \partial_t u_j(\cdot,t) \right\|_{L^2(\Omega)}^2 \leq c\mathcal{G}(U,0) \quad \forall j \in \{1,2,3\} \ . \tag{6.1.12}$$

By multiplying the equation (6.1.10) by $\ell(x) U(x,t)$ and integrating by parts over $\Omega \times [-T,T] \times [0,L]$, we have that for all $(x_o,\xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z}+1)$ and all $h \in (0,1]$, $L \geq 1$, $T > 0$,

$$\begin{aligned}
0 = \ & -i \int_\Omega \int_{-T}^{T} \left(\mathbb{V}(x_o,\xi_{o3}) F\right)(\cdot,\cdot,0) \cdot \ell U dxdt \\
& + i \int_\Omega \int_{-T}^{T} \left(\mathbb{V}(x_o,\xi_{o3}) F\right)(\cdot,\cdot,L) \cdot \ell U dxdt \\
& - h \int_{\Gamma_0} \int_{-T}^{T} \left\{ \left(\int_0^L \mathbb{A}(x_o,\xi_{o3}) f_1 ds \right) \ell \partial_\nu u_1 + \left(\int_0^L \mathbb{A}(x_o,\xi_{o3}) f_2 ds \right) \ell \partial_\nu u_2 \right\} d\sigma dt \\
& + h \int_{\Gamma_0} \int_{-T}^{T} \left(\int_0^L \partial_\nu \left(\mathbb{B}(x_o,\xi_{o3}) f_3\right) ds \right) \ell u_3 d\sigma dt \\
& - h \int\int_{\Gamma_0 \cap \partial\omega} \int_{-T}^{T} \left(\int_0^L \mathbb{B}(x_o,\xi_{o3}) f_3 ds \right) \partial_\nu \ell u_3 d\sigma dt \\
& - h \int_\Omega \left[\left(\int_0^L \partial_t \left(\mathbb{V}(x_o,\xi_{o3}) F\right) ds \right) \cdot \ell U - \left(\int_0^L \left(\mathbb{V}(x_o,\xi_{o3}) F\right) ds \right) \cdot \ell \partial_t U \right]_{-T}^{T} dx \\
& + h \int_\omega \int_{-T}^{T} \left(\int_0^L \mathbb{V}(x_o,\xi_{o3}) F ds \right) \cdot \left[2 (\nabla \ell \cdot \nabla) U + \Delta \ell U \right] dxdt \\
\equiv \ & \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6 + \mathcal{I}_7 \ .
\end{aligned} \tag{6.1.13}$$

The different terms of the last equality will be estimated separately. The quantity $\mathcal{I}_1$ will allow us to recover (6.5). The dispersion property for the one dimensional Schrödinger operator will be used for making $\mathcal{I}_2$ small for large L. We treat $\mathcal{I}_3$ (resp. $\mathcal{I}_4$) by applying the formula (6.1.7) (resp. (6.1.8)). The quantity $\mathcal{I}_5$ and $\mathcal{I}_7$ will correspond to a term localized in ω. Finally, an appropriate choice of T will bound $\mathcal{I}_6$ and give the desired inequality (6.9.2) below.

6.2 Estimate for $\mathcal{I}_1$ (the term at $s = 0$)

We estimate $\mathcal{I}_1 = -i \int_\Omega \int_{-T}^{T} \left(\mathbb{V}(x_o,\xi_{o3}) F\right)(x,t,0) \cdot \ell(x) U(x,t) \, dxdt$ as follows.

Lemma 6.1 .- *There exists $c>0$ such that for any $(x_o,\xi_{o3})\in\overline{\omega_o}\times(2\mathbb{Z}+1)$ and $h\in(0,1]$, $\lambda\geq 1$, $T>0$, we have*

$$\begin{array}{l}\left|\mathcal{I}_1+i\displaystyle\int_{\Omega\times\mathbb{R}}\left(\tfrac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x\xi+t\tau)}\widehat{\varphi F}(\xi,\tau)\,d\xi d\tau\right)\cdot a(x-x_o,t,0)\,\ell(x)\,U(x,t)\,dxdt\right|\\ \quad\leq c\left(e^{-\frac{1}{ch}}+e^{-\frac{T^2}{c}}\right)\sqrt{\mathcal{G}(U,0)}\left(\displaystyle\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi F}(\xi,\tau)\right|d\xi d\tau\right)\ .\end{array}\tag{6.2.1}$$

Proof .- We start with the third component of $\mathbb{V}(x_o,\xi_{o3})\,F$. First, from (6.1.1) and (6.1.4) whenever $s=0$,

$$\begin{array}{l}\quad(\mathbb{B}(x_o,\xi_{o3})\,f)(x,t,0)\\ =\displaystyle\sum_{n=-2Q}^{2P+1}\Big[a\left(x_1-x_{o1},x_2-x_{o2},(-1)^n x_3+2n\tfrac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3},t,0\right)\\ \qquad\tfrac{1}{(2\pi)^4}\displaystyle\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x_1\xi_1+x_2\xi_2+t\tau)}e^{i\left[(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho\right]\xi_3}\widehat{\varphi f}(\xi,\tau)\,d\xi d\tau\Big]\\ =a(x-x_o,t,0)\,\tfrac{1}{(2\pi)^4}\displaystyle\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x\xi+t\tau)}\widehat{\varphi f}(\xi,\tau)\,d\xi d\tau\\ \quad+\displaystyle\sum_{n\in\{-2Q,\cdots,2P+1\}\backslash\{0\}}\Big[a\left(x_1-x_{o1},x_2-x_{o2},(-1)^n x_3+2n\tfrac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3},t,0\right)\\ \qquad\qquad\tfrac{1}{(2\pi)^4}\displaystyle\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x_1\xi_1+x_2\xi_2+t\tau)}e^{i\left[(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho\right]\xi_3}\widehat{\varphi f}(\xi,\tau)\,d\xi d\tau\Big]\ .\end{array}\tag{6.2.2}$$

Next, we estimate the discrete sum over $\{-2Q,\cdots,2P+1\}\backslash\{0\}$. By (6.1.2),

$$\begin{array}{l}\Bigg|\displaystyle\sum_{n\in\{-2Q,\cdots,2P+1\}\backslash\{0\}}\Big[a\left(x_1-x_{o1},x_2-x_{o2},(-1)^n x_3+2n\tfrac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3},t,0\right)\\ \qquad\tfrac{1}{(2\pi)^4}\displaystyle\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x_1\xi_1+x_2\xi_2+t\tau)}e^{i\left[(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho\right]\xi_3}\widehat{\varphi f}(\xi,\tau)\,d\xi d\tau\Big]\Bigg|\\ \leq\tfrac{1}{(2\pi)^4}\displaystyle\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|d\xi d\tau\, e^{-\frac{t^2}{4}}e^{-\frac{|(x_1-x_{o1},x_2-x_{o2})|^2}{4h}}\sum_{n\in\mathbb{Z}\backslash\{0\}}e^{-\frac{\left((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}\right)^2}{4h}}\ .\end{array}\tag{6.2.3}$$

Remark that for any $n\in\mathbb{Z}\backslash\{0\}$, $x_{o3}\in[\rho-2r_o,\rho-r_o]$ and $x_3\in[-\rho,\rho]$,

$$-\left((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}\right)^2\leq-4\rho^2(|n|-1)^2-r_o^2\ .\tag{6.2.4}$$

Indeed,

$$\begin{array}{ll}2\rho\,|n|&=\left|2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho\right|\\ &\leq\left|(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}\right|+|(-1)^n x_3-x_{o3}|\\ &\leq\left|(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}\right|+2\rho-r_o\ .\end{array}\tag{6.2.5}$$

Therefore, for some $c>0$,

$$e^{-\frac{|(x_1-x_{o1},x_2-x_{o2})|^2}{4h}}\sum_{n\in\mathbb{Z}\backslash\{0\}}e^{-\frac{\left((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}\right)^2}{4h}}\leq ce^{-\frac{1}{ch}}\ .\tag{6.2.6}$$

Now, we deduce from (6.2.2), (6.2.3) and (6.2.6) that

$$\begin{array}{l}\left|-i\int_{\Omega}\int_{-T}^{T}\left(\mathbb{B}\left(x_o,\xi_{o3}\right)f\right)(x,t,0)\,\ell(x)\,u_3(x,t)\,dxdt\right.\\ \left.\quad+i\int_{\Omega}\int_{-T}^{T}\left(\frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x\xi+t\tau)}\widehat{\varphi f}(\xi,\tau)\,d\xi d\tau\right)a(x-x_o,t,0)\,\ell(x)\,u_3(x,t)\,dxdt\right|\\ =\left|-i\int_{\Omega}\int_{-T}^{T}\left(\sum_{n\in\{-2Q,\cdots,2P+1\}\backslash\{0\}}\left[a\left(x_1-x_{o1},x_2-x_{o2},(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3},t,0\right)\right.\right.\right.\\ \left.\left.\left.\quad\frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x_1\xi_1+x_2\xi_2+t\tau)}e^{i\left[(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho\right]\xi_3}\widehat{\varphi f}(\xi,\tau)\,d\xi d\tau\right]\right)\ell(x)\,u_3(x,t)\,dxdt\right|\\ \leq\frac{c}{(2\pi)^4}e^{-\frac{1}{ch}}\int_{\Omega}\int_{-T}^{T}e^{-\frac{t^2}{4}}\left|\ell(x)\,u_3(x,t)\right|dxdt\left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|d\xi d\tau\right)\\ \leq ce^{-\frac{1}{ch}}\sqrt{\mathcal{G}(U,0)}\left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|d\xi d\tau\right)\end{array}\tag{6.2.7}$$

where in the last line we have used, from Cauchy-Schwarz inequality and conservation of energy (3.3), the fact that the solution U has the following property

$$\begin{aligned}\int_{-T}^{T}e^{-\frac{t^2}{4}}\int_{\Omega}\left|\ell(x)\,u_3(x,t)\right|dxdt&\leq c\sqrt{|\Omega|}\left(\int_{-\infty}^{\infty}e^{-\frac{t^2}{4}}dt\right)\sqrt{\mathcal{G}(U,0)}\\&\leq c\sqrt{\mathcal{G}(U,0)}\,.\end{aligned}\tag{6.2.8}$$

Here and hereafter, c will be used to denote a generic constant, not necessarily the same in any two places. On the other hand, by using Cauchy-Schwarz inequality and conservation of energy, we have

$$\begin{array}{l}\left|i\int_{\Omega}\int_{\mathbb{R}\backslash(-T,T)}\left(\frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x\xi+t\tau)}\widehat{\varphi f}(\xi,\tau)\,d\xi d\tau\right)e^{-\frac{1}{4}\left(\frac{1}{h}|x-x_o|^2+t^2\right)}\ell(x)\,u_3(x,t)\,dxdt\right|\\ \leq ce^{-\frac{1}{8}T^2}\sqrt{\mathcal{G}(U,0)}\left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|d\xi d\tau\right)\;.\end{array}\tag{6.2.9}$$

Now, we cut the integral on time into two parts to obtain

$$\begin{array}{l}\int_{\Omega\times\mathbb{R}}\left(\frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x\xi+t\tau)}\widehat{\varphi f}(\xi,\tau)\,d\xi d\tau\right)a(x-x_o,t,0)\,\ell(x)\,u_3(x,t)\,dxdt\\ =\int_{\Omega}\int_{-T}^{T}\left(\frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x\xi+t\tau)}\widehat{\varphi f}(\xi,\tau)\,d\xi d\tau\right)a(x-x_o,t,0)\,\ell(x)\,u_3(x,t)\,dxdt\\ +\int_{\Omega}\int_{\mathbb{R}\backslash(-T,T)}\left(\frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x\xi+t\tau)}\widehat{\varphi f}(\xi,\tau)\,d\xi d\tau\right)e^{-\frac{1}{4}\left(\frac{1}{h}|x-x_o|^2+t^2\right)}\ell(x)\,u_3(x,t)\,dxdt\;.\end{array}\tag{6.2.10}$$

We conclude from (6.2.7), (6.2.9) and (6.2.10) that

$$\begin{array}{l}\left|-i\int_{\Omega}\int_{-T}^{T}\left(\mathbb{B}\left(x_o,\xi_{o3}\right)f\right)(x,t,0)\,\ell(x)\,u_3(x,t)\,dxdt\right.\\ \left.\quad+i\int_{\Omega\times\mathbb{R}}\left(\frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x\xi+t\tau)}\widehat{\varphi f}(\xi,\tau)\,d\xi d\tau\right)a(x-x_o,t,0)\,\ell(x)\,u_3(x,t)\,dxdt\right|\\ \leq c\left(e^{-\frac{1}{ch}}+e^{-\frac{1}{8}T^2}\right)\sqrt{\mathcal{G}(U,0)}\left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|d\xi d\tau\right)\;.\end{array}\tag{6.2.11}$$

Similarly,

$$\begin{array}{l}\left|-i\int_{\Omega}\int_{-T}^{T}\left(\mathbb{A}\left(x_o,\xi_{o3}\right)f\right)\left(x,t,0\right)\ell\left(x\right)u_j\left(x,t\right)dxdt\right.\\ \quad\left.+i\int_{\Omega\times\mathbb{R}}\left(\frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}e^{i(x\xi+t\tau)}\ \widehat{\varphi f}\left(\xi,\tau\right)d\xi d\tau\right)a\left(x-x_o,t,0\right)\ell\left(x\right)u_j\left(x,t\right)dxdt\right|\\ \leq c\left(e^{-\frac{1}{ch}}+e^{-\frac{1}{8}T^2}\right)\sqrt{\mathcal{G}\left(U,0\right)}\left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|d\xi d\tau\right)\ .\end{array} \tag{6.2.12}$$

This completes the proof.

6.3 Estimate for $\mathcal{I}_2$ (the term at $s=L$)

We estimate $\mathcal{I}_2=i\int_{\Omega}\int_{-T}^{T}\left(\mathbb{V}\left(x_o,\xi_{o3}\right)F\right)\left(x,t,L\right)\cdot\ell\left(x\right)U\left(x,t\right)dxdt$ as follows.

Lemma 6.2 .- *There exists $c>0$ such that for any $(x_o,\xi_{o3})\in\overline{\omega_o}\times(2\mathbb{Z}+1)$ and $h\in(0,1]$, $L\geq1$, $\lambda\geq1$, $T>0$, we have*

$$\left|\mathcal{I}_2\right|\leq c\left(\frac{1}{\sqrt{L}}+e^{-\frac{1}{ch}}\right)\sqrt{\mathcal{G}\left(U,0\right)}\left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi F}\left(\xi,\tau\right)\right|d\xi d\tau\right)\ . \tag{6.3.1}$$

Proof .- We start with the third component of $\mathbb{V}\left(x_o,\xi_{o3}\right)F$. First,

$$\begin{array}{l}\left|\left(\mathbb{B}\left(x_o,\xi_{o3}\right)f\right)\left(x,t,L\right)\right|\\ \leq\sum\limits_{n\in\mathbb{Z}\backslash\{-2Q,\cdots,2P+1\}}\left|\left(\mathcal{A}\left(x_o,\xi_{o3},n\right)f\right)\left(x,t,L\right)\right|+\left|\sum\limits_{n\in\mathbb{Z}}\left(\mathcal{A}\left(x_o,\xi_{o3},n\right)f\right)\left(x,t,L\right)\right|\ .\end{array} \tag{6.3.2}$$

Next, by (6.1.1) and (6.1.2),

$$\begin{array}{l}\sum\limits_{n\in\mathbb{Z}\backslash\{-2Q,\cdots,2P+1\}}\left|\left(\mathcal{A}\left(x_o,\xi_{o3},n\right)f\right)\left(x,t,L\right)\right|\\ \leq\frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\left(\frac{1}{\left(\sqrt{(hL)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hL)^2}{(hL)^2+1}}\right)\\ \qquad\left(\frac{1}{\sqrt{L^2+1}}e^{-\frac{1}{4h}\frac{\left|(x_1-x_{o1}-2\xi_1hL,x_2-x_{o2}-2\xi_2hL)\right|^2}{L^2+1}}\right)\\ \qquad\frac{1}{\left(\sqrt{L^2+1}\right)^{1/2}}\left(\sum\limits_{n\in\mathbb{Z}\backslash\{-2Q,\cdots,2P+1\}}e^{-\frac{1}{4h}\frac{\left((-1)^nx_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3hL\right)^2}{L^2+1}}\right)d\xi d\tau\ .\end{array} \tag{6.3.3}$$

When $\xi_3\in(\xi_{o3}-1,\xi_{o3}+1)$ with $\xi_{o3}\in(2\mathbb{Z}+1)$,

$$\begin{array}{ll}\sqrt{L^2+1} & \leq4P\rho-2\left(\left|\xi_{o3}\right|+1\right)L\quad\text{from our choice of }P\\ & \leq4P\rho-2\left|\xi_3\right|hL+2\rho-\left|x_3\right|-\left|x_{o3}\right|\quad\text{because }\left|x_3\right|+\left|x_{o3}\right|\leq2\rho-r_o\\ & \leq4P\rho-2\left|\xi_3\right|hL+2\rho+\frac{\xi_{o3}}{|\xi_{o3}|}\left[-\left(-1\right)^nx_3-x_{o3}\right]\quad\forall n\in\mathbb{Z}\end{array} \tag{6.3.4}$$

thus

$$
\begin{aligned}
&\sum_{n\geq 2P+2} e^{-\frac{1}{4h}\frac{\left((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3 hL\right)^2}{L^2+1}} = \sum_{n\geq 2P+2} e^{-\frac{1}{4h}\frac{\left(2n\rho-2|\xi_3|hL+\frac{\xi_{o3}}{|\xi_{o3}|}\left[(-1)^n x_3-x_{o3}\right]\right)^2}{L^2+1}} \\
&= \sum_{n\geq 1} e^{-\frac{1}{4h}\frac{\left(2n\rho+4P\rho-2|\xi_3|hL+2\rho+\frac{\xi_{o3}}{|\xi_{o3}|}\left[-(-1)^n x_3-x_{o3}\right]\right)^2}{L^2+1}} \\
&\leq \sum_{n\geq 1} e^{-\frac{1}{4h}\frac{(2n\rho)^2}{L^2+1}} e^{-\frac{1}{4h}\frac{\left(4P\rho-2|\xi_3|hL+2\rho+\frac{\xi_{o3}}{|\xi_{o3}|}\left[-(-1)^n x_3-x_{o3}\right]\right)^2}{L^2+1}} \\
&\leq e^{-\frac{1}{4h}} \sum_{n\geq 1} e^{-\frac{1}{h}\frac{(n\rho)^2}{L^2+1}} \leq e^{-\frac{1}{4h}} \left(\frac{\sqrt{\pi}}{2}\frac{\sqrt{h}\sqrt{L^2+1}}{\rho}\right) \ .
\end{aligned}
\tag{6.3.5}
$$

Also,

$$
\begin{aligned}
\sqrt{L^2+1} \quad &\leq 4Q\rho-2\rho+r_o \quad \text{from our choice of } Q \\
&\leq 4Q\rho+2|\xi_3|hL-|x_3|-|x_{o3}| \quad \text{because } |x_3|+|x_{o3}|\leq 2\rho-r_o \\
&\leq 4Q\rho+2|\xi_3|hL-\frac{\xi_{o3}}{|\xi_{o3}|}\left[(-1)^n x_3-x_{o3}\right] \quad \forall n\in\mathbb{Z}
\end{aligned}
\tag{6.3.6}
$$

thus

$$
\begin{aligned}
&\sum_{n\leq -2Q-1} e^{-\frac{1}{4h}\frac{\left((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3 hL\right)^2}{L^2+1}} = \sum_{n\geq 2Q+1} e^{-\frac{1}{4h}\frac{\left(2n\rho+2|\xi_3|hL-\frac{\xi_{o3}}{|\xi_{o3}|}\left[(-1)^n x_3-x_{o3}\right]\right)^2}{L^2+1}} \\
&\leq \sum_{n\geq 1} e^{-\frac{1}{4h}\frac{\left(2n\rho+4Q\rho+2|\xi_3|hL-\frac{\xi_{o3}}{|\xi_{o3}|}\left[(-1)^n x_3-x_{o3}\right]\right)^2}{L^2+1}} \\
&\leq \sum_{n\geq 1} e^{-\frac{1}{4h}\frac{(2n\rho)^2}{L^2+1}} e^{-\frac{1}{4h}\frac{\left(4Q\rho+2|\xi_3|hL-\frac{\xi_{o3}}{|\xi_{o3}|}\left[(-1)^n x_3-x_{o3}\right]\right)^2}{L^2+1}} \\
&\leq e^{-\frac{1}{4h}} \sum_{n\geq 1} e^{-\frac{1}{h}\frac{(n\rho)^2}{L^2+1}} \leq e^{-\frac{1}{4h}} \left(\frac{\sqrt{\pi}}{2}\frac{\sqrt{h}\sqrt{L^2+1}}{\rho}\right) \ .
\end{aligned}
\tag{6.3.7}
$$

Therefore, from (6.3.3), (6.3.5) and (6.3.7), we get that

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}\setminus\{-2Q,\cdots,2P+1\}} \left|\left(\mathcal{A}\left(x_o,\xi_{o3},n\right)f\right)(x,t,L)\right| \\
&\leq \frac{1}{(2\pi)^4}\frac{1}{\sqrt{L^2+1}}\frac{1}{\left(\sqrt{L^2+1}\right)^{1/2}}\left(\frac{\sqrt{\pi}\sqrt{h}\sqrt{L^2+1}}{\rho}\right)e^{-\frac{1}{4h}} \\
&\quad \int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|\left(\frac{1}{\left(\sqrt{(hL)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hL)^2}{(hL)^2+1}}\right)d\xi d\tau \ .
\end{aligned}
\tag{6.3.8}
$$

Now, by (6.1.1) and (6.1.2),

$$
\begin{aligned}
&\left|\sum_{n\in\mathbb{Z}}\left(\mathcal{A}\left(x_o,\xi_{o3},n\right)f\right)(x,t,L)\right| \\
&\leq \frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|\left(\frac{1}{\left(\sqrt{(hL)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hL)^2}{(hL)^2+1}}\right) \\
&\quad\left(\frac{1}{\sqrt{L^2+1}}e^{-\frac{1}{4h}\frac{|(x_1-x_{o1}-2\xi_1 hL,x_2-x_{o2}-2\xi_2 hL)|^2}{L^2+1}}\right) \\
&\quad\left|\sum_{n\in\mathbb{Z}}\left(\frac{1}{\sqrt{iL+1}}e^{i\left[(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho\right]\xi_3}e^{-\frac{1}{4h}\frac{\left((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3 hL\right)^2}{iL+1}}\right)\right|d\xi d\tau \\
&\leq \frac{c}{\sqrt{L^2+1}}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|\left(\frac{1}{\left(\sqrt{(hL)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hL)^2}{(hL)^2+1}}\right)d\xi d\tau
\end{aligned}
\tag{6.3.9}
$$

because from Appendix B with $z=\frac{4h}{\rho^2}(iL+1)$ we know that

$$
\left|\frac{1}{\sqrt{iL+1}}\sum_{n\in\mathbb{Z}}e^{i\left[(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho\right]\xi_3}e^{-\frac{1}{4h}\frac{\left((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3 hL\right)^2}{iL+1}}\right| \leq \frac{2\sqrt{h}}{\rho}\left(\frac{\sqrt{\pi}}{2}+\frac{\rho}{\sqrt{h}}\right) \ . \tag{6.3.10}
$$

Finally, (6.3.2), (6.3.8) and (6.3.9) imply that

$$\begin{aligned}
&\left|\left(\mathbb{B}\left(x_o,\xi_{o3}\right)f\right)(x,t,L)\right| \\
&\leq c\left(\tfrac{1}{\sqrt{L^2+1}}+\tfrac{1}{\left(\sqrt{L^2+1}\right)^{1/2}}e^{-\frac{1}{4h}}\right)\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|\left(\tfrac{1}{\left(\sqrt{(hL)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hL)^2}{(hL)^2+1}}\right)d\xi d\tau
\end{aligned} \tag{6.3.11}$$

and we conclude that

$$\begin{aligned}
&\left|i\int_\Omega\int_{-T}^{T}\left(\mathbb{B}\left(x_o,\xi_{o3}\right)f\right)(x,t,L)\,\ell(x)\,u_3(x,t)\,dxdt\right| \\
&\leq c\left(\tfrac{1}{\sqrt{L^2+1}}+\tfrac{1}{\left(\sqrt{L^2+1}\right)^{1/2}}e^{-\frac{1}{4h}}\right) \\
&\quad\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|d\xi\int_{-T}^{T}\left(\tfrac{1}{\left(\sqrt{(hL)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hL)^2}{(hL)^2+1}}\right)\int_\Omega\left|\ell(x)\,u_3(x,t)\right|dxdtd\tau \\
&\leq c\left(\tfrac{1}{\sqrt{L}}+e^{-\frac{1}{4h}}\right)\sqrt{\mathcal{G}(U,0)}\left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|d\xi d\tau\right)
\end{aligned} \tag{6.3.12}$$

where in the last line we have used the fact that the solution U has the following property, from Cauchy-Schwarz inequality and conservation of energy (3.3),

$$\begin{aligned}
\int_{-T}^{T}e^{-\frac{1}{4}\frac{(t+2\tau hL)^2}{(hL)^2+1}}\int_\Omega\left|\ell(x)\,u_3(x,t)\right|dxdt &\leq c\sqrt{|\Omega|}\left(\int_{-\infty}^{\infty}e^{-\frac{1}{4}\frac{t^2}{(hL)^2+1}}dt\right)\sqrt{\mathcal{G}(U,0)} \\
&\leq c\sqrt{|\Omega|}\left(2\sqrt{\pi}\sqrt{(hL)^2+1}\right)\sqrt{\mathcal{G}(U,0)}\ .
\end{aligned} \tag{6.3.13}$$

Similarly,

$$\begin{aligned}
&\left|i\int_\Omega\int_{-T}^{T}\left(\mathbb{A}\left(x_o,\xi_{o3}\right)f\right)(x,t,L)\,\ell(x)\,u_j(x,t)\,dxdt\right| \\
&\leq c\left(\tfrac{1}{\sqrt{L}}+e^{-\frac{1}{4h}}\right)\sqrt{\mathcal{G}(U,0)}\left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|d\xi d\tau\right)\ ,
\end{aligned} \tag{6.3.14}$$

using the estimate

$$\begin{aligned}
&\left|\left(\mathbb{A}\left(x_o,\xi_{o3}\right)f\right)(x,t,L)\right| \\
&\leq \sum_{n\in\mathbb{Z}\setminus\{-2Q,\cdots,2P+1\}}\left|\left(\mathcal{A}\left(x_o,\xi_{o3},n\right)f\right)(x,t,L)\right|+\left|\sum_{n\in\mathbb{Z}}(-1)^n\left(\mathcal{A}\left(x_o,\xi_{o3},n\right)f\right)(x,t,L)\right|
\end{aligned} \tag{6.3.15}$$

and

$$\left|\frac{1}{\sqrt{iL+1}}\sum_{n\in\mathbb{Z}}(-1)^n e^{i\left[(-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho\right]\xi_3}e^{-\frac{1}{4h}\frac{\left((-1)^n x_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3 hL\right)^2}{iL+1}}\right|\leq\frac{2\sqrt{h}}{\rho}\left(\frac{\sqrt{\pi}}{2}+\frac{\rho}{\sqrt{h}}\right) \tag{6.3.16}$$

deduced from Appendix B with $z=\frac{4h}{\rho^2}(iL+1)$. This completes the proof.

6.4 Estimate for $\mathcal{I}_3$ (the boundary term with $\mathbb{A}$)

We estimate

$$\begin{aligned}
\mathcal{I}_3 &= -h\int_{\Gamma_0}\int_{-T}^{T}\left(\int_0^L\mathbb{A}\left(x_o,\xi_{o3}\right)f_1(x,t,s)\,ds\right)\ell(x)\,\partial_\nu u_1(x,t)\,d\sigma dt \\
&\quad+h\int_{\Gamma_0}\int_{-T}^{T}\left(\int_0^L\mathbb{A}\left(x_o,\xi_{o3}\right)f_2(x,t,s)\,ds\right)\ell(x)\,\partial_\nu u_2(x,t)\,d\sigma dt
\end{aligned}$$

as follows.

Lemma 6.3 .- *There exists $c>0$ such that for any $(x_o,\xi_{o3})\in\overline{\omega_o}\times(2\mathbb{Z}+1)$ and $h\in(0,1]$, $L\geq 1$, $\lambda\geq 1$, $T>0$, we have*

$$\left|\mathcal{I}_3\right|\leq chLe^{-\frac{1}{4h}}\sqrt{\mathcal{G}\left(U,0\right)}\left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi F}\left(\xi,\tau\right)\right|d\xi d\tau\right) . \tag{6.4.1}$$

Proof .- First, by (6.1.7), we deduce that

$$\begin{array}{ll}\left|\int_{\Gamma_0}\left(\mathbb{A}\left(x_o,\xi_{o3}\right)f\right)\left(x,t,s\right)\ell\left(x\right)\partial_{x_3}u_j\left(x,t\right)d\sigma\right| & \\ \leq\int_{\Gamma_0}\left|\left(\mathcal{A}\left(x_o,\xi_{o3},-2Q\right)f\right)\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t,s\right)\right|\left|\ell\partial_{x_3}u_j\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\right|d\sigma & (6.4.2)\\ \quad+\int_{\Gamma_0}\left|\left(\mathcal{A}\left(x_o,\xi_{o3},2P+1\right)f\right)\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t,s\right)\right|\left|\ell\partial_{x_3}u_j\left(x_1,x_2,-\frac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\right|d\sigma\ . & \end{array}$$

Next, recall that

$$\begin{array}{l}\left|\left(\mathcal{A}\left(x_o,\xi_{o3},n\right)f\right)\left(x,t,s\right)\right|\\ \leq\frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\left(\frac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}\right)\\ \qquad\left(\frac{1}{\sqrt{s^2+1}}e^{-\frac{1}{4h}\frac{|(x_1-x_{o1}-2\xi_1 hs,x_2-x_{o2}-2\xi_2 hs)|^2}{s^2+1}}\right)\\ \qquad\frac{1}{\left(\sqrt{s^2+1}\right)^{1/2}}\left(e^{-\frac{1}{4h}\frac{\left(2n\rho-2|\xi_3|hs+\frac{\xi_{o3}}{|\xi_{o3}|}\left[(-1)^n x_3-x_{o3}\right]\right)^2}{s^2+1}}\right)d\xi d\tau\ .\end{array} \tag{6.4.3}$$

Here, for any $s\in[0,L]$, $h\in(0,1]$, $x_3\in[-\rho,\rho]$, $x_{o3}\in[\rho-2r_o,\rho-r_o]$, $\xi_3\in(\xi_{o3}-1,\xi_{o3}+1)$, $\xi_{o3}\in(2\mathbb{Z}+1)$, we have chosen $(P,Q)\in\mathbb{N}^2$ (only depending on (ξ_{o3},L) see (6.1.5)) such that

$$s^2+1\leq\left(2n\rho-2\left|\xi_3\right|hs+\frac{\xi_{o3}}{|\xi_{o3}|}\left[(-1)^n x_3-x_{o3}\right]\right)^2\quad\text{when } n\in\{-2Q,2P+1\}\ . \tag{6.4.4}$$

Indeed, for any $x_3\in[-\rho,\rho]$ and $x_{o3}\in[\rho-2r_o,\rho-r_o]$,

$$\begin{array}{ll}\sqrt{s^2+1} & \leq\sqrt{L^2+1}\leq 4P\rho-2\left(|\xi_{o3}|+1\right)L\quad\text{from our choice of } P\\ & \leq 4P\rho-2\left|\xi_3\right|hs+2\rho-\left|x_3+x_{o3}\right|\quad\text{because } r_o\leq 2\rho-\left|x_3+x_{o3}\right|\\ & \leq\left|4P\rho-2\left|\xi_3\right|hs+2\rho+\frac{\xi_{o3}}{|\xi_{o3}|}\left[-x_3-x_{o3}\right]\right|\end{array} \tag{6.4.5}$$

and

$$\begin{array}{ll}\sqrt{s^2+1} & \leq\sqrt{L^2+1}\leq 4Q\rho-2\rho+r_o\quad\text{from our choice of } Q\\ & \leq 4Q\rho+2\left|\xi_3\right|hs-\left|x_3-x_{o3}\right|\quad\text{because } \left|x_3-x_{o3}\right|\leq 2\rho-r_o\\ & \leq\left|-4Q\rho-2\left|\xi_3\right|hs+\frac{\xi_{o3}}{|\xi_{o3}|}\left[x_3-x_{o3}\right]\right|\ .\end{array} \tag{6.4.6}$$

So (6.4.4) implies that

$$e^{-\frac{1}{4h}\frac{\left(2n\rho-2|\xi_3|hs+\frac{\xi_{o3}}{|\xi_{o3}|}\left[(-1)^n x_3-x_{o3}\right]\right)^2}{s^2+1}}\leq e^{-\frac{1}{4h}}\quad\text{when } n\in\{-2Q,2P+1\}\ . \tag{6.4.7}$$

Therefore, from (6.4.3) and (6.4.7), for any $s\in[0,L]$, $h\in(0,1]$, $x_3\in[-\rho,\rho]$, $x_{o3}\in[\rho-2r_o,\rho-r_o]$, $\xi_3\in(\xi_{o3}-1,\xi_{o3}+1)$,

$$\begin{array}{l}\left|\left(\mathcal{A}\left(x_o,\xi_{o3},n\right)f\right)\left(x,t,s\right)\right|\\ \leq\frac{1}{(2\pi)^4}\frac{1}{\sqrt{s^2+1}}\frac{1}{\left(\sqrt{s^2+1}\right)^{1/2}}e^{-\frac{1}{4h}}\\ \quad\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\left(\frac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}\right)d\xi d\tau\quad\text{when } n\in\{-2Q,2P+1\}\ .\end{array} \tag{6.4.8}$$

On the other hand, by Cauchy-Schwarz inequality

$$\begin{aligned}
&\int_{-T}^{T} e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}} \left(\int_{\Gamma_0} \left|\ell(x)\,\partial_{x_3} u_j(x,t)\right| d\sigma\right) dt \\
&\leq c\left(\int_{-T}^{T} e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}} dt\right)^{1/2} \left(\int_{-T}^{T}\int_{\Gamma_0} e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}} \left|\ell(x)\,\partial_{x_3} u_j(x,t)\right|^2 d\sigma dt\right)^{1/2} \\
&\leq c\left(2\sqrt{\pi}\sqrt{(hs)^2+1}\right)^{1/2} \left(\int_{-T}^{T}\int_{\Gamma_0} e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}} \left|\ell(x)\,\partial_{x_3} u_j(x,t)\right|^2 d\sigma dt\right)^{1/2} .
\end{aligned} \tag{6.4.9}$$

Next, by multiplying the equation $\partial_t^2 u_j - \Delta u_j = 0$ by $g\ell^2\nabla u_j \cdot W$ where $g(t) = e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}$ and $W = W(x)$ is a smooth vector field such that $W = \nu$ on $\partial\Omega$ (see [9, page 29]), we get, after integrations by parts and by Cauchy-Schwarz inequality, observing that $\ell u_j = 0$ on $\partial\Omega$,

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\Gamma_0} g(t)\left|\ell(x)\,\partial_{x_3} u_j(x,t)\right|^2 d\sigma dt &\leq c\int_{\mathbb{R}} \left(g + \left|\tfrac{d}{dt} g\right|\right) \int_{\Omega} \left(|u_j|^2 + |\nabla u_j|^2 + |\partial_t u_j|^2\right) dx dt \\
&\leq c\sqrt{(hs)^2 + 1}\mathcal{G}(U,0) \ .
\end{aligned} \tag{6.4.10}$$

Therefore, (6.4.9) and (6.4.10) imply that

$$\int_{-T}^{T} e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}} \left(\int_{\Gamma_0} \left|\ell(x)\,\partial_{x_3} u_j(x,t)\right| d\sigma\right) dt \leq c\sqrt{(hs)^2+1}\sqrt{\mathcal{E}(u,0)} \ . \tag{6.4.11}$$

We conclude from (6.4.2) and (6.4.8) that

$$\begin{aligned}
&\left| h\int_{\Gamma_0}\int_{-T}^{T} \left(\int_0^L \mathbb{A}(x_o,\xi_{o3})\, f(x,t,s)\, ds\right) \ell(x)\,\partial_\nu u_j(x,t)\, d\sigma dt \right| \\
&\leq \tfrac{h}{(2\pi)^4} e^{-\frac{1}{4h}} \int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda} \left|\widehat{\varphi f}(\xi,\tau)\right| d\xi d\tau \\
&\quad \int_0^L \tfrac{1}{\sqrt{s^2+1}} \tfrac{1}{\left(\sqrt{s^2+1}\right)^{1/2}} \tfrac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}} \int_{-T}^{T} e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}} \int_{\Gamma_0} \left|\ell(x)\,\partial_{x_3} u_j(x,t)\right| d\sigma dt ds \\
&\leq chLe^{-\frac{1}{4h}}\sqrt{\mathcal{G}(U,0)} \left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda} \left|\widehat{\varphi f}(\xi,\tau)\right| d\xi d\tau\right) \ ,
\end{aligned} \tag{6.4.12}$$

where in the last inequality, we used (6.4.11). This completes the proof.

6.5 Estimate for $\mathcal{I}_4$ (the boundary term with $\partial_\nu \mathbb{B}$)

We estimate $\mathcal{I}_4 = h\int_{\Gamma_0}\int_{-T}^{T} \left(\int_0^L \partial_\nu \mathbb{B}(x_o,\xi_{o3})\, f_3(x,t,s)\, ds\right) \ell(x)\, u_3(x,t)\, d\sigma dt$ as follows.

Lemma 6.4 .- *There exists $c > 0$ such that for any $(x_o,\xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z}+1)$ and $h \in (0,1]$, $L \geq 1$, $\lambda \geq 1$, $T > 0$, we have*

$$|\mathcal{I}_4| \leq chLe^{-\frac{1}{4h}}\sqrt{\mathcal{G}(U,0)} \left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda} \left|\widehat{\varphi F}(\xi,\tau)\right| d\xi d\tau\right) \ . \tag{6.5.1}$$

Proof .- First, by (6.1.8), we deduce that

$$\begin{aligned}
&\left|\int_{\Gamma_0} \left(\partial_\nu \mathbb{B}(x_o,\xi_{o3})\, f\right)(x,t,s)\, \ell(x)\, u_3(x,t)\, d\sigma\right| \\
&\leq \int_{\Gamma_0} \left|\partial_{x_3}\left(\mathcal{A}(x_o,\xi_{o3},-2Q)\, f\right)\left(x_1,x_2,-\tfrac{\xi_{o3}}{|\xi_{o3}|}\rho,t,s\right)\right| \left|\ell u_3\left(x_1,x_2,-\tfrac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\right| d\sigma \\
&\quad + \int_{\Gamma_0} \left|\partial_{x_3}\left(\mathcal{A}(x_o,\xi_{o3},2P+1)\, f\right)\left(x_1,x_2,-\tfrac{\xi_{o3}}{|\xi_{o3}|}\rho,t,s\right)\right| \left|\ell u_3\left(x_1,x_2,-\tfrac{\xi_{o3}}{|\xi_{o3}|}\rho,t\right)\right| d\sigma \ .
\end{aligned} \tag{6.5.2}$$

Next, recall that by (6.1.1) and (6.1.2),

$$\begin{array}{l}\left|\partial_{x_3}\left(\mathcal{A}\left(x_o,\xi_{o3},n\right)f\right)(x,t,s)\right| \\ \leq \frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\left(\frac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}\right) \\ \qquad \left(\frac{1}{\sqrt{s^2+1}}e^{-\frac{1}{4h}\frac{\left|(x_1-x_{o1}-2\xi_1 hs,x_2-x_{o2}-2\xi_2 hs)\right|^2}{s^2+1}}\right) \\ \qquad \frac{1}{\left(\sqrt{s^2+1}\right)^{1/2}}\left[|\xi_3|+\frac{1}{\sqrt{h}}\right]\left(e^{-\frac{1}{4h}\frac{\left(2n\rho-2|\xi_3|hs+\frac{\xi_{o3}}{|\xi_{o3}|}\left[(-1)^n x_3-x_{o3}\right]\right)^2}{s^2+1}}\right)d\xi d\tau\ .\end{array} \tag{6.5.3}$$

Here, for any $s\in[0,L]$, $h\in(0,1]$, $x_3\in[-\rho,\rho]$, $x_{o3}\in[\rho-2r_o,\rho-r_o]$, $\xi_3\in(\xi_{o3}-1,\xi_{o3}+1)$, $\xi_{o3}\in(2\mathbb{Z}+1)$, we have chosen $(P,Q)\in\mathbb{N}^2$ (only depending on (ξ_{o3},L) see (6.1.5)) such that

$$\left(h\left|\xi_3\right|+1\right)\left(s^2+1\right)\leq\left(2n\rho-2\left|\xi_3\right|hs+\frac{\xi_{o3}}{\left|\xi_{o3}\right|}\left[(-1)^n x_3-x_{o3}\right]\right)^2\quad\text{when } n\in\{-2Q,2P+1\}\ . \tag{6.5.4}$$

Indeed, for any $x_3\in[-\rho,\rho]$ and $x_{o3}\in[\rho-2r_o,\rho-r_o]$

$$\begin{array}{ll}\sqrt{\left(h\left|\xi_3\right|+1\right)\left(s^2+1\right)} & \leq\sqrt{\left(\left|\xi_{o3}\right|+2\right)\left(L^2+1\right)}\leq 4P\rho-2\left(\left|\xi_{o3}\right|+1\right)L \\ & \leq 4P\rho-2\left|\xi_3\right|hs+2\rho-\left|x_3+x_{o3}\right|\quad\text{because } r_o\leq 2\rho-\left|x_3+x_{o3}\right| \\ & \leq\left|4P\rho-2\left|\xi_3\right|hs+2\rho+\frac{\xi_{o3}}{\left|\xi_{o3}\right|}\left[-x_3-x_{o3}\right]\right|\end{array} \tag{6.5.5}$$

and

$$\begin{array}{ll}\sqrt{\left(h\left|\xi_3\right|+1\right)\left(s^2+1\right)} & \leq\sqrt{\left(\left|\xi_{o3}\right|+2\right)\left(L^2+1\right)}\leq 4Q\rho-2\rho+r_o \\ & \leq 4Q\rho+2\left|\xi_3\right|hs-\left|x_3-x_{o3}\right|\quad\text{because } \left|x_3-x_{o3}\right|\leq 2\rho-r_o \\ & \leq\left|-4Q\rho-2\left|\xi_3\right|hs+\frac{\xi_{o3}}{\left|\xi_{o3}\right|}\left[x_3-x_{o3}\right]\right|\ .\end{array} \tag{6.5.6}$$

So (6.5.4) implies that when $n\in\{-2Q,2P+1\}$

$$\left[\left|\xi_3\right|+\frac{1}{\sqrt{h}}\right]e^{-\frac{1}{8h}\frac{\left(2n\rho-2|\xi_3|hs+\frac{\xi_{o3}}{|\xi_{o3}|}\left[(-1)^n x_3-x_{o3}\right]\right)^2}{s^2+1}}\leq\left[\left|\xi_3\right|+\frac{1}{h}\right]e^{-\frac{1}{8}\left(|\xi_3|+\frac{1}{h}\right)}\leq 16e^{-\frac{1}{16h}}\ . \tag{6.5.7}$$

Therefore, from (6.5.3) and (6.5.7), for any $s\in[0,L]$, $h\in(0,1]$, $x_3\in[-\rho,\rho]$, $x_{o3}\in[\rho-2r_o,\rho-r_o]$, $\xi_3\in(\xi_{o3}-1,\xi_{o3}+1)$,

$$\begin{array}{l}\left|\partial_{x_3}\left(\mathcal{A}\left(x_o,\xi_{o3},n\right)f\right)(x,t,s)\right| \\ \leq\frac{1}{(2\pi)^4}\frac{1}{\sqrt{s^2+1}}\frac{1}{\left(\sqrt{s^2+1}\right)^{1/2}}ce^{-\frac{1}{ch}} \\ \qquad\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\left(\frac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}\right)d\xi d\tau\quad\text{when } n\in\{-2Q,2P+1\}\ .\end{array} \tag{6.5.8}$$

On the other hand, by Cauchy-Schwarz inequality, a trace theorem and conservation of energy (3.3), we have

$$\begin{array}{l}\int_{-T}^{T}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}\left(\int_{\Gamma_0}\left|\ell(x)\,u_3(x,t)\right|d\sigma\right)dt \\ \leq c\left(\int_{-T}^{T}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}dt\right)^{1/2}\left(\int_{-T}^{T}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}\int_{\Gamma_0}\left|u_3(x,t)\right|^2d\sigma dt\right)^{1/2} \\ \leq c\left(2\sqrt{\pi}\sqrt{(hs)^2+1}\right)^{1/2}\left(\int_{-T}^{T}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}\left\|u_3(\cdot,t)\right\|^2_{H^1(\Omega)}dt\right)^{1/2} \\ \leq c\left(\sqrt{(hs)^2+1}\right)\sqrt{\mathcal{G}(U,0)}\ .\end{array} \tag{6.5.9}$$

We conclude from (6.5.2) and (6.5.8) that

$$\begin{aligned}
&\left|h\int_{\Gamma_0}\int_{-T}^{T}\left(\int_0^L \partial_\nu\left(\mathbb{B}\left(x_o,\xi_{o3}\right)f\right)(x,t,s)\,ds\right)\ell(x)\,u_3(x,t)\,d\sigma dt\right| \\
&\leq \tfrac{h}{(2\pi)^4}ce^{-\frac{1}{ch}}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|d\xi d\tau \\
&\int_0^L \tfrac{1}{\sqrt{s^2+1}}\tfrac{1}{\left(\sqrt{s^2+1}\right)^{1/2}}\tfrac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}\int_{-T}^{T}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}\int_{\Gamma_0}|\ell(x)\,u_3(x,t)|\,d\sigma dt ds \\
&\leq chLe^{-\frac{1}{ch}}\sqrt{\mathcal{G}(U,0)}\left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|d\xi d\tau\right).
\end{aligned} \tag{6.5.10}$$

where in the last inequality, we used (6.5.9). This completes the proof.

6.6 Estimate for $\mathcal{I}_5$ (the boundary term on $\Gamma_0\cap\partial\omega$)

We estimate $\mathcal{I}_5=-h\int_{\Gamma_0\cap\partial\omega}\int_{-T}^{T}\left(\int_0^L \mathbb{B}\left(x_o,\xi_{o3}\right)f_3(x,t,s)\,ds\right)\partial_\nu\ell(x)\,u_3(x,t)\,d\sigma dt$ as follows.

Lemma 6.5 .- *There exists $c>0$ such that for any $(x_o,\xi_{o3})\in\overline{\omega_o}\times(2\mathbb{Z}+1)$ and $h\in(0,1]$, $L\geq1$, $\lambda\geq1$, $T>0$, we have*

$$|\mathcal{I}_5|\leq ch\left(1+\sqrt{hL}\right)\|(u_3,\partial_t u_3)\|_{L^2(\omega\times(-1-T,T+1))^2}\left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi F}(\xi,\tau)\right|d\xi d\tau\right). \tag{6.6.1}$$

Proof .- Since

$$\begin{aligned}
&\left|\left(\mathbb{B}\left(x_o,\xi_{o3}\right)f\right)(x,t,s)\right|\leq\sum_{n\in\mathbb{Z}}\left|\left(\mathcal{A}\left(x_o,\xi_{o3},n\right)f\right)(x,t,s)\right| \\
&\leq\tfrac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right| \\
&\left(\tfrac{1}{\sqrt{s^2+1}}e^{-\frac{1}{4h}\frac{|(x_1-x_{o1}-2\xi_1hs,x_2-x_{o2}-2\xi_2hs)|^2}{s^2+1}}\right)\left(\tfrac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}\right) \\
&\left(\tfrac{1}{\left(\sqrt{s^2+1}\right)^{1/2}}\sum_{n\in\mathbb{Z}}e^{-\frac{1}{4h}\frac{\left((-1)^nx_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3hs\right)^2}{s^2+1}}\right)d\xi d\tau
\end{aligned} \tag{6.6.2}$$

and

$$\sum_{n\in\mathbb{Z}}e^{-\frac{1}{4h}\frac{\left((-1)^nx_3+2n\frac{\xi_{o3}}{|\xi_{o3}|}\rho-x_{o3}-2\xi_3hs\right)^2}{s^2+1}}\leq2+\frac{\sqrt{\pi}}{\rho}\sqrt{h}\sqrt{s^2+1} \tag{6.6.3}$$

(see Appendix B with $z=\frac{4h}{\rho^2}\left(s^2+1\right)$), we have

$$\begin{aligned}
&\left|h\int_{\Gamma_0\cap\partial\omega}\int_{-T}^{T}\left(\int_0^L \mathbb{B}\left(x_o,\xi_{o3}\right)f ds\right)\partial_\nu\ell\left(x\right)u_3\left(x,t\right)d\sigma dt\right| \\
&\leq h\tfrac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\int_0^L \tfrac{1}{\sqrt{s^2+1}}\tfrac{\left(2+\frac{\sqrt{\pi}}{\rho}\sqrt{h}\sqrt{s^2+1}\right)}{\left(\sqrt{s^2+1}\right)^{1/2}} \\
&\quad\left(\tfrac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}\int_{-T}^{T}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}\int_{\Gamma_0\cap\partial\omega}\left|\partial_\nu\ell u_3\left(\cdot,t\right)\right|d\sigma dt\right)dsd\xi d\tau \\
&\leq ch\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|d\xi d\tau \\
&\quad\int_0^L \tfrac{1}{\sqrt{s^2+1}}\tfrac{\left(1+\sqrt{h}\sqrt{s^2+1}\right)}{\left(\sqrt{s^2+1}\right)^{1/2}}\tfrac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}\left(\int_{-\infty}^{\infty}e^{-\frac{1}{2}\frac{t^2}{(hs)^2+1}}dt\right)^{1/2}ds \\
&\quad\left(\int_{-T}^{T}\int_{\Gamma_0\cap\partial\omega}\left|\partial_\nu\ell u_3\right|^2 d\sigma dt\right)^{1/2} .
\end{aligned}\tag{6.6.4}$$

Now, we shall treat the term $\left(\int_{-T}^{T}\int_{\Gamma_0\cap\partial\omega}\left|\partial_\nu\ell u_3\right|^2 d\sigma dt\right)^{1/2}$ as follows. Let $W=W\left(x\right)$ be a smooth vector field such that $W=\nu$ on $\partial\Omega$ (see [9, p. 29]). Since

$$\operatorname{div}\left(Wu^2\left(\nabla\ell\cdot W\right)^2\right)=2u\left(\nabla u\cdot W\right)\left(\nabla\ell\cdot W\right)^2+u^2\nabla\left[\left(\nabla\ell\cdot W\right)^2\right]\cdot W+u^2\left(\nabla\ell\cdot W\right)^2\operatorname{div}W\ ,\tag{6.6.5}$$

we have the following trace theorem

$$\int_{\partial\Omega}\left|\partial_\nu\ell u\right|^2 d\sigma\leq c\int_\omega\left|u\right|^2 dx+c\int_\omega\left|\nabla u\right|^2\left(\nabla\ell\cdot W\right)^2 dx\ .\tag{6.6.6}$$

Next, by multiplying the equation $\partial_t^2 u_3-\Delta u_3=0$ by $u_3\left(\nabla\ell\cdot W\right)^2 g$ where $g\in C_0^\infty\left(-1-T,T+1\right)$ and $g=1$ in $(-T,T)$, we get, after integrations by parts and by Cauchy-Schwarz inequality, observing that $\partial_\nu u_3\partial_\nu\ell=0$ on $\partial\Omega$,

$$\int_\Omega\int_{-T}^{T}\left|\nabla u_3\right|^2\left(\nabla\ell\cdot W\right)^2 dxdt\leq c\int_\omega\int_{-1-T}^{T+1}\left(\left|u_3\right|^2+\left|\partial_t u_3\right|^2\right)dxdt\ .\tag{6.6.7}$$

Therefore, combining (6.6.6), (6.6.7) and (6.6.4), we conclude that

$$\begin{aligned}
&\left|h\int_{\Gamma_0\cap\partial\omega}\int_{-T}^{T}\left(\int_0^L \mathbb{B}\left(x_o,\xi_{o3}\right)f ds\right)\partial_\nu\ell\left(x\right)u_3\left(x,t\right)d\sigma dt\right| \\
&\leq ch\left(1+\sqrt{hL}\right)\left\|\left(u_3,\partial_t u_3\right)\right\|_{L^2(\omega\times(-1-T,T+1))^2}\left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|d\xi d\tau\right)\ .
\end{aligned}\tag{6.6.8}$$

This completes the proof.

6.7 Estimate for $\mathcal{I}_6$ (the term at $t=\pm T$)

We estimate the quantity

$$\begin{aligned}
\mathcal{I}_6 &= -h\int_\Omega\left[\left(\int_0^L \partial_t\left(\mathbb{V}\left(x_o,\xi_{o3}\right)F\right)\left(x,t,s\right)ds\right)\cdot\ell\left(x\right)U\left(x,t\right)\right]_{t=-T}^{t=T}dx \\
&\quad+h\int_\Omega\left[\left(\int_0^L \left(\mathbb{V}\left(x_o,\xi_{o3}\right)F\right)\left(x,t,s\right)ds\right)\cdot\ell\left(x\right)\partial_t U\left(x,t\right)\right]_{t=-T}^{t=T}dx
\end{aligned}$$

as follows.

Lemma 6.6 .- *There exists $c>0$ such that for any $(x_o,\xi_{o3})\in\overline{\omega_o}\times(2\mathbb{Z}+1)$ and $h\in(0,1]$, $L\geq 1$, $\lambda\geq 1$, we have*

$$\left|\mathcal{I}_4\right|\leq chL\lambda e^{-\frac{1}{h}}\sqrt{\mathcal{G}\left(U,0\right)}\left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{\left|\tau\right|<\lambda}\left|\widehat{\varphi F}\left(\xi,\tau\right)\right|d\xi d\tau\right) \tag{6.7.1}$$

when

$$T=4\left(\frac{\lambda hL}{\sqrt{2}}+\sqrt{h}L+\frac{1}{\sqrt{h}}\right)\ . \tag{6.7.2}$$

Proof .- Since

$$\begin{array}{l}\left|\partial_t\left(\mathcal{A}\left(x_o,\xi_{o3},n\right)f\right)\left(x,t,s\right)\right|\\ \leq\frac{1}{(2\pi)^4}\displaystyle\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{\left|\tau\right|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\\ \quad\left(\frac{1}{\sqrt{s^2+1}}e^{-\frac{1}{4h}\frac{\left|\left(x_1-x_{o1}-2\xi_1hs,x_2-x_{o2}-2\xi_2hs\right)\right|^2}{s^2+1}}\right)\left(\frac{1}{\left(\sqrt{s^2+1}\right)^{1/2}}e^{-\frac{1}{4h}\frac{\left((-1)^nx_3+2n\frac{\xi_{o3}}{\left|\xi_{o3}\right|}\rho-x_{o3}-2\xi_3hs\right)^2}{s^2+1}}\right)\\ \quad\left[\left|\tau\right|+\frac{1}{2}\frac{\left|t+2\tau hs\right|}{\sqrt{(hs)^2+1}}\right]\left(\frac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}\right)d\xi d\tau\ ,\end{array} \tag{6.7.3}$$

we have

$$\begin{array}{l}\left|\left(\mathbb{A}\left(x_o,\xi_{o3}\right)f\right)\left(x,\pm T,s\right)\right|+\left|\partial_t\left(\mathbb{A}\left(x_o,\xi_{o3}\right)f\right)\left(x,\pm T,s\right)\right|\\ +\left|\left(\mathbb{B}\left(x_o,\xi_{o3}\right)f\right)\left(x,\pm T,s\right)\right|+\left|\partial_t\left(\mathbb{B}\left(x_o,\xi_{o3}\right)f\right)\left(x,\pm T,s\right)\right|\\ \leq c\displaystyle\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{\left|\tau\right|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\frac{1}{\sqrt{s^2+1}}\left(\frac{1}{\left(\sqrt{s^2+1}\right)^{1/2}}\sum_{n\in\mathbb{Z}}e^{-\frac{1}{4h}\frac{\left((-1)^nx_3+2n\frac{\xi_{o3}}{\left|\xi_{o3}\right|}\rho-x_{o3}-2\xi_3hs\right)^2}{s^2+1}}\right)\\ \quad\left(1+\lambda+\frac{1}{2}\frac{\left|\pm T+2\tau hs\right|}{\sqrt{(hs)^2+1}}\right)\left(\frac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(\pm T+2\tau hs)^2}{(hs)^2+1}}\right)d\xi d\tau\\ \leq c\left(\frac{1}{\sqrt{s^2+1}}\frac{1+\sqrt{h}\sqrt{s^2+1}}{\left(\sqrt{s^2+1}\right)^{1/2}}\frac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}\right)\lambda\left(\displaystyle\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{\left|\tau\right|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|e^{-\frac{1}{8}\frac{(\pm T+2\tau hs)^2}{(hs)^2+1}}d\xi d\tau\right)\ .\end{array} \tag{6.7.4}$$

On the other hand,

$$e^{-\frac{1}{8}\frac{(\pm T+2\tau hs)^2}{(hs)^2+1}}\leq e^{-\frac{T^2}{16}\frac{1}{(hL)^2+1}}e^{\frac{1}{8}\frac{(2\lambda hL)^2}{(hL)^2+1}}\quad\forall s\in\left[0,L\right],\left|\tau\right|<\lambda\ . \tag{6.7.5}$$

Now, when $T=4\left(\frac{\lambda hL}{\sqrt{2}}+\sqrt{h}L+\frac{1}{\sqrt{h}}\right)$, then $\frac{1}{2}\left(\lambda hL\right)^2+\frac{1}{h}\left(\left(hL\right)^2+1\right)\leq\frac{T^2}{16}$ which implies that

$$e^{-\frac{T^2}{16}\frac{1}{(hL)^2+1}}e^{\frac{1}{8}\frac{(2\lambda hL)^2}{(hL)^2+1}}\leq e^{-\frac{1}{h}}\ . \tag{6.7.6}$$

In conclusion, combining (6.7.4), (6.7.5), (6.7.6) and conservation of energy (3.3), we get

$$\begin{array}{l}\left|-h\displaystyle\int_\Omega\left[\left(\int_0^L\partial_t\left(\mathbb{V}\left(x_o,\xi_{o3}\right)F\right)\left(\cdot,t,\cdot\right)ds\right)\cdot\ell U\left(\cdot,t\right)\right]_{-T}^{T}dx\right.\\ \qquad\left.-h\displaystyle\int_\Omega\left[-\left(\int_0^L\left(\mathbb{V}\left(x_o,\xi_{o3}\right)F\right)\left(\cdot,t,\cdot\right)ds\right)\cdot\ell\partial_tU\left(\cdot,t\right)\right]_{-T}^{T}dx\right|\\ \leq chL\lambda e^{-\frac{1}{h}}\sqrt{\mathcal{G}\left(U,0\right)}\left(\displaystyle\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{\left|\tau\right|<\lambda}\left|\widehat{\varphi F}\left(\xi,\tau\right)\right|d\xi d\tau\right)\ .\end{array} \tag{6.7.7}$$

This completes the proof.

6.8 Estimate for $\mathcal{I}_7$ (the internal term in ω)

We estimate $\mathcal{I}_7 = h\int_{\omega}\int_{-T}^{T}\left(\int_0^L \mathbb{V}\left(x_o,\xi_{o3}\right)F\left(x,t,s\right)ds\right)\cdot\left[2\left(\nabla\ell\left(x\right)\cdot\nabla\right)U\left(x,t\right)+\Delta\ell\left(x\right)U\left(x,t\right)\right]dxdt$ as follows.

Lemma 6.7 .- *There exists $c>0$ such that for any $(x_o,\xi_{o3})\in\overline{\omega_o}\times(2\mathbb{Z}+1)$ and $h\in(0,1]$, $L\geq 1$, $\lambda\geq 1$, $T>0$, we have*

$$\left|\mathcal{I}_7\right|\leq ch\left(1+\sqrt{hL}\right)\left\|(U,\partial_t U)\right\|_{L^2(\omega\times(-1-T,T+1))^6}\left(\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{\left|\tau\right|<\lambda}\left|\widehat{\varphi F}\left(\xi,\tau\right)\right|d\xi d\tau\right)\ . \tag{6.8.1}$$

Proof .- We start with the third component of $\mathbb{V}\left(x_o,\xi_{o3}\right)F$. Since

$$\begin{aligned}
&\left|\left(\mathbb{B}\left(x_o,\xi_{o3}\right)f\right)\left(x,t,s\right)\right|\leq\sum_{n\in\mathbb{Z}}\left|\left(\mathcal{A}\left(x_o,\xi_{o3},n\right)f\right)\left(x,t,s\right)\right|\\
&\leq\tfrac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{\left|\tau\right|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\\
&\quad\left(\tfrac{1}{\sqrt{s^2+1}}e^{-\frac{1}{4h}\frac{\left|\left(x_1-x_{o1}-2\xi_1 hs,x_2-x_{o2}-2\xi_2 hs\right)\right|^2}{s^2+1}}\right)\left(\tfrac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}\right)\\
&\quad\left(\tfrac{1}{\left(\sqrt{s^2+1}\right)^{1/2}}\sum_{n\in\mathbb{Z}}e^{-\frac{1}{4h}\frac{\left((-1)^n x_3+2n\frac{\xi_{o3}}{\left|\xi_{o3}\right|}\rho-x_{o3}-2\xi_3 hs\right)^2}{s^2+1}}\right)d\xi d\tau
\end{aligned}\tag{6.8.2}$$

and

$$\sum_{n\in\mathbb{Z}}e^{-\frac{1}{4h}\frac{\left((-1)^n x_3+2n\frac{\xi_{o3}}{\left|\xi_{o3}\right|}\rho-x_{o3}-2\xi_3 hs\right)^2}{s^2+1}}\leq 2+\frac{\sqrt{\pi}}{\rho}\sqrt{h}\sqrt{s^2+1} \tag{6.8.3}$$

(see Appendix B with $z=\frac{4h}{\rho^2}\left(s^2+1\right)$), we have

$$\begin{aligned}
&\left|\left(\mathbb{B}\left(x_o,\xi_{o3}\right)f\right)\left(x,t,s\right)\right|\\
&\leq\tfrac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{\left|\tau\right|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\left(\tfrac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}\right)d\xi d\tau\\
&\quad\left(\tfrac{1}{\sqrt{s^2+1}}\right)\left(\tfrac{2+\frac{\sqrt{\pi}}{\rho}\sqrt{h}\sqrt{s^2+1}}{\left(\sqrt{s^2+1}\right)^{1/2}}\right)\ .
\end{aligned}\tag{6.8.4}$$

Consequently, we deduce from the above that

$$\begin{aligned}
&\left|h\int_{\omega}\int_{-T}^{T}\left(\int_0^L\left(\mathbb{B}\left(x_o,\xi_{o3}\right)f\right)\left(x,t,s\right)ds\right)\left[2\nabla\ell\nabla u_3+\Delta\ell u_3\right]\left(x,t\right)dxdt\right|\\
&\leq ch\int_{\omega}\int_{-T}^{T}\left(\int_0^L\left|\left(\mathbb{B}\left(x_o,\xi_{o3}\right)f\right)\left(x,t,s\right)\right|ds\right)\left(\left|\nabla\ell\right|\left|\nabla u_3\right|+\left|u_3\right|\right)\left(x,t\right)dxdt\\
&\leq ch\tfrac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\xi_{o3}-1}^{\xi_{o3}+1}\int_{\left|\tau\right|<\lambda}\left|\widehat{\varphi f}\left(\xi,\tau\right)\right|\int_0^L\tfrac{1}{\sqrt{s^2+1}}\tfrac{\left(2+\frac{\sqrt{\pi}}{\rho}\sqrt{h}\sqrt{s^2+1}\right)}{\left(\sqrt{s^2+1}\right)^{1/2}}\\
&\quad\left(\int_{-T}^{T}\left(\tfrac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}}e^{-\frac{1}{4}\frac{(t+2\tau hs)^2}{(hs)^2+1}}\right)\int_{\omega}\left(\left|\nabla\ell\right|\left|\nabla u_3\right|+\left|u_3\right|\right)\left(x,t\right)dxdt\right)dsd\xi d\tau
\end{aligned}\tag{6.8.5}$$

which implies using Cauchy-Schwarz inequality

$$\begin{aligned}
&\left| h\int_\omega \int_{-T}^{T} \left(\int_0^L \left(\mathbb{B}\left(x_o,\xi_{o3}\right) f\right)(x,t,s)\, ds \right) \left[2\nabla\ell\nabla u_3 + \Delta\ell u_3\right](x,t)\, dxdt \right| \\
&\leq ch \tfrac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} \left| \widehat{\varphi f}\left(\xi,\tau\right)\right| d\xi d\tau \\
&\quad \int_0^L \tfrac{1}{\sqrt{s^2+1}} \tfrac{\left(1+\sqrt{h}\sqrt{s^2+1}\right)}{\left(\sqrt{s^2+1}\right)^{1/2}} \left(\tfrac{1}{\left(\sqrt{(hs)^2+1}\right)^{1/2}} \right) \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{t^2}{(hs)^2+1}} dt \right)^{1/2} ds \\
&\quad \left(\int_\omega \int_{-T}^{T} \left(\left|\nabla\ell\right|^2 \left|\nabla u_3\right|^2 + \left|u_3\right|^2 \right) dxdt \right)^{1/2} \\
&\leq ch\left(1+\sqrt{hL}\right) \left(\int_\omega \int_{-T}^{T} \left(\left|\nabla\ell\right|^2 \left|\nabla u_3\right|^2 + \left|u_3\right|^2 \right) dxdt \right)^{1/2} \left(\int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} \left| \widehat{\varphi f}\left(\xi,\tau\right)\right| d\xi d\tau \right) .
\end{aligned} \tag{6.8.6}$$

Similarly, for any $j \in \{1,2\}$,

$$\begin{aligned}
&\left| h\int_\omega \int_{-T}^{T} \left(\int_0^L \left(\mathbb{A}\left(x_o,\xi_{o3}\right) f\right)(x,t,s)\, ds \right) \left[2\nabla\ell\nabla u_j + \Delta\ell u_j\right](x,t)\, dxdt \right| \\
&\leq ch\left(1+\sqrt{hL}\right) \left(\int_\omega \int_{-T}^{T} \left(\left|\nabla\ell\right|^2 \left|\nabla u_j\right|^2 + \left|u_j\right|^2 \right) dxdt \right)^{1/2} \left(\int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} \left| \widehat{\varphi f}\left(\xi,\tau\right)\right| d\xi d\tau \right) .
\end{aligned} \tag{6.8.7}$$

Now, we shall bound the term $\left(\int_\omega \int_{-T}^{T} \left(\left|\nabla\ell\right|^2 \left|\nabla u_j\right|^2 + \left|u_j\right|^2 \right) dxdt \right)^{1/2}$ for any $j \in \{1,2,3\}$ by the quantity $\left\|(U,\partial_t U)\right\|_{L^2(\omega\times(-1-T,T+1))^6}$. By multiplying the equation $\partial_t^2 u_j - \Delta u_j = 0$ by $u_j \left|\nabla\ell\right|^2 g$ where $g \in C_0^\infty\left(-1-T,T+1\right)$ and $g=1$ in $(-T,T)$, we get, after integrations by parts and by Cauchy-Schwarz inequality, observing that $u_j\partial_\nu u_j \left|\nabla\ell\right| = 0$ on $\partial\Omega$,

$$\int_\Omega \int_{-T}^{T} \left|\nabla u_j\right|^2 \left|\nabla\ell\right|^2 dxdt \leq c \int_\omega \int_{-1-T}^{T+1} \left(\left|u_j\right|^2 + \left|\partial_t u_j\right|^2 \right) dxdt \ , \tag{6.8.8}$$

for any $j \in \{1,2,3\}$. This completes the proof.

6.9 Key inequality

From now,

$$T = 4\left(\frac{\lambda h L}{\sqrt{2}} + \sqrt{h}L + \frac{1}{\sqrt{h}} \right) . \tag{6.9.1}$$

By (6.1.7), (6.2.1), (6.3.1), (6.4.1), (6.5.1), (6.6.1), (6.7.1) and (6.8.1), there exists $c>0$ such that for any $(x_o,\xi_{o3}) \in \overline{\omega_o} \times (2\mathbb{Z}+1)$ and $h \in (0,1]$, $L \geq 1$, $\lambda \geq 1$, we have

$$\begin{aligned}
&\left| \int_{\Omega\times\mathbb{R}} \left(\tfrac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} e^{i(x\xi+t\tau)} \widehat{\varphi F}\left(\xi,\tau\right) d\xi d\tau \right) \cdot a\left(x-x_o,t,0\right) \ell\left(x\right) U\left(x,t\right) dxdt \right| \\
&\leq c\left[\left(1+hL\lambda\right) e^{-\frac{1}{ch}} + \tfrac{1}{\sqrt{L}} \right] \sqrt{\mathcal{G}\left(U,0\right)} \left(\int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} \left| \widehat{\varphi F}\left(\xi,\tau\right)\right| d\xi d\tau \right) \\
&\quad + ch\left(1+\sqrt{hL}\right) \left(\left\|(U,\partial_t U)\right\|_{L^2(\omega\times(-1-T,T+1))^6} \right) \left(\int_{\mathbb{R}^2} \int_{\xi_{o3}-1}^{\xi_{o3}+1} \int_{|\tau|<\lambda} \left| \widehat{\varphi F}\left(\xi,\tau\right)\right| d\xi d\tau \right) .
\end{aligned} \tag{6.9.2}$$

By summing over $\xi_{o3} \in (2\mathbb{Z}+1)$, it implies that

$$\begin{aligned}
&\left|\int_{\Omega\times\mathbb{R}} \left(\tfrac{1}{(2\pi)^4}\int_{\mathbb{R}^3}\int_{|\tau|<\lambda} e^{i(x\xi+t\tau)}\widehat{\varphi F}(\xi,\tau)\, d\xi d\tau\right)\cdot a(x-x_o,t,0)\,\ell(x)\,U(x,t)\,dxdt\right| \\
&\leq c\left[(1+hL\lambda)\,e^{-\frac{1}{ch}}+\tfrac{1}{\sqrt{L}}\right]\sqrt{\mathcal{G}(U,0)}\left(\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi F}(\xi,\tau)\right|d\xi d\tau\right) \\
&\quad +ch\left(1+\sqrt{hL}\right)\left(\|(U,\partial_t U)\|_{L^2(\omega\times(-1-T,T+1))^6}\right)\left(\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi F}(\xi,\tau)\right|d\xi d\tau\right)\ .
\end{aligned} \tag{6.9.3}$$

On the other hand, from (A2) of Appendix A,

$$\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi F}(\xi,\tau)\right|d\xi d\tau \leq c\sqrt{\lambda}\left(\lambda^2+\frac{1}{h}\right)\sqrt{\mathcal{G}(U,0)} \tag{6.9.4}$$

whenever $F=(f_1,f_2,f_3)$ with $f_j=u_j$. Therefore, by (6.9.3) with (6.9.4), we obtain that

$$\begin{aligned}
&\left|\int_{\Omega\times\mathbb{R}} \left(\tfrac{1}{(2\pi)^4}\int_{\mathbb{R}^3}\int_{|\tau|<\lambda} e^{i(x\xi+t\tau)}\widehat{\varphi U}(\xi,\tau)\, d\xi d\tau\right)\cdot a(x-x_o,t,0)\,\ell(x)\,U(x,t)\,dxdt\right| \\
&\leq c\left[(1+hL\lambda)\,e^{-\frac{1}{ch}}+\tfrac{1}{\sqrt{L}}\right]\sqrt{\lambda}\left(\lambda^2+\tfrac{1}{h}\right)\mathcal{G}(U,0) \\
&\quad +ch\left(1+\sqrt{hL}\right)\left(\|(U,\partial_t U)\|_{L^2(\omega\times(-1-T,T+1))^6}\right)\sqrt{\lambda}\left(\lambda^2+\tfrac{1}{h}\right)\sqrt{\mathcal{G}(U,0)} \\
&\leq c\tfrac{1}{\sqrt{L}}\sqrt{\lambda}\left(\lambda^2+\tfrac{1}{h}\right)\mathcal{G}(U,0)+ce^{-\frac{1}{ch}}\sqrt{\lambda}\,(1+hL\lambda)\left(\lambda^2+\tfrac{1}{h}\right)\mathcal{G}(U,0) \\
&\quad +ch\left(1+\sqrt{hL}\right)^2\sqrt{L\lambda}\left(1+h\lambda^2\right)\|(U,\partial_t U)\|^2_{L^2(\omega\times(-1-T,T+1))^6}\ ,
\end{aligned} \tag{6.9.5}$$

where in the last inequality we used Cauchy-Schwarz. Next, we choose $L\geq 1$ so that $\frac{1}{\sqrt{L}}\sqrt{\lambda}\left(\lambda^2+\frac{1}{h}\right)=\frac{1}{\sqrt{\lambda}}$ i.e., $L=\lambda^2\left(\lambda^2+\frac{1}{h}\right)^2$. Then there exists $c,\delta,\eta>0$ such that for any $x_o\in\overline{\omega_o}$ and $h\in(0,1]$, $\lambda\geq 1$, we have

$$\begin{aligned}
&\left|\int_{\Omega\times\mathbb{R}} \left(\tfrac{1}{(2\pi)^4}\int_{\mathbb{R}^3}\int_{|\tau|<\lambda} e^{i(x\xi+t\tau)}\widehat{\varphi U}(\xi,\tau)\, d\xi d\tau\right)\cdot a(x-x_o,t,0)\,\ell(x)\,U(x,t)\,dxdt\right| \\
&\leq c\tfrac{1}{\sqrt{\lambda}}\mathcal{G}(U,0)+ce^{-\frac{1}{ch}}\tfrac{\lambda^\delta}{h^\eta}\mathcal{G}(U,0)+c\tfrac{\lambda^\delta}{h^\eta}\|(U,\partial_t U)\|^2_{L^2(\omega\times(-1-T,T+1))^6}\ ,
\end{aligned} \tag{6.9.6}$$

and further

$$1+T=1+4\left(\frac{\lambda hL}{\sqrt{2}}+\sqrt{h}L+\frac{1}{\sqrt{h}}\right)\leq c\frac{\lambda^\delta}{h^\eta}\ . \tag{6.9.7}$$

This implies our claim (6.6) since $a_o(x,t)=a(x-x_o,t,0)$. This completes the proof.

Appendix A

The goal of this Appendix A is to prove the two following inequalities (A1) and (A2) below.

Lemma A .- *Let*

$$a_o(x,t)=e^{-\frac{1}{c_1 h}|x-x_o|^2}e^{-\frac{1}{c_2}t^2} \quad \text{and} \quad \varphi(x,t)=\phi(x)\,e^{-\frac{1}{c_3 h}|x-x_o|^2}e^{-\frac{1}{c_4}t^2}$$

for some $c_1,c_2,c_3,c_4>0$ *and* $\phi\in C_0^\infty(\Omega)$. *Let* $\ell\in C^\infty(\mathbb{R}^3)$ *be such that* $0\leq\ell(x)\leq 1$. *There exists* $c>0$ *such that for any* $h\in(0,1]$, $\lambda\geq 1$ *and any* $u\in C^1(\mathbb{R},H^1(\Omega))\cap C^2(\mathbb{R},L^2(\Omega))$ *satisfying*

$$\partial_t^2 u-\Delta u=0 \quad \text{in } \Omega\times\mathbb{R}\ ,$$

and

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le R_0, \quad \|\partial_t u(\cdot,t)\|_{L^2(\Omega)} + \|\nabla u(\cdot,t)\|_{L^2(\Omega)} \le R_1 \ ,$$

we have

$$\begin{aligned}&\left|\int_{\Omega\times\mathbb{R}} \left(\tfrac{1}{(2\pi)^4}\int_{\mathbb{R}^3}\int_{|\tau|\ge\lambda} e^{i(x\xi+t\tau)}\widehat{\varphi u}(\xi,\tau)\,d\xi d\tau\right) a_o(x,t)\,\ell(x)\,u(x,t)\,dxdt\right| \\ &\le c\sqrt{\tfrac{1}{\lambda}}R_0(R_0+R_1)\end{aligned} \tag{A1}$$

and

$$\int_{\mathbb{R}^3}\int_{|\tau|<\lambda} |\widehat{\varphi u}(\xi,\tau)|\,d\xi d\tau \le c\sqrt{\lambda}\left(\lambda^2+\frac{1}{h}\right)R_0 + c\sqrt{\lambda}\frac{1}{\sqrt{h}}R_1 \ . \tag{A2}$$

Proof of (A1). Introduce

$$\mathcal{R}(f) = \int_{\Omega\times\mathbb{R}}\left(\frac{1}{(2\pi)^4}\int_{\mathbb{R}^3}\int_{|\tau|\ge\lambda} e^{i(x\xi+t\tau)}\widehat{\varphi f}(\xi,\tau)\,d\xi d\tau\right) a_o(x,t)\,\ell(x)\,u(x,t)\,dxdt \ .$$

Thus,

$$\begin{aligned}|\mathcal{R}(f)| &= \left|\int_{\Omega\times\mathbb{R}} a_o(x,t)\,\ell(x)\,u(x,t)\,\partial_t\left(\tfrac{1}{(2\pi)^4}\int_{\mathbb{R}^3}\int_{|\tau|\ge\lambda}\tfrac{1}{i\tau}e^{i(x\xi+t\tau)}\widehat{\varphi f}(\xi,\tau)\,d\xi d\tau\right)dxdt\right| \ , \\ &= \left|\int_{\Omega\times\mathbb{R}} \ell(x)\,\partial_t(a_o u(x,t))\left(\tfrac{1}{2\pi}\int_{|\tau|\ge\lambda}\tfrac{1}{i\tau}e^{it\tau}\left[\int_{\mathbb{R}} e^{-i\theta\tau}(\varphi f)(x,\theta)\,d\theta\right]d\tau\right)dxdt\right| \ .\end{aligned}$$

It follows using Cauchy-Schwarz inequality and Parseval identity that

$$\begin{aligned}|\mathcal{R}(f)| &\le \int_{\Omega\times\mathbb{R}} |\ell(x)\,\partial_t(a_o u(x,t))|\left(\tfrac{1}{2\pi}\left[\int_{|\tau|\ge\lambda}\tfrac{1}{\tau^2}d\tau\right]^{1/2}\left[\int_{\mathbb{R}}\left|\int_{\mathbb{R}} e^{-i\theta\tau}(\varphi f)(x,\theta)\,d\theta\right|^2 d\tau\right]^{1/2}\right)dxdt \\ &\le \int_{\Omega\times\mathbb{R}} |\partial_t(a_o u(x,t))|\left(\tfrac{1}{2\pi}\left[\int_{|\tau|\ge\lambda}\tfrac{1}{\tau^2}d\tau\right]^{1/2}\left[2\pi\int_{\mathbb{R}}|(\varphi f)(x,\theta)|^2 d\theta\right]^{1/2}\right)dxdt \\ &\le \int_{\Omega\times\mathbb{R}} |\partial_t(a_o u(x,t))|\left(\tfrac{1}{\sqrt{2\pi}}\sqrt{\tfrac{2}{\lambda}}\,\|(\varphi f)(x,\cdot)\|_{L^2(\mathbb{R})}\right)dxdt \\ &\le \tfrac{1}{\sqrt{\pi}}\sqrt{\tfrac{1}{\lambda}}\int_{\mathbb{R}}\|\partial_t(a_o u)(\cdot,t)\|_{L^2(\Omega)}\,dt\,\|\varphi f\|_{L^2(\Omega\times\mathbb{R})} \ .\end{aligned}$$

It remains to estimate $\int_{\mathbb{R}}\|\partial_t(a_o u)(\cdot,t)\|_{L^2(\Omega)}$. We have

$$\begin{aligned}\int_{\mathbb{R}}\|\partial_t(a_o u)(\cdot,t)\|_{L^2(\Omega)}\,dt &\le \int_{\mathbb{R}}\left[\int_{\Omega}|\partial_t a_o u(x,t)|^2 dx\right]^{1/2}dt + \int_{\mathbb{R}}\left[\int_{\Omega}|a_o\partial_t u(x,t)|^2 dx\right]^{1/2}dt \\ &\le \int_{\mathbb{R}} e^{-\frac{1}{c_2}t^2}\left(\tfrac{2|t|}{c_2}\left[\int_{\Omega}|u(x,t)|^2 dx\right]^{1/2} + \left[\int_{\Omega}|\partial_t u(x,t)|^2 dx\right]^{1/2}\right)dt \\ &\le c(R_0+R_1) \ ,\end{aligned}$$

where in the last line we used

$$\left(\int_{\mathbb{R}} e^{-\frac{t^2}{c}}\int_{\Omega}|u(x,t)|^2 dxdt\right)^{1/2} \le cR_0 \ .$$

We conclude that

$$|\mathcal{R}(u)| \le c\sqrt{\frac{1}{\lambda}}R_0(R_0+R_1) \ .$$

That completes the proof of (A1).

Proof of (A2). By Cauchy-Schwarz inequality,

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\widehat{\varphi f}(\xi,\tau)\right|d\xi d\tau &= \int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\tfrac{1}{1+|\xi|^2}\left|\left(1+|\xi|^2\right)\widehat{\varphi f}(\xi,\tau)\right|d\xi d\tau \\
&\leq \int_{|\tau|<\lambda}\left[\int_{\mathbb{R}^3}\tfrac{1}{\left(1+|\xi|^2\right)^2}d\xi\right]^{1/2}\left[\int_{\mathbb{R}^3}\left|\left(\widehat{(1-\Delta)(\varphi f)}\right)(\xi,\tau)\right|^2 d\xi\right]^{1/2} d\tau \\
&\leq \pi^2\sqrt{\lambda}\left[\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}\left|\left(\widehat{(1-\Delta)(\varphi f)}\right)(\xi,\tau)\right|^2 d\xi d\tau\right]^{1/2} .
\end{aligned}$$

Since

$$\begin{aligned}
(1-\Delta)(\varphi u) &= \varphi u-\varphi\Delta u+\Delta\varphi u+2\nabla\varphi\nabla u \\
&= -\varphi\partial_t^2 u+\varphi u+\Delta\varphi u+2\nabla\varphi\nabla u \\
&= -\varphi\partial_t^2 u+\varphi u+\Delta\varphi u+2\nabla\varphi\nabla u \\
&= -\partial_t^2(\varphi u)+\partial_t^2\varphi u+2\partial_t\varphi\partial_t u+\varphi u+\Delta\varphi u+2\nabla\varphi\nabla u
\end{aligned}$$

and $\widehat{\partial_t^2(\varphi u)}=-\tau^2\widehat{\varphi u}$, we get when $f=u$, using Parseval identity

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}|\widehat{\varphi u}(\xi,\tau)|\,d\xi d\tau \\
&\leq c\sqrt{\lambda}\lambda^2\|\varphi u\|_{L^2(\Omega\times\mathbb{R})}+c\sqrt{\lambda}\left(\left\|\partial_t^2\varphi u\right\|_{L^2(\Omega\times\mathbb{R})}+\|\partial_t\varphi\partial_t u\|_{L^2(\Omega\times\mathbb{R})}\right) \\
&\quad +c\sqrt{\lambda}\left(\|\varphi u\|_{L^2(\Omega\times\mathbb{R})}+\|\Delta\varphi u\|_{L^2(\Omega\times\mathbb{R})}+\|\nabla\varphi\nabla u\|_{L^2(\Omega\times\mathbb{R})}\right) .
\end{aligned}$$

On the other hand, remark that $\partial_x^j\varphi(x,t)=\frac{1}{h^{j/2}}\phi_j(x)\,e^{-\frac{1}{c_3 h}|x-x_o|^2}e^{-\frac{1}{c_4}t^2}$ for some $\phi_j\in C_0^\infty(\Omega)$ and $\left|\partial_t^2\varphi(x,t)\right|+|\partial_t\varphi(x,t)|\leq c\phi(x)\,e^{-\frac{1}{c_3 h}|x-x_o|^2}e^{-\frac{1}{c}t^2}$. It implies that

$$\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}|\widehat{\varphi u}(\xi,\tau)|\,d\xi d\tau\leq c\sqrt{\lambda}\lambda^2 R_0+c\sqrt{\lambda}\left(\frac{1}{h}R_0+\frac{1}{\sqrt{h}}R_1\right) .$$

We conclude that there exists $c>0$ such that for any $h\in(0,1]$ and $\lambda\geq 1$,

$$\int_{\mathbb{R}^3}\int_{|\tau|<\lambda}|\widehat{\varphi u}(\xi,\tau)|\,d\xi d\tau\leq c\sqrt{\lambda}\left(\lambda^2+\frac{1}{h}\right)R_0+c\sqrt{\lambda}\frac{1}{\sqrt{h}}R_1 .$$

That completes the proof of (A2).

Appendix B

The goal of this Appendix B is to prove the two following inequalities.

Lemma B .- *For any $x,y,C,R\in\mathbb{R}$, any $z\in\mathbb{C},\operatorname{Re}z>0$,*

$$\left|\frac{1}{\sqrt{z}}\sum_{n\in\mathbb{Z}}(-1)^n e^{-\frac{1}{z}(2n+C(-1)^n+R)^2}e^{inx}e^{i(-1)^n y}\right|\leq\frac{\sqrt{\pi}}{2}+\frac{2}{\sqrt{\operatorname{Re}z}} ,$$

$$\left|\frac{1}{\sqrt{z}}\sum_{n\in\mathbb{Z}}e^{-\frac{1}{z}(2n+C(-1)^n+R)^2}e^{inx}e^{i(-1)^n y}\right|\leq\frac{\sqrt{\pi}}{2}+\frac{2}{\sqrt{\operatorname{Re}z}} .$$

Proof .- First we recall the Poisson summation formula. Let $u\in C^2(\mathbb{R},\mathbb{C})$ be such that for any $k\in\{0,1,2\}$, the functions $x\longmapsto\left(1+x^2\right)u^{(k)}(x)$ are bounded on $\mathbb{R}$. Then for any $x\in\mathbb{R}$,

$$\sum_{n\in\mathbb{Z}}u(x+n)=\sum_{n\in\mathbb{Z}}\widehat{u}(2\pi n)\,e^{2\pi inx}\quad\text{where}\quad\widehat{u}(2\pi n)=\int_{\mathbb{R}}u(t)\,e^{-2\pi int}dt .$$

Next, by choosing $u(x) = v(x)\, e^{-2\pi i B x}$ for some $B \in \mathbb{R}$ and $v \in C^2(\mathbb{R}, \mathbb{C})$ such that for any $k \in \{0, 1, 2\}$, the functions $x \longmapsto (1 + x^2)\, v^{(k)}(x)$ are bounded on $\mathbb{R}$, we obtain that for any $x, B \in \mathbb{R}$,

$$\sum_{n\in\mathbb{Z}} \widehat{v}(2\pi(n+B))\, e^{2\pi i n x} = \sum_{n\in\mathbb{Z}} v(x+n)\, e^{-2\pi i B(x+n)} \quad \text{where} \quad \widehat{v}(2\pi(n+B)) = \int_{\mathbb{R}} v(t)\, e^{-2\pi i(n+B)t} dt\ .$$

Now, for any $z \in \mathbb{C}$ such that $\operatorname{Re} z > 0$, we take $v(x) = e^{-\frac{z}{2}x^2}$ in order that $\widehat{v}(2\pi(n+B)) = \frac{\sqrt{2\pi}}{\sqrt{z}} e^{-\frac{1}{2z}(2\pi(n+B))^2}$. Thus, after simple changes, the following formula holds for any $x, B \in \mathbb{R}$, any $z \in \mathbb{C}, \operatorname{Re} z > 0, a > 0$,

$$\frac{1}{\sqrt{z}} \sum_{n\in\mathbb{Z}} e^{-\frac{a}{z}(n+B)^2} e^{i2nx} = \frac{\sqrt{\pi}}{\sqrt{a}} \sum_{n\in\mathbb{Z}} e^{-\frac{z}{a}(x+\pi n)^2} e^{-i2B(x+\pi n)}\ .$$

Finally, we deduce that for any $x, y, C, R \in \mathbb{R}$, any $z \in \mathbb{C}, \operatorname{Re} z > 0$,

$$\begin{aligned}
& \left| \tfrac{1}{\sqrt{z}} \textstyle\sum_{n\in\mathbb{Z}} (-1)^n\, e^{-\frac{1}{z}(2n+C(-1)^n+R)^2} e^{inx} e^{i(-1)^n y} \right| \\
&= \left| \tfrac{1}{\sqrt{z}} \textstyle\sum_{n\in\mathbb{Z}} e^{-\frac{1}{z}(4n+C+R)^2} e^{i2nx} e^{iy} - \tfrac{1}{\sqrt{z}} \textstyle\sum_{n\in\mathbb{Z}} e^{-\frac{1}{z}(4n+2-C+R)^2} e^{i(2n+1)x} e^{-iy} \right| \\
&= \left| e^{iy} \tfrac{1}{\sqrt{z}} \textstyle\sum_{n\in\mathbb{Z}} e^{-\frac{4^2}{z}\left(n+\frac{C+R}{4}\right)^2} e^{i2nx} - e^{-iy} e^{ix} \tfrac{1}{\sqrt{z}} \textstyle\sum_{n\in\mathbb{Z}} e^{-\frac{4^2}{z}\left(n+\frac{2-C+R}{4}\right)^2} e^{i2nx} \right| \\
&= \left| e^{iy} \tfrac{\sqrt{\pi}}{4} \textstyle\sum_{n\in\mathbb{Z}} e^{-\frac{z}{4^2}(x+\pi n)^2} e^{-i\frac{C+R}{2}(x+\pi n)} - e^{-iy} \tfrac{\sqrt{\pi}}{4} \textstyle\sum_{n\in\mathbb{Z}} e^{-\frac{z}{4^2}(x+\pi n)^2} e^{-i\frac{2-C+R}{2}(x+\pi n)} \right| \\
&\leq \tfrac{\sqrt{\pi}}{2} \textstyle\sum_{n\in\mathbb{Z}} e^{-\frac{\operatorname{Re} z}{4^2}\pi^2\left(\frac{x}{\pi}+n\right)^2} \leq \tfrac{\sqrt{\pi}}{2} + \tfrac{2}{\sqrt{\operatorname{Re} z}}\ ,
\end{aligned}$$

and similarly

$$\begin{aligned}
& \left| \tfrac{1}{\sqrt{z}} \textstyle\sum_{n\in\mathbb{Z}} e^{-\frac{1}{z}(2n+C(-1)^n+R)^2} e^{inx} e^{i(-1)^n y} \right| \\
&= \left| e^{iy} \tfrac{1}{\sqrt{z}} \textstyle\sum_{n\in\mathbb{Z}} e^{-\frac{1}{z}(4n+C+R)^2} e^{i2nx} + e^{-iy} e^{ix} \tfrac{1}{\sqrt{z}} \textstyle\sum_{n\in\mathbb{Z}} e^{-\frac{1}{z}(4n+2-C+R)^2} e^{i2nx} \right| \\
&\leq \tfrac{\sqrt{\pi}}{2} + \tfrac{2}{\sqrt{\operatorname{Re} z}}\ .
\end{aligned}$$

References

[1] C. Amrouche, C. Bernardi, M. Dauge, V. Girault, Vector potentials in three-dimensional non-smooth domains, Math. Methods Appl. Sci. 21 (1998) 823–864.

[2] C. Bardos, G. Lebeau, J. Rauch, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, SIAM J. Control Optim. 30 (1992) 1024–1065.

[3] P. Boissoles, Problèmes mathématiques et numériques issus de l'imagerie par résonance magnétique nucléaire, Doctoral thesis, Université de Rennes 1, 2005.

[4] N. Burq, M. Hitrik, Energy decay for damped wave equations on partially rectangular domains, Math. Res. Lett. 14 (2007) 35–47.

[5] M. Cessenat, Mathematical method in electromagnetism, linear theory and applications. World Scientific, Singapore, 1996.

[6] R. Dautray, J.-L. Lions, Analyse mathématique et calcul numérique pour les sciences et les techniques, Volume 5, Spectre des opérateurs. Masson, Paris, 1988.

[7] G. Duvaut, J.-L. Lions, Les inéquations en mécanique et en physique. Dunod, Paris, 1972.

[8] S.S. Krigman, C.E. Wayne, Boundary controllability of Maxwell's equations with nonzero conductivity inside a cube, I: Spectral controllability, J. Math. Anal. Appl. 329 (2007) 1375–1396.

[9] J.-L. Lions, Contrôlabilité exacte, perturbations et stabilisation des systèmes distribués I. Masson, Paris, 1988.

[10] H. Nishiyama, Polynomial decay for damped wave equations on partially rectangular domains, Math. Res. Lett. 16 (2009) 881–894.

[11] K. D. Phung, Contrôle et stabilisation d'ondes électromagnétiques, ESAIM Control Optim. Calc. Var. 5 (2000) 87–137.

[12] K. D. Phung, Polynomial decay rate for the dissipative wave equation, J. Diff. Eq. 240 (2007) 92–124.

[13] J. Ralston, Gaussian beams and propagation of singularities, in Studies in Partial Differential equations, Littman éd., MAA studies in Mathematics 23 (1982) 206–248.

[14] W. Wei, H-M. Yin, J. Tang, An optimal control problem for microwave heating, Nonlinear Analysis 75 (2012) 2024–2036.

[15] R. Ziolkowski, Exact solutions of the wave equation with complex source locations, J. Math. Phys. 26, 4 (1985) 861–863.