NONLOCAL REFUGE MODEL WITH A PARTIAL CONTROL
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In this paper, we analyse the structure of the set of positive solutions of an heterogeneous nonlocal equation of the form:

\[
\int_{\Omega} K(x, y)u(y) \, dy - \int_{\Omega} K(y, x)u(x) \, dy + a_0 u + \lambda a_1(x) u - \beta(x) u^p = 0 \quad \text{in} \quad \Omega,
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded open set, \( K \in C(\mathbb{R}^n \times \mathbb{R}^n) \) is nonnegative, \( a, \beta \in C(\Omega) \) and \( \lambda \in \mathbb{R} \). Such type of equation appears in some studies of population dynamics where the above solutions are the stationary states of the dynamic of a spatially structured population evolving in a heterogeneous partially controlled landscape and submitted to a long range dispersal. Under some fairly general assumptions on \( K, a, \beta \) we first establish a necessary and sufficient criterium for the existence of a unique positive solution. Then we analyse the structure of the set of positive solutions \((\lambda, u_\lambda)\) with respect to the presence or absence of a refuge zone (i.e. \( \omega \) so that \( \beta|_\omega \equiv 0 \)).

1. Introduction

In this article we are interested in the positive bounded solutions of the nonlinear nonlocal equation

\[
\int_{\Omega} K(x, y)u(y) \, dy - k(x)u + a_0(x)u + \lambda a_1(x) u - \beta(x) u^p = 0 \quad \text{in} \quad \Omega,
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded open set, \( K \in C(\mathbb{R}^n \times \mathbb{R}^n) \) is nonnegative, \( k(x) := \int_{\Omega} K(y, x) \, dy \) \( \lambda \in \mathbb{R}, \), and \( a, \beta \) are continuous functions. Our aim is to describe the properties of the positive bounded solutions of (1.2), in terms of the properties of \( K, a, \beta \) and \( \lambda \). That is, we look for existence criteria of positive bounded solutions of (1.1) and we describe some bifurcation diagrams i.e. depending on \( a \) and \( \beta \) we analyse the properties of the curve \((\lambda, u_\lambda)\).

The study of these kind of problems finds its justification in the ecological problematics related to the erosion of Biodiversity. In particular, some recent studies have focused on a better understanding of the impact of some agricultural practises on non targeted species [1, 7, 21, 27, 28, 29]. Such problematic can be addressed through the analysis of the asymptotic behaviour of the positive solution of a reaction diffusion equation:

\[
\frac{\partial u(t, x)}{\partial t} = \int_{\Omega} K(x, y)u(t, y) \, dy - k(x)u(t, x) + a_0(x)u + \lambda a_1(x) u - \beta(x) u^p \quad \text{in} \quad \mathbb{R}^+ \times \Omega
\]

\[
u(0, x) = u_0(x) \quad \text{in} \quad \Omega
\]

where \( u \) represents a population density evolving in a partial controlled heterogeneous. Here the parameter \( \lambda \) is a control related to the practise and \( a_1 \) represents the region where the control is exerted.

In the literature the characterisation of the positive bounded solutions has been extensively studied for the elliptic equations

\[
\mathcal{E}[u] + a_0 u + \lambda a_1(x) u - \beta(x) u^p = 0 \quad \text{in} \quad \Omega,
\]

\[
u(x) = 0, \quad \text{in} \quad \partial \Omega.
\]

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where $E[u] := a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)$ is uniform elliptic [3, 4, 5, 6, 8, 9, 10, 17, 18, 26]. Nowadays, the structure of the positive bounded solutions $u_\lambda$ to (1.4)–(1.5) is well understood. More precisely, a positive bounded solution $u$ to (1.4)–(1.5) exists if and only if
\begin{equation}
\mu_1(E + a_0 + \lambda a_1, \Omega) < 0 < \mu_1(E + a_0 + \lambda a_1, \omega),
\end{equation}
where $\omega$ denotes the refuge zone, i.e. $\omega := \{x \in \Omega | \beta(x) = 0\}$ and $\mu_1(\Omega)$ denotes the first eigenvalue of the spectral problem $\mathcal{E}[\phi] + a_0\phi + \lambda a_1(\phi + \mu \phi = 0, \phi = 0$ on $\partial\Omega$. Depending on the properties of $\beta$ and $a_1$ a description of the curves $(\lambda, u_\lambda)$ can be found in [16, 17, 18, 26].

For nonlocal equations such as (1.1), less is known and the analysis of the existence, uniqueness and the bifurcation diagram have been only studied in particular situations [2, 12, 14, 15, 19, 22, 25, 31]. A large part of the literature is devoted to the existence of positive solution to (1.1) in situations where no refuge zone exists and for a fixed $\lambda$ [2, 12, 14, 15, 22, 31]. To our knowledge [19] is the first paper which considers a nonlocal logistic equation with a refuge zone and analyses the curves $(\lambda, u_\lambda)$. More precisely, the authors investigate the existence, uniqueness of a positive bounded solution of
\begin{align}
J \ast u - u + \lambda u - \beta(x) u^p &= 0 \quad \text{in } \Omega, \\
u &\equiv 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{align}
where $J$ is a symmetric density of probability. They prove that a positive solution of the above problem exists if and only if
\begin{equation}
\mu_1(J \ast u - u, \Omega) < \lambda < \mu_1(J \ast u - u, \omega).
\end{equation}
Moreover, they have showed that this solution is unique and have established the following asymptotic behaviours:
\begin{align}
limit_{\lambda \to \mu_1(J \ast u - u, \Omega)} u_\lambda(x) &= 0 \quad \text{for all } x \in \Omega, \\
limit_{\lambda \to \mu_1(J \ast u - u, \omega)} u_\lambda(x) &= +\infty \quad \text{for all } x \in \Omega.
\end{align}

These results have been recently extended to the more general equation (1.1) with a quadratic nonlinearity $(s(a(x) - b(x)s))$ and under some assumptions on the symmetry of the kernel $K$ and some extra conditions on $a$ and $\lambda$, see [25].

Here we address these questions of existence, uniqueness and the description of some bifurcation diagrams for a general kernel $K$ and with no restriction on the coefficients $a_i$, $\lambda$ and $\beta$.

In what follows we will always assume that the functions $a_i$ and $\beta$ satisfy:
\begin{equation}
\begin{cases}
a_i, \beta \in C(\Omega), a_1 \geq 0, \beta \geq 0 \\
\Omega \setminus \text{supp}(a_1) \text{ is a open set of } \mathbb{R}^n \\
K \in C(\mathbb{R}^n \times \mathbb{R}^n), K \geq 0
\end{cases}
\end{equation}
For the dispersal kernel, we will also require that $K$ satisfies:
\begin{equation}
\exists c_0 > 0, \epsilon_0 > 0 \quad \text{such that } \inf_{x \in \Omega} \left( \inf_{y \in B(x, \epsilon_0)} K(x, y) \right) > c_0.
\end{equation}

A typical example of such dispersal kernel is given by
\[
K(x, y) = J \left( \frac{x_1 - y_1}{g_1(y) h_1(x)}; \frac{x_2 - y_2}{g_2(y) h_2(x)}; \ldots; \frac{x_n - y_n}{g_n(y) h_n(x)} \right),
\]
with $J \in C(\mathbb{R}^n)$ continuous, $J(0) > 0$ and $0 < a_1 \leq g_1 \leq \beta_1$ and $0 \leq h_1 \leq \beta_1$. Such type of kernel have been recently introduced in [11] to model a nonlocal heterogeneous dispersal process. To simplify the presentation of our results, we also introduce the notation $L_{a_i}[u]$ for the continuous linear operator
\[
L_{a_i}[u] := \int_{\Omega} K(x, y) u(y) \, dy - k(x) u(x).
\]
In [19, 25] the analysis essentially relies on the existence of positive eigenfunction associated with a principal eigenvalue \( \mu_1 \) and a \( L^2 \) variational characterisation of \( \mu_1 \). However, such properties (existence of a positive eigenfunction and a \( L^2 \) variational characterisation of \( \mu_1 \)) does not hold for general kernels \( K \) and \( a_1 \) [13] and a new approach and characterisation of the principal eigenvalue has to be developed.

In the past few years, the spectral properties of nonlocal operators such as \( L_{a_1} + a \) have been intensively studied [2, 12, 13, 14, 15, 20, 22, 23, 24]. In particular a notion of generalized principal eigenvalue \( \mu_p \) of a linear operator \( L_{a_1} + a \) has been introduced [12, 15] and is defined by

\[
\mu_p(L_{a_1} + a) := \sup\{ \mu \in \mathbb{R} | \exists \phi \in C(\Omega), \phi > 0, \text{ so that } L_{a_1}[\phi] + (a + \lambda)\phi < 0 \}.
\]

\( \mu_p \) is called a generalized principal eigenvalue because \( \mu_p \) is not necessarily associated with a \( L^1 \) positive eigenfunction [12, 13, 24, 30]. Such notion has been successfully used to derive an optimal criterium for the existence of a unique positive solution of \((1.1)\) in absence of a refuge zone [12, 15].

Equipped with this notion of generalised eigenvalue, we can now state our results. We first present an optimal criterium for the existence of a unique positive bounded solution to \((1.1)\). Namely, we show

**Theorem 1.1.** Let \( K, a_i, \beta \) satisfy the assumptions \((1.9)-(1.10)\) and let \( \omega \) be the refuge set

\[
\omega := \{ x \in \Omega | \beta(x) = 0 \}.
\]

Then a positive continuous bounded solution \( u \) of \((1.1)\) exists if and only if

\[
\mu_p(L_{a_1} + a_0 + \lambda a_1) < 0 < \mu_p(L_{a_1} + a_0 + \lambda a_1),
\]

where we set \( \mu_p(L_{a_1} + a_0 + \lambda a_1) = - \sup \{ a_0 + \lambda a_1 \} \) when \( \omega = \emptyset \). Moreover the solution is unique.

Next we analyse the partially controlled problem \((1.1)\) i.e. we describe the set \( \{ \lambda, u_\lambda \} \) where \( u_\lambda \) is a positive bounded continuous solution to \((1.1)\). We start by describing \( \{ \lambda, u_\lambda \} \) in a case of the absence of a refuge zone. We prove the following

**Theorem 1.2.** Assume that \( K, a_i, \beta \) satisfy \((1.9)-(1.10)\). Assume further that \( \beta > 0 \) in \( \Omega \) then there exists \( \lambda^* \in [-\infty, \infty) \), so that for all \( \lambda > \lambda^* \) there exists a unique positive continuous solution \( u_\lambda \) to \((1.1)\). When \( \lambda^* \in \mathbb{R} \), there is no positive solution to \((1.1)\) for all \( \lambda \leq \lambda^* \). Moreover, we have the following trichotomy:

- \( \lambda^* = -\infty \) when \( \mu_p(L_{a_1} + a_0) < 0 \),
- \( \lambda^* \in [-\infty, \infty) \) when \( \mu_p(L_{a_1} + a_0) = 0 \),
- \( \lambda^* \in \mathbb{R} \) when \( \mu_p(L_{a_1} + a_0) > 0 \).

In addition, the map \( \lambda \to u_\lambda \) is monotone increasing and we have

\[
\forall x \in \Omega \lim_{\lambda \to +\infty} u_\lambda(x) = +\infty,
\]

\[
\forall x \in \Omega \lim_{\lambda \to \lambda^*} u_\lambda(x) = u_\infty(x),
\]

where \( u_\infty \equiv 0 \) on \( \Omega_1 := \{ x \in \Omega | a_1(x) > 0 \} \) and \( u_\infty \) is a nonnegative solution to

\[
\int_{\Omega \setminus \Omega_1} K(x, y)u(y)dy - k(x)u + a_0(x)u - \beta u^p = 0 \quad \text{in} \quad \Omega \setminus \Omega_1.
\]

Furthermore, \( u_\infty \) is non trivial when \( \mu_p(L_{a_1} + a_0) < 0 \).

Finally, we describe the set \( \{ \lambda, u_\lambda \} \) in the situation where a refuge zone exists. We prove the following

**Theorem 1.3.** Assume that \( K, a_i, \beta \) satisfy \((1.9)-(1.10)\). Assume further that \( \omega \neq \emptyset \), then there exists two quantities \( \lambda^*, \lambda^{**} \in [-\infty, +\infty] \) so that we have the following dichotomy:

- Either \( \lambda^{**} < \lambda^* \) and there exists no positive bounded solution to \((1.1)\).
- Or \( \lambda^* < \lambda^{**} \), and for all \( \lambda \in (\lambda^*, \lambda^{**}) \) there exists a unique positive bounded continuous solution to \((1.1)\). When \( \lambda^*, \lambda^{**} \in \mathbb{R} \), there is no positive bounded solution to \((1.1)\) for all \( \lambda \leq \lambda^* \) and for all \( \lambda \geq \lambda^{**} \). Moreover, the map \( \lambda \to u_\lambda \) is monotone increasing and we have
The assumption can be relaxed and we can get a full description of the curves when
$$\mu$$
and
$$(2.1)$$
analyse the bifurcation diagram of (1.1) in the presence of a refuge zone (Theorem 1.3).

Before going to the proofs of these results we would like to make some additional comments.

The paper is organised as follows. In a preliminary section we recall some known results on
$$\mu$$
and on the positive solution of a KPP equation. Then in Section 3 we prove the existence
criterium of Theorem 1.1. The proof of Theorem 1.2 is done in Section 4. Finally, in Section 5 we
analyse the bifurcation diagram of (1.1) in the presence of a refuge zone (Theorem 1.3).

2. Preliminaries

In this section, we recall some results on the principal eigenvalue of a linear nonlocal operator
and some known results about the KPP equation below
$$(2.1)$$
where
$$\mathcal{L}_\Omega[u] + f(x,u) = 0 \quad \text{in} \quad \Omega$$
and $$f(x,s)$$ is satisfying
$$f \in C(\Omega \times [0,\infty)) \quad \text{and is differentiable with respect to} \ s$$
$$f_u(\cdot,0) \in C(\Omega)$$
$$f(\cdot,0) \equiv 0 \quad \text{and} \quad f(x,s)/s \quad \text{is decreasing with respect to} \ s$$
there exists $$M > 0$$ such that $$f(x,s) \leq 0$$ for all $$s \geq M$$ and all $$x$$.

The simplest example of such a nonlinearity is
$$f(x,u) = u(a(x) - u),$$
where $$a(x) \in C(\Omega)$$.

It has been shown in [2, 12] that the existence of a positive solution of (2.1) is conditioned to
the sign of the principal eigenvalue $$\mu_p$$ of the linear operator $$\mathcal{L}_\Omega + f_u(x,0)$$ where $$\mu_p$$ is defined by

$$\mu_p(\mathcal{L}_\Omega + f_u(x,0)) := \sup_{\phi \in \mathcal{C}(\Omega)} \phi$$

That is to say

Theorem 2.1 ([2, 12]). Let $$\Omega$$ be a bounded open set and assume that $$K$$ and $$f$$ satisfy respectively
$$(1.9)$$–$$(1.10)$$ and $$(2.2)$$. Then there exists a unique positive continuous solution to (2.1) if and only if
$$\mu_p(\mathcal{L}_\Omega + f_u(x,0)) < 0.$$ Moreover, if $$\mu_p \geq 0$$ then any non negative uniformly bounded solution of (2.1)
is identically zero.

Also noted in [12] the principal eigenvalue is not always achieved. This means that there is
not always a positive continuous eigenfunction associated with $$\mu_p$$. However as shown in [13],
we can always associate a positive measure $$d\mu$$ with $$\mu_p$$. More precisely,
Theorem 2.2 ([13]). Let \( \Omega \) be an open bounded set and assume that \( K \) and \( f_u(x,0) \) satisfy the assumptions (1.9) and (2.2). Then there exists a positive measure \( d\mu \in M^+(\Omega) \), so that for any \( \phi \in C_c(\Omega) \) we have

\[
\int_{\Omega} \phi(x) \left( \int_{\Omega} K(x,y) d\mu(y) \right) dx + \int_{\Omega} \phi(x)(f_u(x,0) - k(x) + \mu_p) d\mu(x) = 0.
\]

Moreover, there exists a positive function \( \phi_p \in L^1(\Omega) \cap C(\Omega \setminus \Sigma) \) so that \( \int \phi_p > 0 \) and \( d\mu(x) = \phi_p(x) dx + d\mu_u(x) \) where \( d\mu_u(x) \) is a non-negative singular measure with respect to the Lebesgue measure whose support lies in the set \( \Sigma := \{ y \in \Omega | f_x(y,0) - k(y) = \sup_{x \in \Omega} (f_u(x,0) - k(x)) \} \).

As proved in [12, 24, 30], when \( \Omega \) is an open bounded set we can find a condition on the coefficients which guarantees that \( d\mu_u(x) \equiv 0 \) and the existence of a positive continuous eigenfunction. For example the existence of principal eigenfunction is guaranteed, if we assume that the function \( a(x) := f_u(x,0) - \int_{\Omega} K(y,x) dy \) satisfies

\[
\frac{1}{\sup_{\Omega} a - a(x)} \not\in L^1(\Omega') \quad \text{for some open bounded domain} \quad \Omega' \subset \Omega.
\]

For the existence of principal eigenfunction as remark in [15] we also have this useful criteria:

Proposition 2.3. Let \( \Omega \) be a bounded open set, then there exists a positive continuous eigenfunction associated to \( \mu_p \) if and only if \( \mu_p(\mathcal{L}_\Omega + a) < -\sup_{\Omega} a \).

Next we recall some properties of the principal eigenvalue \( \mu_p \) that we will constantly use along this paper:

**Proposition 2.4.**

(i) Assume \( \Omega_1 \subset \Omega_2 \), then for the two operators

\[
\mathcal{L}_{\Omega_1}[u] + a(x) u := \int_{\Omega_1} K(x,y) u(y) dy - k(x) u + a(x) u
\]

\[
\mathcal{L}_{\Omega_2}[u] + a(x) u := \int_{\Omega_2} K(x,y) u(y) dy - k(x) u + a(x) u
\]

respectively defined on \( C(\Omega_1) \) and \( C(\Omega_2) \) we have

\[
\mu_p(\mathcal{L}_{\Omega_1} + a(x)) \geq \mu_p(\mathcal{L}_{\Omega_2} + a(x)).
\]

(ii) Fix \( \Omega \) and assume that \( a_1(x) \geq a_2(x) \), then

\[
\mu_p(\mathcal{L}_{\Omega_1} + a_2(x)) \geq \mu_p(\mathcal{L}_{\Omega_1} + a_1(x)).
\]

Moreover, if \( a_1(x) \geq a_2(x) + \delta \) for some \( \delta > 0 \) then

\[
\mu_p(\mathcal{L}_{\Omega_1} + a_2(x)) > \mu_p(\mathcal{L}_{\Omega_1} + a_1(x)).
\]

(iii) \( \mu_p(\mathcal{L}_\Omega + a(x)) \) is Lipschitz continuous in \( a(x) \). More precisely,

\[
|\mu_p(\mathcal{L}_\Omega + a(x)) - \mu_p(\mathcal{L}_\Omega + b(x))| \leq \|a(x) - b(x)\|_{\infty}
\]

(iv) Assume \( \Omega_1 \subset \Omega_2 \), then for the two operators

\[
\mathcal{L}_{\Omega_1}[u] + a(x) u := \int_{\Omega_1} K(x,y) u(y) dy - k(x) u + a(x) u
\]

\[
\mathcal{L}_{\Omega_2}[u] + a(x) u := \int_{\Omega_2} K(x,y) u(y) dy - k(x) u + a(x) u
\]

respectively defined on \( C(\Omega_1) \) and \( C(\Omega_2) \). Assume that the corresponding principal eigenvalue are associated to a positive continuous principal eigenfunction. Then we have

\[
|\mu_p(\mathcal{L}_{\Omega_1} + a(x)) - \mu_p(\mathcal{L}_{\Omega_2} + a(x))| \leq C_0|\Omega_2 \setminus \Omega_1|,
\]

where \( C_0 \) depends on \( K \) and \( \phi_2 \).

(v) We always have the following estimate

\[
-\sup_{\Omega} \left( a(x) + \int_{\Omega} K(x,y) dy \right) \leq \mu_p(\mathcal{L}_{\Omega_1} + a) \leq -\sup_{\Omega} a.
\]
Proof:

We refer to [12] for the proofs of (i) – (iii) and (v), so we will be concerned only with (iv). Let us introduce the following quantity:

\[
\mu _p(L_\alpha + a) := \inf \{ \mu \in \mathbb{R} \mid \exists \phi \in C(\Omega), \phi > 0 \text{ so that } L_\alpha[\phi] + a\phi + \mu \phi \geq 0 \}.
\]

One can check that \( \mu _p(L_\alpha + a) = \mu _p'(L_\alpha + a) \). Indeed since there is a positive eigenfunction associated with \( \mu _p(L_\alpha + a) \) one has \( \mu _p'(L_\alpha + a) \leq \mu _p(L_\alpha + a) \) by definition of \( \mu _p'(L_\alpha + a) \). We obtain the equality by arguing as follows. Assume by contradiction that \( \mu _p(L_\alpha + a) > \mu _p'(L_\alpha + a) \). Then there exists \( \mu \) so that \( \mu _p'(L_\alpha + a) < \mu < \mu _p(L_\alpha + a) \) and from the definition of \( \mu _p \) and \( \mu _p' \) there exists two positive continuous functions \( \psi \) and \( \phi \) so that

\[
L_\alpha[\phi] + a(x)\phi + \mu \phi \geq 0,
\]

\[
\mu L_\alpha[\psi] + a(x)\psi + \mu \psi < 0.
\]

From the last inequalities we deduce that \( \psi > 0 \) in \( \bar{\Omega} \) and by setting \( w := \frac{\phi}{\psi} \) it follows that

\[
0 \leq \mu L_\alpha[\phi] + (a(x) + \mu)\phi = \mu L_\alpha[\phi] + (a(x) + \phi) \frac{\phi}{\psi} \psi,
\]

\[
\leq \mu L_\alpha[\phi] - \frac{\phi}{\psi}(x)L_\alpha[\psi],
\]

\[
\leq \int_\Omega K(x,y)\psi(y)(w(y) - w(x)) \, dy.
\]

Thus \( w \) cannot achieve a maximum in \( \bar{\Omega} \) without being constant. \( w \) being continuous in \( \bar{\Omega} \), it follows that \( \phi = cw \) for some positive constant \( c \). Thus we get the contradiction

\[
0 \leq \mu L_\alpha[\phi] + (a(x) + \mu)\phi = c(L_\alpha[\psi] + (a(x) + \mu)\psi) < 0.
\]

We are now in position to prove (iv). Let \( \phi_2 \) be the eigenfunction associated to \( \mu _p(L_{\alpha_2} + a(x)) \) normalized by \( \| \phi_2 \|_\infty = 1 \) and let us set \( C_0 := \frac{\| K(\cdot, \cdot) \|_\infty}{\min_{\Omega_2} \phi_2} \).

Now, let us show that \( (\phi_2, \mu _p(L_{\alpha_2} + a) + C_0|\Omega_2 \setminus \Omega_1|) \) is an adequate test function for \( \mu _p'(L_{\alpha_1} + a) \).

By a direct computation and by using the normalisation of \( \phi_2 \) we have

\[
L_{\alpha_1}[\phi_2] + (a + \mu _p(L_{\alpha_2} + a) + C_0|\Omega_2 \setminus \Omega_1|)\phi_2 = -\int_{\Omega_2 \setminus \Omega_1} K(x, y)\phi_2(y) \, dy + C_0|\Omega_2 \setminus \Omega_1|\phi_2,
\]

\[
\geq -\| K(\cdot, \cdot) \|_\infty|\Omega_2 \setminus \Omega_1| + C_0|\Omega_2 \setminus \Omega_1|\phi_2,
\]

\[
\geq \left( \frac{\phi_2}{\min_{\Omega_2} \phi_2} - 1 \right) \| K(\cdot, \cdot) \|_\infty|\Omega_2 \setminus \Omega_1|.
\]

Therefore \( \mu _p'(L_{\alpha_1} + a) \leq \mu _p(L_{\alpha_2} + a) + C_0|\Omega_2 \setminus \Omega_1| \), which combined with (i) and using that \( \mu _p'(L_{\alpha_1} + a) = \mu _p(L_{\alpha_1} + a) \) leads to

\[
| \mu _p(L_{\alpha_1} + a(x)) - \mu _p(L_{\alpha_2} + a(x)) | \leq C_0|\Omega_2 \setminus \Omega_1|.
\]

\[
\Box
\]

3. Optimal existence criterium

In this section we establish an optimal criterium for the existence of a positive continuous bounded solution to

\[
L_{\alpha_1}[u] + a(x)u - \beta(x)u^p = 0 \quad \text{in} \quad \Omega,
\]

when there exists \( \omega \subset \Omega \) so that \( \beta |\omega| \equiv 0 \). Note that (3.1) is a particular case of (2.1) with \( f(x, s) := a(x)s - \beta(x)s^p \). However, due to the presence of refuge zone (i.e. \( \beta |\omega| \equiv 0 \)) the function \( f(x, s) \) does not satisfy the assumptions (2.2) and the Theorem 2.1 does not apply. But we still have a complete characterisation of the existence of a bounded positive solution. Namely we can show the following Theorem:
Theorem 3.1. Assume that $K$, $a$ and $\beta$ satisfy (1.9)–(1.10). Assume further that there exists $\omega \subset \Omega$ so that $\beta_\omega \equiv 0$. Then there exists a bounded positive continuous solution to (3.1) if and only if
\[ \mu_p(L_\omega + a) > 0 > \mu_p(\mathcal{L}_\alpha + a), \]
where we set $\mu_p(L_\omega + a) = -\sup_\omega a$ when $\omega = \emptyset$.

Proof:
First let us assume that $\mu_p(L_\omega + a) \leq 0$, we will show that there is no positive bounded solution to (3.1). Let us suppose by contradiction that there exists $u$, a positive bounded solution to (3.1). So in $\omega$, $u$ satisfies
\[ L_\omega[u] + au = 0, \]
which implies that $\max_\omega a < 0$ and $u$ is continuous on $\omega$. Furthermore, we have
\[ L_\omega[u] + au = -\int_{\Omega \setminus \omega} K(x,y)u(y)dy \leq 0. \]
If $\hat{\omega} = \emptyset$ then we obtain easily a contradiction. Indeed in such case, we have $\mu_p(L_\omega + a) = -\sup_\omega a$ which leads to the contradiction
\[ 0 < -\sup_\omega a = \mu_p(L_\omega + a) \leq 0. \]
In the other situations, $\hat{\omega} \neq \emptyset$ and to obtain our desired contradiction we argue as follows. Since $\mu_p(L_\omega + a) \leq 0 < -\max_\omega a$, by Proposition 2.3 there exists a positive continuous eigenfunction associated with $\mu_p(L_\omega + a)$. As a consequence there exists also a positive continuous eigenfunction associated with $\mu_p(L_\omega^* + a)$ where $L_\omega^* + a$ is defined by
\[ L_\omega^*[\phi] + a\phi := \int_\omega K(x,y)\phi(y)dy - k(x)\phi + a(x)\phi. \]
We can easily check that $\mu_p(L_\omega + a) = \mu_p(L_\omega^* + a)$. Let us denote by $\phi^*$ the positive continuous principal eigenfunction associated with $\mu_p(L_\omega^* + a)$. Now by multiplying (3.2) by $\phi^*$ and then integrating over $\omega$, it follows
\[ \int_\omega \phi^*(x)L_\omega[u](x)dx + au\phi^*(x)dx \leq -c_0 \int_\omega \phi^* \left( \int_{\Omega \setminus \omega} K(x,y)dy \right). \]
By using Fubini’s Theorem in the above inequality we get the contradiction
\[ 0 \leq -\mu_p(L_\omega^* + a) \int_\omega \phi^* u \leq -c_0 \int_\omega \phi^* \left( \int_{\Omega \setminus \omega} K(x,y)dy \right) < 0. \]
Thus in both cases, there is no bounded solution to (3.1) when $\mu_p(L_\omega + a) \leq 0$.

Next we see that there is no positive bounded solution for (3.1) when $\mu_p(L_\alpha + a) \geq 0$. In this situation, with some modifications we can reproduce the argumentation developed in [12] (Subsection 6.2). Let us assume that a positive solution of (3.1) exists and let us denote $u$ this solution. We first observe that by following the argument developed in [2] we can see that $u$ is continuous in $\Omega$ and there exists positive constants $\delta$ and $c_0$ so that
\[ \left\{ \begin{array}{l}
\inf_\Omega u \geq c_0, \\
\inf_{x \in \Omega}(k(x) - a(x) + \beta(x)u^{p-1}) \geq \delta.
\end{array} \right. \]
From the monotone behaviour with respect to the $s$ of the function $g(x,s) := (a - \beta(x)s^{p-1})$, we deduce that $a - \beta u^{p-1} \leq a(x) - \beta c_0^{p-1} \leq a$. Now let us denote $\gamma(x) = a(x) - \beta(x)c_0^{p-1}$. By construction, we have $\gamma(x) \leq a(x)$ and we see by (ii) of Proposition 2.4 that
\[ \mu_p(L_\alpha + \gamma(x)) \geq \mu_p(L_\alpha + a(x)) \geq 0. \]
Moreover, since $u$ is a solution of (3.1), we have
\[ L_\alpha[u] + \gamma u \geq L_\alpha[u] + au - \beta u^p = 0, \]
with a strict inequality for any \( x \in \Omega \setminus \omega \).

We claim that

**Claim – 3.1.** There exists \( \delta > 0 \) and a positive continuous function \( \phi \) so that \( \inf_{\Omega} \phi > \delta \) and

\[
\mathcal{L}_\alpha[\phi] + \gamma \phi \leq 0.
\]

Assume for the moment that the Claim holds true then we get our desired contradiction by arguing as follows. Since \( \phi > \delta \) we can define the following quantity

\[
\tau^* := \inf\{ \tau > 0 | u \leq \tau \phi \}.
\]

Obviously, by proving that \( \tau^* = 0 \) we get the contradiction

\[
c_0 \leq u \leq 0.
\]

Assume by contradiction that \( \tau^* > 0 \) and let us denote \( w := \tau^* \phi - u \). By definition of \( \tau^* \), there exists \( x_0 \in \Omega \) such that \( \tau^* \phi(x_0) = u(x_0) > 0 \) and from (3.3) we see that \( w \) satisfies

\[
\mathcal{L}_\alpha[w] + \gamma w \leq 0.
\]

By evaluating the above expression at \( x_0 \), since \( w \geq 0 \) we see that

\[
0 \leq \int_{\Omega} K(x_0, y)w(y) \, dy \leq 0.
\]

Therefore, since \( K \) satisfies (1.10) we must have \( w(y) = 0 \) for almost every \( y \in \bar{\Omega} \). Thus, we end up with \( \tau^* \phi \equiv u \) and we get the following contradiction

\[
0 < \mathcal{L}_\alpha[u] + \gamma u = \mathcal{L}_\alpha[\tau^* \phi] + \gamma \tau^* \phi \leq 0 \quad \text{on} \quad \Omega \setminus \omega.
\]

Hence \( \tau^* = 0 \).

**Proof of the Claim:**

When \( \mu_p(\mathcal{L}_\alpha + \gamma) > 0 \) then by definition of the principal eigenvalue for all positive \( 0 < \mu < \mu_p(\mathcal{L}_\alpha + \gamma) \) there exists a positive continuous function \( \phi \) such that

\[
\mathcal{L}_\alpha[\phi] + \gamma \phi \leq -\mu \phi < 0.
\]

Observe that \( \phi \geq \delta \) for some positive \( \delta \) since otherwise there exists \( x_0 \in \bar{\Omega} \) so that \( \phi(x_0) = 0 \) and we get the contradiction

\[
0 < \mathcal{L}_\alpha[\phi](x_0) + \gamma(x_0)\phi(x_0) \leq 0.
\]

When \( \mu_p(\mathcal{L}_\alpha + \gamma) = 0 \) we argue as follows. By construction, \( a \geq \gamma \) and on \( \Omega \setminus \omega \) we have

(3.4) \[
\gamma < a \leq \sup_{\Omega} a \leq -\mu_p(\mathcal{L}_\alpha + a) \leq 0.
\]

And another hand on \( \omega \) since \( \beta_{1,\omega} \equiv 0 \), we have

\[
\mathcal{L}_\alpha[u] + a(\bar{\omega})u = 0,
\]

which leads to \( \sup_{\omega} a < 0 \). So on \( \omega \) we also have

(3.5) \[
\gamma \leq \sup_{\omega} a < 0.
\]

By combining (3.4) and (3.5) it follows that \( \sup_{\Omega} \gamma < 0 \). Now, since \( 0 = \mu_p(\mathcal{L}_\alpha + \gamma) < -\sup_{\Omega} \gamma \) we deduce from Proposition 2.3 that there exists a continuous positive principal eigenfunction \( \phi \) associated with \( \mu_p(\mathcal{L}_\alpha + \gamma) \). As above, we have \( \inf_{\Omega} \phi > \delta \) for some positive \( \delta \). \qed

Lastly, let us construct a positive bounded solution to (3.1) when the condition

(3.6) \[
\mu_p(\mathcal{L}_\alpha + a) > 0 > \mu_p(\mathcal{L}_\alpha + a)
\]

is satisfied. The uniqueness of this solution follows form a similar argumentation as in [2, 12], so we will omit the proof here.

From the condition \( 0 > \mu_p(\mathcal{L}_\alpha + a) \), by reproducing the argument in [12] we can find a positive bounded subsolution \( \phi_0 \) of the problem (3.1) so that \( \kappa \phi_0 \) is still a subsolution for any \( \kappa \) small and positive. Here the main difficulty is to find a positive supersolution \( \psi \). Indeed, due to the existence of a refuge zone, the large positive constants are not supersolutions of (3.1). We claim
**Claim — 3.2.** When the condition (3.6) is satisfied, then there exists \( \psi > 0, \psi \in C(\bar{\Omega}) \) supersolution of (3.1).

Note that by proving the claim we end the construction of the solution to (3.1). Indeed, since for \( \kappa \) small we have \( \kappa \phi \leq \psi \), by the monotone iterative scheme there exists a solution \( u \) to (3.1) so that \( \kappa \phi \leq u \leq \psi \).

Now, let us turn our attention to the proof of the Claim.

**Proof of the Claim:**

Let us first assume that \( \tilde{\omega} \neq \emptyset \).

In this situation, by following the argument in [12] (Subsection 6.1) we can introduce a regularisation \( a_\varepsilon \in C(\Omega) \) of \( a - k \) so that the following operator

\[
\mathcal{L}_{\varepsilon,\omega}[u] := \int_\omega K(x,y)u(y)dy + a_\varepsilon(x)u
\]

has a positive continuous principal eigenfunction. By continuity of \( \mu_p(\mathcal{L}_{\varepsilon,\omega}) \) with respect to \( a_\varepsilon \) ((iii) of Proposition 2.4) we can find \( \varepsilon \) small so that

\[
\mu_p(\mathcal{L}_{\varepsilon,\omega}) - \|a_\varepsilon - a + k\|_\infty \geq \frac{\mu_p(\mathcal{L}_\omega + a)}{2}.
\]

Let \( \varepsilon \) be fixed and let us denote \( \omega_\delta \) the following set

\[
\omega_\delta := \{x \in \Omega \mid d(x;\omega) < \delta\}.
\]

By continuity of the function \( \sup_{\omega_\delta} a_\varepsilon \) with respect to \( \delta \), there exists \( \delta_0 \) so that for all \( \delta \leq \delta_0 \) we have

\[
|\sup_{\omega_\delta} a_\varepsilon - \sup_{\omega_\delta} a\| \leq \frac{-\mu_p(\mathcal{L}_\omega + a_\varepsilon) - \sup_{\omega_\delta} a_\varepsilon}{2}.
\]

So, by (i) of Proposition 2.4, we deduce from the above inequality that we have for all \( \delta \leq \delta_0 \),

\[
\mu_p(\mathcal{L}_{\omega_\delta} + a_\varepsilon) \leq \mu_p(\mathcal{L}_\omega + a_\varepsilon) < -\sup_{\omega_\delta} a_\varepsilon.
\]

Therefore, thanks to Proposition 2.3 for all \( \delta \leq \delta_0 \) there exists a positive continuous eigenfunction associated with \( \mu_p(\mathcal{L}_{\varepsilon,\omega_\delta}) \).

By continuity of \( \mu_p(\mathcal{L}_{\varepsilon,\omega_\delta}) \) with respect to the domain ((iv) of Proposition 2.4) we achieve for \( \delta \) small enough, say \( \delta \leq \delta_1 \),

\[
\mu_p(\mathcal{L}_{\varepsilon,\omega_\delta}) \geq \mu_p(\mathcal{L}_{\omega_\delta}) \geq \mu_p(\mathcal{L}_{\omega_{\delta_1}}) \geq \|a_\varepsilon - a + k\|_\infty + \frac{\mu_p(\mathcal{L}_\omega + a)}{8}.
\]

By construction \( \tilde{\Omega} \setminus \omega_\delta \) and \( \omega_{\delta_1} \) are two disjoints bounded closed set, so by the Urysohn Lemma there exists a nonnegative continuous function \( \eta_1 \) such that \( 0 \leq \eta_1 \leq 1, \eta_1(x) = 1 \in \tilde{\Omega} \setminus \omega_\delta, \eta_1(x) = 0 \) in \( \omega_{\delta_1} \).

Let \( \psi_1, \psi_2 \) be the following continuous functions

\[
\psi_1 := \begin{cases} C_1 \eta_1 \text{ in } \Omega \setminus \omega_{\delta_1} \\ 0 \text{ elsewhere} \end{cases} \quad \psi_2 := \begin{cases} C_2 (1 - \eta_1) \Psi_\delta \text{ in } \omega_\delta \\ 0 \text{ elsewhere} \end{cases}
\]

where \( \Psi_\delta \) denotes the positive continuous eigenfunction associated with \( \mu_p(\mathcal{L}_{\varepsilon,\omega_\delta}) \) normalized by \( \|\Psi_\delta\|_\infty = 1 \) and \( C_1 \) and \( C_2 \) are positive constants to be specified later on.

Consider now the function \( \psi := \sup(\psi_1, \psi_2) \), we will prove that for well chosen \( C_1 \) and \( C_2, \psi \) is a supersolution of (3.1).

On \( \Omega \setminus \omega_{\delta_1} \), a short computation shows that for \( C_1 \) large

\[
\mathcal{L}_\Omega[\psi] + a_\psi - \beta \psi^p \leq \frac{C_1^{\frac{p+1}{p}}}{2} \left( \int_\Omega K(x,y) dy + a - \beta \frac{C_1^{\frac{p+1}{p}}}{} \right),
\]

\[
\leq \frac{C_1^{\frac{p+1}{p}}}{2} \left( \int_\Omega K(x,y) dy + a - \inf_{\Omega \setminus \omega_{\delta_1}} (\beta) C_1^{\frac{p+1}{p}} \right).
\]
By construction $\inf_{\Omega \cap \omega_\frac{1}{2}} (\beta) > 0$ and $\frac{\beta - 1}{2} > 0$, so for $C_1$ large enough we have on $\Omega \setminus \omega_\frac{1}{2}$

$$(3.8) \quad \mathcal{L}_\alpha[\psi] + a\psi - \beta \psi^p \leq 0.$$ 

Now, observe that for $C_2 \geq C_1$ large, on $\omega_\frac{1}{2}$ we have

$$\mathcal{L}_\alpha[\psi] + a\psi - \beta \psi^p \leq C_1 \int_{\Omega \setminus \omega_\delta} K(x, y) \, dy + C_2 \int_{\omega_\delta \setminus \omega_\frac{1}{2}} K(x, y) \, dy + C_2 \left( \int_{\omega_\delta} K(x, y) \psi(y) \, dy - k(x) \psi_\delta + a(x) \psi_\delta \right).$$

Since $\Psi_\delta > 0$ in $\omega_\delta$, we have

$$\mathcal{L}_\alpha[\psi] + a\psi - \beta \psi^p \leq C_2 \left( \frac{c_1}{C_2} K_1 + K_2 |\omega_\delta \setminus \omega_\frac{1}{2}| \right) + C_2 \left( \int_{\omega_\delta} K(x, y) \psi(y) \, dy - k(x) \psi_\delta + a(x) \psi_\delta \right),$$

where $K_1 := \sup_{x \in \Omega} \int_{\Omega} K(x, y) \, dy$ and $K_2 := \|K(\cdot, \cdot)\|_{\infty}$.

Recall that $\Psi_\delta$ is the eigenfunction associated to $\mu_p(\mathcal{L}_{\omega_\delta})$, so it follows that

$$\mathcal{L}_\alpha[\psi] + a\psi - \beta \psi^p \leq C_2 \left( \frac{c_1}{C_2} K_1 + K_2 |\omega_\delta \setminus \omega_\frac{1}{2}| + (a - k - a_\epsilon - \mu_p(\mathcal{L}_{\omega_\delta})) \psi_\delta \right)$$

which combined with (3.7) reduces to

$$\mathcal{L}_\alpha[\psi] + a\psi - \beta \psi^p \leq C_2 \left( \frac{c_1}{C_2} K_1 + K_2 |\omega_\delta \setminus \omega_\frac{1}{2}| - \frac{\mu_p(\mathcal{L}_{\omega_\delta})}{8} \psi_\delta \right).$$

By using that for all $\delta \in [0, \delta_0]$, the principal eigenfunction $\Psi_\delta$ associated to $\mu_p(\mathcal{L}_{\omega_\delta})$ is positive and continuous, we can see that

$$\inf_{\delta \in [0, \delta_0]} \inf_{\omega_\delta} \psi_\delta > c,$$

for some positive constant $c$. Moreover we can find $\delta$ small, say $\delta < \delta_1$ so that for all $\delta \leq \delta_1$

$$K_2 |\omega_\delta \setminus \omega_\frac{1}{2}| - \frac{\mu_p(\mathcal{L}_{\omega_\delta})}{8} \psi_\delta \leq - \frac{\mu_p(\mathcal{L}_{\omega_\delta}) + a}{16} c.$$ 

Thus for $\delta \leq \delta_1$, we achieve on $\omega_\frac{1}{2}$

$$\mathcal{L}_\alpha[\psi] + a\psi - \beta \psi^p \leq C_2 \left( \frac{c_1}{C_2} K_1 - \frac{\mu_p(\mathcal{L}_{\omega_\delta}) + a}{16} c \right).$$

Now by choosing $C_2 := C_1^{\frac{p+1}{p}}$ we have $\lim_{C_1 \to +\infty} \frac{C_1}{C_2} = 0$ since $p > 1$. So for $C_1$ large enough, say $C_1 \geq C_1^* := \left( \frac{2K_1}{\mu_p(\mathcal{L}_{\omega_\delta})^{\frac{p-1}{p}} a} \right)^{\frac{1}{p+1}}$ we achieve on $\omega_\delta$

$$(3.9) \quad \mathcal{L}_\alpha[\psi] + a\psi - \beta \psi^p \leq -cC_1^{\frac{p+1}{p}} \frac{\mu_p(\mathcal{L}_{\omega_\delta}) + a}{32} < 0.$$ 

Hence from (3.8) and (3.9) we see that the function $\psi$ is a positive continuous supersolution of (3.1).

Let us now assume that $\omega := \emptyset$. In this situation, we have $0 < \mu_p(\mathcal{L}_{\omega}) + a = \inf_{\omega} a$. By continuity of $a$ and $K$ there exists $\delta$ small so that

$$\sup_{x \in \omega_\delta} \int_{\omega_\delta} K(x, y) \, dy \leq \frac{\mu_p(\mathcal{L}_{\omega}) + a}{2} < -\sup_{\omega_\delta} a,$$

where as above $\omega_\delta := \{ x \in \Omega | d(x, \omega) < \delta \}$.

From the above inequality it follows from $(v)$ of Proposition 2.4 that $0 < \mu_p(\mathcal{L}_{\omega_\delta}) + a$. Let us consider $\beta_\delta := \beta \eta_1$, where $\eta_1$ is constructed above, then we have $\beta_\delta \leq \beta$ and $\omega_\frac{1}{2} := \{ x \in \Omega | d(x, \omega) < \delta \}$.
\( \Omega | \beta_0(x) = 0 \). By construction \( \omega^\circ \neq \emptyset \) and \( 0 < \mu_p(L_{\omega_1}) + a \leq \mu_p(L_{\omega_1} + a) \), therefore by using the above arguments there exists a positive continuous supersolution \( \psi \) to

\[
L_\mu[u] + \alpha(x)u - \beta_8u^p = 0 \quad \text{in} \quad \Omega.
\]

Thanks to \( \beta_8 \leq \beta \), we have

\[
L_\mu[\psi] + \alpha(x)\psi - \beta_8\psi^p \leq L_\mu[\psi] + \alpha(x)\psi - \beta_8\psi^p \leq 0 \quad \text{in} \quad \Omega
\]

and \( \psi \) is our desired supersolution.

\( \square \)

4. The partially controlled problem: The KPP case

In this section we analyse the dependence in \( \lambda \) of the positive continuous solutions to (1.1) in absence of a refuge zone and we prove the Theorem 1.2 that we recall below. More precisely, we look for positive continuous solution of the partially controlled problem:

(4.1)  \[
\int_\Omega K(x,y)u(y)\,dy - k(x)u(x) + a_0(x)u + \lambda a_1(x)u - \beta u^p = 0
\]

when \( \beta > 0 \) and \( \lambda \in \mathbb{R} \).

In absence of a refuge zone, we can show that there exists a critical value \( \lambda^* \) characterising completely the existence/non existence of a positive stationary solution. More precisely we have,

Theorem 4.1. Assume that \( K, a_i \) and \( \beta \) satisfy (1.9)–(1.10). Assume further that \( \beta > 0 \) in \( \Omega \) then there exists \( \lambda^* \in [-\infty, \infty) \), so that for all \( \lambda > \lambda^* \) there exists a unique positive continuous solution \( u_\lambda \) to (4.1). When \( \lambda^* \in \mathbb{R} \), there is no positive solution to (1.1) for all \( \lambda \leq \lambda^* \). Moreover, we have the following trichotomy:

\begin{itemize}
  \item \( \lambda^* = -\infty \) when \( \mu_p(L_{a_1} + a_0) < 0 \),
  \item \( \lambda^* \in [-\infty, \infty) \) when \( \mu_p(L_{a_1} + a_0) = 0 \),
  \item \( \lambda^* \in \mathbb{R} \) when \( \mu_p(L_{a_1} + a_0) > 0 \).
\end{itemize}

In addition, the map \( \lambda \to u_\lambda \) is monotone increasing and we have

\[
\forall x \in \Omega \quad \lim_{\lambda \to +\infty} u_\lambda(x) = +\infty,
\]

\[
\forall x \in \Omega \quad \lim_{\lambda \to -\lambda^*} u_\lambda(x) = u_\infty(x),
\]

where \( u_\infty \equiv 0 \) on \( \Omega_1 := \{ x \in \Omega | a_1(x) > 0 \} \) and \( u_\infty \) is a nonnegative solution to

\[
\int_{\Omega \setminus \Omega_1} K(x,y)u(y)\,dy - k(x)u(x) + a_0(x)u - \beta u^p = 0 \quad \text{in} \quad \Omega \setminus \Omega_1.
\]

Furthermore, \( u_\infty \) is non trivial when \( \mu_p(L_{a_1} + a_0) < 0 \).

Proof:

In absence of a refuge zone, we observe that a problem (4.1) is a particular case of the KPP equation (2.1) where the nonlinearity \( f \) is given by \( f(x,s) := a_0s + \lambda a_1s - \beta s^p \). Therefore by the Theorem 2.1, for each \( \lambda \in \mathbb{R} \) the existence of a positive solution to (4.1) is conditioned by the sign of \( \mu_p(L_{a_1} + a_0 + \lambda a_1) \).

First let us observe that for \( \lambda > \frac{\|k\|_{L^\infty} - \|a_0\|_{L^\infty}}{\|a_1\|_{L^\infty}} \) we have \( \sup_{x \in \Omega}(a_0(x) + \lambda a_1(x) - k(x)) > 0 \) and by (v) of Proposition 2.4 we have \( \mu_p(L_{a_1} + a_0 + \lambda a_1) \leq -\sup_{x \in \Omega}(a_0(x) + \lambda a_1(x) - k(x)) < 0 \). Therefore by Theorem 2.1, there exists a positive solution to (4.1) for all \( \lambda > \frac{\|k\|_{L^\infty} - \|a_0\|_{L^\infty}}{\|a_1\|_{L^\infty}} \). Let us consider the following set \( \{ \lambda | \mu_p(L_{a_1} + a_0 + \lambda a_1) = 0 \} \). When \( \{ \lambda | \mu_p(L_{a_1} + a_0 + \lambda a_1) = 0 \} \neq \emptyset \), by monotonicity of \( \mu_p \) with respect to \( \lambda \) (of Proposition 2.4), we can see that \( \{ \lambda | \mu_p(L_{a_1} + a_0 + \lambda a_1) = 0 \} \) is bounded from above. Therefore we can define \( \lambda^* \in [-\infty, +\infty) \) by the following formula

\[
\lambda^* := \sup \{ \lambda | \mu_p(L_{a_1} + a_0 + \lambda a_1) = 0 \},
\]

where we set \( \lambda^* = -\infty \) when \( \{ \lambda | \mu_p(L_{a_1} + a_0 + \lambda a_1) = 0 \} = \emptyset \).
By construction, thanks to Theorem 2.1 for all $\lambda > \lambda^*$ there exists a unique positive continuous solution to (4.1) and when $\lambda^* \in \mathbb{R}$, there is no positive solution to (1.1) for all $\lambda \leq \lambda^*$.

Before proving the trichotomy, let us look at the asymptotic behaviours with respect to $\lambda$ of the unique solution $u_\lambda$. First let us observe that the map $\lambda \mapsto u_\lambda$ is monotone non decreasing. Indeed, thanks to the nonnegativity of $a_1$, for any $\lambda \geq \lambda'$, the continuous bounded function $u_{\lambda'}$ is a subsolution of the problem
\begin{equation}
\int_{\Omega} K(x, y) u(y) \, dy - k(x) u(x) + a_0(x) u + \lambda a_1(x) u - \beta u^p = 0. \tag{4.2}
\end{equation}

Observe that any large constant $M$ is a super-solution of (4.2). Therefore by taking $M$ large enough we have $u_{\lambda'} \leq M$ and by the monotone iteration scheme we can construct a positive bounded solution of (4.2) which satisfies $u_{\lambda'} \leq u \leq M$. We conclude by using the uniqueness of the solution of problem (4.2). Hence, $u_{\lambda'} \leq u_\lambda \equiv u$.

The asymptotic behaviour of $u_\lambda$ when $\lambda \to +\infty$ is obtained by establishing a bound from below for the solution $u_\lambda$ when $\lambda \to +\infty$. More precisely we show that for all $x \in \Omega_1$ we have for $\lambda$ large enough
\begin{equation}
|u_\lambda(x)| \geq \left( \frac{\lambda a_1 + a_0 - k(x)}{\sup_{\Omega} \beta} \right)^{\frac{1}{p-1}}. \tag{4.3}
\end{equation}
Indeed from (4.1) using that $u_\lambda$ is non negative we have
\begin{equation*}
\beta(x) u_\lambda^p(x) \geq |k(x) + a_0(x) + \lambda a_1(x)| u_\lambda.
\end{equation*}
Thus for $x \in \Omega_1$ (4.3) holds for $\lambda$ large enough. From (4.3) we get trivially that for all $x \in \Omega_1$
\begin{equation*}
\lim_{\lambda \to +\infty} u_\lambda(x) \geq \lim_{\lambda \to +\infty} \left( \frac{\lambda a_1 + a_0 - k(x)}{\sup_{\Omega} \beta} \right)^{\frac{1}{p-1}} = +\infty.
\end{equation*}
So for $x \in \Omega \setminus \Omega_1$ so that $|B_{c_0}(x) \cap \Omega_1| > 0$ where $c_0$ is given by (1.10) we conclude that
\begin{equation*}
\lim_{\lambda \to +\infty} \int_{B_{c_0}(x) \cap \Omega_1} u_\lambda(y) \, dy = +\infty.
\end{equation*}

Therefore from (1.10), (4.1) and $u_{\lambda} \geq 0$ we deduce that
\begin{equation*}
\left( \beta(x) u_\lambda^{p-1} + k(x) \right) u_\lambda(x) \geq \int_{\Omega} K(x, y) u_\lambda(y) \, dy \geq c_0 \int_{B_{c_0}(x) \cap \Omega_1} u_\lambda(y) \, dy
\end{equation*}
which leads to
\begin{equation*}
\lim_{\lambda \to +\infty} u_\lambda \left( \beta(x) u_\lambda^{p-1} + k(x) \right) \geq \lim_{\lambda \to +\infty} c_0 \int_{B_{c_0}(x) \cap \Omega_1} u_\lambda(y) \, dy = +\infty \quad \text{for all} \quad x \in \bigcup_{z \in \Omega_1} B_{c_0}(z).
\end{equation*}
The later implies that
\begin{equation*}
\lim_{\lambda \to +\infty} u_\lambda(x) = +\infty \quad \text{for all} \quad x \in \bigcup_{z \in \Omega_1} B_{c_0}(z).
\end{equation*}

By repeating the above argument with $\bigcup_{z \in \Omega_1} B_{c_0}(z)$ instead $\Omega_1$, we show that
\begin{equation*}
\lim_{\lambda \to +\infty} u_\lambda(x) = +\infty \quad \text{for all} \quad x \in \bigcup_{z \in \Omega_1} B_{2c_0}(z).
\end{equation*}

By a finite iteration of the above argumentation, we get
\begin{equation*}
\lim_{\lambda \to +\infty} u_\lambda(x) = +\infty \quad \text{for all} \quad x \in \bar{\Omega}.
\end{equation*}

Let us now deal with the limit of $u_\lambda$ when $\lambda \to \lambda^*$. First let us assume that $\lambda^{*+} \in \mathbb{R}$. In this situation by using the positivity of $u_\lambda$ and the monotonicity of $u_\lambda$ with respect to $\lambda$, we deduce that $u_\lambda$ converges pointwise to $u_{\lambda^{*+}}$ when $\lambda \to \lambda^{*+}$. Moreover thanks to the Lebesgue
dominated convergence Theorem by passing to the limit in (4.1), we see that \( u_{\lambda^*} \) is a non negative solution of (4.1) with \( \lambda = \lambda^{*+} \). Therefore by Theorem 2.1 we deduce that \( u_{\lambda^*} = 0 \) since \( \mu_p(\mathcal{L}_\alpha + a_0 + \lambda a_1) = 0 \). Thus in this case
\[
\lim_{\lambda \to \lambda^{*+}} u_\lambda(x) = 0 \quad \text{for all} \quad x \in \Omega.
\]
Lastly assume that \( \lambda^{*+} = -\infty \). Again by using the positivity of \( u_\lambda \) and the monotonicity of \( u_\lambda \) with respect to \( \lambda \), we deduce that \( u_\lambda \) converges pointwise to \( u_{\infty} \) when \( \lambda \to -\infty \). Now observe that by the monotonicity of \( u_\lambda \), we have for all \( \lambda \leq 0, u_\lambda \leq M_0 := \|u_0\|_\infty \) and
\[
a_1(\lambda|u_\lambda| \leq C_0,
\]
where
\[
C_0 := M_0 \left\| \int_\Omega K(\cdot, y)dy + k + a_0 \right\|_\infty + \|\beta\||M_0|^p.
\]
Therefore for \( x \in \Omega_1 \), we deduce that
\[
0 \leq u_{\infty}(x) = \lim_{\lambda \to -\infty} u_\lambda(x) \leq \lim_{\lambda \to -\infty} \frac{C_0}{a_1(\lambda|u_\lambda|} = 0.
\]
By passing to the limit in the equation (4.1), thanks to the Lebesgue dominated convergence Theorem we see that \( u_{\infty} \) satisfies the equation below
\[
(4.4) \quad \mathcal{L}_{\alpha, \Omega_1}[u] + a_0(x)u + \lambda \alpha_1(x)u - \beta u^p = 0.
\]
By Theorem 2.1, the existence of a positive solution to the above equation is governed by the sign of \( \mu_p(\mathcal{L}_{\alpha, \Omega_1} + a_0) \). Therefore when \( \mu_p(\mathcal{L}_{\alpha, \Omega_1} + a_0) < 0 \) there is a unique positive solution whereas for \( \mu_p(\mathcal{L}_{\alpha, \Omega_1} + a_0) \geq 0 \) there is none. In the later case, we deduce that
\[
\lim_{\lambda \to -\infty} u_\lambda(x) = 0 \quad \text{for all} \quad x \in \Omega.
\]
Now let us look more closely at the properties of \( \lambda^* \) and prove the trichotomy
(1) \( \lambda^* = -\infty \) when \( \mu_p(\mathcal{L}_{\alpha, \Omega_1} + a_0) < 0 \),
(2) \( \lambda^* \in [-\infty, \infty) \) when \( \mu_p(\mathcal{L}_{\alpha, \Omega_1} + a_0) = 0 \),
(3) \( \lambda^* \in \mathbb{R} \) when \( \mu_p(\mathcal{L}_{\alpha, \Omega_1} + a_0) > 0 \).

Case 1: \( \mu_p(\mathcal{L}_{\alpha, \Omega_1} + a_0) < 0 \). In this situation, observe that by (i) of Proposition 2.4, we have for all \( \lambda \)
\[
0 > \mu_p(\mathcal{L}_{\alpha, \Omega_1} + a_0) = \mu_p(\mathcal{L}_{\alpha, \Omega_1} + a_0 + \lambda a_1) \geq \mu_p(\mathcal{L}_\alpha + a_0 + \lambda a_1).
\]
Therefore, thanks to Theorem 2.1 there exists a positive non trivial solution to (4.1) for all \( \lambda \in \mathbb{R} \). Thus \( \lambda^* = -\infty \).

Case 2: \( \mu_p(\mathcal{L}_{\alpha, \Omega_1} + a_0) = 0 \). In this situation, by monotonicity of \( \mu_p \) with respect to \( \lambda \) (ii) of Proposition 2.4) and (i) of Proposition 2.4 either \( \mu_p(\mathcal{L}_\alpha + a_0 + \lambda a_1) < 0 \) for all \( \lambda \leq 0 \) or there exists \( \lambda_0 \leq 0 \) so that \( \mu_p(\mathcal{L}_\alpha + a_0 + \lambda_0 a_1) = 0 \). In the first situation, as above there exists a positive solution to (4.1) for any \( \lambda \) and \( \lambda^* = -\infty \). In the other case, \( \lambda^* \geq \lambda_0 \) and \( \lambda^* \in \mathbb{R} \).

Case 3: \( \mu_p(\mathcal{L}_{\alpha, \Omega_1} + a_0) > 0 \). In this last situation, we claim that

Claim - 4.1.
\[
\lim_{\lambda \to \lambda^*} \inf \mu_p(\mathcal{L}_\alpha + a_0 + \lambda a_1) > 0.
\]
Assume the claim holds true then this implies that \( \{\lambda \mid \mu_p(\mathcal{L}_\alpha + a_0 + \lambda a_1) = 0\} \) is non empty and therefore \( \lambda^* \in \mathbb{R} \). Indeed, since \( \mu_p(\mathcal{L}_\alpha + a_0 + \lambda a_1) < 0 \) for any \( \lambda > \frac{\|K\|_\infty}{\|a_1\|_\infty} \) and by the claim there exists \( \lambda \) so that \( \mu_p(\mathcal{L}_\alpha + a_0 + \lambda a_1) > 0 \), by continuity of \( \mu_p \) with respect to \( \lambda \) there exists a \( \lambda < \lambda_0 < \frac{\|K\|_\infty - \|a_0\|_\infty}{\|a_1\|_\infty} \) so that \( \mu_p(\mathcal{L}_\alpha + a_0 + \lambda_0 a_1) = 0 \).

Proof of the Claim:
The proof of this claim relies on the construction of an adequate test function. By arguing as in the proof of Claim 3.2 we can introduce a regularisation $a_\varepsilon$ of $a_0 - k$ so that the following operator

$$ L_{\cdot,\cdot,m_\varepsilon}[u] := \int_{\Omega \times \Omega} K(x, y) u(y) dy + a_\varepsilon(x) u $$

has a positive continuous principal eigenfunction. By continuity of $\mu_p(L_{\cdot,\cdot,m_\varepsilon})$ with respect to $a_\varepsilon$ \((iii)\) of Proposition 2.4) we can find $\varepsilon$ small so that

$$ \mu_p(L_{\cdot,\cdot,m_\varepsilon}) - \|a_\varepsilon - a_0 + k\|_\infty \geq \frac{\mu_p(L_{\cdot,\cdot,m_\varepsilon} + a_0)}{2}. $$

Let $\varepsilon$ be fixed and let $\Omega_\delta$ be the set

$$ \Omega_\delta := \{x \in \Omega_1 \mid d(x; \partial \Omega_1) > \delta\}. $$

As in the proof of the Claim 3.2, by continuity of $\sup_{\Omega \times \Omega_\delta}$ and $\mu_p(L_{\cdot,\cdot,m_\varepsilon})$ with respect to the domain we achieve for $\delta$ small enough, say $\delta \leq \delta_0$,

$$ \mu_p(L_{\cdot,\cdot,m_\varepsilon}) \geq \mu_p(L_{\cdot,\cdot,m_\varepsilon}) \geq \mu_p(L_{\cdot,\cdot,m_\varepsilon}) \geq \|a_\varepsilon - a_0 + k\|_\infty + \frac{\mu_p(L_{\cdot,\cdot,m_\varepsilon} + a_0)}{8}. \tag{4.5} $$

By construction $\bar{\Omega} \setminus \Omega_2$ and $\Omega_\delta$ are two disjoint bounded closed set, so by the Urysohn Lemma there exists a nonnegative continuous function $\eta_1$ such that $0 \leq \eta_1 \leq 1, \eta_1(x) = 1$ in $\Omega \setminus \Omega_2, \eta_1(x) = 0$ in $\Omega_\delta$.

Let $\psi_1, \psi_2$ be the following continuous functions

$$ \psi_1 := \begin{cases} \Psi_\delta \eta_1 & \text{in } \Omega \setminus \Omega_\delta \\ 0 & \text{elsewhere}, \end{cases} \quad \psi_2 := \begin{cases} c_0 \eta_2 & \text{in } \Omega_1 \\ 0 & \text{elsewhere}. \end{cases} $$

where $\Psi_\delta$ is the positive continuous eigenfunction associated with $\mu_p(L_{\cdot,\cdot,m_\varepsilon})$ normalized by $\|\Psi_\delta\|_\infty = 1$ and $c_0$ is positive constant to be specified later on. Consider now the function $\psi := \sup(\psi_1, \psi_2)$ and let $\gamma$ be a positive constant to be fixed later on. We will prove that for $\gamma, \delta, \lambda$ and $c_0$ well chosen the function $\psi$ is an adequate test function for $L_{\alpha_1 + a_0 + \lambda a_1 + \gamma}$. So let us compute

$$ L_{\alpha_1}[\psi] + a_\varepsilon \psi + \lambda a_1 \psi + \gamma \psi \leq \int_{\Omega \setminus \Omega_2} K(x, y) \Psi_\delta(y) dy + \|K(\cdot, \cdot)\|_\infty \left(\Omega_2 \setminus \Omega_\delta + c_0 |\Omega_\delta|\right) + a_\varepsilon \Psi_\delta + \|a_\varepsilon + k - a_0\|_\infty \Psi_\delta + \lambda a_1 \Psi_\delta + \gamma \Psi_\delta. $$

Therefore by (4.5) we see that

$$ L_{\alpha_1}[\psi] + a_\varepsilon \psi + \lambda a_1 \psi + \gamma \psi \leq \left(-\frac{\mu_p(L_{\cdot,\cdot,m_\varepsilon} + a_0)}{8} + \gamma\right) \Psi_\delta + \|K(\cdot, \cdot)\|_\infty \left(\Omega_2 \setminus \Omega_\delta + c_0 |\Omega_\delta|\right). \tag{4.6} $$

Let $m_0$ be the constant

$$ m_0 := \inf_{\delta \in [0, \delta_0]} \left( \inf_{x \in \Omega \setminus \Omega_\delta} \Psi_\delta(x) \right). $$

We have $m_0 > 0$ since for all $\delta \in [0, \delta_0]$ $\Psi_\delta$ is positive and continuous in $\Omega \setminus \Omega_\delta$.

Now let us fix $\gamma := \frac{\mu_p(L_{\cdot,\cdot,m_\varepsilon} + a_0)}{16}$ and choose $\delta$ and $c_0$ such that

$$ |\Omega_2 \setminus \Omega_\delta| \leq \frac{m_0 \mu_p(L_{\cdot,\cdot,m_\varepsilon} + a_0)}{64\|K(\cdot, \cdot)\|_\infty}, \tag{4.7} $$

$$ c_0 \leq \frac{m_0 \mu_p(L_{\cdot,\cdot,m_\varepsilon} + a_0)}{64\|K(\cdot, \cdot)\|_\infty |\Omega_1|}. \tag{4.8} $$
By combining (4.6), (4.7) and (4.8) we see that
\[
\left( -\frac{\mu p(\mathcal{L}_\Omega U + a_0)}{8} + \gamma \right) \psi \delta + \|K(\cdot, \cdot)\|_{\infty} \left( |\Omega| - |\Omega_\delta| + c_0 |\Omega_\delta| \right) \leq 0.
\]

Therefore on \( \Omega \setminus \Omega_\delta \), we achieve for all \( \lambda \leq 0 \)
\[\mathcal{L}_{\Omega}(\psi) + (a_0 + \lambda a_1 + \gamma)\psi \leq 0. \tag{4.9}\]

Now on \( \Omega_\delta \) we have by construction
\[
\mathcal{L}_{\Omega}(\psi) + (a_0 + \lambda a_1 + \gamma)\psi \leq \|K(\cdot, \cdot)\|_{\infty} |\Omega| + \|a_0\|_{\infty} + \lambda c_0 \left( \inf_{\Omega} a_1 \right).
\]

Since \( a_1 > 0 \) in \( \Omega_\delta \), by choosing \( \lambda \leq -\frac{\|K(\cdot, \cdot)\|_{\infty} |\Omega| + \|a_0\|_{\infty} + \gamma}{\inf_{\Omega} a_1} \) we get
\[
\mathcal{L}_{\Omega}(\psi) + (a_0 + \lambda a_1 + \gamma)\psi \leq 0 \quad \text{in} \quad \Omega_\delta. \tag{4.10}
\]

Hence from (4.9) and (4.10) we see that for \( \lambda \) negative enough, the function \((\psi, \gamma)\) is an adequate test function for the operator \( \mathcal{L}_{\Omega} + a_0 + \lambda a_1 \). That is: \( \psi \) is a positive continuous function on \( \Omega \) which satisfies
\[
\mathcal{L}_{\Omega}(\psi) + (a_0 + \lambda a_1 + \gamma)\psi \leq 0.
\]

So by definition of \( \mu p(\mathcal{L}_{\Omega} + a_0 + \lambda a_1) \) we deduce that for \( \lambda \) negative enough we have \( \mu p(\mathcal{L}_{\Omega} + a_0 + \lambda a_1) \geq \gamma > 0 \).

\[\Box\]

5. The partially controlled problem: The refuge case

In this Section, we analyse (1.1) in the presence of a refuge zone, i.e. when there exists \( \omega \subset \Omega \) so that \( \beta_\omega \equiv 0 \). In a presence of a refuge zone, the analysis of (1.1) is more involved and the characterisation of the existence/non-existence of a positive solution of (1.1) cannot always be summarised to a single critical value \( \lambda^* \). In this situation, we prove the Theorem 1.3 that we recall below:

**Theorem 5.1.** Assume that \( K, a, \) and \( \beta \) satisfy (1.9)–(1.10). Assume further that \( \omega \neq \emptyset \), then there exists two quantities \( \lambda^*, \lambda^{**} \in [-\infty, +\infty) \) so that we have the following dichotomy:

- Either \( \lambda^{**} \leq \lambda^* \) and there exists no positive bounded solution to (1.1).
- Or \( \lambda^* < \lambda^{**} \), and for all \( \lambda \in (\lambda^*, \lambda^{**}) \) there exists a unique positive bounded continuous solution to (1.1). When \( \lambda^*, \lambda^{**} \in \mathbb{R} \), there is no positive bounded solution to (1.1) for all \( \lambda \leq \lambda^* \) and for all \( \lambda \geq \lambda^{**} \). Moreover, the map \( \lambda \rightarrow u_\lambda \) is monotone increasing and we have

\[(i) \quad \lim_{\lambda \rightarrow \lambda^{**}-} \|u_\lambda\|_{\infty, \omega} = +\infty,
\]

where \( \|u_\lambda\|_{\infty, \omega} := \sup_{x \in \omega} |u_\lambda(x)| \).

\[(ii) \text{If } \mu p(\mathcal{L}_{\omega} + a_0 + \lambda^{**} a_1) \text{ is an eigenvalue in } L^1(\omega) \text{ or } \lambda^{**} = +\infty \text{ then}
\]

\( \forall x \in \Omega, \lim_{\lambda \rightarrow \lambda^{**}-} u_\lambda(x) = +\infty \).

\[(iii) \text{For all } x \in \Omega \text{ we have } \lim_{\lambda \rightarrow \lambda^{**}+} u_\lambda(x) = u_{\infty}(x), \text{ where } u_{\infty} \text{ is a function satisfying on } \Omega_i := \{x \in \Omega | a_1(x) > 0 \} \equiv 0 \text{ and } u_{\infty} \text{ is a nonnegative solution to}
\]

\[
\int_{\Omega \setminus \Omega_1} K(x, y)u(y)dy - k(x)u + a_0(x)u - \beta u^p = 0 \quad \text{in} \quad \Omega \setminus \Omega_1.
\]

Furthermore, \( u_{\infty} \) is non trivial when \( \mu p(\mathcal{L}_{\Omega \setminus \Omega_1} + a_0) < 0 \).
Proof:

Thanks to Theorem 3.1 the existence of a positive unique bounded solution to (1.1) in presence of a refuge zone is conditioned by the following inequality

\[ \mu_p(\mathcal{L}_a + a_0 + \lambda a_1) > 0 > \mu_p(\mathcal{L}_a + a_0 + \lambda a_1). \]

Let us introduce the following quantities:

\[ \lambda^* := \sup \{ \lambda \mid \mu_p(\mathcal{L}_a + a_0 + \lambda a_1) \geq 0 \}, \]

\[ \lambda^{**} := \inf \{ \lambda \mid \mu_p(\mathcal{L}_a + a_0 + \lambda a_1) \leq 0 \}. \]

We can see that the description of the set of positive bounded solutions of (1.1) is then equivalent to show whether or not we have \( \lambda^* < \lambda^{**} \). To answer this question, we analyse separately the three different situations:

1. \( \omega \subset \Omega \setminus \Omega_1 \),
2. \( \omega \subset \subset \Omega_1 \),
3. \( \omega \not\subset \Omega \setminus \Omega_1 \) and \( \omega \not\subset \Omega_1 \).

Let us start with the analysis the first situation.

**Case 1:** \( \omega \subset \Omega \setminus \Omega_1 \). In this situation, since \( \omega \subset \Omega \setminus \Omega_1 \), we have \( \mu_p(\mathcal{L}_a + a_0 + \lambda a_1) = \mu_p(\mathcal{L}_a + a_0) \). So from (5.1) we see that for all \( \lambda \) there is no bounded solution to (4.1) when \( \mu_p(\mathcal{L}_a + a_0) \leq 0 \) whereas the existence of a bounded solution will be conditioned only by the sign of \( \mu_p(\mathcal{L}_a + a_0 + \lambda a_1) \) when \( \mu_p(\mathcal{L}_a + a_0) > 0 \). In the later case, the analysis of the Section 4 can be reproduced, so we get \( \lambda^* \in [-\infty, \infty) < \lambda^{**} = +\infty \).

**Case 2:** \( \omega \subset \subset \Omega_1 \). In this situation, since \( \omega \subset \Omega_1 \), by (v) of Proposition 2.4 we can see that for some positive constant \( C \)

\[ -\lambda \sup \omega a_1 - C \leq \mu_p(\mathcal{L}_a + a_0 + \lambda a_1) \leq C - \lambda \sup \omega a_1. \]

Therefore, we see that \( \lambda^{**} \in \mathbb{R} \) and by definition of \( \lambda^{**} \) and (i) of Proposition 2.4 we have

\[ \mu_p(\mathcal{L}_a + a_0 + \lambda^{**} a_1) \leq \mu_p(\mathcal{L}_a + a_0 + \lambda a_1) = 0. \]

From the above inequality, we get the following dichotomy:

- Either \( \mu_p(\mathcal{L}_a + a_0 + \lambda^{**} a_1) < 0 \) and by (ii) of Proposition 2.4, we deduce that \( \lambda^* < \lambda^{**} \).
- Or \( \mu_p(\mathcal{L}_a + a_0 + \lambda a_1) = 0 \) and by definition of \( \lambda^* \) we have \( \lambda^* \geq \lambda^{**} \) and for all \( \lambda \) there is no positive bounded solution to (1.1).

**Case 3:** \( \omega \not\subset \Omega \setminus \Omega_1 \) and \( \omega \not\subset \Omega_1 \). In this situation, since \( \omega \cap \Omega_1 \neq \emptyset \), by (v) of Proposition 2.4 we can see that for some positive constant \( C \)

\[ \mu_p(\mathcal{L}_a + a_0 + \lambda a_1) \leq C - \lambda \sup \omega a_1. \]

Therefore \( \lambda^{**} \in [-\infty, +\infty) \). Now let us observe that in this situation we also have for all \( \lambda \)

\[ \mu_p(\mathcal{L}_a + a_0 + \lambda a_1) \leq \mu_p(\mathcal{L}_{\cap a1} + a_0 + \lambda a_1) = \mu_p(\mathcal{L}_{\cap a1} + a_0). \]

We then have two case to analyse:

(i) \( \mu_p(\mathcal{L}_{\cap a1} + a_0) \leq 0 \):

In this situation, from the above inequality, we can already conclude that \( \lambda^{**} = -\infty \). Hence in this situation, for all \( \lambda \) there is no positive bounded solution of (1.1).

(ii) \( \mu_p(\mathcal{L}_{\cap a1} + a_0) > 0 \):

In this situation, by working as in Section 4 we see that there exists \( \lambda << -1 \) so that \( \mu_p(\mathcal{L}_a + a_0 + \lambda a_1) > 0 \). Therefore, \( \mu^{**} \in \mathbb{R} \) and we can argue as in the Case 2.
Let us now look at the asymptotic behaviour of $u_\lambda$ with respect to $\lambda$. The monotone behaviour of $u_\lambda$ and its limit as $\lambda \to \lambda^*$ (i.e. (iii)) can be obtained by following the arguments in Section 4, so we drop the proof here and prove only (i) and (ii) i.e. we analyse the limits of $u_\lambda$ as $\lambda \to \lambda^*$.

When $\lambda^* = +\infty$, the behaviour of $u_\lambda$ can be obtained by reproducing the arguments of Section 4 and we get for all $x \in \Omega$

$$\lim_{\lambda \to +\infty} u_\lambda(x) = +\infty.$$ 

Now, let us assume that $\lambda^* \in \mathbb{R}$. By definition of $\lambda^*$, we must have $$\mu_p(\mathcal{L}_\omega + a_0 + \lambda^* a_1) = 0.$$ 

As a consequence we also have $\mu_p(\mathcal{L}_\omega^* + a_0 + \lambda^* a_1) = 0$ where $\mathcal{L}_\omega^* + a_0 + \lambda^* a_1$ is defined by

$$\mathcal{L}_\omega^* [\phi] + a_0 \phi + \lambda^* a_1 \phi := \int_\omega K(y, x) \phi(y) dy - k(x) \phi + a_0 \phi + \lambda^* a_1 \phi.$$ 

Let us start with the proof of (i). First assume that $\mathcal{O} \neq \emptyset$, then by Theorem 2.2 there exists a positive measure $d\mu^*$ associated with $\mu_p(\mathcal{L}_\omega^* + a_0 + \lambda^* a_1)$. Moreover, $\phi^*_p(x)dx$ the regular part of $d\mu^*$ satisfies $\inf_{\mathcal{O}} \phi^*_p > 0$. By integrating the equation (1.1) over $\omega$ with respect to the measure $d\mu^*$, we get for any $\lambda^* < \lambda < \lambda^*$

$$\int_\omega \left( \int_\omega K(x, y) u_\lambda(y) dy \right) d\mu^* + \int_\omega (-k(x) + a_0 + \lambda a_1) u_\lambda d\mu^* = 0.$$ 

By definition of $\mu_p(\mathcal{L}_\omega^* + a_0 + \lambda^*)$, it follows from the above equality

$$\int_\omega \left( \int_\omega K(x, y) u_\lambda(y) dy \right) d\mu^* = (\lambda^* - \lambda) \int_\omega a_1 u_\lambda d\mu^*.$$ 

Therefore, by using the monotonicity of the map $u_\lambda$ we have for $\lambda_0 < \lambda < \lambda^*$

$$\frac{1}{\lambda^* - \lambda} \int_\omega \left( \int_\omega K(x, y) u_\lambda(y) dy \right) d\mu^* \leq \| a_1 \|_{\infty, \omega} \int_\omega d\mu^*,$$

which enforces

$$\lim_{\lambda \to \lambda^*} \| u_\lambda \|_{\infty, \omega} = +\infty.$$ 

Assume now that $\mathcal{O} = \emptyset$. In this situation, we have $\mu_p(\mathcal{L}_\omega + a_0 + \lambda^* a_1) = -\sup_{\mathcal{O}} (-k + a_0 + \lambda^* a_1)$. Since $\mathcal{O}$ is a compact set there exists $x_0 \in \mathcal{O}$ so that $-k(x_0) + a_0(x_0) + \lambda^* a_1(x_0) = 0$. We can check that $x_0 \in \Omega_1$, otherwise we have $\sup_{\mathcal{O} \cap (\Omega \setminus \Omega_1)} (-k + a_0) = \sup_{\mathcal{O} \cap (\Omega \setminus \Omega_1)} (-k + a_0 + \lambda^* a_1) = 0$ and $\mu_p(\mathcal{L}_{\mathcal{O} \cap (\Omega \setminus \Omega_1)} + a_0) = 0$. The latter equality leads to the contradiction $-\infty < \lambda^* = -\infty$, since for all $\lambda$ we have

$$\mu_p(\mathcal{L}_\omega + a_0 + \lambda a_1) \leq \mu_p(\mathcal{L}_{\mathcal{O} \cap (\Omega \setminus \Omega_1)} + a_0) = 0.$$ 

Now at $x_0$, we have

$$\int_\Omega K(x_0, y) u_\lambda(y) dy = (\lambda^* - \lambda) a_1(x_0) u_\lambda(x_0).$$ 

By using that $u_\lambda$ is monotone with respect to $\lambda$ we get for all $\lambda_0 \leq \lambda < \lambda^*$

$$\frac{1}{(\lambda^* - \lambda) a_1(x_0)} \int_\Omega K(x_0, y) u_\lambda(y) dy = u_\lambda(x_0),$$

which implies

$$\lim_{\lambda \to \lambda^*} u_\lambda(x_0) = +\infty.$$ 

Let us now prove (ii). When $\mu_p(\mathcal{L}_\omega + a_0 + \lambda^* a_1)$ is associated with a positive $L^1(\omega)$ eigenfunction we claim that

**Claim** – 5.1.

$$\lim_{\lambda \to \lambda^*} \int_\Omega u_\lambda(x) dx = +\infty.$$
Assume for the moment that the claim holds then we get (ii) by arguing as follows. Since $\Omega$ is compact, in view of the claim there exists $x \in \Omega$ so that
\[
\lim_{\lambda \to \lambda^{**}} \int_{B(x, \epsilon_0) \cap \Omega} u_\lambda(x) \, dx = +\infty.
\]
From the equation (1.1) and by the assumption (1.10) we always have
\[
c_0 \int_{B(x, \epsilon_0) \cap \Omega} u_\lambda(x) \, dx \leq (k(x) - a_0(x) - \lambda a_1(x)) u_\lambda(x) + \beta(x) u_\lambda^p(x).
\]
Therefore for all $x \in B(\bar{x}, \epsilon / 2)$, we have $B(\bar{x}, \epsilon / 2) \subset B(\bar{x}, \epsilon_0)$ which combined with the above inequalities implies that
\[
\lim_{\lambda \to \lambda^{**}} (k(x) - a_0(x) - \lambda a_1(x)) u_\lambda(x) + \beta(x) u_\lambda^p(x) = +\infty.
\]
Thus
\[
\lim_{\lambda \to \lambda^{**}} u_\lambda(x) = +\infty \quad \text{for all} \quad x \in B(\bar{x}, \epsilon / 2) \cap \Omega.
\]
In the above arguments by replacing $\bar{x}$ by any $x \in B(\bar{x}, \epsilon / 2) \cap \Omega$, we achieve
\[
\lim_{\lambda \to \lambda^{**}} u_\lambda(x) = +\infty \quad \text{for all} \quad x \in \bigcup_{x \in B(\bar{x}, \epsilon / 2)} B(x, \epsilon_0) \cap \Omega.
\]
Since $\Omega$ is compact we achieve $\lim_{\lambda \to \lambda^{**}} u_\lambda(x) = +\infty$ for all $x \in \Omega$ after a finite iteration of this argument.

\[\square\]

**Proof of the Claim**

Assume by contradiction that $\sup_\lambda \int_\Omega u_\lambda(x) \, dx < +\infty$. Since $u_\lambda$ is monotone, by Lebesgue monotone convergence Theorem we have $u_\lambda \to \bar{u}$ in $L^1(\Omega)$ when $\lambda \to \lambda^{**}$ and $\bar{u} > u_{\lambda_0} > 0$ satisfies the equation
\[
\mathcal{L}_\Omega[\bar{u}](x) + (a_0 + \lambda^{**} a_1) \bar{u}(x) - \beta(x) \bar{u}^p(x) = 0 \quad \text{for almost every} \ x \in \Omega.
\]
Therefore we have
\[
(5.2) \quad \mathcal{L}_\Omega[\bar{u}](x) + (a_0 + \lambda^{**} a_1) \bar{u}(x) = 0 \quad \text{for almost every} \ x \in \omega.
\]

By assumption there exists a positive $L^1$ eigenfunction $\phi_p$ associated with $\mu_p(\mathcal{L}_\omega + a_0 + \lambda^{**} a_1)$. Moreover the positive function $\frac{1}{k(x) - a_0(x) - \lambda^{**} a_1(x)} \in L^1(\omega)$ and the compact operator $\mathcal{K}$:
\[
\frac{C(\omega)}{v} \mapsto \mathcal{K}[v] := \int_\omega K(y, x) \frac{v(y) \, dy}{k(y) - a_0(y) - \lambda^{**} a_1(y)}
\]
is well defined. By construction, we can check that $\mu_p(\mathcal{K}) = -1$. Indeed, let $v_p := (k(x) - a_0(x) - \lambda^{**} a_1(x)) \phi_p$ then we can see that $v_p$ is positive and continuous, since by assumption we have
\[
\mathcal{L}_\omega[\phi_p] - v_p = 0.
\]

Moreover, $v_p$ satisfies $\mathcal{K}[v_p] = v_p$. Thus by the Krein-Rutman theory, we have $\mu_p(\mathcal{K}) = -1$ and $\psi_1 := v_p$ where $\psi_1$ is the principal positive continuous eigenfunction associated with $\mu_p(\mathcal{K})$.

Let us now consider $\mathcal{K}$ the following compact operator
\[
\frac{L^1(\omega)}{v} \mapsto \mathcal{K}^*[v] := \frac{1}{k(x) - a_0(x) - \lambda^{**} a_1(x)} \int_\omega K(y, x) v(y) \, dy.
\]

By the Krein-Rutman Theory there exists an eigenvalue $\nu_1$ associated with a positive $L^1(\omega)$ function $\phi^*$. Furthermore we can check that $\nu_1 = -1$. Indeed, since $\phi^*$ is associated with $\nu_1$ we have
\[
\mathcal{K}^*[\phi^*_p] = -\nu_1 \phi^*_p.
\]
By multiplying the above equation by $v_p$ and then integrating over $\omega$ it follows that
\[
\nu_1 \int_\omega v_p(x) \phi^*(x) dx = -\int_\omega \frac{v_p(x)}{k(x) - a_0(x) - \lambda^{**}a_1(x)} \left( \int_\omega K(y, x) \phi^*(y) dy \right) dx
\]
\[
= -\int_\omega v_p(y) \phi^*(y) dy,
\]
which implies that $\nu_1 = -1$.

Let $\bar{v}$ be the $L^1(\omega)$ function $\bar{v} := (k(x) - a_0(x) - \lambda^{**})\bar{u}$ then we get by (5.2)
\[
K[\bar{v}] - \bar{v} = -\int_{\Omega \setminus \omega} K(x, y)\bar{u}(y) dy \quad \text{for almost every } x \in \omega.
\]
Since $K[\bar{v}]$ is continuous, we deduce from (5.3) that $\bar{v}$ is continuous in $\omega$. So by multiplying (5.2) by $\phi^*$ and then integrating over $\omega$ we get
\[
\int_\omega \phi^*_1(x) \left( \int_{\Omega \setminus \omega} K(x, y)\bar{u}(y) dy \right) dx = \int_\omega \phi^*_1(x)[K[\bar{v}] - \bar{v}] dx
\]
Since $\nu_1 = -1$ and $\bar{u} > 0$ we end up with the contradiction
\[
0 < c_0 \leq \int_\omega \phi^*_1(x) \int_{\Omega \setminus \omega} K(x, y)\bar{u}(y) dy dx = 0.
\]

References


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