Optimization problem under change of regime of interest rate

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Abstract

In this paper, we study the classical problem of maximization of the sum of the utility of the terminal wealth and the utility of the consumption, in a case where a sudden jump in the risk-free interest rate creates incompleteness. The value function of the dual problem is proved to be solution of a BSDE and the duality between the primal and the dual value functions is exploited to study the BSDE associated to the primal problem.

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1 Introduction

Many studies in the field of Mathematical Finance are devoted to portfolio and/or consumption optimization problems. In the case of a complete market, with several risky assets and a savings account adapted to a Brownian filtration, the problem is fully solved in the monography of Karatzas and Shreve [11]. The situation in incomplete markets is more delicate, and it is not easy to give closed form solutions (see, e.g., Menoncin [15]). The incompleteness of the market may arise from a number of risky assets smaller than the dimension of the driving noise, from constraints on the portfolio, or from an interest rate which depends on an extra noise, which will be the case in our setting. The literature about the two first cases of incompleteness is important,

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on the other hand the literature about the third case of incompleteness is reduced. We can cite [15] for the case of a multidimensional incomplete market (and constant interest rate) and a Brownian filtration under Markovian framework, where the author solves the problem using HJB equation. The case where the measurability of the interest rate creates incompleteness is presented in Bauerle and Rieder [1] in which the dynamics of the interest rate is driven by a Markov chain.

A classical tool to solve utility maximization problem is the dual approach. This one consists in solving an auxiliary optimization problem, called the dual problem, which is defined on the set of all equivalent martingale measures. The list of papers studying that problem is long and we quote only few of them. This approach is used in the case of incomplete markets generated by a savings account (with constant interest rate) and several stocks (represented by general semi-martingales) for HARA utility, by Kramkov and Schachermayer [13]. They state an existence and uniqueness result for the final optimal wealth (associated to an investment problem), but no explicit formulas are provided. Rogers [16] formulates an abstract theorem in which the value function of the utility maximization problem and the value function for the associated dual problem satisfy a bidual relation. As it is mentioned, this procedure can be applied for a wide class of portfolio and/or consumption optimization problems. Castañeda-Leyva and Hernández-Hernández [2] deal with a combined investment and consumption optimization problem with a single risky asset, in a Brownian framework, and where the coefficients of the model (including the interest rate) are deterministic functions of some external economic factor process.

Here, we are concerned with the problem of maximization of expected power utility of both terminal wealth and consumption, in a market with investment opportunities in a savings account with a stochastic interest rate, which suffers an unexpected shock at some random time τ, and a stock modeled by a semi-martingale driven by a Brownian motion. The unexpected shock can for example be due to some serious macroeconomic issue. This one implies that the market is incomplete. The problem will be solved in the filtration generated by prices (of stock and savings account) so that the change of regime time τ is a stopping time, under the immersion hypothesis between the filtration generated by the stock and the general filtration.

Using standard results of duality, the original optimization problem (called the *primal* problem) is linked to the *dual* problem, in which the control parameters take value in the set of equivalent martingale measures. Then, we prove, by using a similar approach to the one used in Hu *et al.*[7] for the case of the primal problem without consumption and more recently in Cheridito and Hu [3] for the case with consumption, that the value function of that problem is solution of a particular BSDE, involving one jump. Using a recent result of Kharroubi and Lim [12], we show that this BSDE has a unique solution. Then, we give the optimal portfolio and consumption in terms of the solution of this BSDE, and explicit formula for the optimal wealth process. We also establish a duality result for the dynamic versions of the value functions associated to primal and dual optimization problems which allows us to prove that the BSDE associated to the primal problem has a unique solution. To the best of our knowledge, the BSDE methodology has not been used yet for dual problems in the literature.

The paper is organized as follows. In Sections 2 and 3, we describe the set up and model. In Section 4, we characterize the set of the equivalent martingale measures, then we derive and solve the dual optimization problem. Finally, Section 5 is dedicated to the link between the value functions associated to the primal and dual optimization problems and to the computation of explicit formulas for the optimal wealth process, optimal trading and consumption policies.

2 Set up

Throughout this paper $(\Omega, \mathcal{G}, \mathbb{P})$ is a probability space on which is defined a one dimensional Brownian motion $(B_t)_{t\in[0,T]}$ where $T < \infty$ is the terminal time. We denote by $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ the natural filtration of B (augmented by the $\mathbb{P}$-null sets) and we assume that $\mathcal{F}_T \subsetneq \mathcal{G}$. On the same probability space is given a finite positive $\mathcal{G}$-measurable random variable τ which is interpreted as a random time associated to some unpredicted evolution (with respect to the filtration $\mathbb{F}$) in the dynamics of the interest rate or to a switching regime. Let H be the càdlàg process equal to 0 before τ and 1 after τ, i.e., $H_t := \mathbb{1}_{\tau\leq t}$. We introduce the filtration $\mathbb{G}$ which is the smallest right-continuous extension of $\mathbb{F}$ that makes τ a $\mathbb{G}$-stopping time. More precisely $\mathbb{G} := (\mathcal{G}_t)_{t\in[0,T]}$, where $\mathcal{G}_t$ is defined for any $t \in [0, T]$ by

$$\mathcal{G}_t := \bigcap_{\epsilon>0} \mathcal{G}^0_{t+\epsilon} ,$$

where $\mathcal{G}^0_t := \mathcal{F}_t \vee \sigma(H_u , u \in [0, t])$, for any $t \in [0, T]$. Throughout the sequel, we assume the following classical hypotheses.

(H1) Any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale, i.e., $\mathbb{F}$ is immersed in $\mathbb{G}$.

(H2) The process H admits an absolutely continuous compensator, i.e., there exists a non-negative $\mathbb{G}$-adapted process $\lambda^{\mathbb{G}}$, called the $\mathbb{G}$-intensity, such that the compensated process M defined by

$$M_t := H_t - \int_0^t \lambda^{\mathbb{G}}_s ds ,$$

is a $\mathbb{G}$-martingale. Note that the process $\lambda^{\mathbb{G}}$ vanishes after τ, and we can write $\lambda^{\mathbb{G}}_t = \lambda^{\mathbb{F}}_t \mathbb{1}_{t<\tau}$ where $\lambda^{\mathbb{F}}$ is an $\mathbb{F}$-adapted process, called the $\mathbb{F}$-intensity of the process H. We assume that $\lambda^{\mathbb{G}}$ is uniformly bounded, hence $\lambda^{\mathbb{F}}$ is also uniformly bounded. The existence of $\lambda^{\mathbb{G}}$ implies that τ is not an $\mathbb{F}$-stopping time (in fact, τ avoids $\mathbb{F}$-stopping times and is a totally inaccessible $\mathbb{G}$-stopping time).

We recall in this framework the standard decomposition of any $\mathbb{G}$-predictable process ψ which is given by Jeulin [8, Lemma 4.4].

Lemma 1. *Any $\mathbb{G}$-predictable process ψ can be decomposed under the following form*

$$\psi_t = \psi^0_t \mathbb{1}_{t\leq\tau} + \psi^1_t(\tau)\mathbb{1}_{t>\tau} ,$$

where the process ψ^0 is $\mathbb{F}$-predictable, and for fixed non-negative u, the process $\psi^1_\cdot(u)$ is $\mathbb{F}$-predictable. Furthermore, for any fixed $t \in [0, T]$, the mapping $\psi^1_t(\cdot)$ is $\mathcal{F}_t \otimes \mathcal{B}([0,\infty))$-measurable. Moreover, if the process ψ is uniformly bounded, then it is possible to choose bounded processes ψ^0 and $\psi^1(u)$.

Remark 1. *The process $(\exp(aB_t - \frac{1}{2}a^2 t))_{t\in[0,T]}$ being an $\mathbb{F}$-continuous martingale for every real number a, the immersion property implies that it is a $\mathbb{G}$-continuous martingale, hence B is a $\mathbb{G}$-Brownian motion. It follows that the stochastic integral $\int \vartheta_s dB_s$ is well defined for a $\mathbb{G}$-adapted process ϑ (up to integrability conditions, e.g. if ϑ is bounded) and that this integral is a $\mathbb{G}$ local-martingale.*

We define the following spaces which will be used throughout this paper.

- $\mathcal{S}^\infty_{\mathbb{F}}(u,T)$ (resp. $\mathcal{S}^\infty_{\mathbb{G}}(u,T)$) denotes the set of $\mathbb{F}$ (resp. $\mathbb{G}$)-progressively measurable processes X which are essentially bounded on $[u,T]$, i.e., such that

$$\operatorname*{ess\,sup}_{t\in[u,T]} |X_t| < \infty \ ;$$

- $\mathcal{S}^{\infty,+}_{\mathbb{F}}(u,T)$ (resp. $\mathcal{S}^{\infty,+}_{\mathbb{G}}(u,T)$) denotes the subset of $\mathcal{S}^\infty_{\mathbb{F}}(u,T)$ (resp. $\mathcal{S}^\infty_{\mathbb{G}}(u,T)$) such that $X_t \geq C$ for a positive constant C;
- $\mathcal{H}^2_{\mathbb{F}}(u,T)$ (resp. $\mathcal{H}^2_{\mathbb{G}}(u,T)$) denotes the set of square integrable $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable processes X on $[u,T]$, i.e.,

$$\|X\|^2_{\mathcal{H}^2(u,T)} := \mathbb{E}\Big(\int_u^T |X_t|^2 dt\Big) < \infty \ ;$$

- $\mathcal{H}^2_{\mathbb{G}}(M)$ denotes the set of $\mathbb{G}$-predictable processes X on $[0,T]$ such that

$$\|X\|^2_{\mathcal{H}^2_{\mathbb{G}}(M)} := \mathbb{E}\Big(\int_0^T \lambda^{\mathbb{G}}_t |X_t|^2 dt\Big) < \infty \ .$$

3 Model

The financial market consists in a savings account with a stochastic interest rate with dynamics

$$dS^0_t = r_t S^0_t dt \ , \quad S^0_0 = 1 \ ,$$

where r is a non-negative $\mathbb{G}$-adapted process, and a risky asset whose price process S follows the dynamics

$$dS_t = S_t(\nu_t dt + \sigma_t dB_t) \ .$$

Our assumptions about the market are the following

(H3) r is a $\mathbb{G}$-adapted process of the form

$$r_t = r^0_t \mathbb{1}_{t<\tau} + r^1_t(\tau)\mathbb{1}_{t\geq\tau} \ ,$$

where r^0 is a non-negative uniformly bounded $\mathbb{F}$-adapted process, and for any fixed non-negative u, $r^1_\cdot(u)$ is a non-negative uniformly bounded $\mathbb{F}$-adapted process, and for fixed $t \in [0,T]$, the mapping $r^1_t(\cdot)$ is $\mathcal{F}_t \otimes \mathcal{B}([0,\infty))$-measurable.

(H4) ν and σ are $\mathbb{F}$-adapted processes, and there exists a positive constant C such that $|\nu_t| \leq C$ and $\frac{1}{C} \leq \sigma_t \leq C$, $t \in [0,T]$, $\mathbb{P}$ - a.s.

Throughout the sequel, we use the notation R for the discount factor defined by $R_t := e^{-\int_0^t r_s ds}$ for any $t \in [0,T]$.

We now consider an investor acting in this market, starting with an initial amount $x > 0$ and we denote by π^0 and π the part of wealth invested in the savings account and in the risky asset, and by c the associated instantaneous consumption process. Obviously we have the relation $\pi^0_t = 1 - \pi_t$. We denote by $X^{x,\pi,c}$ the wealth process associated to the strategy (π, c) and the initial wealth x, and we assume that the strategy is self-financing, which leads to the equation

$$\left\{ \begin{array}{rcl} X^{x,\pi,c}_0 &=& x \ , \\ dX^{x,\pi,c}_t &=& X^{x,\pi,c}_t\big[(r_t + \pi_t(\nu_t - r_t))dt + \pi_t\sigma_t dB_t\big] - c_t dt \ . \end{array} \right. \qquad (1)$$

We consider the set $\mathcal{A}(x)$ of the admissible strategies defined below.

Definition 1. *The set $\mathcal{A}(x)$ of admissible strategies (π, c) consists in $\mathbb{G}$-predictable processes (π, c) such that $\mathbb{E}(\int_0^T |\pi_s\sigma_s|^2 ds) < \infty$, $c_t \geq 0$ and $X_t^{x,\pi,c} > 0$ for any $t \in [0,T]$.*

We are interested in solving the classical problem of utility maximization defined by

$$V(x) := \sup_{(\pi,c)\in\mathcal{A}(x)} \mathbb{E}\Big[\int_0^T U(c_s)ds + U(X_T^{x,\pi,c})\Big] , \tag{2}$$

where the utility function U is $U(x) = x^p/p$ with $p \in (0,1)$.

4 Dual approach

To prove that there exists an optimal strategy to the problem (2), we use the dual approach introduced by Karatzas *et al.* [10] or Cox and Huang [4].

For that, we introduce the convex conjugate function $\widetilde{U}$ of the utility function U, which is defined by

$$\widetilde{U}(y) := \sup_{x>0}(U(x) - xy) , \quad y > 0 .$$

The supremum is attained at the point $I(y) := (U')^{-1}(y)$ and a direct computation shows that $I(y) = y^{\frac{1}{p-1}}$ and $\widetilde{U}(y) = -\frac{y^q}{q}$ where $q := \frac{p}{p-1} < 0$. We also have the conjugate relation

$$U(x) = \inf_{y>0}(\widetilde{U}(y) + xy) , \quad x > 0 . \tag{3}$$

Before studying the dual problem, we characterize the set of equivalent martingale measures which is used to introduce the dual problem.

4.1 Characterization of the set of equivalent martingale measures

The set $\mathcal{M}(\mathbb{P})$ of equivalent martingale measures (e.m.m.) is

$$\mathcal{M}(\mathbb{P}) := \{\mathbb{Q} \mid \mathbb{Q} \sim \mathbb{P}, RS \text{ is a } (\mathbb{Q},\mathbb{G}) - \text{local martingale } \}.$$

The dynamics of the discounted price of the risky asset $\tilde{S} := RS$ is given by

$$d\tilde{S}_t = \sigma_t \tilde{S}_t(dB_t + \theta_t dt) , \tag{4}$$

where $\theta_t := \frac{\nu_t - r_t}{\sigma_t}$ is the risk premium.

Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$, defined by its Radon-Nikodym density

$$d\mathbb{Q}\big|_{\mathcal{G}_t} = L_t^{\mathbb{Q}} d\mathbb{P}\big|_{\mathcal{G}_t} ,$$

where $L^{\mathbb{Q}}$ is a positive $\mathbb{G}$-martingale with $L_0^{\mathbb{Q}} = 1$.

According to the Predictable Representation Theorem (see Kusuoka [14]), and using the fact that $L^{\mathbb{Q}}$ is positive, there exists a pair (a,γ) of $\mathbb{G}$-predictable processes satisfying $\gamma_t > -1$ for any $t \in [0,T]$ such that

$$dL_t^{\mathbb{Q}} = L_{t^-}^{\mathbb{Q}}(a_t dB_t + \gamma_t dM_t) .$$

From Girsanov's theorem, the process $\widehat{B}$ defined by

$$\widehat{B}_t := B_t - \int_0^t a_s ds$$

is a $(\mathbb{Q}, \mathbb{G})$-Brownian motion, and the process $\widehat{M}$ defined by

$$\widehat{M}_t := M_t - \int_0^t \gamma_s \lambda_s^{\mathbb{G}} ds = H_t - \int_0^t (1+\gamma_s)\lambda_s^{\mathbb{G}} ds$$

is a $(\mathbb{Q}, \mathbb{G})$-discontinuous martingale, orthogonal to $\widehat{B}$.

Using (4), we notice that if a probability measure $\mathbb{Q}$ is an e.m.m., then $a_t = -\theta_t$ for any $t \in [0, T]$.

Lemma 2. *The set $\mathcal{M}(\mathbb{P})$ is determined by all the probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$, whose Radon-Nikodym density process has the form*

$$L_t^{\mathbb{Q}} = \exp\Big(- \int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds + \int_0^t \ln(1+\gamma_s) dH_s - \int_0^t \gamma_s \lambda_s^{\mathbb{G}} ds \Big) \ ,$$

where γ is a $\mathbb{G}$-predictable process satisfying $\gamma_t > -1$.

To alleviate the notations, for any $\mathbb{Q} \in \mathcal{M}(\mathbb{P})$, we write L^γ for $L^{\mathbb{Q}}$ where γ is the process associated to $\mathbb{Q}$, i.e.,

$$dL_t^\gamma = L_{t^-}^\gamma (-\theta_t dB_t + \gamma_t dM_t) \ , \quad L_0^\gamma = 1.$$

For any $\mathbb{Q} \in \mathcal{M}(\mathbb{P})$, we remark that $RX^{x,\pi,c} + \int_0^\cdot R_s c_s ds$ is a positive $(\mathbb{Q}, \mathbb{G})$-local martingale, hence a supermartingale, so we have

$$\mathbb{E}^{\mathbb{Q}}\Big(R_T X_T^{x,\pi,c} + \int_0^T R_s c_s ds \Big) \le x \ , \quad \forall\, (\pi, c) \in \mathcal{A}(x) \ ,$$

where $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation w.r.t. the probability measure $\mathbb{Q}$ or equivalently

$$\mathbb{E}\Big(R_T L_T^\gamma X_T^{x,\pi,c} + \int_0^T R_s L_s^\gamma c_s ds \Big) \le x \ , \quad \forall\, (\pi, c) \in \mathcal{A}(x) \ . \tag{5}$$

4.2 Dual optimization problem

We now define the dual problem associated to (2) according to the standard theory of convex duality. For that, we consider the set Γ of dual admissible processes.

Definition 2. *The set Γ of dual admissible processes is the set of $\mathbb{G}$-predictable processes γ such that there exists two constants A and C satisfying $-1 < A \le \gamma_t \le C$ for any $t \in [0, \tau]$ and $\gamma_t = 0$ for any $t \in (\tau, T]$.*

It is interesting to work with this admissible set Γ throughout the sequel since, for any $\gamma \in \Gamma$, the process L^γ is a positive $\mathbb{G}$-martingale (indeed, due to the bounds on γ, the process L^γ is a true martingale), and it satisfies the following integrability property which simplifies some proofs in the sequel. Moreover, we consider that γ is null after the time τ since the value of γ after τ does not interfere in the calculus, thus it is possible to fix any value for γ after τ.

Lemma 3. *For any $\gamma \in \Gamma$, the process L^γ satisfies*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} (L_t^\gamma)^q\Big] < \infty\ .$$

Proof. From Itô's formula, we get

$$d(L_t^\gamma)^q = (L_{t^-}^\gamma)^q\Big[\Big(\frac{1}{2}q(q-1)|\theta_t|^2 - q\lambda_t^{\mathbb{G}}\gamma_t + \lambda_t^{\mathbb{G}}\big((1+\gamma_t)^q - 1\big)\Big)dt - q\theta_t dB_t + \big((1+\gamma_t)^q - 1\big)dM_t\Big]\ .$$

This can be written under the following form [1]

$$(L_t^\gamma)^q = K_t\mathcal{E}\Big(-\int_0^\cdot q\theta_s dB_s + \int_0^\cdot \big((1+\gamma_s)^q - 1\big)dM_s\Big)_t\ ,$$

where K is the bounded process defined by

$$K_t := \exp\Big(\int_0^t \Big(\frac{1}{2}q(q-1)|\theta_s|^2 - q\lambda_s^{\mathbb{G}}\gamma_s + \lambda_s^{\mathbb{G}}\big((1+\gamma_s)^q - 1\big)\Big)ds\Big)\ .$$

Therefore, there exists a positive constant C such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}(L_t^\gamma)^q\Big] \le C\mathbb{E}\Big[\sup_{t\in[0,T]}\mathcal{E}\Big(-\int_0^\cdot q\theta_s dB_s\Big)_t\Big]\ .$$

We conclude by using the Burkholder-Davis-Gundy inequality. □

From the conjugate relation (3), we get for any $\eta > 0$, $\gamma \in \Gamma$ and $(\pi, c) \in \mathcal{A}(x)$

$$\mathbb{E}\Big[\int_0^T U(c_s)ds + U(X_T^{x,\pi,c})\Big] \le \mathbb{E}\Big[\int_0^T \widetilde{U}(\eta R_s L_s^\gamma)ds + \widetilde{U}(\eta R_T L_T^\gamma)\Big] + \eta\mathbb{E}\Big[\int_0^T R_s L_s^\gamma c_s ds + R_T L_T^\gamma X_T^{x,\pi,c}\Big]\ .$$

Using (5), the previous inequality gives for any $\eta > 0$, $\gamma \in \Gamma$ and $(\pi, c) \in \mathcal{A}(x)$

$$\mathbb{E}\Big[\int_0^T U(c_s)ds + U(X_T^{x,\pi,c})\Big] \le \mathbb{E}\Big[\int_0^T \widetilde{U}(\eta R_s L_s^\gamma)ds + \widetilde{U}(\eta R_T L_T^\gamma)\Big] + \eta x\ .$$

Therefore, the following inequality holds for any $(\pi, c) \in \mathcal{A}(x)$

$$\mathbb{E}\Big[\int_0^T U(c_s)ds + U(X_T^{x,\pi,c})\Big] \le \inf_{\eta>0,\gamma\in\Gamma}\Big(\mathbb{E}\Big[\int_0^T \widetilde{U}(\eta R_s L_s^\gamma)ds + \widetilde{U}(\eta R_T L_T^\gamma)\Big] + \eta x\Big)\ .$$

We thus obtain

$$\sup_{(\pi,c)\in\mathcal{A}(x)}\mathbb{E}\Big[\int_0^T U(c_s)ds + U(X_T^{x,\pi,c})\Big] \le \inf_{\eta>0,\gamma\in\Gamma}\Big(\mathbb{E}\Big[\int_0^T \widetilde{U}(\eta R_s L_s^\gamma)ds + \widetilde{U}(\eta R_T L_T^\gamma)\Big] + \eta x\Big)\ . \quad (6)$$

[1] $\mathcal{E}(Y)$ denotes the Doléans-Dade stochastic exponential process associated to a generic martingale Y.

We introduce the dual problem for any $\eta > 0$

$$\begin{aligned}\tilde{V}(\eta) &= \inf_{\gamma\in\Gamma}\mathbb{E}\Big[\int_0^T \tilde{U}(\eta R_s L_s^\gamma)ds + \tilde{U}(\eta R_T L_T^\gamma)\Big] \\ &= -\frac{\eta^q}{q}\inf_{\gamma\in\Gamma}\mathbb{E}\Big[\int_0^T (R_s L_s^\gamma)^q ds + (R_T L_T^\gamma)^q\Big]\ .\end{aligned}$$

We thus consider the following optimization problem

$$\inf_{\gamma\in\Gamma}\mathbb{E}\Big[\int_0^T (R_s L_s^\gamma)^q ds + (R_T L_T^\gamma)^q\Big]\ .$$

To solve this problem we use a similar approach to the one used in Cheridito and Hu [3] which is linked to the dynamic programming principle. More precisely, we look for a family of processes $\{(J_t^{(d)}(\gamma))_{t\in[0,T]}\ :\ \gamma\in\Gamma\}$, called the conditional gains, satisfying the following conditions

(i) $J_T^{(d)}(\gamma) = (R_T L_T^\gamma)^q + \int_0^T (R_s L_s^\gamma)^q ds$, for any $\gamma\in\Gamma$.

(ii) $J_0^{(d)}(\gamma_1) = J_0^{(d)}(\gamma_2)$, for any $\gamma_1, \gamma_2\in\Gamma$.

(iii) $J^{(d)}(\gamma)$ is a $\mathbb{G}$-submartingale for any $\gamma\in\Gamma$.

(iv) There exists some $\gamma^*\in\Gamma$ such that $J^{(d)}(\gamma^*)$ is a $\mathbb{G}$-martingale.

Under these conditions, we have

$$J_0^{(d)}(\gamma^*) = \inf_{\gamma\in\Gamma}\mathbb{E}\Big[\int_0^T (R_s L_s^\gamma)^q ds + (R_T L_T^\gamma)^q\Big]\ .$$

Indeed, using (i) and (iii), we have

$$J_0^{(d)}(\gamma) \le \mathbb{E}\big[J_T^{(d)}(\gamma)\big]\ =\ \mathbb{E}\Big[\int_0^T (R_s L_s^\gamma)^q ds + (R_T L_T^\gamma)^q\Big]\ , \tag{7}$$

for any $\gamma\in\Gamma$. Then, using (i) and (iv), we have

$$J_0^{(d)}(\gamma^*) = \mathbb{E}\big[J_T^{(d)}(\gamma^*)\big]\ =\ \mathbb{E}\Big[\int_0^T (R_s L_s^{\gamma^*})^q ds + (R_T L_T^{\gamma^*})^q\Big]\ . \tag{8}$$

Therefore, from (ii), (7) and (8), we get for any $\gamma\in\Gamma$

$$\mathbb{E}\Big[\int_0^T (R_s L_s^{\gamma^*})^q ds + (R_T L_T^{\gamma^*})^q\Big] = J_0^{(d)}(\gamma^*) = J_0^{(d)}(\gamma) \le \mathbb{E}\Big[\int_0^T (R_s L_s^\gamma)^q ds + (R_T L_T^\gamma)^q\Big]\ .$$

We can see that

$$J_0^{(d)}(\gamma^*) = \inf_{\gamma\in\Gamma}\mathbb{E}\Big[\int_0^T (R_s L_s^\gamma)^q ds + (R_T L_T^\gamma)^q\Big]\ .$$

We now construct a family of processes $\{(J_t^{(d)}(\gamma))_{t\in[0,T]}\ :\ \gamma\in\Gamma\}$ satisfying the previous conditions using BSDEs. For that we look for $J^{(d)}(\gamma)$ under the following form, which is based on the dynamic programming principle,

$$J_t^{(d)}(\gamma) = \int_0^t (R_s L_s^\gamma)^q ds + (R_t L_t^\gamma)^q \Phi_t\ , \quad t\in[0,T]\ , \tag{9}$$

where $(\Phi, \widehat{\varphi}, \tilde{\varphi})$ is solution in $\mathcal{S}^\infty_{\mathbb{G}}(0,T) \times \mathcal{H}^2_{\mathbb{G}}(0,T) \times \mathcal{H}^2_{\mathbb{G}}(M)$ to

$$\Phi_t = 1 - \int_t^T f(s, \Phi_s, \widehat{\varphi}_s, \tilde{\varphi}_s) ds - \int_t^T \widehat{\varphi}_s dB_s - \int_t^T \tilde{\varphi}_s dH_s \; , \tag{10}$$

where f is to be determined such that (iii) and (iv) above hold. In order to determine f, we write $J^{(d)}(\gamma)$ as the sum of a martingale and a non-decreasing process that is null for some $\gamma^* \in \Gamma$.

Applying integration by parts formula leads us to

$$\begin{aligned} d[(R_t L^\gamma_t)^q] = (R_t L^\gamma_{t-})^q \Big[\Big(\frac{1}{2} q(q-1)|\theta_t|^2 + \lambda_t \big((1+\gamma_t)^q - 1 \big) - q(\lambda^{\mathbb{G}}_t \gamma_t + r_t) \Big) dt \\ - q\theta_t dB_t + \big((1+\gamma_t)^q - 1 \big) dM_t \Big] \; . \end{aligned} \tag{11}$$

Taking into account (11) and applying integration by parts formula for the product of processes $(RL^\gamma)^q$ and Φ, we get

$$dJ^{(d)}_t(\gamma) = (R_t L^\gamma_{t-})^q A^\gamma_t dt + (R_t L^\gamma_{t-})^q \Big((\widehat{\varphi}_t - q\theta_t \Phi_t) dB_t + \big[(\Phi_{t-} + \tilde{\varphi}_t)(1+\gamma_t)^q - \Phi_{t-} \big] dM_t \Big) \; ,$$

where the predictable finite variation part of $J^{(d)}(\gamma)$ is given by $\int_0^\cdot (R_s L^\gamma_{s-})^q A^\gamma_s ds$, where

$$A^\gamma_t := \lambda^{\mathbb{G}}_t a_t(\gamma_t) + 1 + f(t, \Phi_t, \widehat{\varphi}_t, \tilde{\varphi}_t) - q r_t \Phi_{t-} + \frac{1}{2} q(q-1)|\theta_t|^2 \Phi_{t-} - \lambda^{\mathbb{G}}_t \Phi_{t-} - q\theta_t \widehat{\varphi}_t \; ,$$

with

$$a_t(x) := (\Phi_{t-} + \tilde{\varphi}_t)(1+x)^q - q\Phi_{t-} x \; , \quad t \in [0,T] \; . \tag{12}$$

In order to obtain a non-negative process A^γ for any $\gamma \in \Gamma$ (to satisfy the condition (iii)) and that is null for some $\gamma^* \in \Gamma$ (to satisfy the condition (iv)), it is obvious that the family $\{(A^\gamma_t)_{t\in[0,T]} \; : \; \gamma \in \Gamma\}$ has to satisfy $\min_{\gamma\in\Gamma} A^\gamma_t = 0$. Assuming that there exists a positive constant C such that $\Phi_t \geq C$ and $\Phi_{t-} + \tilde{\varphi}_t \geq C$ for any $t \in [0,\tau)$, we remark that the minimum is attained for γ^* defined by

$$\gamma^*_t := \Big(\frac{\Phi_{t-}}{\Phi_{t-} + \tilde{\varphi}_t} \Big)^{\frac{1}{q-1}} - 1$$

so that

$$\underline{a}_t := \min_{x > -1} a_t(x) = (1-q)\Phi^p_{t-} (\Phi_{t-} + \tilde{\varphi}_t)^{1-p} + q\Phi_{t-} \; .$$

This leads to the following choice for the generator f

$$\begin{aligned} f(t,y,z,u) \quad = \quad & \Big(q r_t - \frac{1}{2} q(q-1)|\theta_t|^2 + (1-q)\lambda^{\mathbb{G}}_t \Big) y + q\theta_t z \\ & -(1-q)\lambda^{\mathbb{G}}_t (y+u)^{1-p} \, y^p - 1 \; . \end{aligned} \tag{13}$$

4.3 Solution of the BSDE (10)

We remark that the obtained generator (13) is non standard since it involves in particular the term $(y+u)^{1-p} y^p$. We shall prove the following result

Theorem 1. *The BSDE*

$$\begin{aligned}\Phi_t = 1 - \int_t^T \Big(& \big(qr_s - \frac{1}{2}q(q-1)|\theta_s|^2 + (1-q)\lambda_s^{\mathbb{G}}\big)\Phi_s + q\theta_s\widehat{\varphi}_s \\ & - (1-q)\lambda_s^{\mathbb{G}}(\Phi_s + \tilde{\varphi}_s)^{1-p}\ \Phi_s^p - 1\Big)ds - \int_t^T \widehat{\varphi}_s dB_s - \int_t^T \tilde{\varphi}_s dH_s\ ,\end{aligned} \tag{14}$$

admits a solution $(\Phi, \widehat{\varphi}, \tilde{\varphi})$ *belonging to* $\mathcal{S}_{\mathbb{G}}^{\infty,+}(0,T) \times \mathcal{H}_{\mathbb{G}}^2(0,T) \times \mathcal{H}_{\mathbb{G}}^2(M)$*, such that* $\Phi_{t^-} + \tilde{\varphi}_t \geq 1$.

We use the decomposition procedure introduced in [12] to prove Theorem 1. For that, we transform the BSDE (14) into a recursive system of Brownian BSDEs. In a first step, for each $u \in [0,T]$, we prove that the following BSDE has a solution on the time interval $[u,T]$

$$\begin{cases} d\Phi_t^1(u) &= \Big[\big(qr_t^1(u) - \frac{1}{2}q(q-1)|\theta_t^1(u)|^2\big)\Phi_t^1(u) + q\theta_t^1(u)\widehat{\varphi}_t^1(u) - 1\Big]dt + \widehat{\varphi}_t^1(u)dB_t\ , \\ \Phi_T^1(u) &= 1\ , \end{cases} \tag{15}$$

and that the initial value $\Phi_u^1(u)$ of this BSDE is $\mathcal{F}_u$-measurable. Then, in a second step, we prove that the following BSDE has a solution on the time interval $[0,T]$

$$\begin{cases} d\Phi_t^0 &= \Big[\big(qr_t^0 - \frac{1}{2}q(q-1)|\theta_t^0|^2 + (1-q)\lambda_t^{\mathbb{F}}\big)\Phi_t^0 + q\theta_t^0\widehat{\varphi}_t^0 \\ & \qquad -(1-q)\lambda_t^{\mathbb{F}}(\Phi_t^1(t))^{1-p}(\Phi_t^0)^p\big) - 1\Big]dt + \widehat{\varphi}_t^0 dB_t\ , \\ \Phi_T^0 &= 1\ , \end{cases} \tag{16}$$

where Φ^1 is part of the solution of the BSDE (15).

Proposition 1. *For any* $u \in [0,T]$*, the BSDE (15) admits a unique solution* $(\Phi^1(u), \widehat{\varphi}^1(u)) \in \mathcal{S}_{\mathbb{F}}^\infty(u,T) \times \mathcal{H}_{\mathbb{F}}^2(u,T)$*. Furthermore,* $1 \leq \Phi_t^1(u) \leq C$ *for any* $t \in [u,T]$ *where* C *is a constant which does not depend on* u.

Proof. Let us fix $u \in [0,T]$. Since the BSDE (15) is linear with bounded coefficients, the solution $(\Phi^1(u), \widehat{\varphi}^1(u)) \in \mathcal{S}_{\mathbb{F}}^\infty(u,T) \times \mathcal{H}_{\mathbb{F}}^2(u,T)$ is given by

$$\Phi_t^1(u) = \mathbb{E}\Big[\Gamma_T^t(u) + \int_t^T \Gamma_s^t(u)ds\Big|\mathcal{F}_t\Big]\ , \quad t \in [u,T]\ , \tag{17}$$

where for a fixed $t \in [u,T]$, $(\Gamma_s^t(u))_{t\leq s\leq T}$ stands for the adjoint process defined by

$$\Gamma_s^t(u) = \exp\Big(\int_t^s \big(-qr_v^1(u) + \frac{1}{2}q(q-1)|\theta_v^1(u)|^2\big)dv\Big)\mathcal{E}\Big(-\int_t^\cdot q\theta_v^1(u)dB_v\Big)_s.$$

To prove that Φ^1 is uniformly bounded, we introduce the probability measure $\mathbb{P}^u$, defined on $\mathcal{F}_t$, for $t \leq T$, by its Radon-Nikodym density $Z_t(u) := \mathcal{E}(-\int_0^\cdot q\theta_v^1(u)dB_v)_t$, which is a true martingale, and we denote by $\mathbb{E}^u$ the expectation under this probability. Then, by virtue of the formula (17) and Bayes' rule, we get

$$\begin{aligned}\Phi_t^1(u) &= \mathbb{E}^u\Big[\exp\Big(\int_t^T \big(-qr_s^1(u) + \frac{1}{2}q(q-1)|\theta_s^1(u)|^2\big)ds\Big)\Big|\mathcal{F}_t\Big] \\ &\quad + \mathbb{E}^u\Big[\int_t^T \exp\Big(\int_t^s \big(-qr_v^1(u) + \frac{1}{2}q(q-1)|\theta_v^1(u)|^2\big)dv\Big)ds\Big|\mathcal{F}_t\Big]\ .\end{aligned}$$

From **(H3)** and **(H4)**, and since $q < 0$, there exists a positive constant C which is independent of u such that $1 \leq \Phi_t^1(u) \leq C$ for any $t \in [u, T]$. $\square$

Proposition 2. *The BSDE (16) admits a unique solution* $(\Phi^0, \widehat{\varphi}^0) \in \mathcal{S}_{\mathbb{F}}^{\infty,+}(0,T) \times \mathcal{H}_{\mathbb{F}}^2(0,T)$.

Proof. The generator of the BSDE (16) is not defined on the whole space $[0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}$ and the generator is not classical. So the proof of this proposition will be performed in several steps. We first introduce a modified BSDE where the term y^p is replaced by $(y \vee m)^p$ (where m is a positive constant which is defined later) to ensure that the generator is well defined on the whole space $[0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}$. We then prove via a comparison theorem that the solution of the modified BSDE satisfies the initial BSDE. In the last step, we prove the uniqueness of the solution.

Step 1. *Introduction of the modified BSDE.*

We consider

$$\left\{ \begin{array}{rcl} -dY_t & = & \bar{g}(t, Y_t, \widehat{y}_t)dt - \widehat{y}_t dB_t \ , \\ Y_T & = & 1 \ , \end{array} \right. \tag{18}$$

where the generator $\bar{g}$ is given by

$$\bar{g}(t,y,z) := 1 + \Big(\frac{1}{2}q(q-1)|\theta_t^0|^2 - qr_t^0 - (1-q)\lambda_t^{\mathbb{F}}\Big)y - q\theta_t^0 z + (1-q)\lambda_t^{\mathbb{F}}(\Phi_t^1(t))^{1-p}(y \vee m)^p \ ,$$

with $m := \exp\big((q-1)\Lambda\big)$, and Λ is a constant such that $\lambda_t^{\mathbb{F}} \leq \Lambda$ for any $t \in [0,T]$.

Since $p \in (0,1)$, there exists a positive constant C such that $(y \vee m)^p \leq C(1 + |y|)$. We also have $\Phi^1(.)$ is uniformly bounded, and using assumptions $(\mathbf{H2})$, $(\mathbf{H3})$ and $(\mathbf{H4})$ we obtain that $\bar{g}$ has linear growth uniformly w.r.t. y. It follows from Fan and Jiang [6] that the BSDE (18) has a unique solution $(Y, \widehat{y}) \in \mathcal{S}_{\mathbb{F}}^{\infty}(0,T) \times \mathcal{H}_{\mathbb{F}}^2(0,T)$.

For the convenience of the reader, we recall the Fan and Jiang conditions, which, in our setting, are obviously satisfied. The solution of the BSDE

$$-dY_t = f(t, Y_t, \widehat{y}_t)dt - \widehat{y}_t dB_t \ , \quad Y_T = 1$$

is unique if:

(1) the process $(f(t,0,0))_{t\in[0,T]} \in L^2(0,T)$,
(2) $(d\mathbb{P} \times dt)$ a.s., $(y,z) \to f(\omega,t,y,z)$ is continuous,
(3) f is monotonic in y, i.e., there exists a constant $\mu \geq 0$, such that, $(d\mathbb{P} \times dt)$ a.s.,

$$\forall y_1, y_2, z, \Big(f(\omega,t,y_1,z) - f(\omega,t,y_2,z)\Big)(y_1 - y_2) \leq \mu(y_1 - y_2)^2 \ ,$$

(4) f has a general growth with respect to y, i.e., $(dP \times dt)$ .a.s.,

$$\forall y, |f(\omega,t,y,0)| \leq |f(\omega,t,0,0)| + \varphi(|y|)$$

where $\varphi : \mathbb{R} \to \mathbb{R}^+$ is an increasing continuous function,

(5) f is uniformly continuous in z and uniform w.r.t. (ω, t, y), i.e., there exists a continuous, non-decreasing function ϕ from $\mathbb{R}^+$ to itself with at most linear growth and $\phi(0) = 0$ such that $(d\mathbb{P} \times dt)$ a.s.,

$$\forall y, z_1, z_2, \quad |f(\omega,t,y,z_1) - f(\omega,t,y,z_2)| \leq \phi(|z_1 - z_2|) \ .$$

Step 2. *Comparison.*

We now show that the solution of the BSDE (18) is lower bounded by m, and this is accomplished via a comparison result for solutions of Brownian BSDEs. We remark that the following inequality holds

$$\bar{g}(t,y,z) \geq \Big(\frac{1}{2}q(q-1)|\theta^0_t|^2 - qr^0_t - (1-q)\lambda^{\mathbb{F}}_t\Big)y - q\theta^0_t z =: g(t,y,z)\ .$$

Therefore, we introduce the following linear BSDE

$$\left\{\begin{array}{rcl} -dZ_t &=& g(t,Z_t,\widehat{z}_t)dt - \widehat{z}_t dB_t\ , \\ Z_T &=& 1\ . \end{array}\right. \tag{19}$$

In the same way as we proceed with the BSDE (15), we have an explicit form of the solution of the BSDE (19) given by

$$Z_t = \mathbb{E}\big(\Upsilon^t_T\big|\mathcal{F}_t\big)\ ,$$

where $(\Upsilon^t_s)_{t\leq s\leq T}$ stands for the solution of the linear SDE

$$d\Upsilon^t_s = \Upsilon^t_s\Big[\Big(\frac{1}{2}q(q-1)|\theta^0_s|^2 - qr^0_s - (1-q)\lambda^{\mathbb{F}}_s\Big)ds - q\theta^0_s dB_s\Big]\ , \quad \Upsilon^t_t = 1\ .$$

We can rewrite the solution of the BSDE (19) under the following form

$$Z_t = \mathbb{E}^*\Big[\exp\Big(\int_t^T \big(\frac{1}{2}q(q-1)|\theta^0_s|^2 - qr^0_s - (1-q)\lambda^{\mathbb{F}}_s\big)ds\Big)\Big|\mathcal{F}_t\Big]\ ,$$

where $\mathbb{E}^*$ is the expectation under the probability $\mathbb{P}^*$ defined by its Radon-Nikodym density $d\mathbb{P}^*|_{\mathcal{F}_t} = \mathcal{E}(-\int_0^. q\theta^0_v dB_v)_t d\mathbb{P}|_{\mathcal{F}_t}$ for any $t\in[0,T]$. By virtue of the assumption **(H4)**, it follows that

$$Z_t \geq \mathbb{E}^*\Big[\exp\Big(-\int_t^T (1-q)\lambda^{\mathbb{F}}_s ds\Big)\Big|\mathcal{F}_t\Big] \geq m\ .$$

From the comparison theorem for Brownian BSDEs, we obtain

$$Y_t \geq Z_t \geq m\ ,$$

which implies that $Y_t \vee m = Y_t$ for any $t\in[0,T]$. Therefore, $(Y,\widehat{y})$ is a solution of the BSDE (16) in $\mathcal{S}^{\infty,+}_{\mathbb{F}}(0,T)\times\mathcal{H}^2_{\mathbb{F}}(0,T)$.

Step 3. *Uniqueness of the solution.* Suppose that the BSDE (16) has two solutions (Y^1,Z^1) and (Y^2,Z^2) in $\mathcal{S}^{\infty,+}_{\mathbb{F}}(0,T)\times\mathcal{H}^2_{\mathbb{F}}(0,T)$. Thus, there exists a positive constant c such that $Y^1_t\geq c$ and $Y^2_t\geq c$ for any $t\in[0,T]$. In this case, (Y^1,Z^1) and (Y^2,Z^2) are solutions of the following BSDE

$$\left\{\begin{array}{rcl} -dY_t &=& h(t,Y_t,\widehat{y}_t)dt - \widehat{y}_t dB_t\ , \\ Y_T &=& 1\ , \end{array}\right.$$

where the generator h is given by

$$h(t,y,z) := 1+\Big(-qr^0_t+\frac{1}{2}q(q-1)|\theta^0_t|^2-(1-q)\lambda^{\mathbb{F}}_t\Big)y-q\theta^0_t z+(1-q)\lambda^{\mathbb{F}}_t(\Phi^1_t(t))^{1-p}(y\vee c)^p\ .$$

From [6], we know that this BSDE admits a unique solution, therefore we get $Y^1=Y^2$. □

We are now able to prove Theorem 1.

Proof. From Propositions 1 and 2 and Theorem 3.1 in [12], we obtain that the BSDE (14) admits a solution $(\Phi, \widehat{\varphi}, \tilde{\varphi})$ belonging to $\mathcal{S}^\infty_{\mathbb{G}}(0,T) \times \mathcal{H}^2_{\mathbb{G}}(0,T) \times \mathcal{H}^2_{\mathbb{G}}(M)$ given by

$$\begin{aligned} \Phi_t &= \Phi^0_t \mathbb{1}_{t<\tau} + \Phi^1_t(\tau)\mathbb{1}_{t\geq\tau} \;, \\ \widehat{\varphi}_t &= \widehat{\varphi}^0_t \mathbb{1}_{t\leq\tau} + \widehat{\varphi}^1_t(\tau)\mathbb{1}_{t>\tau} \;, \\ \tilde{\varphi}_t &= (\Phi^1_t(t) - \Phi^0_t)\mathbb{1}_{t\leq\tau} \;. \end{aligned} \tag{20}$$

Note that $\widehat{\varphi}$ and $\tilde{\varphi}$ are $\mathbb{G}$-predictable processes. Moreover, from Propositions 1 and 2, there exists a positive constant C such that $\Phi_t \geq C$. We also remark that

$$\Phi_{t^-} + \tilde{\varphi}_t = \Phi^1_t(t)\mathbb{1}_{t\leq\tau} + \Phi^1_t(\tau)\mathbb{1}_{t>\tau} = \Phi^1_t(t\wedge\tau) \;,$$

which implies that $\Phi_{t^-} + \tilde{\varphi}_t \geq 1$. □

Remark 2. *We remark that if r^1 and θ^1 are deterministic, Φ^1 is deterministic. Moreover, if r^0, θ^0 and $\lambda^{\mathbb{F}}$ are deterministic, Φ^0 is deterministic. More precisely, $\widehat{\varphi}^1_t(u) = \widehat{\varphi}^0_t = 0$, and the BSDEs (15) and (16) turn into ODEs*

$$\begin{cases} d\Phi^1_t(u) &= \Big[\big(qr^1_t(u) - \frac{1}{2}q(q-1)|\theta^1_t(u)|^2\big)\Phi^1_t(u)) - 1\Big]dt \;, \\ \\ \Phi^1_T(u) &= 1 \;, \end{cases}$$

and

$$\begin{cases} d\Phi^0_t &= \Big[\Big(qr^0_t - \frac{1}{2}q(q-1)|\theta^0_t|^2 + (1-q)\lambda^{\mathbb{F}}_t\Big)\Phi^0_t - (1-q)\lambda^{\mathbb{F}}_t(\Phi^1_t(t))^{1-p}(\Phi^0_t)^p\big) - 1\Big]dt \;, \\ \Phi^0_T &= 1 \;, \end{cases}$$

with an explicit solution for the first equation.

Remark 3. *If $r^1_t(u) = r^0_t$ for any $0 \leq u \leq t \leq T$, there is no change of regime. Our result is coherent with that obvious observation, since, in that case, we have that $\theta^1_t(u) = \theta^0_t$ for any $0 \leq u \leq t \leq T$ which implies $\Phi^0_t = \Phi^1_t(t)$ for any $t \in [0,T]$.*

4.4 A verification Theorem

We now turn to the sufficient condition of optimality. In this part, we prove that the family of processes $\{(J^{(d)}_t(\gamma))_{t\in[0,T]} \,:\, \gamma\in\Gamma\}$ defined by $J^{(d)}(\gamma) := \int_0^\cdot (R_s L^\gamma_s)^q ds + (RL^\gamma)^q\Phi$ with Φ defined by (20) satisfies the conditions (i), (ii), (iii) and (iv). By construction, $J^{(d)}(\gamma)$ satisfies the conditions (i) and (ii). As explained previously a candidate to be an optimal γ is a process γ^* such that $J^{(d)}(\gamma^*)$ is a $\mathbb{G}$-martingale, hence this one is

$$\gamma^*_t := \Big(\frac{\Phi_{t^-}}{\Phi_{t^-} + \tilde{\varphi}_t}\Big)^{\frac{1}{q-1}} - 1 \;. \tag{21}$$

Lemma 4. *The process γ^* defined by (21) is admissible.*

Proof. By construction, γ^* is $\mathbb{G}$-predictable. Moreover, from Theorem 1, we remark that there exists two constants A and C such that $-1 < A \leq \gamma^*_t \leq C$ for any $t\in[0,T]$ which implies that $\gamma^* \in \Gamma$. □

From the above results, $J^{(d)}(\gamma)$ is a semi-martingale with a local martingale part and a non-decreasing predictable variation part and $J^{(d)}(\gamma^*)$ is a local martingale.

Proposition 3. *The process $J^{(d)}(\gamma)$ is a $\mathbb{G}$-submartingale for any admissible process $\gamma \in \Gamma$ and is a $\mathbb{G}$-martingale for γ^* given by* (21).

Proof. From (11) and (14), we can rewrite the dynamics of $J^{(d)}(\gamma)$ under the following form

$$dJ_t^{(d)}(\gamma) = (R_t L_{t-}^\gamma)^q \big(dM_t^\gamma + A_t^\gamma dt\big) ,$$

where

$$dM_t^\gamma = \big(\widehat{\varphi}_t - q\theta_t \Phi_{t-}\big)dB_t + \Big((1+\gamma_t)^q\big(\Phi_{t-} + \tilde{\varphi}_t\big) - \Phi_{t-}\Big)dM_t ,$$

and

$$A_t^\gamma = \lambda_t^{\mathbb{G}}\big[a_t(\gamma_t) - a_t(\gamma_t^*)\big] ,$$

with $a(.)$ defined by (12).
From (9), Lemma 3 and since $\Phi \in \mathcal{S}_{\mathbb{G}}^\infty(0,T)$, we remark that for any $\gamma \in \Gamma$

$$\mathbb{E}\Big[\sup_{t\in[0,T]} J_t^{(d)}(\gamma)\Big] < \infty . \tag{22}$$

For any $\gamma \in \Gamma$, we have that $\int_0^\cdot (R_s L_{s-}^\gamma)^q dM_s^\gamma$ is a $\mathbb{G}$-local martingale. Hence, there exists an increasing sequence of $\mathbb{G}$-stopping times $(T_n)_{n\in\mathbb{N}}$ valued in $[0,T]$ satisfying $\lim_{n\to\infty} T_n = T$, $\mathbb{P}-a.s.$ such that $\int_0^{\cdot\wedge T_n} (R_s L_{s-}^\gamma)^q dM_s^\gamma$ is a $\mathbb{G}$-martingale for any $n \in \mathbb{N}$. Therefore, we obtain for any $t \in [0,T]$

$$\mathbb{E}\Big[J_{t\wedge T_n}^{(d)}(\gamma)\Big] = J_0^{(d)}(\gamma) + \mathbb{E}\Big[\int_0^{t\wedge T_n} (R_s L_{s-}^\gamma)^q A_s^\gamma ds\Big] .$$

Since $(RL^\gamma)^q A^\gamma \geq 0$, from (22) and using the monotone convergence theorem, we obtain

$$\mathbb{E}\Big[\int_0^T (R_t L_{t-}^\gamma)^q A_t^\gamma dt\Big] < \infty .$$

From (22) and the previous inequality, we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \Big|\int_0^t (R_s L_{s-}^\gamma)^q dM_s^\gamma\Big|\Big] < \infty .$$

It follows that the local martingale $\int_0^\cdot (R_s L_{s-}^\gamma)^q dM_s^\gamma$ is a true martingale and the process $J^{(d)}(\gamma)$ is a $\mathbb{G}$-submartingale for any $\gamma \in \Gamma$. We obtain with the same arguments that the process $J^{(d)}(\gamma^*)$ is a martingale.

□

4.5 Uniqueness of the solution of the BSDE (10)

To solve the dual problem it is not necessary to prove the uniqueness of the solution of the BSDE (10) but this one is useful to characterize the value function of the primal problem in the last part of this paper. To prove the uniqueness we do not use a comparison theorem for BSDE but the following dynamic programming principle.

Lemma 5. *Let Y be a process with $Y_T = 1$ such that $\int_0^{\cdot}(R_s L_s^{\gamma})^q ds + (RL^{\gamma})Y$ is a $\mathbb{G}$-submartingale for any $\gamma \in \Gamma$ and there exists $\gamma' \in \Gamma$ such that $\int_0^{\cdot}(R_s L_s^{\gamma'})^q ds + (RL^{\gamma'})Y$ is a $\mathbb{G}$-martingale. Then, we have*

$$Y_t = \operatorname*{ess\,inf}_{\gamma\in\Gamma}\Big\{\frac{1}{(R_t L_t^{\gamma})^q}\mathbb{E}\Big[\int_t^T (R_s L_s^{\gamma})^q ds + (R_T L_T^{\gamma})^q \Big| \mathcal{G}_t\Big]\Big\}\ .$$

Proof. The following inequality holds for any $\gamma \in \Gamma$

$$Y_t \leq \frac{1}{(R_t L_t^{\gamma})^q}\mathbb{E}\Big[\int_t^T (R_s L_s^{\gamma})^q ds + (R_T L_T^{\gamma})^q \Big| \mathcal{G}_t\Big]\ .$$

Moreover, we know that

$$Y_t = \frac{1}{(R_t L_t^{\gamma'})^q}\mathbb{E}\Big[\int_t^T (R_s L_s^{\gamma'})^q ds + (R_T L_T^{\gamma'})^q \Big| \mathcal{G}_t\Big]\ .$$

Therefore, we get

$$Y_t = \operatorname*{ess\,inf}_{\gamma\in\Gamma}\Big\{\frac{1}{(R_t L_t^{\gamma})^q}\mathbb{E}\Big[\int_t^T (R_s L_s^{\gamma})^q ds + (R_T L_T^{\gamma})^q \Big| \mathcal{G}_t\Big]\Big\}\ .$$

□

We now prove that any solution of the BSDE (10) satisfies the properties of Lemma 5.

Lemma 6. *Let $(\Phi, \widehat{\varphi}, \tilde{\varphi}) \in \mathcal{S}_{\mathbb{G}}^{\infty,+}(0,T) \times \mathcal{H}_{\mathbb{G}}^2(0,T) \times \mathcal{H}_{\mathbb{G}}^2(M)$ be a solution of the BSDE (10). Then, the process $\int_0^{\cdot}(R_s L_s^{\gamma})^q ds + (RL^{\gamma})\Phi$ is a $\mathbb{G}$-submartingale for any $\gamma \in \Gamma$ and there exists $\gamma' \in \Gamma$ such that $\int_0^{\cdot}(R_s L_s^{\gamma'})^q ds + (RL^{\gamma'})\Phi$ is a $\mathbb{G}$-martingale.*

Proof. To simplify the notation we denote $\mathcal{W}^{\gamma} := \int_0^{\cdot}(R_s L_s^{\gamma})^q ds + (RL^{\gamma})\Phi$. From Itô's formula, we get for any $\gamma \in \Gamma$

$$\mathcal{W}_t^{\gamma} = (R_t L_{t-}^{\gamma})^q\Big\{\lambda_t\big((a_t(\gamma_t) - a_t(\gamma_t'))\big)dt + \big(\widehat{\varphi}_t - q\theta_t\Phi_t\big)dB_t + \big((1+\gamma_t)^q(\Phi_t + \tilde{\phi}_t) - \Phi_t\big)dM_t\Big\}\ ,$$

where $a(.)$ is defined by (12) and $\gamma_t' := \Big(\frac{\Phi_{t-}}{\Phi_{t-} + \tilde{\phi}_t}\Big)^{p-1} - 1$.
We know that $\mathbb{E}[\sup_{0\leq t\leq T}\mathcal{W}_t^{\gamma}] < \infty$ from Lemma 3 and $a_t(\gamma_t) \geq a_t(\gamma_t')$ for any $\gamma \in \Gamma$ by definition of γ'. Therefore, using the same arguments as for the proof of Proposition 3 we can prove that, for any $\gamma \in \Gamma$, the process $\int_0^{\cdot}(R_s L_s^{\gamma})^q ds + (RL^{\gamma})\Phi$ is a $\mathbb{G}$-submartingale and $\int_0^{\cdot}(R_s L_s^{\gamma'})^q ds + (RL^{\gamma'})\Phi$ is a $\mathbb{G}$-martingale. □

We can conclude from Lemmas 5 and 6 that there exists a unique solution of the BSDE (10) in $\mathcal{S}_{\mathbb{G}}^{\infty,+}(0,T) \times \mathcal{H}_{\mathbb{G}}^2(0,T) \times \mathcal{H}_{\mathbb{G}}^2(M)$.

5 Primal problem and optimal strategy

In this section, we deduce the solution of the primal problem (2) using the duality result of the previous section, and we characterize the value function associated to the primal problem by the solution of a BSDE which is in relationship with the BSDE (14) associated to the dual problem.

The following proposition shows the existence of an optimal solution for the primal problem and characterizes this solution in terms of the solution of the dual problem.

Proposition 4. *The optimal strategy is given by*

$$c_t^* = \left(\eta^* R_t L_{t^-}^{\gamma^*}\right)^{\frac{1}{p-1}}, \quad \pi_t^* = \frac{1}{\sigma_t}\left(\frac{\widehat{\varphi}_t}{\Phi_{t^-}} + \frac{\theta_t}{1-p}\right), \quad t \in [0,T], \tag{23}$$

where η^* is defined by

$$\eta^* := \left(\frac{x}{\mathbb{E}\left[\int_0^T (R_t L_t^{\gamma^*})^q dt + (R_T L_T^{\gamma^*})^q\right]}\right)^{p-1}, \tag{24}$$

and γ^* is given by (21).

Before proving Proposition 4, we prove that the strategy (π^*, c^*) is admissible.

Lemma 7. *The strategy (π^*, c^*) given by (23) is admissible and the wealth associated to (π^*, c^*) is*

$$X_t^{x,\pi^*,c^*} = \left(\eta^* R_t L_t^{\gamma^*}\right)^{\frac{1}{p-1}} \Phi_t. \tag{25}$$

Proof. Using assumptions (**H3**) and (**H4**), and the properties of $(\Phi, \widehat{\varphi}, \tilde{\varphi})$ given by Theorem 1, we obtain that $\mathbb{E}(\int_0^T |\pi_s^* \sigma_s|^2 ds) < \infty$ and π^* is $\mathbb{G}$-predictable. Moreover, from (1), the wealth process X^{x,π^*,c^*} associated to the strategy (π^*, c^*) is defined by the SDE

$$\begin{cases} X_0^{x,\pi^*,c^*} &= x, \\ dX_t^{x,\pi^*,c^*} &= X_t^{x,\pi^*,c^*}\left[\left(r_t - (q-1)|\theta_t|^2 + \theta_t \frac{\widehat{\varphi}_t}{\Phi_{t^-}}\right)dt + \left(\frac{\widehat{\varphi}_t}{\Phi_{t^-}} - (q-1)\theta_t\right)dB_t\right] \\ & \quad -\left(\eta^* R_t L_{t^-}^{\gamma^*}\right)^{\frac{1}{p-1}} dt. \end{cases} \tag{26}$$

From Proposition 3, the process $\int_0^\cdot (R_s L_s^{\gamma^*})^q ds + (RL^{\gamma^*})^q \Phi$ is a $\mathbb{G}$-martingale, which implies

$$\Phi_0 = \mathbb{E}\left[\int_0^T (R_t L_t^{\gamma^*})^q dt + (R_T L_T^{\gamma^*})^q\right].$$

From the previous equality and (24), we remark that $(\eta^*)^{\frac{1}{p-1}} \Phi_0 = x$. Using Itô's formula and (14), we check that $(\eta^* R L^{\gamma^*})^{\frac{1}{p-1}} \Phi$ is a solution of the SDE (26). Moreover, this SDE admits a unique solution. Therefore, we have

$$X_t^{x,\pi^*,c^*} = (\eta^* R_t L_t^{\gamma^*})^{\frac{1}{p-1}} \Phi_t. \tag{27}$$

Using the fact that $c_t^* \geq 0$ and $X_t^{x,\pi^*,c^*} > 0$ for any $t \in [0,T]$, we conclude the proof. In particular, $(\eta^* R_T L_T^{\gamma^*})^{\frac{1}{p-1}} = X_T^{x,\pi^*,c^*}$ is hedgeable. □

We now prove Proposition 4.

Proof. From (6), we obtain

$$\sup_{(\pi,c)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^T U(c_s)ds + U(X_T^{x,\pi,c})\right] \leq \inf_{\eta>0,\gamma\in\Gamma}\left(-\frac{\eta^q}{q}\mathbb{E}\left[\int_0^T (R_s L_s^{\gamma})^q ds + (R_T L_T^{\gamma})^q\right] + \eta x\right).$$

By the definition of γ^* and η^*, the previous inequality is equivalent to

$$\sup_{(\pi,c)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^T U(c_s)ds + U(X_T^{x,\pi,c})\right] \leq \frac{x^p}{p}\left(\mathbb{E}\left[\int_0^T (R_s L_s^{\gamma^*})^q ds + (R_T L_T^{\gamma^*})^q\right]\right)^{1-p}. \tag{28}$$

By definition of (π^*, c^*) and Lemma 7, we remark that

$$\mathbb{E}\Big[\int_0^T U(c_s^*)ds + U(X_T^{x,\pi^*,c^*})\Big] = \frac{x^p}{p}\Big(\mathbb{E}\Big[\int_0^T (R_s L_s^{\gamma^*})^q ds + (R_T L_T^{\gamma^*})^q\Big]\Big)^{1-p} . \quad (29)$$

Since (π^*, c^*) is admissible, from (28) and (29), we obtain

$$\mathbb{E}\Big[\int_0^T U(c_s^*)ds + U(X_T^{x,\pi^*,c^*})\Big] = \sup_{(\pi,c)\in\mathcal{A}(x)} \mathbb{E}\Big[\int_0^T U(c_s)ds + U(X_T^{x,\pi,c})\Big] .$$

Therefore, (π^*, c^*) is an optimal solution of the primal problem (2). $\square$

We now characterize the value function associated to the primal problem using the dynamic programming principle. For fixed $t \in [0,T]$ we denote by (π^t, c^t) a strategy defined on the time interval $[t,T]$ and $(X_s^{t,x,\pi^t,c^t})_{s\in[t,T]}$ the wealth process associated to this strategy given the initial value at time t is $x > 0$. We first define the set of control for a fixed $t \leq T$.

Definition 3. *The set $\mathcal{A}_t(x)$ of admissible strategies (π^t, c^t) from time t consists in the set of $\mathbb{G}$-predictable processes $(\pi_s^t, c_s^t)_{s\in[t,T]}$ such that $\mathbb{E}(\int_t^T |\pi_s^t \sigma_s|^2 ds) < \infty$, $c_s^t \geq 0$ and $X_s^{t,x,\pi^t,c^t} > 0$ for any $s \in [t,T]$.*

We define the value function at time $t \leq T$ for the primal problem as follows

$$V(t,x) := \frac{x^p}{p}\Psi_t(x) , \quad (30)$$

where

$$\Psi_t(x) := \frac{1}{x^p} \operatorname*{ess\,sup}_{(\pi^t,c^t)\in\mathcal{A}_t(x)} \mathbb{E}\Big[\int_t^T (c_s^t)^p ds + (X_T^{t,x,\pi^t,c^t})^p \Big| \mathcal{G}_t\Big] .$$

For any $(\pi^t, c^t) \in \mathcal{A}_t(x)$, we define the strategy $(\hat{\pi}^t, \hat{c}^t)$ by $\hat{\pi}^t := \pi^t$ and $\hat{c}^t := c^t/x$. We remark that $(\hat{\pi}^t, \hat{c}^t) \in \mathcal{A}_t(1)$ and $X_T^{t,x,\pi^t,c^t} = xX_T^{t,1,\hat{\pi}^t,\hat{c}^t}$ from (1). Combining the previous relations with the definition of $\Psi_t(x)$, we obtain $\Psi_t(x) = \Psi_t(1)$. For the sake of brevity, we shall denote Ψ_t instead of $\Psi_t(1)$. The value function at time $t \leq T$ can be rewritten as follows $V(t,x) = x^p\Psi_t/p$. From (30) and Proposition 4, we have

$$V(0,x) = \mathbb{E}\Big[\int_0^T \frac{(c_s^*)^p}{p} ds + \frac{(X_T^{x,\pi^*,c^*})^p}{p}\Big] = V(x) . \quad (31)$$

Using dynamic control techniques, we derive the following characterization of the value function.

Proposition 5. *For any $(\pi, c) \in \mathcal{A}(x)$, $\int_0^{\cdot} \frac{(c_s)^p}{p} ds + V(., X^{x,\pi,c})$ is a $\mathbb{G}$-supermartingale and there exists $(\pi^*, c^*) \in \mathcal{A}(x)$ such that $\int_0^{\cdot} \frac{(c_s^*)^p}{p} ds + V(., X^{x,\pi^*,c^*})$ is a $\mathbb{G}$-martingale.*

The proof of this proposition is given in El Karoui [5].

Using these properties, we can characterize the value function with a BSDE.

Proposition 6. *The process* Ψ *satisfies the equality* $\Psi = \Phi^{1-p}$*. Moreover, the process* Ψ *is solution of the BSDE*

$$\begin{aligned}\Psi_t = 1 - \int_t^T \Big(-(1-p)\Psi_s^q - pr_s\Psi_s + \frac{1}{2}\frac{p}{p-1}(|\theta_s|^2\Psi_s + \frac{\widehat{\psi}_s^2}{\Psi_s}) + \frac{p}{p-1}\theta_s\widehat{\psi}_s\Big)ds \\ - \int_t^T \widehat{\psi}_s dB_s - \int_t^T \widetilde{\psi}_s dH_s \ .\end{aligned} \tag{32}$$

Proof. From (23), (25) and (30), we have

$$\int_0^T \frac{(c_s^*)^p}{p}ds + V(T, X_T^{x,\pi^*,c^*}) = \frac{(\eta^*)^q}{p}\Big(\int_0^T (R_sL_s^{\gamma^*})^q ds + (R_TL_T^{\gamma^*})^q\Phi_T\Big) \ .$$

From Propositions 3 and 5, the process $\int_0^{\cdot} \frac{(c_s^*)^p}{p}ds + V(., X_{\cdot}^{x,\pi^*,c^*})$ and $\int_0^{\cdot}(R_sL_s^{\gamma^*})^q ds + (RL^{\gamma^*})^q\Phi$ are $\mathbb{G}$-martingales. Therefore, taking the conditional expectation for the above equality, we obtain

$$\int_0^t (c_s^*)^p ds + (X_t^{x,\pi^*,c^*})^p\Psi_t \quad = \quad (\eta^*)^q\Big(\int_0^t (R_sL_s^{\gamma^*})^q ds + (R_tL_t^{\gamma^*})^q\Phi_t\Big) \ .$$

Since $c_t^* = (\eta^* R_t L_t^{\gamma^*})^{\frac{1}{p-1}}$, the following relation holds for any $t \in [0, T]$

$$(X_t^{x,\pi^*,c^*})^p\Psi_t = (\eta^*)^q(R_tL_t^{\gamma*})^q\Phi_t \ . \tag{33}$$

Therefore, from (27) and (33), we obtain

$$\Phi_t^{1-p} = \Psi_t \ .$$

Applying Itô's formula to Φ^{1-p}, we obtain

$$\begin{aligned}d\Phi_t^{1-p} = (1-p)\Phi_{t^-}^{-p}\Big(\big(qr_t - \frac{1}{2}q(q-1)|\theta_t|^2 + \lambda_t^{\mathbb{G}}(1-q)\big)\Phi_t + q\theta_t\widehat{\varphi}_t - (1-q)\lambda_t^{\mathbb{G}}(\Phi_{t^-} + \tilde{\varphi}_t)^{1-p}\Phi_t^p \\ -1 - \frac{1}{2}p\Phi_t^{-1}\widehat{\varphi}_t^2\Big)dt + (1-p)\widehat{\varphi}_t\Phi_{t^-}^{-p}dB_t + \Big((\Phi_{t^-} + \widetilde{\varphi}_t)^{1-p} - \Phi_{t^-}^{1-p}\Big)dH_t \ .\end{aligned}$$

Setting $\widehat{\psi} = (1-p)\widehat{\varphi}\Phi^{-p}$ and $\widetilde{\psi} = (\Phi + \widetilde{\varphi})^{1-p} - \Phi^{1-p}$, we get that $(\Psi, \widehat{\psi}, \widetilde{\psi})$ satisfies (32). □

The above result is not sufficient to characterize the value function of the primal problem since it is not obvious that the BSDE (32) admits a unique solution. But thanks to the uniqueness of the solution of the BSDE (14) we get the following characterization of the value function of the primal problem.

Theorem 2. Ψ *is the unique solution of the BSDE* (32) *in* $\mathcal{S}_{\mathbb{G}}^{\infty,+}(0,T) \times \mathcal{H}_{\mathbb{G}}^2(0,T) \times \mathcal{H}_{\mathbb{G}}^2(M)$.

Proof. Let $(y, z, u) \in \mathcal{S}_{\mathbb{G}}^{\infty,+}(0,T) \times \mathcal{H}_{\mathbb{G}}^2(0,T) \times \mathcal{H}_{\mathbb{G}}^2(M)$ be a solution of the BSDE (32). We define $Y_t := y_t^{1-q}$, $Z_t := (1-q)y_t^{-q}z_t$ and $U_t := (y_{t^-} + u_t)^{1-q} - y_{t^-}^{1-q}$ for any $t \in [0, T]$. From Itô's formula, we get that

$$\begin{aligned} dY_t \quad = \quad & \Big[\big(qr_t - \frac{q(q-1)}{2}|\theta_t|^2 + (1-q)\lambda_t^{\mathbb{G}}\big)Y_t - (1-q)\lambda_t^{\mathbb{G}}(U_t + Y_t)^{1-p}Y_t^p \\ & + q\theta_tZ_t - 1\Big]dt + Z_tdB_t + U_tdH_t \ .\end{aligned}$$

Therefore, (Y, Z, U) is solution of the BSDE (14) and, from Subsection 4.5, we have by uniqueness of the solution $Y = \Phi$ and from Proposition 6 we get that $y = \Psi$. $\square$

Remark 4. *From Theorems 1 and 2 and Proposition 6, we can conclude that the BSDE (32), which is associated to the primal problem (2), admits a unique solution $(\Psi, \widehat{\psi}, \widetilde{\psi})$ belonging to $\mathcal{S}_{\mathbb{G}}^{\infty,+}(0,T) \times \mathcal{H}_{\mathbb{G}}^2(0,T) \times \mathcal{H}_{\mathbb{G}}^2(M)$. Solving this BSDE directly is not evident because of the terms Ψ^q and $\frac{\widehat{\psi}^2}{\Psi}$.*

Remark 5. *We point out that, in the case where the coefficients of the model are deterministic functions of some external economic factors and in a Brownian setting, the optimal control processes (π^*, c^*) have the same expressions that those obtained by Castañeda-Leyva and Hernández-Hernández [2] (see Proposition 3.1. in [2]).*
We also remark that, in a default-density setting, the optimal control processes (π^*, c^*) have the same expressions that those obtained by Jiao and Pham [9]. In these two papers, the optimal portfolio is given in terms of the value function of the primal problem or in terms of the solution of the primal BSDE as

$$\pi_t^* = \frac{1}{(1-p)\sigma_t}\Big(\frac{\widehat{\psi}_t}{\Psi_{t^-}} + \theta_t\Big) .$$

We have proved that $\frac{1}{1-p}\frac{\widehat{\psi}_t}{\Psi_{t^-}} = \frac{\widehat{\varphi}_t}{\Phi_{t^-}}$, hence, the two solutions have the same form. In particular, after τ, our setting is that one of a complete market, and our formula is rather standard. In particular, in the case where r^1 and θ^1 are deterministic, the investor is myopic, and the optimal portfolio is $\pi^* = \frac{1}{1-p}\frac{\theta}{\sigma}$. Before τ, the investor takes into account the fact that the interest rate will change, since the after default value function appears in the before default value function (this is the term $\Phi_t^1(t)$ in the associated BSDE).

6 Conclusion

In this paper, we have studied the problem of maximization of expected power utility of both terminal wealth and consumption in a market with a stochastic interest rate in a model where immersion holds. We have derived the optimal strategy solving the associated dual problem. Then, we have given the link between the value functions associated to the primal and dual problems, which has allowed to characterize the value function of the primal problem by a BSDE.

If one assumes that B is a $\mathbb{G}$-semi-martingale with canonical decomposition of B in $\mathbb{G}$ of the form

$$B_t = B_t^{\mathbb{G}} + \int_0^t \mu_s ds ,$$

with a bounded process μ and $B^{\mathbb{G}}$ a $\mathbb{G}$-Brownian motion, the price dynamics of the risky asset can be rewritten as follows

$$dS_t = S_t\Big((\nu_t + \sigma_t\mu_t)dt + \sigma_t dB_t^{\mathbb{G}}\Big) ,$$

where the coefficients ν and σ can be chosen $\mathbb{G}$-predictable and bounded. In this case, the e.m.m. can be written on the form (indeed, a predictable representation property holds for the pair $B^{\mathbb{G}}, M$)

$$dL_t = L_{t^-}(a_t dB_t^{\mathbb{G}} + \gamma_t dM_t)$$

and using the same methods and arguments, we can obtain similar results. The real difficulty is that one has to assume that the process μ is bounded, and we do not know any condition on τ which implies that fact.

Without any theoretical difficulty, we can generalize this paper to the case where there are several ordered changes of regime of interest rate.

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