Local Symmetry Group in the General Theory of Elastic Shells

Victor A. Eremeyev, Wojciech Pietraszkiewicz

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Abstract We establish the local symmetry group of the dynamically and kinematically exact theory of elastic shells. The group consists of an ordered triple of tensors which make the shell strain energy density invariant under change of the reference placement. Definitions of the fluid shell, the solid shell, and the membrane shell are introduced in terms of members of the symmetry group. Within solid shells we discuss in more detail the isotropic, hemitropic, and orthotropic shells and corresponding invariant properties of the strain energy density. For the physically linear shells, when the density becomes a quadratic function of the shell strain and bending tensors, reduced representations of the density are established for orthotropic, cubic-symmetric, and isotropic shells. The reduced representations contain much less independent material constants to be found from experiments.

1. Introduction

In the non-linear theory of elastic shells it is important to formulate restrictions put on the strain energy density following from a reasonably defined shell material symmetry. Such restrictions are necessary to considerably lower the number of

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material constants to be found from experiments. For example, in the general case of shell anisotropy the quadratic function proposed by Libai and Simmonds [24] to describe the strain energy density of the linearly elastic shell contains more than 100 constants. It is practically impossible to determine all of them from any kind of experiments. Shell material symmetries can also be used to formulate and efficiently analyse the non-linear shell problem by the semi-inverse approach.

The material symmetry of elastic shells was discussed in the literature primarily within the shell model consisting of material surface with one deformable director. Let us mention here early results by Caroll and Naghdi [6] and Ericksen [16, 17] developed by Murdoch and Cohen [30, 31], Adeleke [1], and Cohen and Wang [9]. Gurtin and Murdoch [21] formulated the mechanical isotropy group while Murdoch [29] the thermodynamic isotropy group for the non-linear theory of elastic membranes. Wang [45] discussed the symmetry group for two-layer elastic membranes, Steigmann [40] and Steigmann and Ogden [41] for the Kirchhoff–Love type elastic shells, while Murdoch and Cohen [30, 31] for the second grade elastic material surfaces.

In the general non-linear theory of shells initiated by Reissner [36] and developed by Libai and Simmonds [23, 24], Makowski and Stumpf [26], Chróścielewski et al. [7, 8], and Pietraszkiewicz et al. [35] the gross deformation of the shell cross-section is represented by the translation vector and work-averaged rotation tensor fields, both defined at the material base surface. The kinematic structure of such a dynamically and kinematically exact theory of shells is identical with that of the Cosserat surface originally proposed by Cosserat and Cosserat [10] and developed by Altenbach and Zhilin [2], Zubov [48], and Eremeyev and Zubov [15]. Unfortunately, this kinematic structure differs from that of the material surface with one or more deformable directors often also called the Cosserat surface, see Ericksen and Truesdell [18], Naghdi [32], and Rubin [37]. In particular, the kinematics used in [32, 37] leaves indefinite the drilling rotation about the director. Thus, the results on material symmetries for the shell model with one deformable director given for example by Murdoch and Cohen [30, 31] cannot be used in the general theory of shells. Altenbach and Zhilin [2, 3], Altenbach et al. [4], and Zhilin [47] discussed the symmetry group and the corresponding 2D elasticity tensors within the reduced Timoshenko–Reissner type shell model with transverse shears and assuming the quadratic strain energy density. Four-field local symmetry group of the micropolar shell model was discussed by Eremeyev and Zubov [15] and Eremeyev [12], where alternative surface strain measures were used, the strain energy density was assumed to depend on the curvature of the reference base surface, and differences in transforming the axial and polar tensors were not accounted for.

The aim of this paper is to discuss the local symmetry group for the general theory of elastic shells developed in [8, 24, 35] and to derive several consistently simplified forms of the 2D strain energy density. In Section 2 we recall some relations of the general theory of shells. The local resultant equilibrium conditions (1) are derived by exact through-the-thickness integration of equilibrium conditions of continuum mechanics. The natural definitions (10) of the shell strain and bending tensors are uniquely established as work-conjugate dual fields in the 2D virtual work identity (3). As a result, the shell deformation is described by the translation vector and rotation tensor fields. The corresponding shell strain energy density (16) depends on the 2D shell strain and bending tensors and is also sensitive to change of the structure curvature tensor.

Transformation properties of various tensor fields under change of the reference placement are discussed in Section 3. Applying the polar decomposition theorem (19) of the shell deformation gradient we introduce an orthogonal tensor field and several tangential surface fields describing stretching and bending parts of deformation. Properties of various shell strain measures under the orthogonal transformation are discussed taking into account that polar tensors may change their signs according to the rule (28). However, under change of the reference placement the elastic strain energy of any part of the shell should remain unchanged, (29).

The local symmetry group of the elastic shell is defined in Section 4 as an ordered triple of transformation tensors which leave unchanged the strain energy density, (33). The structure of the group is analysed and its behaviour under change of the reference placement is discussed. We classify in Section 6 various shell models in terms of the triples of tensors of the local symmetry group. In this way the liquid shell, the solid shell, and the membrane shell are characterized. We then propose definitions of isotropic, hemitropic, and orthotropic solid shells. For physically linear elastic material, when the strain energy density is a quadratic function of the shell strain and bending tensors, we derive consistently reduced forms of the density for the surface isotropic, hemitropic, and orthotropic shells.

2. Notation and Preliminary Relations

The system of notation used here follows that of Libai and Simmonds [24], Eremeyev and Pietraszkiewicz [13], Chróścielewski et al. [8], and Pietraszkiewicz et al. [35].

In the general theory of shells developed in [8, 12, 23, 24, 35] the global, exact, resultant equilibrium conditions, formulated at the base surface M in the reference placements κ, are derived by direct through-the-thickness integration of corresponding global equilibrium conditions of continuum mechanics. Then applying the surface Cauchy postulate and the Stokes theorem we obtain the usual local, exact, resultant equilibrium equations and dynamic boundary conditions [8, 13]

$$\begin{aligned} &\mathrm{Div}_s\, \boldsymbol{N} + \boldsymbol{f} = \boldsymbol{0}, \quad \mathrm{Div}_s\, \boldsymbol{M} + \mathrm{ax}\left(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T\right) + \boldsymbol{c} = \boldsymbol{0} \quad \text{in } M, \\ &\boldsymbol{N}\boldsymbol{\nu} - \boldsymbol{n}^* = \boldsymbol{0}, \quad \boldsymbol{M}\boldsymbol{\nu} - \boldsymbol{m}^* = \boldsymbol{0} \quad \text{along } \partial M_f. \end{aligned} \tag{1}$$

In Equation (1), $\boldsymbol{N}$, $\boldsymbol{M} \in E \otimes T_x M$ are the internal surface stress resultant and stress couple tensors of the 1^{st} Piola–Kirchhoff type, $\boldsymbol{f}, \boldsymbol{c} \in E$ are the external surface resultant force and couple vectors applied at any point $y = \chi(x)$ of the deformed base surface $N = \chi(M)$, $x \in M$, but measured per unit area of M, while $\boldsymbol{n}^*, \boldsymbol{m}^* \in E$ are the external boundary resultant force and couple vectors applied along the part of the deformed boundary ∂N_f, but measured per unit length of ∂M_f, respectively. Here E is the 3D translation vector space of the physical space $\mathcal{E}$, $T_x M$ is the tangent space to M at $x \in M$, and $\otimes$ is the tensor product. Additionally, $\boldsymbol{\nu}$ in (1) means the unit vector externally normal to ∂M, $\boldsymbol{F} = \mathrm{Grad}_s\, \boldsymbol{y} \in E \otimes T_x M$ is the shell deformation gradient tensor, Grad_s and Div_s are the respective surface gradient and divergence operators on M defined intrinsically in [21, 25, 27, 28], and $\mathrm{ax}(\cdot)$ denotes the axial vector associated with the skew tensor $(\cdot)$.

Let $\boldsymbol{v}, \boldsymbol{w} \in E$ be two arbitrary smooth vector fields given on M. We can set the integral identity

$$\iint_M \left\{(\mathrm{Div}_s\, \boldsymbol{N} + \boldsymbol{f}) \cdot \boldsymbol{v} + \left(\mathrm{Div}_s\, \boldsymbol{M} + \mathrm{ax}\left(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T\right) + \boldsymbol{c}\right) \cdot \boldsymbol{w}\right\} da$$

$$- \int_{\partial M_f} \left\{\left(\boldsymbol{N}\boldsymbol{v} - \boldsymbol{n}^*\right) \cdot \boldsymbol{v} + \left(\boldsymbol{M}\boldsymbol{v} - \boldsymbol{m}^*\right) \cdot \boldsymbol{w}\right\} ds = 0, \tag{2}$$

which transformed with the help of the Stokes theorem leads to

$$\iint_M \{\boldsymbol{N} \cdot (\mathrm{Grad}_s\, \boldsymbol{v} - \boldsymbol{W}\boldsymbol{F}) + \boldsymbol{M} \cdot \mathrm{Grad}_s\, \boldsymbol{w}\}\, da$$

$$= \int\int_M (\boldsymbol{f} \cdot \boldsymbol{v} + \boldsymbol{c} \cdot \boldsymbol{w})\, da + \int_{\partial M_f} \left(\boldsymbol{n}^* \cdot \boldsymbol{v} + \boldsymbol{m}^* \cdot \boldsymbol{w}\right) ds$$

$$+ \int_{\partial M_d} (\boldsymbol{N}\boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{M}\boldsymbol{v} \cdot \boldsymbol{w})\, ds, \tag{3}$$

where $\cdot$ means the scalar product in the vector and tensor spaces, respectively, $\boldsymbol{w} = \mathrm{ax}\boldsymbol{W}$, and $\partial M_d = \partial M \backslash \partial M_f$.

The vector field $\boldsymbol{v}$ may be interpreted, in particular, as the kinematically admissible virtual translation of N while the vector field $\boldsymbol{w}$ as the kinematically admissible virtual rotation of the shell cross-section at N, such that $\boldsymbol{v} = \boldsymbol{w} = \boldsymbol{0}$ along ∂M_d. Then the last line integral identically vanishes, two integrals in the second row of (3) describe the external virtual work, while the first surface integral in (3) describes the internal virtual work, and the formula (3) represents the principle of virtual work for the shell.

The principle (3) allows one to interpret [8, 35] that in the reference placement κ the shell is represented by the position vector $\boldsymbol{x} \in E$ (relative to a point $o \in \mathcal{E}$) of the base surface M *plus* the non-singular structure tensor $\boldsymbol{T} \in E \otimes E$, $\det \boldsymbol{T} \neq 0$, attached to any $x \in M$. The tensor $\boldsymbol{T}$ can be introduced through three non-coplanar directors $\boldsymbol{t}_i, i = 1, 2, 3$, such that $\boldsymbol{t}_i = \boldsymbol{T}\boldsymbol{e}_i$, with $\boldsymbol{e}_i$ an orthonormal base of the 3D inertial frame $(o, \boldsymbol{e}_i)$, so that $\boldsymbol{T} = \boldsymbol{t}_i \otimes \boldsymbol{e}_i$, Figure 1.

In some special cases of shell geometry (skew lateral boundary surface, folded, branching or intersecting shells, for example [22]), it is convenient to use the non-orthogonal set of directors $\boldsymbol{t}_i$ indeed. However, it has been shown in [8, 35] that the orthonormal triad $\boldsymbol{t}_i$ entirely describes those changes of $\boldsymbol{T}$ along an arbitrary path C on M which are compatible with (3) thus allowing to describe completely the shell geometry in the reference placement. In this paper the directors $\boldsymbol{t}_i$ are assumed to be orthonormal, so that $\boldsymbol{T}$ is the proper orthogonal tensor, $\boldsymbol{T}^T = \boldsymbol{T}^{-1}$, $\det \boldsymbol{T} = +1$.

In the deformed placement γ the shell can be represented by the relations

$$y = \chi(x) = x + \boldsymbol{u}, \quad \boldsymbol{d}_i = \varphi(x) = \boldsymbol{Q}\boldsymbol{t}_i, \tag{4}$$

where $y \in N$, $\boldsymbol{d}_i$ are three orthonormal directors attached to y, $\boldsymbol{u}(x)$ is the translation field of the base surface, and $\boldsymbol{Q}(x)$ is the work-averaged rotation field of the shell cross-sections, $\boldsymbol{Q} \in Orth^+$, Figure 1.

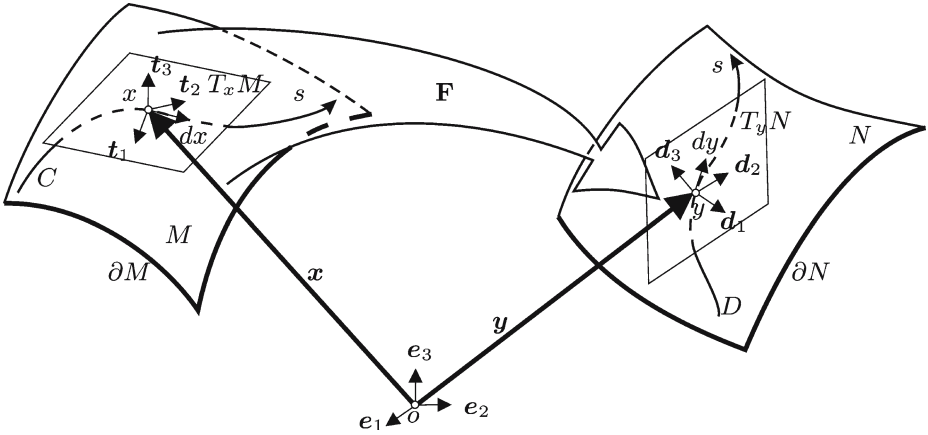

Figure 1 Shell kinematics.

Let C be the smooth curve on M given by $x = x(s)$, where s is the arc length parameter. Then $\boldsymbol{x} = \boldsymbol{x}(s)$ and $\boldsymbol{T} = \boldsymbol{T}(s)$ along C and their differentials are [35]

$$\begin{aligned} d\boldsymbol{x} &= \boldsymbol{x}'ds = (\mathrm{Grad}_s\, \boldsymbol{x})dx, \quad d\boldsymbol{x} \in E, \\ d\boldsymbol{T} &= \boldsymbol{T}'ds = (\mathrm{Grad}_s\, \boldsymbol{T})dx, \quad dx \in T_xM, \\ \mathrm{Grad}_s\, \boldsymbol{x} &= \boldsymbol{I} \in E \otimes T_xM, \quad \mathrm{Grad}_s\, \boldsymbol{T} \in E \otimes E \otimes T_xM, \end{aligned} \tag{5}$$

where $\boldsymbol{I}$ is the inclusion operator at $x \in M$, see Gurtin and Murdoch [21].

Since $d\left(\boldsymbol{T}\boldsymbol{T}^{-1}\right) = \boldsymbol{0}$, the skew tensor $d\boldsymbol{T}\boldsymbol{T}^{-1}$ can be represented by its axial vector depending linearly on dx, so that

$$\mathrm{ax}\left(d\boldsymbol{T}\boldsymbol{T}^{-1}\right) = \boldsymbol{B}dx, \quad \boldsymbol{B} \in E \otimes T_xM, \tag{6}$$

where $\boldsymbol{B}$ is the structure curvature tensor in κ. The two tensors $\boldsymbol{I}$, $\boldsymbol{B}$ are the basic measures of local geometry of the shell base surface M with the attached structure tensor $\boldsymbol{T}$.

In the deformed placement γ the shell is represented by the position vector $\boldsymbol{y} \in E$ (relative to the same $o \in \mathcal{E}$) of the base surface $N = \chi(M)$ and by the structure tensor $\boldsymbol{D} = \boldsymbol{Q}\boldsymbol{T} = \boldsymbol{d}_i \otimes \boldsymbol{e}_i$. Differentials of $\boldsymbol{y}(s)$ and $\boldsymbol{D}(s)$ along $D = \chi(C)$ are given by

$$\begin{aligned} d\boldsymbol{y} &= \boldsymbol{y}'ds = (\mathrm{grad}_s\, \boldsymbol{y})dy = (\mathrm{Grad}_s\, \boldsymbol{y})dx, \\ d\boldsymbol{D} &= \boldsymbol{D}'ds = (\mathrm{grad}_s\, \boldsymbol{D})dy = (\mathrm{Grad}_s\, \boldsymbol{D})dx, \\ \mathrm{grad}_s\, \boldsymbol{y} &= \boldsymbol{J} \in E \otimes T_yN, \quad \mathrm{grad}_s\, \boldsymbol{D} \in E \otimes E \otimes T_yN, \end{aligned} \tag{7}$$

where grad_s is the surface gradient operator and $\boldsymbol{J}$ the inclusion operator at N.

Again, the skew tensor $d\boldsymbol{D}\boldsymbol{D}^{-1}$ can be represented by its axial vector depending linearly on dy, so that

$$\mathrm{ax}\left(d\boldsymbol{D}\boldsymbol{D}^{-1}\right) = Cdy, \quad \boldsymbol{C} \in E \otimes T_yN, \tag{8}$$

where $\boldsymbol{C}$ is the structure curvature tensor in γ.

Since $dy = \mathbf{F}dx$, where $\mathbf{F} \in T_yN \otimes T_xM$, $\det \mathbf{F} > 0$, is the tangential surface deformation gradient, $\boldsymbol{F} = \boldsymbol{J}\mathbf{F}$, from (5) to (8) we obtain

$$\begin{aligned} d\boldsymbol{y} - \boldsymbol{Q}d\boldsymbol{x} &= (\boldsymbol{J}\mathbf{F} - \boldsymbol{Q}\boldsymbol{I})dx, \\ \boldsymbol{C}dy - \boldsymbol{Q}\boldsymbol{B}dx &= (\boldsymbol{C}\mathbf{F} - \boldsymbol{Q}\boldsymbol{B})dx, \end{aligned} \tag{9}$$

where

$$\boldsymbol{E} = \boldsymbol{J}\mathbf{F} - \boldsymbol{Q}\boldsymbol{I}, \quad \boldsymbol{K} = \boldsymbol{C}\mathbf{F} - \boldsymbol{Q}\boldsymbol{B} \tag{10}$$

are the natural strain and bending tensors in the spatial representation describing the local deformation of the shell base surface with the attached structure tensor.

It has been proved in [8, 35] that the co-rotational variations of $\boldsymbol{E}$ and $\boldsymbol{K}$ are given by

$$\begin{aligned} \delta^c \boldsymbol{E} &= \boldsymbol{Q}\left[\delta\left(\boldsymbol{Q}^T\boldsymbol{E}\right)\right] = \mathrm{Grad}_s\, \boldsymbol{v} - \boldsymbol{W}\boldsymbol{F}, \\ \delta^c \boldsymbol{K} &= \boldsymbol{Q}\left[\delta\left(\boldsymbol{Q}^T\boldsymbol{K}\right)\right] = \mathrm{Grad}_s\, \boldsymbol{w}, \end{aligned} \tag{11}$$

where $\boldsymbol{v} \equiv \delta\boldsymbol{u}$ is the virtual translation and $\boldsymbol{w} \equiv \mathrm{ax}\left(\delta\boldsymbol{Q}\boldsymbol{Q}^T\right)$ the virtual rotation vectors appearing in the principle of virtual work (3). Therefore, it follows from (3) that the internal virtual power is given by

$$\sigma = \boldsymbol{N} \cdot \delta^c \boldsymbol{E} + \boldsymbol{M} \cdot \delta^c \boldsymbol{K} = \mathbf{N} \cdot \delta\mathbf{E} + \mathbf{M} \cdot \delta\mathbf{K}, \tag{12}$$

where

$$\begin{aligned} \mathbf{N} &= \boldsymbol{Q}^T\boldsymbol{N}, \quad \mathbf{M} = \boldsymbol{Q}^T\boldsymbol{M}, \\ \mathbf{E} &= \boldsymbol{Q}^T\boldsymbol{E} = \boldsymbol{Q}^T\boldsymbol{J}\mathbf{F} - \boldsymbol{I}, \quad \mathbf{K} = \boldsymbol{Q}^T\boldsymbol{K} = \boldsymbol{Q}^T\boldsymbol{C}\mathbf{F} - \boldsymbol{B} \end{aligned} \tag{13}$$

are the resultant shell stress measures and the corresponding shell strain measures in the material representation, respectively. The measures $\mathbf{N}$, $\mathbf{M}$ and $\mathbf{E}$, $\mathbf{K}$ are more convenient in the discussion of the constitutive equations given below.

In what follows it is convenient to use also the relative strain $\mathbf{U}$ and bending $\mathbf{V}$ measures in the material representation defined by

$$\mathbf{U} = \boldsymbol{Q}^T\boldsymbol{J}\mathbf{F} = \mathbf{E} + \boldsymbol{I}, \quad \mathbf{V} = \boldsymbol{Q}^T\boldsymbol{C}\mathbf{F} = \mathbf{K} + \boldsymbol{B}. \tag{14}$$

For the elastic shells there exists the strain energy density $W_\kappa(\mathbf{E}, \mathbf{K})$ such that $\sigma = \delta W_\kappa$ and the constitutive equations take the form

$$\mathbf{N} = \frac{\partial W_\kappa}{\partial \mathbf{E}}, \quad \mathbf{M} = \frac{\partial W_\kappa}{\partial \mathbf{K}}. \tag{15}$$

The strain energy density W_κ is written here relative to the reference placement κ, so that W_κ depends also on the local geometry of κ, that is on $\boldsymbol{I}$ and $\boldsymbol{B}$. The dependence of W_κ on $\boldsymbol{I}$ is trivial and does not require any further discussion. However, the dependence of W_κ on $\boldsymbol{B}$ may have a considerable influence on the form of the constitutive equations discussed below. Therefore, we explicitly indicate this fact by writing

$$W_\kappa \equiv W = W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}). \tag{16}$$

3. Change of the Reference Placement

Let us introduce another reference placement $\kappa_\star$ consisting of the base surface $M_\star$ described by the position vector $\boldsymbol{x}_\star$ relative to the same $o \in \mathcal{E}$ and three orthonormal directors $\boldsymbol{t}_{\star i}$ attached to any $x_\star \in M_\star$ (Figure 2). Let $\mathbf{P} \in T_{x_\star} M_\star \otimes T_x M$, $\det \mathbf{P} \neq 0$, be the tangential surface deformation gradient transforming dx into $dx_\star$, and $\boldsymbol{R} \in Orth^+$ be the rotation tensor transforming $\boldsymbol{t}_i$ into $\boldsymbol{t}_{\star i}$, so that

$$dx_\star = \mathbf{P} dx, \quad \boldsymbol{t}_{\star i} = \boldsymbol{R}\boldsymbol{t}_i. \tag{17}$$

It is apparent that $\mathbf{P}$ satisfies the relations

$$\mathbf{P}\boldsymbol{I}^T\boldsymbol{n} = \boldsymbol{n}_\star \boldsymbol{I}_\star \mathbf{P} = \mathbf{0}, \tag{18}$$

where $\boldsymbol{n}$ and $\boldsymbol{n}_\star$ are the unit normals orienting M and $M_\star$, respectively.

In what follows all fields associated with deformation relative to the reference placement $\kappa_\star$ will be marked by index $\star$.

Let us analyse how changes the unit normal vector $\boldsymbol{n}$ under change of the reference placement $\kappa \to \kappa_\star$. By the polar decomposition of the 3D nonsingular tensor $\mathbf{P} + \boldsymbol{n}_\star \otimes \boldsymbol{n}$ we obtain

$$\mathbf{P} + \boldsymbol{n}_\star \otimes \boldsymbol{n} = \boldsymbol{H}(\mathbf{Y} + \boldsymbol{n} \otimes \boldsymbol{n}), \tag{19}$$

where $\boldsymbol{H}$ is the orthogonal tensor, $\boldsymbol{H} \in Orth$, and $\mathbf{Y}$ is the tangential surface stretch tensor, $\mathbf{Y} \in T_x M \otimes T_x M$, symmetric and positive definite, which satisfies the relations

$$\mathbf{Y}\boldsymbol{I}^T\boldsymbol{n} = \boldsymbol{n}\boldsymbol{I}\mathbf{Y} = \mathbf{0}. \tag{20}$$

The tensor $\mathbf{Y}$ can be calculated by the formula $\mathbf{Y} = \sqrt{\mathbf{P}^T\mathbf{P}}$.

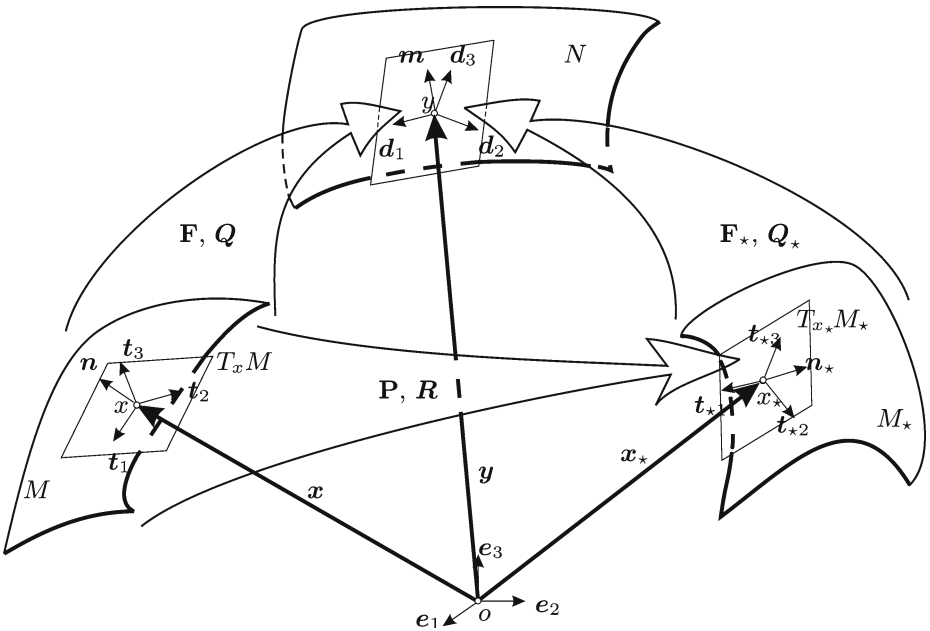

Figure 2 Change of reference placement.

From the polar decomposition (19) it immediately follows that

$$\boldsymbol{n}_\star = \boldsymbol{H}\boldsymbol{n}. \tag{21}$$

Let us choose an arbitrary point $x_\circ \in M$ and discuss all such transformations of the reference placement under which the normal direction to M at the point $x_\circ$ does not change, i.e. we have

$$\boldsymbol{n}_\star(x_\circ) = \pm\boldsymbol{n}(x_\circ), \quad \text{or} \quad \pm\boldsymbol{n}(x_\circ) = \boldsymbol{H}(x_\circ)\,\boldsymbol{n}(x_\circ), \tag{22}$$

when $x_\circ \in M$ and $x_\circ \in M_\star$. Therefore, we analyse the set of all reference placements which have the same common tangent space $T_{x_\circ}M$ at the point $x_\circ \in M$, Figure 3.

It is easy to see that in this case the tensor $\mathbf{P} \in T_{x_\circ}M \otimes T_{x_\circ}M$, and $\boldsymbol{n}(x_\circ)\,\boldsymbol{I}(x_\circ)\,\mathbf{P}(x_\circ) = \mathbf{P}(x_\circ)\,\boldsymbol{I}(x_\circ)^T\,\boldsymbol{n}(x_\circ) = \mathbf{0}$. Indeed, the second of the equations follows from definition of the surface gradient while the first one results from $\boldsymbol{n}\boldsymbol{I}\mathbf{P} = \mathbf{0}$, which follows from (20) and (22).

From (17) and $dy = \mathbf{F}dx = \mathbf{F}_\star dx_\star$ immediately follow the relations between the tangential surface deformation gradients $\mathbf{F}$ and $\mathbf{F}_\star$ and the rotation tensors $\boldsymbol{Q} \equiv \boldsymbol{d}_i \otimes \boldsymbol{t}_i$ and $\boldsymbol{Q}_\star \equiv \boldsymbol{d}_i \otimes \boldsymbol{t}_{\star i}$

$$\mathbf{F} = \mathbf{F}_\star\mathbf{P}, \quad \boldsymbol{Q} = \boldsymbol{Q}_\star\boldsymbol{R}. \tag{23}$$

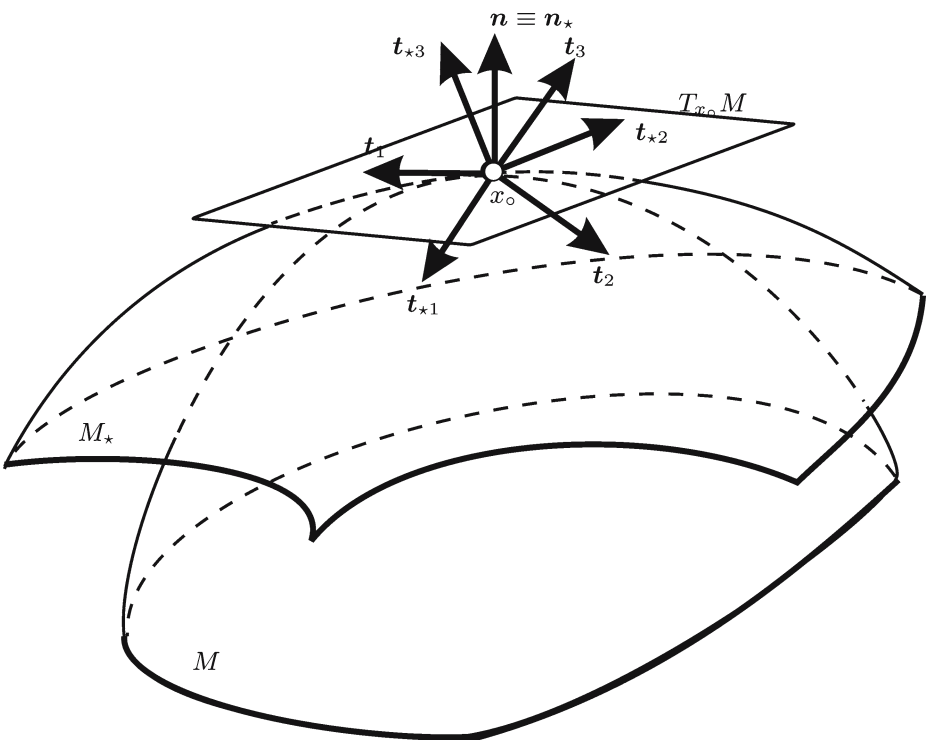

Figure 3 Two base surfaces with the same tangent space.

From (23) we obtain the relations between the relative strain measures $\mathbf{U}$ and $\mathbf{U}_\star \equiv \boldsymbol{Q}_\star^T \boldsymbol{J}\mathbf{F}_\star$ as well as between the strain tensors $\mathbf{E}$ and $\mathbf{E}_\star \equiv \boldsymbol{Q}_\star^T \boldsymbol{J}\mathbf{F}_\star - \boldsymbol{I}_\star$

$$\mathbf{U}_\star = \boldsymbol{R}\mathbf{U}\mathbf{P}^{-1}, \quad \mathbf{E}_\star = \boldsymbol{R}\mathbf{E}\mathbf{P}^{-1} + \boldsymbol{R}\boldsymbol{I}\mathbf{P}^{-1} - \boldsymbol{I}_\star. \tag{24}$$

Here $\mathbf{U} \in E \otimes T_x M$, $\mathbf{U}_\star \in E \otimes T_{x_\star} M_\star$, and by $\mathbf{P}^{-1}$ we understand the inverse tangential surface deformation gradient such that $\mathbf{P}\mathbf{P}^{-1} = \mathbf{A}_\star \equiv \mathbf{1} - \boldsymbol{n}_\star \otimes \boldsymbol{n}_\star$, $\mathbf{P}^{-1}\mathbf{P} = \mathbf{A} \equiv \mathbf{1} - \boldsymbol{n} \otimes \boldsymbol{n}$, where $\mathbf{1}$ is the 3D unit tensor. From (22) we obtain that $\mathbf{A}_\star = \mathbf{A}$ and therefore $\boldsymbol{I}_\star = \boldsymbol{I}$ at the point $x_\circ \in M$.

Let us discuss changes of the relative bending measure $\mathbf{V}$ and the bending tensor $\mathbf{K}$ under change of the reference placement. Since $\mathbf{V}_\star \equiv \boldsymbol{Q}_\star^T \boldsymbol{C}\boldsymbol{F}_\star$, from (23) we obtain

$$\mathbf{V}_\star = \boldsymbol{R}\mathbf{V}\mathbf{P}^{-1}. \tag{25}$$

To derive the relations between $\mathbf{K}_\star \equiv \boldsymbol{Q}_\star^T \boldsymbol{C}\mathbf{F}_\star - \boldsymbol{B}_\star$ and $\mathbf{K}$ defined in $(13)_2$ we still need to know how to express the tensor $\boldsymbol{B}_\star$ through the tensor $\boldsymbol{B}$. Since $d\boldsymbol{T}\boldsymbol{T}^{-1}$ is skew, its axial vector is $\boldsymbol{b} = \operatorname{ax}\left(d\boldsymbol{T}\boldsymbol{T}^{-1}\right) = \dfrac{1}{2}\boldsymbol{t}_i \times d\boldsymbol{t}_i$, where $\times$ is the vector product. Then from (6) and $(5)_2$ we obtain

$$\boldsymbol{B} = \frac{1}{2}\boldsymbol{t}_i \times \operatorname{Grad}_s \boldsymbol{t}_i.$$

Analogously, $\boldsymbol{b}_\star = \operatorname{ax}\left(d\boldsymbol{T}_\star \boldsymbol{T}_\star^{-1}\right) = \dfrac{1}{2}\left(\boldsymbol{t}_{\star j} \times d\boldsymbol{t}_{\star j}\right) = \boldsymbol{B}_\star dx_\star$ and $\boldsymbol{B}_\star = \dfrac{1}{2}\boldsymbol{t}_{\star i} \times \operatorname{Grad}_s^\star \boldsymbol{t}_{\star i}$. Then, using the relation $\boldsymbol{t}_{\star j} = \boldsymbol{R}\boldsymbol{t}_j$ we obtain that

$$\boldsymbol{b}_\star = \frac{1}{2}\left(\boldsymbol{R}\boldsymbol{t}_j \times d\left(\boldsymbol{R}\boldsymbol{t}_j\right)\right) = \frac{1}{2}\left(\boldsymbol{R}\boldsymbol{t}_j \times \boldsymbol{R}d\boldsymbol{t}_j\right) + \frac{1}{2}\left[\boldsymbol{R}\boldsymbol{t}_j \times (d\boldsymbol{R})\boldsymbol{t}_j\right]$$

$$= \boldsymbol{R}\frac{1}{2}\left(\boldsymbol{t}_j \times d\boldsymbol{t}_j\right) - \boldsymbol{R}\frac{1}{2}\left[\boldsymbol{t}_j \times \left(d\boldsymbol{R}^T\boldsymbol{R}\right)\boldsymbol{t}_j\right].$$

If we introduce the axial vector $\boldsymbol{g}$ and the skew tensor $\boldsymbol{G}$ by the formula $\boldsymbol{g} = \operatorname{ax}\left(d\boldsymbol{R}^T\boldsymbol{R}\right) \equiv \boldsymbol{G}dx$ we can perform the following transformations:

$$\left[\boldsymbol{t}_j \times \left(d\boldsymbol{R}^T\boldsymbol{R}\right)\boldsymbol{t}_j\right] = \boldsymbol{t}_j \times \left(\boldsymbol{g} \times \boldsymbol{t}_j\right) = \boldsymbol{g}\left(\boldsymbol{t}_j \cdot \boldsymbol{t}_j\right) - \boldsymbol{t}_j\left(\boldsymbol{g} \cdot \boldsymbol{t}_j\right) = 2\boldsymbol{g}.$$

Therefore, we obtain

$$\boldsymbol{B}_\star = \boldsymbol{R}\boldsymbol{B}\mathbf{P}^{-1} - \boldsymbol{L}, \quad \boldsymbol{L} = \boldsymbol{R}\boldsymbol{G}\mathbf{P}^{-1}. \tag{26}$$

The tensors $\boldsymbol{G} \in E \otimes T_x M$ and $\boldsymbol{L} \in E \otimes T_x M$ describe changes of the structure curvature tensor under change of the reference placement.

As a result, for $\mathbf{K}_\star$ we obtain

$$\mathbf{K}_\star = \boldsymbol{R}\mathbf{K}\mathbf{P}^{-1} + \boldsymbol{L}. \tag{27}$$

Let us note that the tensors $\boldsymbol{B}$, $\boldsymbol{C}$, $\mathbf{K}$, $\mathbf{V}$, and $\mathbf{M}$ are the axial tensors (pseudotensors), not the usual (polar) ones. The resultant surface couple $\boldsymbol{c}$ and the angular velocity $\boldsymbol{\omega} \equiv \operatorname{ax}\left(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T\right)$ are also pseudovectors. Pseudovectors and pseudotensors differ from the usual (polar) ones in that they are sensitive to the change of orientation of the space and change their sign under the inversion transformation of the space. Inversion of the space does not correspond to any real deformation of the reference placement. The transformation simply reflects the requirement of invariance of the constitutive equations under the mirror reflection of the reference placement, or under the change of orientation of the base vectors from the right-hand one to

the left-hand one. The concept of pseudoscalars, pseudovectors and pseudotensors is widely used in modern physics [5], and in the theory of electromagnetism in particular. In shell theory it was used first by Zhilin [47], see also Altenbach and Zhilin [2, 3].

The simple example of psedovector is the vector product $\boldsymbol{a} \times \boldsymbol{b}$ of two usual (polar) vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. In this case we have the formula $(\boldsymbol{Ra}) \times (\boldsymbol{Rb}) = (\det \boldsymbol{R})\boldsymbol{R}(\boldsymbol{a} \times \boldsymbol{b})$, because the vector product changes its sign if $\boldsymbol{R} = -\boldsymbol{1}$. An example of pseudotensor is the tensor $\mathrm{ax}^{-1}(\boldsymbol{\omega})$, that is the skew tensor which axial vector is $\boldsymbol{\omega}$. Another example of pseudotensor is the Levi–Civita pseudotensor used in [5].

These remarks allow us to generalize the transformation properties (25), (26) and (27) using the orthogonal tensors $\boldsymbol{R} \in Orth$ as follows:

$$\mathbf{V}_\star = (\det \boldsymbol{R})\boldsymbol{R}\mathbf{V}\mathbf{P}^{-1}, \quad \boldsymbol{B}_\star = (\det \boldsymbol{R})\boldsymbol{R}\boldsymbol{B}\mathbf{P}^{-1} - \boldsymbol{L}, \tag{28}$$

$$\mathbf{K}_\star = (\det \boldsymbol{R})\boldsymbol{R}\mathbf{K}\mathbf{P}^{-1} + \boldsymbol{L}, \quad \boldsymbol{L} = (\det \boldsymbol{R})\boldsymbol{R}\boldsymbol{G}\mathbf{P}^{-1}.$$

The use of orthogonal tensors $\boldsymbol{R}$ in (28) will allow us to reduce further the number of independent material constants in the constitutive equations.

The form of 2D elastic strain energy density of the shell depends upon the choice of the reference placement, in general. Particularly important are sets of reference placements which leave unchanged the form of the density. Transformations of the reference placement under which the density remains unchanged are called here invariant transformations. Knowing all such invariant transformations allows one to precisely define the fluid shell, the solid shell, or the membrane shell as well as to introduce notions of isotropic, hemitropic, or orthotropic solid shell. Similar approach is used in continuum mechanics [42, 44] and theory of elasticity [46].

The elastic strain energy density $W_\star$ relative to the changed reference placement $\kappa_\star$ depends in each point $x_\circ \in M_\star$ on the strain tensor $\mathbf{E}_\star$, the bending tensor $\mathbf{K}_\star$, and also upon the structure curvature tensor $\boldsymbol{B}_\star$. This dependence may, in general, be different than that of $W(\mathbf{E}, \mathbf{K}; \boldsymbol{B})$. However, the elastic strain energy of any part of the shell should be conserved, so that

$$\iint\limits_{M'} W\, da = \iint\limits_{M'_\star} W_\star\, da_\star \tag{29}$$

for any part of the base surface $M' \subset M$ corresponding to $M'_\star \subset M_\star$, because the functions W and $W_\star$ describe the strain energy density of the same deformed state of $N = \chi(M)$.

From (29) it follows that $W_\star$ and W are related by

$$J(\mathbf{P})W_\star(\mathbf{E}_\star, \mathbf{K}_\star; \boldsymbol{B}_\star) = W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}),$$

where $J(\mathbf{P})$ describes the change of elementary surface element $da_\star = J(\mathbf{P})da$, and J is given by

$$J = |\det \mathbf{P}| = \sqrt{\frac{1}{2}\left[\mathrm{tr}^2\left(\mathbf{P}^T\mathbf{P}\right) - \mathrm{tr}\left(\mathbf{P}^T\mathbf{P}\right)^2\right]}. \tag{30}$$

4. The Local Symmetry Group

First, let us give some physical statements which should be accounted for when discussing invariant transformations of the reference placement. Let us introduce the following hypothesis:

Invariant transformations of the reference placement should preserve the elementary surface element of M.

Indeed, it is difficult to imagine that the constitutive equations may not 'feel' the change of surface area. If we would allow transformations which change the elementary surface element then, without changing the strain energy, the surface area could be changed virtually to zero. In this sense, preserving the elementary surface area corresponds to the analogous requirement of unimodular transformations which constitute the local symmetry group of the 3D simple material [42–44, 46].

The requirement that the elementary surface element should remain constant under the change of the reference placement can be expressed by

$$J(\mathbf{P}) = 1, \tag{31}$$

where $J(\mathbf{P})$ is given by (30).

The assumption that the constitutive relation is insensitive to the change of κ into $\kappa_\star$ means that the strain energy densities W and $W_\star$ should coincide, that is

$$W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) = W(\mathbf{E}_\star, \mathbf{K}_\star; \boldsymbol{B}_\star).$$

Therefore, using (24) and (28) we obtain the following invariance requirement for W under change of the reference placement:

$$\begin{aligned} &W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) \\ &\quad = W\big[\boldsymbol{R}\mathbf{E}\mathbf{P}^{-1} + \boldsymbol{R}\boldsymbol{I}\mathbf{P}^{-1} - \boldsymbol{I}(\det \boldsymbol{R})\boldsymbol{R}\mathbf{K}\mathbf{P}^{-1} + \boldsymbol{L}; \\ &\qquad (\det \boldsymbol{R})\boldsymbol{R}\boldsymbol{B}\mathbf{P}^{-1} - \boldsymbol{L}\big]. \end{aligned} \tag{32}$$

The relation (32) holds locally, i.e. it should be satisfied at any particular point $x_\circ \in M$, and the tensors $\boldsymbol{R}$ and $\boldsymbol{L}$ are independent here. As a result, the local invariance of W under change of the reference placement is described by three tensors $\mathbf{P}$, $\boldsymbol{R}$, and $\boldsymbol{L}$, where $\mathbf{P} \in T_x M \otimes T_x M$, $\boldsymbol{R} \in Orth$, and $\boldsymbol{L} \in E \otimes T_x M$. In what follows we do not explicitly indicate that all the tensors depend also on the point $x_\circ \in M$.

We will use the following nomenclature:

$Orth = \{\boldsymbol{O} : \boldsymbol{O}^{-1} = \boldsymbol{O}^T\}$ – group of orthogonal tensors;
$Orth^+ = \{\boldsymbol{O} : \boldsymbol{O} \in Orth, \det \boldsymbol{O} = 1\}$ – group of rotation tensors;
$Orth_n = \{\boldsymbol{O} : \boldsymbol{O} \in Orth, \boldsymbol{O}\boldsymbol{n} = \pm\boldsymbol{n}\}$ – group of rotations about $\boldsymbol{n}$ and

reflections relative to the tangent plane;

$Orth_n^+ = \{\boldsymbol{O} : \boldsymbol{O} \in Orth^+, \boldsymbol{O}\boldsymbol{n} = \boldsymbol{n}\}$ – group of rotations about $\boldsymbol{n}$;
$Unim_n = \{\mathbf{P} : \boldsymbol{n}\boldsymbol{I}\mathbf{P} = \mathbf{P}\boldsymbol{I}^T\boldsymbol{n} = \mathbf{0}, J(\mathbf{P}) = 1\}$ – two-dimensional analogue of

the unimodular group;

$Lin_n = \{\boldsymbol{L} : \boldsymbol{L}\boldsymbol{I}^T\boldsymbol{n} = \mathbf{0}\}$.

Here $Unim_n$ – group with regard to multiplication, and Lin_n – group with regard to addition.

Now we are able to introduce the following definition:

DEFINITION 1. We call the local symmetry group $\mathcal{G}_\kappa$ of the elastic shell all sets of ordered triples of tensors

$$\mathbb{X} = (\mathbf{P} \in Unim_n, \boldsymbol{R} \in Orth, \boldsymbol{L} \in Lin_n),$$

satisfying the relation

$$\begin{aligned} &W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) \\ &\quad = W\big[\boldsymbol{R}\mathbf{E}\mathbf{P}^{-1} + \boldsymbol{R}\boldsymbol{I}\mathbf{P}^{-1} - \boldsymbol{I},\ (\det \boldsymbol{R})\boldsymbol{R}\mathbf{K}\mathbf{P}^{-1} + \boldsymbol{L}; \\ &\qquad (\det \boldsymbol{R})\boldsymbol{R}\boldsymbol{B}\mathbf{P}^{-1} - \boldsymbol{L}\big] \end{aligned} \tag{33}$$

for any tensors $\mathbf{E}$, $\mathbf{K}$, $\boldsymbol{B}$ in the domain of definition of the function W.

The set $\mathcal{G}_\kappa$ is the group relative to the group operation $\circ$ defined by

$$(\mathbf{P}_1, \boldsymbol{R}_1, \boldsymbol{L}_1) \circ (\mathbf{P}_2, \boldsymbol{R}_2, \boldsymbol{L}_2) = \big[\mathbf{P}_1\mathbf{P}_2,\ \boldsymbol{R}_1\boldsymbol{R}_2,\ \boldsymbol{L}_1 + (\det \boldsymbol{R}_1)\,\boldsymbol{R}_1\boldsymbol{L}_2\mathbf{P}_1^{-1}\big].$$

We will show that this definition allows us to establish an analogue of the Noll rule.

Let us check that if $\mathbb{X}_1 \equiv (\mathbf{P}_1, \boldsymbol{R}_1, \boldsymbol{L}_1) \in \mathcal{G}_\kappa$ and $\mathbb{X}_2 \equiv (\mathbf{P}_2, \boldsymbol{R}_2, \boldsymbol{L}_2) \in \mathcal{G}_\kappa$, then also $\mathbb{X}_1 \circ \mathbb{X}_2 \in \mathcal{G}_\kappa$. Indeed, since $\mathbb{X}_1 \in \mathcal{G}_\kappa$ and $\mathbb{X}_2 \in \mathcal{G}_\kappa$ then

$$\begin{aligned} &W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) \\ &\quad = W\big[\boldsymbol{R}_1\mathbf{E}{\mathbf{P}_1}^{-1} + \boldsymbol{R}_1\boldsymbol{I}\mathbf{P}_1^{-1} - \boldsymbol{I},\ (12 \det \boldsymbol{R}_1)\,\boldsymbol{R}_1\mathbf{K}{\mathbf{P}_1}^{-1} + \boldsymbol{L}_1; \\ &\qquad (\det \boldsymbol{R}_1)\,\boldsymbol{R}_1\boldsymbol{B}\mathbf{P}_1^{-1} - \boldsymbol{L}_1\big] \\ &\quad = W\big[\boldsymbol{R}_2\mathbf{E}{\mathbf{P}_2}^{-1} + \boldsymbol{R}_2\boldsymbol{I}\mathbf{P}_2^{-1} - \boldsymbol{I},\ (\det \boldsymbol{R}_2)\,\boldsymbol{R}_2\mathbf{K}{\mathbf{P}_2}^{-1} + \boldsymbol{L}_2; \\ &\qquad (\det \boldsymbol{R}_2)\,\boldsymbol{R}_2\boldsymbol{B}\mathbf{P}_2^{-1} - \boldsymbol{L}_2\big]. \end{aligned}$$

Taking these relations into account we have

$$\begin{aligned} &W\big[\boldsymbol{R}_1\boldsymbol{R}_2\mathbf{E}{\mathbf{P}_2}^{-1}{\mathbf{P}_1}^{-1} + \boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{I}{\mathbf{P}_2}^{-1}{\mathbf{P}_1}^{-1} - \boldsymbol{I}, \\ &\quad (\det \boldsymbol{R}_1)\,(\det \boldsymbol{R}_2)\,\boldsymbol{R}_1\boldsymbol{R}_2\mathbf{K}{\mathbf{P}_2}^{-1}{\mathbf{P}_1}^{-1} + \boldsymbol{L}_1 + (\det \boldsymbol{R}_1)\,\boldsymbol{R}_1\boldsymbol{L}_2\mathbf{P}_1^{-1}; \\ &\quad (\det \boldsymbol{R}_1)\,(\det \boldsymbol{R}_2)\,\boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{B}{\mathbf{P}_2}^{-1}{\mathbf{P}_1}^{-1} - \boldsymbol{L}_1 - (\det \boldsymbol{R}_1)\,\boldsymbol{R}_1\boldsymbol{L}_2\mathbf{P}_1^{-1}\big] \\ &\quad = W\big\{\boldsymbol{R}_1\left(\boldsymbol{R}_2\mathbf{E}\mathbf{P}_2^{-1} + \boldsymbol{R}_2\boldsymbol{I}\mathbf{P}_2^{-1} - \boldsymbol{I}\right)\mathbf{P}_1^{-1} + \boldsymbol{R}_1\boldsymbol{I}\mathbf{P}_1^{-1} - \boldsymbol{I}, \\ &\qquad (\det \boldsymbol{R}_1)\,\boldsymbol{R}_1\left[(\det \boldsymbol{R}_2)\,\boldsymbol{R}_2\mathbf{K}{\mathbf{P}_2}^{-1} + \boldsymbol{L}_2\right]\mathbf{P}_1^{-1} + \boldsymbol{L}_1; \\ &\qquad (\det \boldsymbol{R}_1)\,\boldsymbol{R}_1\left[(\det \boldsymbol{R}_2)\,\boldsymbol{R}_2\boldsymbol{B}{\mathbf{P}_2}^{-1} - \boldsymbol{L}_2\right]\mathbf{P}_1^{-1} - \boldsymbol{L}_1\big\} \\ &\quad = W\big[\boldsymbol{R}_2\mathbf{E}{\mathbf{P}_2}^{-1} + \boldsymbol{R}_2\boldsymbol{I}\mathbf{P}_2^{-1} - \boldsymbol{I}\,, \\ &\qquad (\det \boldsymbol{R}_2)\,\boldsymbol{R}_2\mathbf{K}{\mathbf{P}_2}^{-1} + \boldsymbol{L}_2;\ (\det \boldsymbol{R}_2)\,\boldsymbol{R}_2\boldsymbol{B}\mathbf{P}_2^{-1} - \boldsymbol{L}_2\big] \\ &\quad = W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}), \end{aligned}$$

which proves that $\mathbb{X}_1 \circ \mathbb{X}_2$ belongs to the symmetry group $\mathcal{G}_\kappa$ indeed.

The unit element of $\mathcal{G}_\kappa$ is $\mathbb{I} = (\mathbf{A}, \mathbf{1}, \mathbf{0})$. The inverse element to $\mathbb{X} \in \mathcal{G}_\kappa$ is given by

$$\mathbb{X}^{-1} \equiv (\mathbf{P}, \boldsymbol{R}, \boldsymbol{L})^{-1} = \left[\mathbf{P}^{-1}, \boldsymbol{R}^T, -(\det \boldsymbol{R})\boldsymbol{R}^T\boldsymbol{L}\mathbf{P}\right].$$

Indeed,

$$\begin{aligned}\mathbb{X} \circ \mathbb{X}^{-1} &\equiv (\mathbf{P}, \boldsymbol{R}, \boldsymbol{L}) \circ (\mathbf{P}, \boldsymbol{R}, \boldsymbol{L})^{-1} \\ &= \left[\mathbf{P}\mathbf{P}^{-1}, \boldsymbol{R}\boldsymbol{R}^T, \boldsymbol{L} - (\det \boldsymbol{R})^2\boldsymbol{R}\boldsymbol{R}^T\boldsymbol{L}\mathbf{P}\mathbf{P}^{-1}\right] = (\mathbf{A}, \mathbf{1}, \mathbf{0}).\end{aligned}$$

The group operation $\circ$ used here is analogous to but not identical with the one introduced by Murdoch and Cohen [30] for the material surface with one deformable director. Our symmetry group also differs from the one of 3D micropolar elastic material introduced by Eringen and Kafadar [19], because we take into account that some arguments of W are pseudotensors and our group consists of more narrow subgroups as compared with the 3D case. The property (28) was not taken into account in [15, 12], where the symmetry group for the micropolar theory of elastic shells was proposed assuming that the strain energy density may depend on curvature of the reference base surface.

The local symmetry group depends not only on the point $x_\circ \in M$ but also upon the choice of the reference placement. Let us analyse how the symmetry groups of different reference placements are related. Let κ_1 and κ_2 be two different reference placements having the common tangent plane at the point $x_\circ \in M$ where the symmetry group is discussed, and $\mathcal{G}_1$ and $\mathcal{G}_2$ be the symmetry groups corresponding to the reference placements κ_1 and κ_2, respectively. In what follows quantities described in the placements κ_1 and κ_2 are marked by indices 1 and 2, respectively.

Let $\mathbf{P}$ be the tangential deformation gradient and $\boldsymbol{R}$ be the rotation tensor associated with deformation $M_1 \to M_2$, as well as $\mathbf{P}^{-1}$ and $\boldsymbol{R}^T$ be the inverse deformation gradient and the inverse rotation tensor associated with the inverse deformation, respectively. Then, by analogy to (24), (28), we can relate the shell strain measures $\mathbf{E}_1$ and $\mathbf{E}_2$, $\mathbf{K}_1$ and $\mathbf{K}_2$, as well as the structure curvature tensors $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ defined relative to different reference placements

$$\mathbf{E}_2 = \boldsymbol{R}\mathbf{E}_1\mathbf{P}^{-1} + \boldsymbol{R}\boldsymbol{I}_1\mathbf{P}^{-1} - \boldsymbol{I}_2, \quad \mathbf{K}_2 = (\det \boldsymbol{R})\boldsymbol{R}\mathbf{K}_1\mathbf{P}^{-1} + \boldsymbol{L}. \tag{34}$$

$$\boldsymbol{B}_2 = (\det \boldsymbol{R})\boldsymbol{R}\boldsymbol{B}_1\mathbf{P}^{-1} - \boldsymbol{L}, \quad \boldsymbol{L} = (\det \boldsymbol{R})\boldsymbol{R}\boldsymbol{G}\mathbf{P}^{-1}, \tag{35}$$

where $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ are the inclusion operators for κ_1 and κ_2, respectively.

Let W_1 and W_2 be the strain energy densities defined relative to the two reference placements. From (29) it follows that W_2 and W_1 are related by

$$J(\mathbf{P})W_2(\mathbf{E}_2, \mathbf{K}_2; \boldsymbol{B}_2) = W_1(\mathbf{E}_1, \mathbf{K}_1; \boldsymbol{B}_1).$$

Taking into account (34) and (35) we have

$$\begin{aligned}J(\mathbf{P})W_2\big[&\boldsymbol{R}\mathbf{E}_1\mathbf{P}^{-1} + \boldsymbol{R}\boldsymbol{I}_1\mathbf{P}^{-1} - \boldsymbol{I}_2, \ (\det \boldsymbol{R})\boldsymbol{R}\mathbf{K}_1\mathbf{P}^{-1} + \boldsymbol{L}; \\ &(\det \boldsymbol{R})\boldsymbol{R}\boldsymbol{B}_1\mathbf{P}^{-1} - \boldsymbol{L}\big] \\ &= W_1(\mathbf{E}_1, \mathbf{K}_1; \boldsymbol{B}_1).\end{aligned} \tag{36}$$

Let the element $\mathbb{X}_1 \equiv (\mathbf{P}_1, \boldsymbol{R}_1, \boldsymbol{L}_1) \in \mathcal{G}_1$. Then using (36) we obtain

$$
\begin{aligned}
J(\mathbf{P})W_2(\mathbf{E}_2, \mathbf{K}_2; \boldsymbol{B}_2) &\\
&= W_1(\mathbf{E}_1, \mathbf{K}_1; \boldsymbol{B}_1)\\
&= W_1\big[\boldsymbol{R}_1\mathbf{E}_1\mathbf{P}_1^{-1} + \boldsymbol{R}_1\boldsymbol{I}_1\mathbf{P}_1^{-1} - \boldsymbol{I}_1,\\
&\qquad (\det \boldsymbol{R}_1)\boldsymbol{R}_1\mathbf{K}_1\mathbf{P}_1^{-1} + \boldsymbol{L}_1;\\
&\qquad (\det \boldsymbol{R}_1)\boldsymbol{R}_1\boldsymbol{B}_1\mathbf{P}_1^{-1} - \boldsymbol{L}_1\big]\\
&= J(\mathbf{P})W_2\big[\boldsymbol{R}\boldsymbol{R}_1\mathbf{E}_1\mathbf{P}_1^{-1}\mathbf{P}^{-1} + \boldsymbol{R}\boldsymbol{R}_1\boldsymbol{I}_1\mathbf{P}_1^{-1}\mathbf{P}^{-1} - \boldsymbol{I}_2,\\
&\qquad (\det \boldsymbol{R})(\det \boldsymbol{R}_1)\boldsymbol{R}\boldsymbol{R}_1\mathbf{K}_1\mathbf{P}_1^{-1}\mathbf{P}^{-1}\\
&\qquad + (\det \boldsymbol{R})\boldsymbol{R}\boldsymbol{L}_1\mathbf{P}^{-1} + \boldsymbol{L};\\
&\qquad (\det \boldsymbol{R})(\det \boldsymbol{R}_1)\boldsymbol{R}\boldsymbol{R}_1\boldsymbol{B}_1\mathbf{P}_1^{-1}\mathbf{P}^{-1}\\
&\qquad -(\det \boldsymbol{R})\boldsymbol{R}\boldsymbol{L}_1\mathbf{P}^{-1} - \boldsymbol{L}\big]\\
&= J(\mathbf{P})W_2\big[\boldsymbol{R}\boldsymbol{R}_1\boldsymbol{R}^T\mathbf{E}_2\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1} + \boldsymbol{R}\boldsymbol{R}_1\boldsymbol{R}^T\boldsymbol{I}_2\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1} - \boldsymbol{I}_2,\\
&\qquad (\det \boldsymbol{R}_1)\boldsymbol{R}\boldsymbol{R}_1\boldsymbol{R}^T\mathbf{K}_2\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1}\\
&\qquad - (\det \boldsymbol{R}_1)\boldsymbol{R}\boldsymbol{R}_1\boldsymbol{R}^T\boldsymbol{L}\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1}\\
&\qquad + (\det \boldsymbol{R}_1)\boldsymbol{R}\boldsymbol{L}_1\mathbf{P}^{-1} + \boldsymbol{L};\\
&\qquad (\det \boldsymbol{R}_1)\boldsymbol{R}\boldsymbol{R}_1\boldsymbol{R}^T\boldsymbol{B}_2\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1}\\
&\qquad\qquad + (\det \boldsymbol{R}_1)\boldsymbol{R}\boldsymbol{R}_1\boldsymbol{R}^T\boldsymbol{L}\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1}\\
&\qquad\qquad -(\det \boldsymbol{R}_1)\boldsymbol{R}\boldsymbol{L}_1\mathbf{P}^{-1} - \boldsymbol{L}\big].
\end{aligned}
\tag{37}
$$

From (37) it follows that the element $\mathbb{X}_2 \equiv (\mathbf{P}_2, \boldsymbol{R}_2, \boldsymbol{L}_2) \in \mathcal{G}_2$, where

$$
\begin{aligned}
&\boldsymbol{R}_2 = \boldsymbol{R}\boldsymbol{R}_1\boldsymbol{R}^T, \mathbf{P}_2 = \mathbf{P}\mathbf{P}_1\mathbf{P}^{-1},\\
&\boldsymbol{L}_2 = \boldsymbol{L} + (\det \boldsymbol{R}_1)\boldsymbol{R}\boldsymbol{L}_1\mathbf{P}^{-1} - (\det \boldsymbol{R}_1)\boldsymbol{R}\boldsymbol{R}_1\boldsymbol{R}^T\boldsymbol{L}\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1}.
\end{aligned}
$$

It is easy to show that $\mathbb{X}_2 = \mathbb{P} \circ \mathbb{X}_1 \circ \mathbb{P}^{-1}$, where $\mathbb{P} \equiv (\boldsymbol{R}, \mathbf{P}, \boldsymbol{L})$. Indeed,

$$
\begin{aligned}
\mathbb{P} \circ \mathbb{X}_1 &\equiv (\mathbf{P}, \boldsymbol{R}, \boldsymbol{L}) \circ (\mathbf{P}_1\boldsymbol{R}_1, \boldsymbol{L}_1)\\
&= \big[\mathbf{P}\mathbf{P}_1, \boldsymbol{R}\boldsymbol{R}_1, \boldsymbol{L} + (\det \boldsymbol{R})\boldsymbol{R}^T\boldsymbol{L}\mathbf{P}\big].
\end{aligned}
$$

Taking into account that $\mathbb{P}^{-1} = \big[\mathbf{P}^{-1}, \boldsymbol{R}^T, -(\det \boldsymbol{R})\boldsymbol{R}^T\boldsymbol{L}\mathbf{P}\big]$, we obtain

$$
\begin{aligned}
\mathbb{P} \circ \mathbb{X}_1 \circ \mathbb{P}^{-1} = \big[&\mathbf{P}\mathbf{P}_1\mathbf{P}^{-1}, \boldsymbol{R}\boldsymbol{R}_1\boldsymbol{R}^T, \boldsymbol{L} + (\det \boldsymbol{R})^2(\det \boldsymbol{R}_1)\boldsymbol{R}\boldsymbol{L}_1\mathbf{P}^{-1}\\
&-(\det \boldsymbol{R})^2(\det \boldsymbol{R}_1)\boldsymbol{R}\boldsymbol{R}_1\boldsymbol{R}^T\boldsymbol{L}\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1}\big],
\end{aligned}
$$

from which follows the sought result.

Then, the local symmetry group under change of the reference placement transforms according to

$$
\mathcal{G}_2 = \mathbb{P} \circ \mathcal{G}_1 \circ \mathbb{P}^{-1}. \tag{38}
$$

The transformation (38) is a counterpart in the general theory of shells of the well known Noll rule [38, 42, 44, 46] for symmetry groups of simple materials in continuum mechanics.

5. Shell Strain Energy Density

The structure of the local symmetry group $\mathcal{G}_\kappa$ puts some constraints on the form of W which allows one to considerably simplify this form. In what follows we discuss some simplifications of W following from the structure of $\mathcal{G}_\kappa$.

The group $\mathcal{G}_\kappa$ allows one to analyse properties of W only at the chosen point $x_\circ \in M$. Therefore, statements that the symmetry group corresponds to a solid or fluid membrane, for example, are valid only at the chosen point. At another point of the shell the symmetry group may be different, in general.

Let us discuss the trivial symmetry group $\mathcal{G}_\kappa = \{\mathbf{A}, \pm\boldsymbol{I}, \mathbf{0}\}$. From definition of $\mathcal{G}_\kappa$ it follows that W should be an even function of $\mathbf{K}$ and $\boldsymbol{B}$, that is

$$W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) = W(\mathbf{E}, -\mathbf{K}; -\boldsymbol{B}). \tag{39}$$

From (39) it follows, in particular, that $W(\mathbf{E}, \mathbf{K}; \boldsymbol{B})$ cannot have terms linear in $\mathbf{K}$ alone. Only when W is assumed to explicitly depend also on the structure curvature tensor $\boldsymbol{B}$ the terms linear in $\mathbf{K}$ of the type $\mathrm{tr}\,(\boldsymbol{B}\mathbf{K}^T)$ are possible.

If the elements of $\mathcal{G}_\kappa$ consist of tensor triples containing an arbitrary tensor $\boldsymbol{L} \in Lin_n$, then the number of arguments in W can be decreased.

Indeed, let $\mathbb{X} = (\mathbf{A}, \boldsymbol{I}, \boldsymbol{L}) \in \mathcal{G}_\kappa$. Then

$$W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) = W(\mathbf{E}, \mathbf{K} + \boldsymbol{L}; \boldsymbol{B} - \boldsymbol{L}), \quad \forall\, \boldsymbol{L} \in Lin_n. \tag{40}$$

Introducing a one-parameter family of transformations

$$W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) = W(\mathbf{E}, \mathbf{K} + t\boldsymbol{L}; \boldsymbol{B} - t\boldsymbol{L}), \quad \forall\, \boldsymbol{L} \in Lin_n, \forall\, t \in R,$$

we can differentiate it relative to t and find that

$$0 = \frac{\partial W}{\partial \mathbf{K}} \cdot \boldsymbol{L} - \frac{\partial W}{\partial \boldsymbol{B}} \cdot \boldsymbol{L}, \quad \forall\, \boldsymbol{L} \in Lin_n,$$

from which we obtain

$$\frac{\partial W}{\partial \mathbf{K}} = \frac{\partial W}{\partial \boldsymbol{B}}.$$

This equation is satisfied when $W = W(\mathbf{E}, \mathbf{K} + \boldsymbol{B}) = W(\mathbf{E}, \mathbf{V})$.

Alternatively, since (40) is satisfied by any $\boldsymbol{L}$ let us take $\boldsymbol{L}$ equal to $\boldsymbol{B}$. Then from Equation (40) it follows that

$$W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) = W(\mathbf{E}, \mathbf{K} + \boldsymbol{B}; \mathbf{0}) = W(\mathbf{E}, \mathbf{V}).$$

If the strain energy density is reduced to $W(\mathbf{E}, \mathbf{K})$, i.e. when we neglect its explicit dependence on $\boldsymbol{B}$, then from the Equation (40) we obtain that $W = W(\mathbf{E})$. This form corresponds to the constitutive equations of the Cosserat membrane. Indeed, in this case we have

$$W(\mathbf{E}, \mathbf{K}) = W(\mathbf{E}, \mathbf{K} + t\boldsymbol{L}), \quad \forall\, \boldsymbol{L} \in Lin_n, \forall\, t \in R,$$

and by the same way as above we obtain that $\dfrac{\partial W}{\partial \mathbf{K}} = \mathbf{0}$.

In the elastic Cosserat membrane the stress couple tensor $\mathbf{M}$ vanishes while the stress resultant tensor $\mathbf{N}$ still remains non-symmetric, in general. The couple equilibrium equation $(1)_2$ is non-trivial here and can be used to derive the field of rotation $\boldsymbol{Q}$. However, it is then not possible to assume the rotation $\boldsymbol{Q}$ or the stress couple $\mathbf{M}$ at the base surface boundary ∂M.

If the local symmetry group of the membrane takes the form

$$\mathcal{G}_\kappa = \{\mathbf{P}, \boldsymbol{R} \in Orth^+, \boldsymbol{L} \in Lin_n\},$$

i.e. when $\mathcal{G}_\kappa$ consists of an arbitrary proper orthogonal tensor $\boldsymbol{R} \in Orth^+$ and an arbitrary tensor $\boldsymbol{L} \in E \otimes T_x M$, then the constitutive equations describe the usual membrane for which the strain energy density is given by $W = W(\mathbf{Y})$, $\mathbf{Y} = \sqrt{\boldsymbol{F}^T \boldsymbol{F}}$, i.e. W depends here on the symmetric tangential stretch tensor $\mathbf{Y}$ alone.

Indeed, let us introduce into the formula

$$W(\mathbf{E}) = W(\boldsymbol{R}\mathbf{E}\mathbf{P}^{-1} + \boldsymbol{R}\boldsymbol{I}\mathbf{P}^{-1} - \boldsymbol{I}) = W(\boldsymbol{R}\boldsymbol{Q}^T \boldsymbol{F}\mathbf{P}^{-1} - \boldsymbol{I})$$

the relation $\mathbf{P} = \mathbf{A}$ and $\boldsymbol{R} = \boldsymbol{H}^T \boldsymbol{Q}$, where $\boldsymbol{H}$ is the proper orthogonal tensor following from the polar decomposition

$$\boldsymbol{F} + \boldsymbol{m} \otimes \boldsymbol{n} = \boldsymbol{H}(\mathbf{Y} + \boldsymbol{n} \otimes \boldsymbol{n}),$$

in analogy to (19), where $\boldsymbol{m}$ is the unit normal to N. Then it follows that $\boldsymbol{F} = \boldsymbol{H}\boldsymbol{I}\mathbf{Y}$ and

$$W(\mathbf{E}) = W\left(\boldsymbol{H}^T \boldsymbol{Q}\boldsymbol{Q}^T \boldsymbol{H}\boldsymbol{I}\mathbf{Y}\mathbf{A} - \boldsymbol{I}\right) = W(\boldsymbol{I}\mathbf{Y} - \boldsymbol{I}) = W^\times(\mathbf{Y}).$$

Then the stress resultant tensor $\mathbf{N}$ becomes symmetric, $\mathbf{N} = \mathbf{N}^T$, and the couple equilibrium equation $(1)_2$ is identically satisfied for the vanishing resultant surface couple vector $\boldsymbol{c}$. The symmetry groups and representations of constitutive equations for membranes were discussed in [21, 45].

More detailed classification of the constitutive equations for membranes can be obtained by the symmetry group containing the tensor $\mathbf{P}$.

Let us recall here the following results [21, 45]:

1. $\mathcal{G}_\kappa = \{\mathbf{P} \in Unim_n, \boldsymbol{R} \in Orth^+, \boldsymbol{L} \in Lin_n\}$: The liquid membrane, $W = W(\det \mathbf{Y})$.
2. $\mathcal{G}_\kappa = \{\mathbf{P} \in \mathcal{S}_n\mathbf{A}, \boldsymbol{R} \in \mathcal{S}_n, \boldsymbol{L} \in Lin_n\}$, where $\mathcal{S}_n$ is a subgroup of $Orth^+$: The solid membrane.
3. $\mathcal{G}_\kappa = \{\mathbf{P} = \boldsymbol{O}\boldsymbol{I}\mathbf{A}, \boldsymbol{R} \in Orth^+, \boldsymbol{L} \in Lin_n\}$, where $\boldsymbol{O} \in Orth^+$: The isotropic membrane, $W = W(\operatorname{tr} \mathbf{Y}, \det \mathbf{Y})$.

6. Liquid Shells

To define the liquid shell we apply, similarly as in 3D bodies, the requirement that the constitutive equations should be insensitive to change of the reference placement, i.e. the Equation (32) should be satisfied by any triple of tensors $\mathbf{P} \in Unim_n$, $\boldsymbol{R} \in Orth$, $\boldsymbol{L} \in Lin_n$.

DEFINITION 2. The shell is called liquid if there exists a reference placement κ, called undistorted, such that the following relation holds:

$$\begin{aligned} &W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) \\ &\quad = W\left[\boldsymbol{R}\mathbf{E}\mathbf{P}^{-1} + \boldsymbol{R}\boldsymbol{I}\mathbf{P}^{-1} - \boldsymbol{I},\ (\det \boldsymbol{R})\boldsymbol{R}\mathbf{K}\mathbf{P}^{-1} + \boldsymbol{L};\right. \\ &\qquad \left.(\det \boldsymbol{R})\boldsymbol{R}\boldsymbol{B}\mathbf{P}^{-1} - \boldsymbol{L}\right], \\ &\forall \mathbf{P} \in Unim_n, \quad \forall \boldsymbol{R} \in Orth, \quad \forall \boldsymbol{L} \in Lin_n. \end{aligned} \tag{41}$$

From the Noll rule (38) it is easy to find that for the liquid shell any reference placement becomes undistorted, similarly as it is for simple 3D bodies, because the symmetry group becomes here to be maximal.

The requirement (41) is satisfied for any $\boldsymbol{n}$, i.e. Equation (41) is true for any 2D unimodular group $Unim_n$ and for any group with regard to addition Lin_n.

From (41) it immediately follows, as it was proved before, that the strain energy density takes the form $W = W(\mathbf{E}, \mathbf{V})$.

If we use the polar decomposition $\boldsymbol{F} = \boldsymbol{H}\boldsymbol{I}\mathbf{Y}$, where $\mathbf{Y}$ is the tangential symmetric stretch tensor and $\boldsymbol{H}$ is the macrorotation tensor, and introduce into Equation (41) that $\mathbf{P} = \mathbf{Y}\det^{-1}\mathbf{F}$, $\boldsymbol{R} = \boldsymbol{H}^T\boldsymbol{Q}$, then we obtain

$$W = W(\mathbf{E}, \mathbf{V}) = W^{\times}(\det\mathbf{F}, \boldsymbol{C}), \tag{42}$$

where $\boldsymbol{C}$ is the structure curvature tensor of the deformed placement defined in (8).

Indeed,

$$\begin{aligned} W(\mathbf{E}, \mathbf{V}) &= W\left[\boldsymbol{Q}^T\boldsymbol{F} - \boldsymbol{I},\ \boldsymbol{Q}^T\boldsymbol{C}\mathbf{F}\right] \\ &= W\left[\boldsymbol{R}\boldsymbol{Q}^T\boldsymbol{F}\mathbf{P}^{-1} - \boldsymbol{I}, (\det\boldsymbol{R})\boldsymbol{R}\boldsymbol{Q}^T\boldsymbol{C}\mathbf{F}\mathbf{P}^{-1}\right] \\ &= W\left[(\det\mathbf{F} - 1)\boldsymbol{I}, \boldsymbol{H}^T\boldsymbol{C}\boldsymbol{I}^T\boldsymbol{H}(\det\mathbf{F})\right] = W^{\times}(\det\mathbf{F}, \boldsymbol{C}). \end{aligned}$$

With the use of the principle of material frame indifference the function $W^{\times}$ satisfies the condition $W^{\times}(\det\mathbf{F}, \boldsymbol{C}) = W^{\times}(\det\mathbf{F}, \boldsymbol{O}^T\boldsymbol{C}\boldsymbol{I}^T\boldsymbol{O})$, $\forall\, \boldsymbol{O} \in Orth_n$, i.e. it is a surface isotropic function with regard to $\boldsymbol{C}$. This allows us to apply the theorem given in the Appendix B.

The strain energy density (42) describes the material surface which energy is insensitive to arbitrary deformations preserving the elementary surface element of the base surface called also inextensional deformations. However, it is sensitive to the change of orientation of the shell particles.

If the density (42) is compared with the 3D strain energy density of the micropolar fluid [14] one can easily note that they coincide up to notation. This allows one to regard the shell described by the strain energy density (42) to be a 2D analogue of the micropolar fluid.

Correct derivation of models of liquid shells is stimulated by their possible applications to describe thin films which show some effects of oriented elasticity similar to that of viscoelastic micropolar fluids and of liquid crystals. Similar constitutive equations also appear when modelling thin films made of low-symmetric smectics (see for example [11]).

7. Solid Shells

Let us discuss the local symmetry group for solid shells consisting of three independent transformations. For 3D solids the symmetry group is constructed with the help of orthogonal transformations describing rotations and reflections of the reference placement [42, 46]. In the case of solid shells it would be difficult to accept that the constitutive equations might be insensitive to independent rotations of the base surface about its normal as well as to rotations of the directors. It is also difficult to

allow invariance with regard to arbitrary changes of the structure curvature tensor $\boldsymbol{B}$. For solid shells an additional hypothesis seems to be physically justified:

The local symmetry group of the solid shell consists of all transformations of the reference placements performed by the same rotations of shell cross-sections and of shell directors.

Taking into account that $\mathbf{P} \in T_x M \otimes T_x M$, $\boldsymbol{R} \in E \otimes E$, this hypothesis restricts values of the orthogonal tensor $\boldsymbol{R}$ to the subgroup $Orth_n$, and values of the tangential deformation gradient tensor $\mathbf{P}$ to 2D orthogonal tensors following from the projection of $Orth_n$ on the tangent plane $T_x M$.

Accepting the hypothesis we can propose the following

DEFINITION 3. The shell is called solid if there exists a reference placement, called undistorted, such that the local symmetry group is given by

$$\mathcal{G}_\kappa = \{(\mathbf{P} = \boldsymbol{O}\boldsymbol{I}\mathbf{A}, \boldsymbol{O}, \mathbf{0}); \quad \boldsymbol{O} \in \mathcal{S}_n \subset Orth_n\}.$$

The group $\mathcal{G}_\kappa$ is described by the subgroup $\mathcal{S}_n$ of rotations about $\boldsymbol{n}$ and reflections with regard to the tangent plane $Orth_n$. It is easy to see that for an arbitrary tensor $\boldsymbol{X} \in Lin_n$ and any $\boldsymbol{O} \in Orth_n$ we have the identity $\boldsymbol{X}\boldsymbol{I}^T\boldsymbol{O}\boldsymbol{I}\mathbf{A} = \boldsymbol{X}\boldsymbol{I}^T\boldsymbol{O}\boldsymbol{I}$, which simplifies the description of transformations of W. The invariance requirement of W leads here to finding the subgroup $\mathcal{S}_n$ of the full orthogonal group, i.e. such $\boldsymbol{O} \in \mathcal{S}_n \subset Orth_n$ that

$$W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) = W\left[\boldsymbol{O}\mathbf{E}\boldsymbol{I}^T\boldsymbol{O}^T, (\det \boldsymbol{O})\boldsymbol{O}\mathbf{K}\boldsymbol{I}^T\boldsymbol{O}^T; (\det \boldsymbol{O})\boldsymbol{O}\boldsymbol{B}\boldsymbol{I}^T\boldsymbol{O}^T\right].$$

8. Liquid Crystal Shells

In general, the strain energy density of an elastic shell may also admit other local symmetry groups. For example, it is possible to construct the local symmetry groups of W in analogy to the symmetry groups of liquid crystals in continuum mechanics of simple materials [42–44, 46]. In continuum mechanics such materials are called liquid crystals or subfluids, but their mechanical properties do not correspond to the ones of real liquid crystals like nematics, smectics and others. Such a density W would then describe the material shell which is neither isotropic nor solid.

In shell theory the number of local symmetry groups corresponding to the so-called liquid crystals would be much larger than in the case of 3D simple materials because the structure of $\mathcal{G}_\kappa$ is more complex for shells. We leave discussion of symmetry groups for liquid crystal shells for future research.

9. Some Local Symmetry Groups for Solid Shells

Let us discuss simplified forms of W for some particular cases of anisotropy.

The most narrow case includes the isotropic shells which material is insensitive to rotation of shell elements by an arbitrary angle about the normal as well as to an inversion transformation of the space.

DEFINITION 4. The solid shell is called isotropic if there exists a reference placement, called undistorted, such that the strain energy density satisfies the relation

$$W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) = W\left[\boldsymbol{O}\mathbf{E}\boldsymbol{I}^T\boldsymbol{O}^T, (\det \boldsymbol{O})\boldsymbol{O}\mathbf{K}\boldsymbol{I}^T\boldsymbol{O}^T; (\det \boldsymbol{O})\boldsymbol{O}\boldsymbol{B}\boldsymbol{I}^T\boldsymbol{O}^T\right], \quad \forall \boldsymbol{O} \in Orth_n. \tag{43}$$

If in the case above we use only proper orthogonal tensors then the resulting constitutive equations correspond to the hemitropic shell.

DEFINITION 5. The solid shell is called hemitropic if there exists a reference placement, called undistorted, such that the strain energy density satisfies the relation

$$W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) = W\left(\boldsymbol{O}\mathbf{E}\boldsymbol{I}^T\boldsymbol{O}^T, \boldsymbol{O}\mathbf{K}\boldsymbol{I}^T\boldsymbol{O}^T; \boldsymbol{O}\boldsymbol{B}\boldsymbol{I}^T\boldsymbol{O}^T\right), \quad \forall \boldsymbol{O} \in Orth_n^+. \tag{44}$$

DEFINITION 6. The solid shell is called orthotropic if the strain energy density for some reference placement satisfies the relation

$$W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) = W\left(\boldsymbol{O}\mathbf{E}\boldsymbol{I}^T\boldsymbol{O}^T, \boldsymbol{O}\mathbf{K}\boldsymbol{I}^T\boldsymbol{O}^T, \boldsymbol{O}\boldsymbol{B}\boldsymbol{I}^T\boldsymbol{O}^T\right), \quad \boldsymbol{O} = 2\boldsymbol{n} \otimes \boldsymbol{n} - \mathbf{1}, \tag{45}$$

where $\boldsymbol{O}$ is the orthogonal tensor performing the rotation of 180° about the normal to the base surface.

By this transformation the tangential surface tensors do not change. Independent polynomial invariants of the tensor $\boldsymbol{X} \in Lin_n$ are components of $\mathbf{X}_\parallel$ and $j_4(\boldsymbol{X})$, $j_5(\boldsymbol{X})$ (see Appendix B). Then the strain energy density of the orthotropic shell can be given in the form

$$W = W(\mathbf{E}_\parallel, \mathbf{K}_\parallel, \mathbf{E}^T\boldsymbol{n}, \mathbf{K}^T\boldsymbol{n}; \boldsymbol{B}_\parallel, \boldsymbol{B}^T\boldsymbol{n}),$$

where W is the even function of its vector arguments $W(\cdot, \cdot, \boldsymbol{a}, \boldsymbol{b}, \cdot, \boldsymbol{z}) = W(\cdot, \cdot, -\boldsymbol{a}, -\boldsymbol{b}, \cdot, -\boldsymbol{z})$.

The strain energy densities following from definitions of isotropic (43) and orthotropic (45) shells are different from those of isotropic and orthotropic materials used in 3D non-linear micropolar elasticity [14, 19, 20, 33]. Here we discuss more narrow symmetry groups containing only rotations about normals to the reference base surface. For example, the isotropy requirement of 3D micropolar body coincides with Equation (43) if we ignore dependence on $\boldsymbol{B}$ and $\boldsymbol{O}$ take to be an orthogonal tensor. Therefore, the relations (43) and (45) are less restrictive than in the 3D case, which results in necessity to use here more material constants. For example, the constitutive equations of the isotropic physically linear 3D Cosserat body contains only six elastic constants [20, 33], but the constitutive equations of the isotropic physically linear shell within the model discussed here contains eight such constants. But definitions used here entirely coincide with physical concepts concerning the surface anisotropy.

As an example of W satisfying Equation (43) we remind the physically linear isotropic shell which the strain energy density is assumed as the quadratic form

$$2W = \alpha_1 \mathrm{tr}^2\mathbf{E}_\parallel + \alpha_2 \mathrm{tr}\,\mathbf{E}_\parallel^2 + \alpha_3 \mathrm{tr}\,\left(\mathbf{E}_\parallel^T\mathbf{E}_\parallel\right) + \alpha_4 \boldsymbol{n}\mathbf{E}\mathbf{E}^T\boldsymbol{n}$$
$$+\beta_1 \mathrm{tr}^2\mathbf{K}_\parallel + \beta_2 \mathrm{tr}\,\mathbf{K}_\parallel^2 + \beta_3 \mathrm{tr}\,\left(\mathbf{K}_\parallel^T\mathbf{K}_\parallel\right) + \beta_4 \boldsymbol{n}\mathbf{K}\mathbf{K}^T\boldsymbol{n},$$
$$\mathbf{E}_\parallel = \mathbf{E} - \boldsymbol{n}\mathbf{E}, \quad \mathbf{K}_\parallel = \mathbf{K} - \boldsymbol{n}\mathbf{K}, \quad \mathbf{A} = \mathbf{1} - \boldsymbol{n}\otimes\boldsymbol{n}.$$

The bilinear terms of $\mathbf{E}$ and $\mathbf{K}$ are not present in (46) because $\mathbf{K}$ is the pseudotensor and changes its sign under the inversion transformation.

In the expression (46) we have eight factors α_k, β_k $(k = 1, 2, 3, 4)$ which can depend on $\boldsymbol{B}$, in general.

The function (46) generates the following constitutive equations:

$$\mathbf{N} = \alpha_1 \mathbf{A}\,\mathrm{tr}\,\mathbf{E}_\parallel + \alpha_2\mathbf{E}_\parallel^T + \alpha_3\mathbf{E}_\parallel + \alpha_4\boldsymbol{n}\otimes\mathbf{E}\boldsymbol{I}^T\boldsymbol{n},$$
$$\mathbf{M} = \beta_1 \mathbf{A}\,\mathrm{tr}\,\mathbf{K}_\parallel + \beta_2\mathbf{K}_\parallel^T + \beta_3\mathbf{K}_\parallel + \beta_4\boldsymbol{n}\otimes\mathbf{K}\boldsymbol{I}^T\boldsymbol{n}. \tag{47}$$

In books [8, 34] the following strain energy density was used:

$$2W = C\left[\nu \mathrm{tr}^2\mathbf{E}_\parallel + (1-\nu)\mathrm{tr}\,(\mathbf{E}_\parallel^T\mathbf{E}_\parallel)\right] + \alpha_s C(1-\nu)\boldsymbol{n}\mathbf{E}\mathbf{E}^T\boldsymbol{n}$$
$$+ D\left[\nu \mathrm{tr}^2\mathbf{K}_\parallel + (1-\nu)\mathrm{tr}\,(\mathbf{K}_\parallel^T\mathbf{K}_\parallel)\right] + \alpha_t D(1-\nu)\boldsymbol{n}\mathbf{K}\mathbf{K}^T\boldsymbol{n},$$

which is a particular case of (46), where now C, D, α_s, α_t, and ν are independent material constants.

In a similar way we can describe other types of anisotropic solid shells as well. In the analysis below we limit ourselves to representing the strain energy density by quadratic functions of $\mathbf{E}$ and $\mathbf{K}$ modelling the physically linear elastic material. Let us remind that W is the even function with respect to $\mathbf{K}$ and $\boldsymbol{B}$: $W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) = W(\mathbf{E}, -\mathbf{K}; -\mathbf{B})$.

For simplicity of discussion let us first assume that W does not depend on $\boldsymbol{B}$. This means that the quadratic relation $W(\mathbf{E}, \mathbf{K})$, which in general has 78 independent constants (see Equation (49) below), cannot have mixed terms containing both $\mathbf{E}$ and $\mathbf{K}$. Therefore, it can be given as the sum of two separate quadratic functions of $\mathbf{E}$ and $\mathbf{K}$: $W = f_1(\mathbf{E}) + f_2(\mathbf{K})$. This representation requires much less independent material constants than in the general case.

To make the presentation concise let us discuss first the quadratic function of one tensor variable $f(\mathbf{X})$, $\mathbf{X} \in Lin_n$. Using the relations $\mathbf{X} = \mathbf{X}_\parallel + \boldsymbol{n}\otimes\boldsymbol{x}$, $\boldsymbol{x} = \boldsymbol{n}\mathbf{X}$, $\mathbf{X}_\parallel = X_{\alpha\beta}\boldsymbol{g}_\alpha\otimes\boldsymbol{g}_\beta$, $\boldsymbol{x} = x_\alpha\boldsymbol{g}_\alpha$, $(\boldsymbol{g}_\alpha\cdot\boldsymbol{n} = 0)$ (see Appendix B) we obtain

$$f = C_{\alpha\beta\gamma\varepsilon}X_{\alpha\beta}X_{\gamma\varepsilon} + D_{\alpha\beta\gamma}X_{\alpha\beta}x_\gamma + G_{\alpha\beta}x_\alpha x_\beta$$
$$= C_{1111}X_{11}^2 + 2C_{1112}X_{11}X_{12} + 2C_{1121}X_{11}X_{21} + 2C_{1122}X_{11}X_{22}$$
$$+C_{1212}X_{12}^2 + 2C_{1221}X_{12}X_{21} + 2C_{1222}X_{12}X_{22}$$
$$+C_{2121}X_{21}^2 + 2C_{2122}X_{21}X_{22} + C_{2222}X_{22}^2$$
$$+D_{111}X_{11}x_1 + D_{112}X_{11}x_2 + D_{121}X_{12}x_1 + D_{122}X_{12}x_2$$
$$+D_{211}X_{21}x_1 + D_{212}X_{21}x_2 + D_{221}X_{22}x_1 + D_{222}X_{22}x_2$$
$$+G_{11}x_1^2 + G_{12}x_1x_2 + G_{22}x_2^2,$$

where $C_{\alpha\beta\gamma\varepsilon}$, $D_{\alpha\beta\gamma}$, $G_{\alpha\beta}$ are material constants. The number of all independent constants is 21 here, for the scalar variables $X_{\alpha\beta}$ and x_α are all independent.

In the case of orthotropy the quadratic function $f(\mathbf{X})$ should be even with regard to the vector $\boldsymbol{x}$. This results in that $D_{\alpha\beta\gamma} = 0$ and only 13 material constants remain independent. The function f takes the form

$$\begin{aligned} f &= C_{\alpha\beta\gamma\varepsilon} X_{\alpha\beta} X_{\gamma\varepsilon} + G_{\alpha\beta} E_\alpha E_\beta \\ &= C_{1111} X_{11}^2 + 2C_{1112} X_{11} X_{12} + 2C_{1121} X_{11} X_{21} + 2C_{1122} X_{11} X_{22} \\ &\quad + C_{1212} X_{12}^2 + 2C_{1221} X_{12} X_{21} + 2C_{1222} X_{12} X_{22} \\ &\quad + C_{2121} X_{21}^2 + 2C_{2122} X_{21} X_{22} + C_{2222} X_{22}^2 \\ &\quad + G_{11} x_1^2 + G_{12} x_1 x_2 + G_{22} x_2^2, \end{aligned}$$

Let us discuss the case of cubic symmetry. This means that the group $\mathcal{S}_n$ consists of rotations of 90° described by the rotation tensors

$$\boldsymbol{O} = (\mathbf{1} - \boldsymbol{n} \otimes \boldsymbol{n}) \cos\varphi + \boldsymbol{n} \otimes \boldsymbol{n} - \boldsymbol{n} \times \mathbf{1} \sin\varphi, \quad \varphi = 0, \pm\frac{\pi}{2}, \pi.$$

Then the following relations should be satisfied:

$$\begin{aligned} & f(X_{11}, X_{12}, X_{21}, X_{22}, x_1, x_2) \\ &\quad = f(X_{22}, -X_{21}, -X_{12}, X_{11}, x_2, -x_1) \\ &\quad = f(X_{11}, X_{12}, X_{21}, X_{22}, -x_1, -x_2) \\ &\quad = f(X_{22}, -X_{21}, -X_{12}, X_{11}, -x_2, x_1). \end{aligned} \tag{48}$$

Under such symmetries the quadratic function can be reduced to

$$\begin{aligned} f &= 2C_{1111}\left(X_{11}^2 + X_{22}^2\right) + 2C_{1212}\left(X_{12}^2 + X_{21}^2\right) + 2C_{1221} X_{12} X_{21} \\ &\quad + 2G_{11}(x_1^2 + x_2^2), \end{aligned}$$

and contains only four independent material constants. It is easy to see that for the quadratic function f the cubic symmetry is equivalent to the condition of isotropy.

Representations of quadratic functions W of two tensors $\mathbf{E}$, $\mathbf{K}$ can be found in the same way. Using the relations $\mathbf{E} = \mathbf{E}_\parallel + \boldsymbol{n} \otimes \boldsymbol{e}$, $\boldsymbol{e} = \boldsymbol{n}\mathbf{E}$, $\mathbf{E}_\parallel = E_{\alpha\beta} \boldsymbol{g}_\alpha \otimes \boldsymbol{g}_\beta$, $\boldsymbol{e} = E_\alpha \boldsymbol{g}_\alpha$, $\mathbf{K} = \mathbf{K}_\parallel + \boldsymbol{n} \otimes \boldsymbol{k}$, $\boldsymbol{k} = \boldsymbol{n}\mathbf{K}$, $\mathbf{K}_\parallel = K_{\alpha\beta} \boldsymbol{g}_\alpha \otimes \boldsymbol{g}_\beta$, $\boldsymbol{k} = K_\alpha \boldsymbol{g}_\alpha$, $(\boldsymbol{g}_\alpha \cdot \boldsymbol{n} = 0)$ we obtain

$$\begin{aligned} W &= C^E_{\alpha\beta\gamma\varepsilon} E_{\alpha\beta} E_{\gamma\varepsilon} + D^E_{\alpha\beta\gamma} E_{\alpha\beta} E_\gamma + G^E_{\alpha\beta} E_\alpha E_\beta \\ &\quad + C^K_{\alpha\beta\gamma\varepsilon} K_{\alpha\beta} K_{\gamma\varepsilon} + D^K_{\alpha\beta\gamma} K_{\alpha\beta} K_\gamma + G^K_{\alpha\beta} K_\alpha K_\beta \\ &\quad + C^{EK}_{\alpha\beta\gamma\varepsilon} E_{\alpha\beta} K_{\gamma\varepsilon} + D^{EK}_{\alpha\beta\gamma} E_{\alpha\beta} K_\gamma + D^{KE}_{\alpha\beta\gamma} K_{\alpha\beta} E_\gamma + G^{EK}_{\alpha\beta} E_\alpha K_\beta, \end{aligned} \tag{49}$$

where $C^E_{\alpha\beta\gamma\varepsilon}$, $C^K_{\alpha\beta\gamma\varepsilon}$, $C^{EK}_{\alpha\beta\gamma\varepsilon}$, $D^E_{\alpha\beta\gamma}$, $D^K_{\alpha\beta\gamma}$, $D^{EK}_{\alpha\beta\gamma}$, $D^{KE}_{\alpha\beta\gamma}$, $G^E_{\alpha\beta}$, $G^K_{\alpha\beta}$, $G^{EK}_{\alpha\beta}$ are material constants. In the general case of anisotropy the function W in (49) contains 78 independent material constants.

If we take into account that $W(\mathbf{E},\mathbf{K}) = W(\mathbf{E},-\mathbf{K})$ then we immediately obtain that $C^{EK}_{\alpha\beta\gamma\varepsilon} = 0$, $D^{EK}_{\alpha\beta\gamma} = 0$, $D^{KE}_{\alpha\beta\gamma} = 0$, $G^{EK}_{\alpha\beta} = 0$. Then W should be the sum of two separate quadratic functions of $\mathbf{E}$ and $\mathbf{K}$: $W = f_1(\mathbf{E}) + f_2(\mathbf{K})$. In the general case this sum contains 42 independent material constants and is given by

$$\begin{aligned} W = {} & C^{E}_{\alpha\beta\gamma\varepsilon} E_{\alpha\beta} E_{\gamma\varepsilon} + D^{E}_{\alpha\beta\gamma} E_{\alpha\beta} E_{\gamma} + G^{E}_{\alpha\beta} E_{\alpha} E_{\beta} \\ & + C^{K}_{\alpha\beta\gamma\varepsilon} K_{\alpha\beta} K_{\gamma\varepsilon} + D^{K}_{\alpha\beta\gamma} K_{\alpha\beta} K_{\gamma} + G^{K}_{\alpha\beta} K_{\alpha} K_{\beta}. \end{aligned}$$

The orthotropic physically linear shell model has the strain energy density with 26 independent material constants. It becomes the sum of two separate quadratic functions of $\mathbf{E}$ and $\mathbf{K}$: $W = f_1(\mathbf{E}) + f_2(\mathbf{K})$.

The cubic symmetry of the linear shell model leads to W with only eight independent material constants, and can be decomposed into the sum $W = f_1(\mathbf{E}) + f_2(\mathbf{K})$, where both $f_1(\mathbf{E})$ and $f_2(\mathbf{K})$ are surface isotropic functions. The strain energy density is thus given by

$$\begin{aligned} W = {} & 2C^{E}_{1111}\left(E^2_{11} + E^2_{22}\right) + 2C^{E}_{1212}\left(E^2_{12} + E^2_{21}\right) + 2C^{E}_{1221} E_{12} E_{21} \\ & + 2G^{E}_{11}(E^2_1 + E^2_2) \\ & + 2C^{K}_{1111}\left(K^2_{11} + K^2_{22}\right) + 2C^{K}_{1212}\left(K^2_{12} + K^2_{21}\right) + 2C^{K}_{1221} K_{12} K_{21} \\ & + 2G^{K}_{11}(K^2_1 + K^2_2). \end{aligned}$$

The analysis of quadratic functions of $\mathbf{E}$, $\mathbf{K}$ which explicitly depend also on $\boldsymbol{B}$ is more complex and leads to a greater number of independent material constants. Additionally, representations of such functions become very complex and some hypotheses are needed on how the function W depends upon the structure curvature tensor $\boldsymbol{B}$.

Let us discuss the quadratic representation for W given in (49) with additional terms linear in $\mathbf{E}$ and $\mathbf{K}$. Let us also assume that W depends on $\boldsymbol{B}$ only through the material tensors. Then

$$\begin{aligned} W = {} & L^{E}_{\alpha\beta} E_{\alpha\beta} + L^{K}_{\alpha\beta} K_{\alpha\beta} + l^{E}_{\alpha} E_{\alpha} + l^{K}_{\alpha} K_{\alpha} + \\ & + C^{E}_{\alpha\beta\gamma\varepsilon} E_{\alpha\beta} E_{\gamma\varepsilon} + D^{E}_{\alpha\beta\gamma} E_{\alpha\beta} E_{\gamma} + G^{E}_{\alpha\beta} E_{\alpha} E_{\beta} \\ & + C^{K}_{\alpha\beta\gamma\varepsilon} K_{\alpha\beta} K_{\gamma\varepsilon} + D^{K}_{\alpha\beta\gamma} K_{\alpha\beta} K_{\gamma} + G^{K}_{\alpha\beta} K_{\alpha} K_{\beta} \\ & + C^{EK}_{\alpha\beta\gamma\varepsilon} E_{\alpha\beta} K_{\gamma\varepsilon} + D^{EK}_{\alpha\beta\gamma} E_{\alpha\beta} K_{\gamma} + D^{KE}_{\alpha\beta\gamma} K_{\alpha\beta} E_{\gamma} + G^{EK}_{\alpha\beta} E_{\alpha} K_{\beta}, \end{aligned} \tag{50}$$

where $L^{E}_{\alpha\beta}$, $L^{K}_{\alpha\beta}$, l^{E}_{α}, l^{K}_{α}, $C^{E}_{\alpha\beta\gamma\varepsilon}$, $C^{K}_{\alpha\beta\gamma\varepsilon}$, $C^{EK}_{\alpha\beta\gamma\varepsilon}$, $D^{E}_{\alpha\beta\gamma}$, $D^{K}_{\alpha\beta\gamma}$, $D^{EK}_{\alpha\beta\gamma}$, $D^{KE}_{\alpha\beta\gamma}$, $G^{E}_{\alpha\beta}$, $G^{K}_{\alpha\beta}$, $G^{EK}_{\alpha\beta}$ are now functions of $\boldsymbol{B}$.

The linear terms of (50) model the existence of initial stress resultants and stress couples in the shell. The initial fields are given by

$$\mathbf{N}^{\circ} = \left.\frac{\partial W}{\partial \mathbf{E}}\right|_{\mathbf{E}=\mathbf{0},\mathbf{K}=\mathbf{0}}, \quad \mathbf{M}^{\circ} = \left.\frac{\partial W}{\partial \mathbf{K}}\right|_{\mathbf{E}=\mathbf{0},\mathbf{K}=\mathbf{0}}. \tag{51}$$

Such initial stress and couple stress states can be associated, for example, with imperfections introduced by realization process of the shell structure.

Using the relation $W(\mathbf{E}, \mathbf{K}; \boldsymbol{B}) = W(\mathbf{E}, -\mathbf{K}; -\mathbf{B})$ one can show that $L^E_{\alpha\beta}, l^E_\alpha, C^E_{\alpha\beta\gamma\varepsilon}$, $D^E_{\alpha\beta\gamma}, G^E_{\alpha\beta}, C^K_{\alpha\beta\gamma\varepsilon}, D^K_{\alpha\beta\gamma}, G^K_{\alpha\beta}$ are even functions of $\boldsymbol{B}$ and $L^K_{\alpha\beta}, l^K_\alpha, C^{EK}_{\alpha\beta\gamma\varepsilon}, D^{EK}_{\alpha\beta\gamma}, D^{KE}_{\alpha\beta\gamma}$, $G^{EK}_{\alpha\beta}$ are odd functions of $\boldsymbol{B}$. Then we can expand these material functions into Taylor series in the neighborhood of $\boldsymbol{B} = \mathbf{0}$ and take into account up to quadratic terms of the series, for example

$$C^E_{\alpha\beta\gamma\varepsilon}(\mathbf{B}) = {}^{(0)}C^E_{\alpha\beta\gamma\varepsilon} + {}^{(1)}C^E_{\alpha\beta\gamma\varepsilon}(\mathbf{B}) + {}^{(2)}C^E_{\alpha\beta\gamma\varepsilon}(\mathbf{B}),$$

where ${}^{(0)}C^E_{\alpha\beta\gamma\varepsilon}$ are constant, ${}^{(1)}C^E_{\alpha\beta\gamma\varepsilon}(\mathbf{B})$ are linear functions of $\boldsymbol{B}$, and ${}^{(2)}C^E_{\alpha\beta\gamma\varepsilon}(\mathbf{B})$ are quadratic functions of $\mathbf{B}$. Then the following material functions should disappear in the expression (50):

$$\begin{aligned}
&{}^{(1)}L^E_{\alpha\beta}(\mathbf{B}),\ {}^{(1)}l^E_\alpha(\mathbf{B}),\ {}^{(0)}L^K_{\alpha\beta},\ {}^{(2)}L^K_{\alpha\beta}(\mathbf{B}),\ {}^{(0)}l^K_\alpha,\ {}^{(2)}l^K_\alpha(\mathbf{B}),\ {}^{(1)}C^E_{\alpha\beta\gamma\varepsilon}(\mathbf{B}),\\
&{}^{(1)}C^K_{\alpha\beta\gamma\varepsilon}(\mathbf{B}),\ {}^{(0)}C^{EK}_{\alpha\beta\gamma\varepsilon},\ {}^{(2)}C^{EK}_{\alpha\beta\gamma\varepsilon}(\mathbf{B}),\ {}^{(1)}D^E_{\alpha\beta\gamma}(\mathbf{B}),\ {}^{(1)}D^K_{\alpha\beta\gamma}(\mathbf{B}),\ {}^{(0)}D^{EK}_{\alpha\beta\gamma},\\
&{}^{(2)}D^{EK}_{\alpha\beta\gamma}(\mathbf{B}),\ {}^{(0)}D^{KE}_{\alpha\beta\gamma},\ {}^{(2)}D^{KE}_{\alpha\beta\gamma}(\mathbf{B}),\ {}^{(1)}G^E_{\alpha\beta}(\mathbf{B}),\ {}^{(1)}G^K_{\alpha\beta}(\mathbf{B}),\ {}^{(0)}G^{EK}_{\alpha\beta},\\
&{}^{(2)}G^{EK}_{\alpha\beta}(\mathbf{B}).
\end{aligned}$$

Even with this important reduction of the number of material constants, there is still a tremendous number of material constants left in Equation (50). Note that there are still in (50) linear expressions of $\mathbf{E}$ and $\mathbf{K}$ of the form $L^E_{\alpha\beta}E_{\alpha\beta} + l^E_\alpha E_\alpha + {}^{(1)}L^K_{\alpha\beta}(\boldsymbol{B})K_{\alpha\beta} + {}^{(1)}l^K_\alpha(\boldsymbol{B})K_\alpha$. Thus, the initial couple stress resultants can only be taken into account if W is supposed to depend on $\boldsymbol{B}$.

Finally, if we assume that W depends explicitly on $\boldsymbol{B}$, then the representation (46) can be generalized to the form

$$\begin{aligned}
2W = {}& \alpha_0 \operatorname{tr}\mathbf{E} + \alpha_{00}\boldsymbol{n}\boldsymbol{B}\boldsymbol{I}^T\mathbf{E}\boldsymbol{B}^T\boldsymbol{n}\\
&+ \beta_0 \operatorname{tr}\boldsymbol{B}\operatorname{tr}\mathbf{K} + \beta_{01}\operatorname{tr}(\boldsymbol{B}\boldsymbol{I}^T\mathbf{K}) + \beta_{10}\operatorname{tr}(\boldsymbol{B}^T\mathbf{K}) + \beta_{00}\boldsymbol{n}\boldsymbol{B}\boldsymbol{I}^T\mathbf{K}\boldsymbol{B}^T\boldsymbol{n}\\
&+ \alpha_1 \operatorname{tr}^2\mathbf{E}_\parallel + \alpha_2 \operatorname{tr}\mathbf{E}_\parallel^2 + \alpha_3 \operatorname{tr}\left(\mathbf{E}_\parallel^T\mathbf{E}_\parallel\right) + \alpha_4\boldsymbol{n}\mathbf{E}\mathbf{E}^T\boldsymbol{n}\\
&+ \alpha_5 \operatorname{tr}^2(\boldsymbol{B}\mathbf{E}_\parallel) + \alpha_6 \operatorname{tr}^2\left(\boldsymbol{B}^T\boldsymbol{I}\mathbf{E}_\parallel\right) + \alpha_7\boldsymbol{n}\mathbf{E}\boldsymbol{I}^T\boldsymbol{B}\mathbf{E}^T\boldsymbol{n}\\
&+ \beta_1 \operatorname{tr}^2\mathbf{K}_\parallel + \beta_2 \operatorname{tr}\mathbf{K}_\parallel^2 + \beta_3 \operatorname{tr}\left(\mathbf{K}_\parallel^T\mathbf{K}_\parallel\right) + \beta_4\boldsymbol{n}\mathbf{K}\mathbf{K}^T\boldsymbol{n}\\
&+ \beta_5 \operatorname{tr}^2(\boldsymbol{B}\mathbf{K}_\parallel) + \beta_6 \operatorname{tr}^2\left(\boldsymbol{B}^T\boldsymbol{I}\mathbf{K}_\parallel\right) + \beta_7\boldsymbol{n}\mathbf{K}\boldsymbol{I}^T\boldsymbol{B}\mathbf{K}^T\boldsymbol{n}\\
&+ \gamma_1 \operatorname{tr}(\boldsymbol{B})\operatorname{tr}(\mathbf{E}\mathbf{K}) + \gamma_2 \operatorname{tr}(\boldsymbol{B})\operatorname{tr}(\mathbf{E}^T\mathbf{K}) + \gamma_3 \operatorname{tr}(\boldsymbol{B}\boldsymbol{I}^T\mathbf{E}\boldsymbol{I}^T\mathbf{K}).
\end{aligned} \tag{52}$$

In (52) the constitutive factors $\alpha_5, \alpha_6, \alpha_7, \alpha_{00}, \beta_{00}$ are odd while other ones are even surface isotropic functions of $\boldsymbol{B}$ up to the second degree, for which the representation given in Appendix B can be used. Note that in (52) there are bilinear terms in $\mathbf{E}$ and $\mathbf{K}$ which have not been present in (46). These terms allow one to model deformation of the shell with $\boldsymbol{B} \neq \mathbf{0}$ for which $\mathbf{N} \neq \mathbf{0}$ when $\mathbf{E} = \mathbf{0}$ and/or $\mathbf{M} \neq \mathbf{0}$ when $\mathbf{K} = \mathbf{0}$.

It is not possible to reduce further the form (52) for an isotropic shell using only the symmetry considerations. For this purpose one may try to apply other requirements like, for example, some additional 2D inequalities analogous to inequalities in nonlinear elasticity [44].

The form (52) is one of many possible representations for the isotropic quadratic function of three nonsymmetric tensors, not the most general one. The problem is that the number of polynomial invariants of several tensors is greater than the

number of their components. This means that from components of two or three tensors one can obtain more combinations which are invariant under orthogonal transformations than the numbers of the components themselves.

11. Conclusions

We have defined the local symmetry group for the general dynamically and kinematically exact theory of elastic shells. The group consists of an ordered triple of tensors under which the shell strain energy density becomes invariant. It has been proved that the group satisfies the Noll rule known for the local symmetry group of continuum mechanics.

The local symmetry group of the shell has allowed us to describe precisely invariant properties of the strain energy density of liquid and solid shells. Within the solid shells we have described more precisely the surface orthotropic, hemitropic, and isotropic shells. Special attention has been paid to define membranes and solid shells made of physically linear elastic material. In the later case W becomes the quadratic function of the strains and bendings. This has allowed us to propose several constitutive equations involving reduced numbers of constants to be established from experiments.

The case of surface isotropic function of several arguments is more complex and is not discussed here. Additionally, as is shown in [39] in this case it can happen that the number of independent invariants is greater than the number of components of components of tensorial arguments of the function.

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Appendix A. Table with constitutive equations and corresponding local symmetry groups

The local symmetry groups and the corresponding strain energy densities of the hyperelastic shells are collected in Table 1.

Appendix B. Representation of the surface isotropic function of one tensor argument

In order to derive explicitly particular forms of the constitutive equations for an elastic material we need the following theorem:

THEOREM 1. Let the scalar-valued function $f : Lin_n \to \mathbb{R}$ be the isotropic function at the surface, that is

$$f(\boldsymbol{X}) = f(\boldsymbol{O}\boldsymbol{X}\boldsymbol{O}^T), \quad \forall\, \boldsymbol{X} \in Lin_n, \quad \forall\, \boldsymbol{O} \in Orth_n.$$

Table 1 Local symmetry groups and corresponding strain energy densities

Elements of the symmetry group	Shell description	Strain energy density
$\mathbf{P} \in Unim_n$ $\boldsymbol{R} \in Orth_n$ $\boldsymbol{L} \in Lin_n$	Liquid shell	$W = W(\det \mathbf{F}, \boldsymbol{C})$ $= W\left[\det \mathbf{F}, (\det \boldsymbol{O})\boldsymbol{O}\boldsymbol{C}\boldsymbol{I}^T\boldsymbol{O}^T\right]$ $\forall\, \boldsymbol{O} \in Orth_n$
$\mathbf{P} = \boldsymbol{O}\boldsymbol{I}\mathbf{A}$ $\boldsymbol{O} \in \mathcal{S}_n \subset Orth_n$ $\boldsymbol{R} = \boldsymbol{O}$ $\boldsymbol{L} = \mathbf{0}$	Solid shell	$W = W(\mathbf{E}, \mathbf{K}; \boldsymbol{B})$ $= W\left[\boldsymbol{O}\mathbf{E}\boldsymbol{I}^T\boldsymbol{O}^T, (\det \boldsymbol{O})\boldsymbol{O}\mathbf{K}\boldsymbol{I}^T\mathbf{O}^T;\right.$ $\left.(\det \boldsymbol{O})\boldsymbol{O}\boldsymbol{B}\boldsymbol{I}^T\boldsymbol{O}^T\right]$ $\forall\, \boldsymbol{O} \in \mathcal{S}_n \subset Orth_n$
$\mathbf{P} = \boldsymbol{O}\boldsymbol{I}\mathbf{A}$ $\boldsymbol{O} \in \mathcal{S}_n = \{\mathbf{1}, -\mathbf{1}, 2\boldsymbol{n} \otimes \boldsymbol{n} - \mathbf{1}\}$ $\boldsymbol{R} = \boldsymbol{O}$ $\boldsymbol{L} = \mathbf{0}$	Orthotropic shell	$W = W(\mathbf{E}, \mathbf{K}; \boldsymbol{B})$ $= W\left[\boldsymbol{O}\mathbf{E}\boldsymbol{I}^T\boldsymbol{O}^T, (\det \boldsymbol{O})\boldsymbol{O}\mathbf{K}\boldsymbol{I}^T\boldsymbol{O}^T;\right.$ $\left.(\det \boldsymbol{O})\boldsymbol{O}\boldsymbol{B}\boldsymbol{I}^T\boldsymbol{O}^T\right]$ $\forall\, \boldsymbol{O} \in \mathcal{S}_n = \{\mathbf{1}, -\mathbf{1}, 2\boldsymbol{n} \otimes \boldsymbol{n} - \mathbf{1}\}$
$\mathbf{P} = \boldsymbol{O}\boldsymbol{I}\mathbf{A}$ $\boldsymbol{O} \in Orth_n^+$ $\boldsymbol{R} = \boldsymbol{O}$ $\boldsymbol{L} = \mathbf{0}$	Hemitropic shell	$W = W(\mathbf{E}, \mathbf{K}; \boldsymbol{B})$ $= W\left[\boldsymbol{O}\mathbf{E}\boldsymbol{I}^T\boldsymbol{O}^T, \boldsymbol{O}\mathbf{K}\boldsymbol{I}^T\boldsymbol{O}^T;\right.$ $\left.\boldsymbol{O}\boldsymbol{B}\boldsymbol{I}^T\boldsymbol{O}^T\right]$ $\forall\, \boldsymbol{O} \in Orth_n^+$
$\mathbf{P} = \boldsymbol{O}\boldsymbol{I}\mathbf{A}$ $\boldsymbol{O} \in Orth_n$ $\boldsymbol{R} = \boldsymbol{O}$ $\boldsymbol{L} = \mathbf{0}$	Isotropic shell	$W = W(\mathbf{E}, \mathbf{K}; \boldsymbol{B})$ $= W\left[\boldsymbol{O}\mathbf{E}\boldsymbol{I}^T\boldsymbol{O}^T, (\det \boldsymbol{O})\boldsymbol{O}\mathbf{K}\boldsymbol{I}^T\mathbf{O}^T;\right.$ $\left.(\det \boldsymbol{O})\boldsymbol{O}\boldsymbol{B}\boldsymbol{I}^T\boldsymbol{O}^T\right]$ $\forall\, \boldsymbol{O} \in Orth_n$

Then f is the function of only five independent invariants

$$f(\boldsymbol{X}) = f\left[j_k(\boldsymbol{X})\right],$$
$$j_1(\boldsymbol{X}) = \operatorname{tr} \boldsymbol{X}, \quad j_2(\boldsymbol{X}) = \operatorname{tr} \mathbf{X}_\parallel^2, \quad j_3(\boldsymbol{X}) = \operatorname{tr}(\mathbf{X}_\parallel^T \mathbf{X}_\parallel),$$
$$j_4(\boldsymbol{X}) = \boldsymbol{n}\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{n}, \quad j_5(\boldsymbol{X}) = \boldsymbol{n}\boldsymbol{X}^2\boldsymbol{X}^T\boldsymbol{n},$$

where $\mathbf{X}_\parallel \equiv \boldsymbol{X} - \boldsymbol{n}\boldsymbol{X} \in T_x M \otimes T_x M$.

Proof. The tensor $\boldsymbol{X}$ can be decomposed into $\boldsymbol{X} = \mathbf{X}_\parallel + \boldsymbol{n} \otimes \boldsymbol{x}$, $\boldsymbol{x} = \boldsymbol{n}\boldsymbol{X}$, $\boldsymbol{x} \in T_x M$ ($\boldsymbol{n} \cdot \boldsymbol{x} = 0$), and $\mathbf{X}_\parallel$ can be represented by its symmetric and skew-symmetric parts: $\mathbf{X}_\parallel = \mathbf{X}_\parallel^s + \mathbf{X}_\parallel^a$, $\mathbf{X}_\parallel^s = \mathbf{X}_\parallel^{s\,T}$, $\mathbf{X}_\parallel^a = -\mathbf{X}_\parallel^{a\,T}$.

Let $\boldsymbol{g}_\alpha$ ($\alpha = 1, 2$), $\boldsymbol{n} \cdot \boldsymbol{g}_\alpha = 0$ be and arbitrary orthonormal surface base. Then $\mathbf{X}_\parallel^s = X_{\alpha\beta}^s \boldsymbol{g}_\alpha \otimes \boldsymbol{g}_\beta$ ($X_{\alpha\beta}^s = X_{\beta\alpha}^s$), $\mathbf{X}_\parallel^a = X^a(\boldsymbol{g}_1 \otimes \boldsymbol{g}_2 - \boldsymbol{g}_2 \otimes \boldsymbol{g}_1)$. Therefore, the function f can be expressed through the surface symmetric tensor $\mathbf{X}_\parallel^s$, the vector $\boldsymbol{x}$, and the scalar X^a:

$$f = f^\times(\mathbf{X}_\parallel^s, \boldsymbol{x}, X^a).$$

From definition of the surface isotropic function it follows that $f^\times$ is the isotropic function of its arguments, i.e. the function with the symmetry group $SO(2)$. Therefore, it is the function of invariants [39] of $\mathbf{X}_\parallel^s$ and $\boldsymbol{x}$: $\operatorname{tr} \mathbf{X}_\parallel^s$, $\operatorname{tr} \mathbf{X}_\parallel^{s\,2}$, $\boldsymbol{x}\mathbf{X}_\parallel^s\boldsymbol{x}$, $\boldsymbol{x} \cdot \boldsymbol{x}$, as

well as of X^a. Taking into account that $\operatorname{tr}\mathbf{X}_\parallel^s = \operatorname{tr}\boldsymbol{X}$, $\boldsymbol{x}\mathbf{X}_\parallel^s\boldsymbol{x} = \boldsymbol{x}\boldsymbol{X}\boldsymbol{x}$, $\operatorname{tr}\mathbf{X}_\parallel^{s\,2} = \operatorname{tr}\mathbf{X}_\parallel{}^2 + 2X^{a\,2} = \operatorname{tr}(\mathbf{X}_\parallel^T\mathbf{X}_\parallel) - 2X^{a\,2}$, the function f depends on the following invariants:

$$\operatorname{tr}\boldsymbol{X}, \quad \operatorname{tr}\mathbf{X}_\parallel^2, \quad \operatorname{tr}(\mathbf{X}_\parallel^T\mathbf{X}_\parallel), \quad \boldsymbol{x}\boldsymbol{X}\boldsymbol{x}, \quad \boldsymbol{x}\cdot\boldsymbol{x},$$

and dependence on X^α is taken here implicitly in the invariant $\operatorname{tr}(\mathbf{X}_\parallel^T\mathbf{X}_\parallel)$.

This completes the proof.

Let us note that for this proof based on results by Spencer [39] it is enough if we use the group $Orth_n^+$.

From the proof above it is seen that in the expression for invariants we can use $\boldsymbol{X}$ in place of $\mathbf{X}_\parallel$ as well.

From the theorem it follows

COROLLARY 1. The surface isotropic linear function $f : Lin_n \to \mathbb{R}$ is given by the formula $f(\boldsymbol{X}) = \alpha_\circ \operatorname{tr}\boldsymbol{X}$.

COROLLARY 2. The surface isotropic quadratic function $f : Lin_n \to \mathbb{R}$ is given by the formula

$$2f(\boldsymbol{X}) = \alpha_1 \operatorname{tr}^2\boldsymbol{X} + \alpha_2 \operatorname{tr}\mathbf{X}_\parallel^2 + \alpha_3 \operatorname{tr}(\mathbf{X}_\parallel^T\mathbf{X}_\parallel) + \alpha_4 \boldsymbol{n}\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{n}.$$

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