OPTIMAL CONTROL OF DISCRETE EVENT SYSTEMS IN CASE OF UPDATED REFERENCE INPUT

E. Menguy, J.L. Boimond, J.L. Hardouin

Laboratoire d'Ingénierie des Systèmes Automatisés
62, avenue Notre-Dame du lac 49000 ANGERS
[eric.menguy, laurent.hardouin, jean-louis.boimond]@istia.univ-angers.fr

Abstract: we present an optimal control under the case of updated reference input in production. The proposed approach uses the concept of optimal control defined for a reference input entirely known. The development is based on a decomposition allowing the separation of past and future control effects. Computation of the control law is given and is illustrated through an example.

Résumé: nous présentons une commande optimale dans le cas d'une mise à jour de la consigne en cours de production. L'approche proposée utilise le concept de commande optimale définie pour une consigne entièrement connue. Le développement est basé sur une décomposition permettant de séparer les effets de commande passés et futurs. Le calcul de la loi de commande est donné et illustré à travers un exemple.

Keywords: Discrete event systems. Timed event graphs. Linear systems. Optimal control. Constraints. Reference input.

1 Introduction

A linear system theory analogous to the conventional theory has been developed for a certain class of Petri nets called event graphs using some particular algebras (Baccelli, *et al.*, 1992; Cohen, *et al.*, 1989). An example of such algebra is $(\max, +)$ -algebra: if the focus is on the date of occurrence of events, the behavior of these systems can be represented by a state representation or by an input-output relation (Max-Plus, 1991; Gaubert, 1992). The control of these systems can be stated as follows:

given a desired output trajectory also called reference input, find the *latest (greatest)* input trajectory which yields an output trajectory less (earlier) than the given one (Baccelli, *et al.*, 1992, §5.6; Cohen, *et al.*, 1989; Cohen, *et al.*, 1993; Libeaut and Loiseau, 1995). This control statement is linked to the notion of Just in Time: in a production context, we minimize the stocks while satisfying the customer's demand.

An optimal control has been defined; its synthesis consists in 'inverting' a dynamic system and appeals to a theory -residuation theory- which defines the notion of 'inversion' in ordered sets (Baccelli, *et al.*, 1992, §4.4; Blyth and Janowitz, 1972; Cuninghame, 1979).

The proposed method needs to entirely know the desired output trajectory before the production launching.

The purpose of this paper is to extend the concept of optimal control in case of reference input updated in production to take into account a new production objective, for example the desired output trajectory:

- is modified in production;
- or is partially known and sequentially completed according to successive customer demands.

Construction of such optimal control implies taking into account both the new desired output trajectory and the effect of past input due to former specification. This control will be called optimal control in case of updated reference input.

The paper is organized as follows: in section 2 we recall a few algebraic and linear system theory elements. The control problem is stated in section 3. After introducing the notion of past and future signals, we define and give the expression of the optimal control in case of updated reference input in section 4. We present in section 5 the practical synthesis of the previous optimal control. To conclude we use a short example in section 6.

2 Preliminaries

2.1 Dioid

For an exhaustive presentation see (Baccelli, *et al.*, 1992; Cuninghame, 1979).

A semiring is a triple $(S, \oplus, \otimes)$ where $(S, \oplus)$ and $(S, \otimes)$ are monoids with neutral elements ε and e respectively. $\oplus$ is commutative. $\otimes$ is distributive with respect to $\oplus$, and the neutral element of $\oplus$ is absorbing for $\otimes$.

A dioid $\mathcal{D}$ is an idempotent semiring ($\forall a, a \oplus a = a$).

$\mathcal{D}$ is said commutative if $\otimes$ is commutative.

In any dioid, a natural order is defined as follows:

$$a \preceq b \Leftrightarrow a \oplus b = b.$$

$(\mathcal{D}, \preceq)$ is a **complete dioid** if each subset A of $\mathcal{D}$ admits a least upper bound denoted $\bigoplus_{x \in A} x$, and if

$\otimes$ distributes over infinite sums.

The greatest element of a complete dioid $\mathcal{D}$ is noted T : $T = \bigoplus_{x \in \mathcal{D}} x$.

In a complete dioid, the greatest lower bound $\wedge$ always exists and is characterized by:

$$a \wedge b = \bigoplus_{x \preceq a, x \preceq b} x.$$

$\mathcal{D}$ is a **distributive dioid** if it is complete and, for all subsets C of $\mathcal{D}$,

$$(\bigwedge_{c \in C} c) \oplus a = \bigwedge_{c \in C} (c \oplus a) \text{ and } (\bigoplus_{c \in C} c) \wedge a = \bigoplus_{c \in C} (c \wedge a).$$

2.2 Application to Timed Event Graphs

Let us consider the timed event graph (TEG) of fig. 1. $x_i(k)$ denotes the firing date numbered k of transition labeled X_i , x_i is called dater function. Let $\overline{\mathbb{Z}}_{max}$ denote the set $\mathbb{Z} \cup \{\pm\infty\}$ endowed with max as $\oplus$ and usual addition as $\otimes$. It is a complete dioid with the neutral elements $\varepsilon = -\infty$, $e = 0$ respectively for $\oplus$ and $\otimes$, and $T = +\infty$.

The following inequalities are obtained:

$$\begin{cases} x_1(k) \geq \max(u(k) + 2, x_2(k-3) + 2) \\ x_2(k) \geq \max(u(k) + 1, x_1(k-1) + 2, \\ x_2(k-4) + 4) \\ y(k) \geq \max(x_1(k), x_2(k-1)) \end{cases} \quad (1)$$

The minimal solution (which achieves equalities in Eqs. (1)) is the one of interest, since we are interested in transition firings occurring at the earliest possible time.

More generally, the dynamic behavior of a TEG with p input transitions and q output transitions can be represented by standard $\overline{\mathbb{Z}}_{max}$ -linear dater equations (Baccelli, *et al.*, 1992, §2.5.):

$$\begin{cases} x(k+1) = A \otimes x(k) \oplus B \otimes u(k+1) \\ y(k) = C \otimes x(k) \end{cases} \quad (2)$$

Fig. 1. A timed event graph

$B \in (\overline{\mathbb{Z}}_{max})^{n \times p}$, $C \in (\overline{\mathbb{Z}}_{max})^{q \times n}$ and $A \in (\overline{\mathbb{Z}}_{max})^{n \times n}$.

Remarks:

1. Another useful dioid is $\overline{\mathbb{Z}}_{min}$, the dual of $\overline{\mathbb{Z}}_{max}$ (max is replaced by min). In this dioid, $\overline{\mathbb{Z}}_{min}$ -linear counter equations can be written, the counter function $x_i(t)$ of transition labeled X_i denotes the number of firings up to time t . In the following we only consider the 'dater case', nevertheless the proposed results can be transposed to the 'counter case'.

2. Starting from a 'scalar' dioid $\mathcal{D}$, let us consider $p \times p$ matrices with entries in $\mathcal{D}$. The sum and product of matrices are defined conventionally from the sum and product of scalars in $\mathcal{D}$. This set of matrices endowed with these two operations is also a dioid denoted $\mathcal{D}^{p \times p}$.

2.3 Representation by Sup-Convolution

For a transition X_i , dater $x_i = \{x_i(k)\}_{k \in \mathbb{Z}}$ is an increasing function from $\mathbb{Z}$ into $\overline{\mathbb{Z}}_{max}$. We introduce the dioid $\Sigma = \text{Croiss}(\overline{\mathbb{Z}}_{max})$ of increasing functions $\mathbb{Z} \rightarrow \overline{\mathbb{Z}}_{max}$ endowed with

1. pointwise max as addition noted $\oplus$:

$$(u \oplus v)(k) = u(k) \oplus v(k) = \max(u(k), v(k))$$

2. the sup-convolution as multiplication noted $*$:

$$\begin{aligned} (v * u)(k) &= \bigoplus_{s \in \mathbb{Z}} [v(k-s) \otimes u(s)] \\ &= \text{Sup}[v(k-s) + u(s)]. \end{aligned}$$

$(\Sigma, \oplus, *)$ is a (complete) distributive dioid with neutral elements of $\oplus$ and $*$ denoted ε and e respectively, defined by:

$$\varepsilon(k) = \varepsilon \quad \forall k \in \mathbb{Z}$$

$$\epsilon(k) = \begin{cases} \epsilon & \text{if } k \geq 0 \\ \epsilon & \text{otherwise} \end{cases}$$

It can be shown that a shift invariant linear system over the set of signals Σ can be represented by the input-output relation:

$$[S(u)](k) = \bigoplus_{s \in \mathbb{Z}} [g(k-s) \otimes u(s)] = (g * u)(k).$$

$g \in \Sigma$ and is called impulse response of the system (Baccelli, *et al.*, 1992, §6.3; Gaubert, 1992; Max-Plus, 1991).

The system described by Eqs. (2) leads to (Markov) parameters:

$$g(k) = \begin{cases} \epsilon & \text{if } k < 0 \\ CA^k B & \text{if } k \geq 0 \end{cases}$$

More generally, a time invariant linear system from $\Sigma^{p \times 1}$ into $\Sigma^{q \times 1}$ can be represented by an input-output relation with $g = (g_{ij})_{(1 \leq i \leq q, 1 \leq j \leq p)} \in \Sigma^{q \times p}$.

Valuation:

Let us define the valuation of g_{ij} as follows: $\forall i \in \{1, \dots, q\}, j \in \{1, \dots, p\}$,

$$\text{val } g_{ij} \stackrel{\text{def}}{=} \text{Inf} \{ k \mid g_{ij}(k) > \epsilon \}.$$

We have the following property:

Property 1 :

Let C the set of paths from input transition U_j to output transition Y_i . If $c \in C$, we denote $n(c)$ the number of tokens contained into this path. Then

$$\text{val } g_{ij} = \text{Inf}_{c \in C} n(c).$$

Proof:

By definition, $\text{val } g_{ij} = \text{Inf} \{ k \mid (CA^k B)_{ij} > \epsilon \}$. $(CA^k B)_{ij} > \epsilon \Leftrightarrow \exists m, r \in \{1, \dots, n\}$ such that $C_{im} \otimes A_{mr}^k \otimes B_{rj} > \epsilon$
 $\Leftrightarrow \exists m, r \in \{1, \dots, n\}$ such that $C_{im} > \epsilon$, $A_{mr}^k > \epsilon$ and $B_{rj} > \epsilon$.

From Eqs. (2) we easily deduce that:

$C_{im} > \epsilon$ iff a path without token exists from the internal transition X_m to Y_i ,

$B_{rj} > \epsilon$ iff a path without token exists from U_j to the internal transition X_r ,

and $A_{mr}^k > \epsilon$ iff a path exists from the internal transition X_r to the internal transition X_m with exactly k tokens.

We deduce that $(CA^k B)_{ij} > \epsilon$ iff a path exists from U_j to Y_i with exactly k tokens. Then proof is easily achieved from the definition of $\text{val } g_{ij}$.

Remark:

$\text{val } g_{ij}$ represents the "minimal event delay" between input U_j and output Y_i : more precisely the firing of transition Y_i numbered k is due to

the firing of transition U_j numbered $k - \nu$ with $\nu \geq \text{val } g_{ij}$.

For TEG of Fig. 1, $\text{val } g = 0$ (consider the path $U \mapsto X_1 \mapsto Y$).

More generally, if $v \in \Sigma$, we define

$$\text{val } v \stackrel{\text{def}}{=} \text{Inf} \{ k \mid v(k) > \epsilon \}.$$

We easily prove the following property by considering the definitions of operations $\oplus$ and $*$ over Σ .

Property 2 :

1. $\text{val}(v \oplus u) = \min(\text{val } v, \text{val } u)$
2. $\text{val}(v * u) = \text{val } v + \text{val } u$

2.4 Residuation Theory

For an exhaustive presentation see (Blyth and Janowitz, 1972).

In ordered set theory, residuation is a general notion which allows to define "pseudo-inverses" of some isotone maps (f isotone if $a \preceq b \Rightarrow f(a) \preceq f(b)$). Let us present some basic results about this theory. An isotone map $f : \mathcal{D} \mapsto \mathcal{C}$ where $\mathcal{D}$ and $\mathcal{C}$ are ordered sets is residuated if $\forall y \in \mathcal{C}$, the least upper bound of the subset $\{x \in \mathcal{D} \mid f(x) \preceq y\}$ ($\preceq$ is the order relation on $\mathcal{C}$) exists and belongs to this subset. This *greatest* 'subsolution' is denoted $f^*(y)$. By definition we have: $f[f^*(y)] \preceq y \quad \forall y \in \mathcal{C}$.

Application:

If $(\mathcal{D}, \preceq)$ is a complete dioid, the greatest subsolution of inequation $a \otimes x \preceq b$ exists and is denoted $(a \setminus b)$ (consider the isotone map $x \mapsto a \otimes x$).

Let us mention three basic properties of quotient (Baccelli, *et al.*, 1992, §4.4.4) which will be used in the following.

Property 3 :

1. $\forall b \in \mathcal{D}, \quad a \otimes (a \setminus b) \preceq b$ and $a \setminus (a \otimes b) \succeq b$.
2. The mapping $x \mapsto a \setminus x$ is isotone.
3. $\forall a, b, c \in \mathcal{D}, \quad a \setminus (b \wedge c) = (a \setminus b) \wedge (a \setminus c)$

Remark:

We generalize residuation to matrices with entries in a complete dioid (Baccelli, *et al.*, 1992, §4.6). The greatest matrix $X \in \mathcal{D}^{p \times n}$ such that $AX \preceq B$ with $A \in \mathcal{D}^{q \times p}$ and $B \in \mathcal{D}^{q \times n}$ will still be denoted $A \setminus B$ defined as:

$$(A \setminus B)_{ij} = \bigwedge_{l=1}^q A_{il} \setminus B_{lj} \quad i \in \{1, \dots, p\}, j \in \{1, \dots, n\}.$$

3 Control Problem Statement

3.1 Optimal Control

Let g the transfer relation of a TEG with p inputs and q outputs, consider the following control

problem:

given reference input $z = (z_1, \dots, z_q)^t$ with $z_i = \{z_i(k)\}_{(k \in \mathbb{Z})}$ the dates of desired output, find the *latest (greatest)* control $u = (u_1, \dots, u_p)^t$ with $u_j = \{u_j(k)\}_{(k \in \mathbb{Z})}$ such that the desired output is matched as closely as possible.

This can be formally written by:

find the optimal control u_{opt} defined as the *latest (greatest)* signal v satisfying:

$$(g * v)(k) \leq z(k) \quad \forall k \in \mathbb{Z}.$$

This solution exists and is given by residuation theory:

$$u_{opt} \stackrel{\text{def}}{=} \bigoplus_{g * v \leq z} v = g \setminus z$$

(consider the mapping $x \mapsto g * x$).

Remarks:

1. The desired objectives $\{z_i(k)\}_{k \in \mathbb{Z}}$ will be given up to a number k_{i_j} , i.e., $z_i(k) = T$ (i.e., $+\infty$) $\forall k > k_{i_j}$, which means that the production is forecast until the event numbered k_{i_j} of the desired output.

2. u_{opt} is the solution of the backward equations:

$$\begin{cases} \xi(k) = A \setminus \xi(k+1) \wedge C \setminus z(k) \\ u(k) = B \setminus \xi(k) \end{cases}$$

These equations exhibit the backward nature of the recursion which implies having all the values of the desired output sequences to compute u_{opt} (Baccelli, et al., 1992, §5.6; Cohen, et al., 1993; Cohen, et al., 1989).

3.2 Case of Updated Reference Input

Here we consider some more general assumptions on the desired output. Indeed, reference input z can be updated in production to take into account a new production objective, for example:

- In production the reference input z is modified at time t ;
- Reference input is completed beyond events numbered k_{i_j} , $i \in \{1, \dots, q\}$, which leads to consider an increasing of the final event number.

Before time t_n , let the optimal control u_n corresponding to reference input $z_n = (z_{n_1}, \dots, z_{n_q})^t$ with $z_{n_i} = \{z_{n_i}(k)\}_{(k \in \mathbb{Z})}$. At time t_n , we consider the reference input z_{n+1} due to a new production objective.

Then, any control v applied after time t_n will verify the constraints:

$$v_j(k) = u_{n_j}(k) \quad \forall k \leq K_{n_j}, \quad \forall j \in \{1, \dots, p\}$$

with $K_{n_j} = \text{Sup}\{k \mid u_{n_j}(k) \leq t_n\}$.

Therefore the optimal control is defined as the *latest (greatest)* signal v which must *ideally* satisfy:

$$\begin{cases} v_j(k) = u_{n_j}(k) \quad \forall k \leq K_{n_j}, \quad \forall j \in \{1, \dots, p\} \\ (g * v)(k) \leq z_{n+1}(k) \quad \forall k \in \mathbb{Z} \end{cases}$$

4 Optimal Control in case of Updated Reference Input

A formal expression of the optimal control satisfying both the former constraints and tracking objectives is firstly given in the S.I.S.O. case.

4.1 Past Signal and Future Signal

Let $v \in \Sigma$, let us define the "past" signal $r_K^-(v)$ and the "future" signal $r_K^+(v)$ of v at the event K by:

$$[r_K^-(v)](k) = \begin{cases} v(k) & \text{if } k < K \\ v(K) & \text{if } k \geq K \end{cases}$$

$$[r_K^+(v)](k) = \begin{cases} \varepsilon & \text{if } k < K \\ v(k) & \text{if } k \geq K \end{cases}$$

By construction:

$$v = r_K^-(v) \oplus r_{K+1}^+(v).$$

These signals verify the following property:

Property 4 :

1. r_K^- and r_K^+ are $\oplus$ -morphisms,
 $\forall u, v \in \Sigma$, $r_K^-(u \oplus v) = r_K^-(u) \oplus r_K^-(v)$ and
 $r_K^+(u \oplus v) = r_K^+(u) \oplus r_K^+(v)$.
2. $r_K^- \circ r_{K'}^- = r_{\min(K, K')}^-$ and $r_K^+ \circ r_{K'}^+ = r_{\max(K, K')}^+$.
In particular, $r_K^- \circ r_K^- = r_K^-$ and $r_K^+ \circ r_K^+ = r_K^+$;
 r_K^- and r_K^+ are projectors.
3. $r_K^-(u) = \text{Inf}\{v \in \Sigma \mid v(k) = u(k) \forall k \leq K\}$
and $r_K^+(u) = \text{Inf}\{v \in \Sigma \mid v(k) = u(k) \forall k \geq K\}$.
4. r_K^- and r_K^+ are residuated mappings, in particular, the residuated mapping $(r_K^-)^2$ is defined by:
 $(r_K^-)^2: x \mapsto r_K^-(x) \oplus d_{K+1}$ where d_{K+1} is the signal defined by:

$$d_{K+1}(k) = \begin{cases} -\infty & \text{if } k \leq K \\ +\infty & \text{if } k \geq K+1 \end{cases}$$

4.2 Control Design

Let us consider a time t at which we update the reference input due to a new objective of production. Let z this updated reference input and u the optimal control applied until t . Then, any control v applied after time t must verify the constraints:

$$\begin{cases} r_K^-(v) = r_K^-(u) \\ g * v \leq z \end{cases}$$

with $K = \text{Sup}\{k \mid u(k) \leq t\}$.

$r_K^-(u)$ is the least signal (if this set is non empty)

verifying the previous conditions (cf. Prop. 4.3), so by isotony of $x \mapsto g * x$ we have:

$$g * v \succeq g * r_K^-(u).$$

If for a time greater than t it exists an event k' ($k' > K$) such that

$$(g * r_K^-(u))(k') > z(k'),$$

the condition $(g * v)(k) \leq z(k) \quad \forall k \in \mathbb{Z}$ will never be satisfied because no control modification is possible before time t .

This leads us to modify the initial tracking problem replacing z by

$$z \oplus g * r_K^-(u).$$

Such signal is as close as possible to z under the constraint $r_K^-(v) = r_K^-(u)$.

Then we define the optimal control $u_{opt,K}$ as the latest (greatest) signal v satisfying:

$$\begin{cases} r_K^-(v) = r_K^-(u) \\ g * v \preceq z \oplus g * r_K^-(u) \end{cases} \quad (3)$$

The following property gives its formal expression.

Property 5 :

$$u_{opt,K} = r_K^-(u) \oplus r_{K+1}^+[g \setminus (z \oplus g * r_K^-(u))].$$

$u_{opt,K}$ is the greatest solution of Syst. (3) composed by an equation and an inequation. We show that $u_{opt,K}$ is the solution of the following system of inequations:

$$\begin{cases} r_K^-(v) \preceq r_K^-(u) \\ g * v \preceq z \oplus g * r_K^-(u) \end{cases} \quad (4)$$

We firstly prove the following result :

Theorem 1 :

Let $\mathcal{D}$ and $\mathcal{C}$ two complete dioids, $F, G : \mathcal{D} \rightarrow \mathcal{C}$ two residuated mappings and $b, c \in \mathcal{C}$.

Let us consider the systems :

$$(I) \begin{cases} F(x) = b \\ G(x) \preceq c \end{cases} \quad \text{and} \quad (II) \begin{cases} F(x) \preceq b \\ G(x) \preceq c \end{cases}$$

Let us note $S_{(I)}$ the set of solutions of (I). $S_{(II)}$ the set of solutions of (II). $u_{opt}^{(I)} = \bigoplus_{v \in S_{(I)}} v$ and

$$u_{opt}^{(II)} = \bigoplus_{v \in S_{(II)}} v. \text{ We have:}$$

$$\text{If } S_{(I)} \text{ is non empty, then } u_{opt}^{(I)} = u_{opt}^{(II)}.$$

Proof:

$S_{(I)}$ is supposed non empty, therefore $u_{opt}^{(I)}$ exists ($\mathcal{D}$ is complete). As $S_{(I)} \subset S_{(II)}$, we have:

$$u_{opt}^{(I)} \preceq u_{opt}^{(II)}.$$

Let $a \in S_{(I)}$; as F is isotone, $b = F(a) \preceq$

$$F(u_{opt}^{(II)}) \preceq b.$$

Then $F(u_{opt}^{(II)}) = b$ and $u_{opt}^{(II)} \in S_{(I)}$. We deduce that $u_{opt}^{(II)} \preceq u_{opt}^{(I)}$.

We conclude that $u_{opt}^{(I)} = u_{opt}^{(II)}$.

Remark:

We easily prove that $u_{opt}^{(II)} = F^2(b) \wedge G^2(c)$.

Proof of property 5:

From theorem 1, $u_{opt,K}$ is the greatest solution of Syst. (4) if Syst. (3) admits at least one solution. We show that $r_K^-(u) \in S_K$, indeed:

$r_K^-(r_K^-(u)) = r_K^-(u)$ (r_K^- is a projection), and $g * r_K^-(u) \preceq g * r_K^-(u) \oplus z$.

Let us note $G : x \mapsto g * x$: in particular $G^2 : x \mapsto g \setminus x$.

$u_{opt,K}$ is the greatest solution of Syst. (4), then:

$$u_{opt,K} = (r_K^-)^2(r_K^-(u)) \wedge G^2(z \oplus g * r_K^-(u)).$$

$$(r_K^-)^2(r_K^-(u)) = r_K^-(r_K^-(u)) \oplus d_{K+1} = r_K^-(u) \oplus d_{K+1}. \text{ Then}$$

$$\begin{aligned} u_{opt,K} &= (r_K^-(u) \oplus d_{K+1}) \wedge g \setminus (z \oplus g * r_K^-(u)) \\ &= [r_K^-(u) \wedge g \setminus (z \oplus g * r_K^-(u))] \\ &\quad \oplus [d_{K+1} \wedge g \setminus (z \oplus g * r_K^-(u))]. \end{aligned}$$

(($\Sigma, \oplus, *$) is a distributive dioid).

- As $z \oplus g * r_K^-(u) \succeq g * r_K^-(u)$, $g \setminus (z \oplus g * r_K^-(u)) \succeq g \setminus (g * r_K^-(u))$ by Prop. 3.2 and $g \setminus (g * r_K^-(u)) \succeq r_K^-(u)$ by Prop. 3.1. We deduce that $r_K^-(u) \wedge g \setminus (z \oplus g * r_K^-(u)) = r_K^-(u)$.

- We have obviously:

$$d_{K+1} \wedge g \setminus (z \oplus g * r_K^-(u)) = r_{K+1}^+(g \setminus (z \oplus g * r_K^-(u))).$$

We conclude that

$$u_{opt,K} = r_K^-(u) \oplus r_{K+1}^+(g \setminus (z \oplus g * r_K^-(u))).$$

◊

We prove that the construction of $u_{opt,K}$ depends only on reference input z beyond event $K + 1 + \text{val } g$. This is displayed by the following expression of $u_{opt,K}$.

Property 6 :

$$u_{opt,K} = r_K^-(u) \oplus g \setminus r_{K+1+\text{val } g}^+[z \oplus g * r_K^-(u)].$$

Proof:

We firstly prove the following lemma:

Lemma 1 :

$$h \setminus r_{K+1+\text{val } h}^+(y) = r_{K+1}^+(h \setminus y).$$

Proof of lemma:

Let $a \in \mathbb{Z}$ and the signal $d_a = \bigoplus \{v \in \Sigma \mid \text{val } v \geq a\}$.

d_a is defined by:

$$d_a(k) = \begin{cases} \varepsilon & \text{if } k < a \\ T & \text{if } k \geq a \end{cases}$$

Then $r_{K+1+\text{val } h}^+(y) = y \wedge d_{K+1+\text{val } h}$

and $h \setminus r_{K+1+\text{val } h}^+(y) = (h \setminus y) \wedge (h \setminus d_{K+1+\text{val } h})$ by Prop. 3.3.

Let us prove now that $h \setminus d_{K+1+\text{val } h} = d_{K+1}$.

$h \setminus d_{K+1+\text{val } h}$ is by definition the greatest solution of inequality:

$$h * x \preceq d_{K+1+\text{val } h} \quad (5)$$

Prop. 2.2 yields $\text{val}(h * d_{K+1}) = K + 1 + \text{val } h$, then we deduce that $h * d_{K+1} \preceq d_{K+1+\text{val } h}$ by definition of $d_{K+1+\text{val } h}$, so d_{K+1} is solution of Ineq. (5).

Moreover, $\forall x$ solution of Ineq. (5), $\text{val}(h * x) = \text{val } h + \text{val } x \geq \text{val } d_{K+1+\text{val } h} = K + 1 + \text{val } h$ (cf. Prop. 2.2).

Then $\text{val } x \geq K + 1$ so $x \preceq d_{K+1}$ which involves that d_{K+1} is the greatest solution of Ineq. (5).

Hence $h \setminus r_{K+1+\text{val } h}^+(y) = (h \setminus y) \wedge d_{K+1} = r_{K+1}^+(h \setminus y)$.

◇

From lemma we easily deduce that:

$$r_{K+1}^+(g \setminus (z \oplus g * r_K^-(u))) = g \setminus r_{K+1+\text{val } g}^+(z \oplus g * r_K^-(u)).$$

The next property shows that if $r_K^-(u)$ has no effect on output beyond event $K + 1 + \text{val } g$, i.e.,

$r_{K+1+\text{val } g}^+(g * r_K^-(u)) \preceq r_{K+1+\text{val } g}^+(z)$ then the optimal control $u_{\text{opt}} = g \setminus z$ is achievable beyond event $K + 1$.

Property 7 :

If $r_{K+1+\text{val } g}^+(g * r_K^-(u)) \preceq r_{K+1+\text{val } g}^+(z)$ then

$$u_{\text{opt},K} = r_K^-(u) \oplus r_{K+1}^+(u_{\text{opt}}).$$

Proof:

If $r_{K+1+\text{val } g}^+(g * r_K^-(u)) \preceq r_{K+1+\text{val } g}^+(z)$,

$$r_{K+1+\text{val } g}^+(z \oplus g * r_K^-(u)) =$$

$$r_{K+1+\text{val } g}^+(z) \oplus r_{K+1+\text{val } g}^+(g * r_K^-(u))$$

because $r_{K+1+\text{val } g}^+$ is a $\oplus$ -morphism (cf. Prop. 4.1.)

$$= r_{K+1+\text{val } g}^+(z).$$

Then

$$\begin{aligned} u_{\text{opt},K} &= r_K^-(u) \oplus g \setminus r_{K+1+\text{val } g}^+(z) \\ &= r_K^-(u) \oplus r_{K+1}^+(g \setminus z) \text{ by lemma 1,} \\ &= r_K^-(u) \oplus r_{K+1}^+(u_{\text{opt}}). \end{aligned}$$

4.3 Extension to M.I.M.O. Case

Let us just present the expression of $u_{\text{opt},K}$ in M.I.M.O. case.

Let $g = (g_{ij})_{(1 \leq i \leq q, 1 \leq j \leq p)}$ the transfer relation of a TEG with p inputs and q outputs, the optimal control $u = (u_1, \dots, u_p)^t$ applied until a time t and $z = (z_1, \dots, z_q)^t$ the updated reference input.

Let $K = (K_1, \dots, K_p)^t$ with $K_j \stackrel{\text{def}}{=} \sup\{k \mid u_j(k) \leq t\}$.

The notions of past signal and future signal are extended as:

$$r_K^-(w) = (r_{K_1}^-(w_1), \dots, r_{K_p}^-(w_p))^t.$$

$$r_{K+1}^+(w) = (r_{K_1+1}^+(w_1), \dots, r_{K_p+1}^+(w_p))^t.$$

As in S.I.S.O. case, the optimal control $u_{\text{opt},K}$ is defined as the *latest (greatest)* signal v satisfying:

$$\begin{cases} r_K^-(v) = r_K^-(u) \\ g * v \preceq z \oplus g * r_K^-(u) \end{cases}$$

By using the previous notation,

$$u_{\text{opt},K} = r_K^-(u) \oplus r_{K+1}^+[g \setminus (z \oplus g * r_K^-(u))]$$

and the expression of its components is (cf. remark §2.4): $\forall j \in \{1, \dots, p\}$,

$$(u_{\text{opt},K})_j = r_{K_j}^-(u_j) \oplus \bigwedge_{i=1}^q r_{K_j+1}^+[g_{ij} \setminus (z_i \oplus (g * r_K^-(u))_i)]$$

$$= r_{K_j}^-(u_j) \oplus \bigwedge_{i=1}^q g_{ij} \setminus r_{K_j+1+\text{val } g_{ij}}^+[z_i \oplus (g * r_K^-(u))_i]$$

by lemma 1.

5 Application

The planification of the production is initially defined from a reference input z_0 . Let us note $u_0 = g \setminus z_0$ the optimal control. At time t_n ($n \geq 1$) we consider reference input z_n due to new objectives of production.

Let

$$K_n = \text{Sup}\{k \mid u_{n-1}(k) \leq t_n\}.$$

$u_{n-1}(K_n)$ is the latest firing date in relation to z_{n-1} .

The optimal control u_n is then defined as the *latest (greatest)* signal v satisfying:

$$\begin{cases} r_{K_n}^-(v) = r_{K_n}^-(u_{n-1}) \\ g * v \preceq z_n \oplus g * r_{K_n}^-(u_{n-1}) \end{cases}$$

By Prop. 5 we deduce that

$$\begin{aligned} u_n &= u_{\text{opt},K_n} \\ &= r_{K_n}^-(u_{n-1}) \oplus r_{K_n+1}^+[g \setminus (z_n \oplus g * r_{K_n}^-(u_{n-1}))]. \end{aligned}$$

6 Example

Let us consider the TEG of fig. 1 and the following production policy:

Initially the desired production rate is equal to 4 parts per 4 time units, i.e.,

$$z_0(k) = \begin{cases} \varepsilon & \text{for } k < 0 \text{ and } T \text{ for } k \geq 20 \\ 4[\frac{k}{4}] & \text{for } 0 \leq k < 20 \end{cases}$$

Fig. 2. Reference inputs z_0 z_1 z_2 and output y

($[x]$ is the truncated part of x).

Then at time $t_1 = 8$, the desired production rate becomes equal to 2 parts per 1 time unit, i.e.,

$$z_1(k) = \begin{cases} \varepsilon & \text{for } k < 8 \text{ and } T \text{ for } k \geq 30 \\ 8 + \left\lceil \frac{k-8}{2} \right\rceil & \text{for } 8 \leq k < 30 \end{cases}$$

Finally at time $t_2 = 13$, the desired production rate becomes equal to 2 parts per 4 time units, i.e.,

$$z_2(k) = \begin{cases} \varepsilon & \text{for } k < 18 \text{ and } T \text{ for } k \geq 35 \\ 13 + 4 \left\lceil \frac{k-18}{2} \right\rceil & \text{for } 18 \leq k < 35 \end{cases}$$

Fig. 2 represents the behavior of reference inputs and output $y = g * u_a$ where

$$u_a(k) = \begin{cases} u_0(k) = (g \setminus z_0)(k) & \text{for } k \leq 9. \\ u_{opt,9}(k) & \text{for } 9 < k \leq 17, \\ u_{opt,17}(k) & \text{for } k > 17. \end{cases}$$

Initially the applied control u_a is identical to $g \setminus z_0$. It would be optimal if no modification of reference input occurred.

At time $t_1 = 8$, the reference input is updated. The production of the new reference input z_1 - being greater than the production rate of the system (equal to 1 part per 1 time unit) - and the previous applied control (u_0) prevent the output from occurring before z_1 (see dotted circle).

From time $t_2 = 13$, the production rate of the system allows to produce enough parts to satisfy as soon as possible the updated reference input z_2 . This is a consequence of Prop. 7, a graphic representation of controls trajectories would show that $r_{22}^+(u_{opt,18}) = r_{22}^+(g \setminus z_2)$.

7 Conclusion

We have extended the concept of optimal control to the case of updated reference input in production. We use a particular formalism (definitions of

past signal and future signal) in order to underline the effect of past and future actions on output system.

This formalism and the concept of optimal control in case of updated reference input can be extended to control statement in case of modeling error which leads to use a feedback control structure (Boimond and Ferrier, 1996; Menguy, et al., 1997).

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