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# NONLOCAL REFUGE MODEL WITH A PARTIAL CONTROL

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ABSTRACT. In this paper, we analyse the structure of the set of positive solutions of an heterogeneous nonlocal equation of the form:

$$\int_\Omega K(x,y)u(y)\,dy - \int_\Omega K(y,x)u(x)\,dy + a_0u + \lambda a_1(x)u - \beta(x)u^p = 0 \quad \text{in} \quad \times\Omega$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open set, $K\in C(\mathbb{R}^n\times\mathbb{R}^n)$ is nonnegative, $a_i,\beta\in C(\Omega)$ and $\lambda\in\mathbb{R}$. Such type of equation appears in some studies of population dynamics where the above solutions are the stationary states of the dynamic of a spatially structured population evolving in a heterogeneous partially controlled landscape and submitted to a long range dispersal. Under some fairly general assumptions on $K, a_i$ and $\beta$ we first establish a necessary and sufficient criterium for the existence of a unique positive solution. Then we analyse the structure of the set of positive solution $(\lambda, u_\lambda)$ with respect to the presence or absence of a refuge zone (i.e $\omega$ so that $\beta_{|\omega}\equiv 0$).

## 1. INTRODUCTION

In this article we are interested in the positive bounded solutions of the nonlinear nonlocal equation

$$\int_\Omega K(x,y)u(y)\,dy - k(x)u + a_0(x)u + \lambda a_1(x)u - \beta(x)u^p = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open set, $K\in C(\mathbb{R}^n\times\mathbb{R}^n)$ is non negative, $k(x):=\int_\Omega K(y,x)\,dy$ $\lambda\in\mathbb{R}$, and $a_i,\beta$ are continuous functions. Our aim is to describe the properties of the positive bounded solutions of (1.2), in terms of the properties of $K, a_i, \beta$ and $\lambda$. That is, we look for existence criteria of positive bounded solutions of (1.1) and we describe some bifurcation diagrams i.e. depending on $a_i$ and $\beta$ we analyse the properties of the curve $(\lambda, u_\lambda)$.

The study of these kind of problems finds its justification in the ecological problematics related to the erosion of Biodiversity. In particular, some recent studies have focused on a better understanding of the impact of some agricultural practises on non targeted species [1, 7, 21, 27, 28, 29]. Such problematic can be addressed through the analysis of the asymptotic behaviour of the positive solution of a reaction diffusion equation :

$$\frac{\partial u(t,x)}{\partial t} = \int_\Omega K(x,y)u(t,y)\,dy - k(x)u(t,x) + a_0(x)u + \lambda a_1(x)u - \beta(x)u^p \quad \text{in} \quad \mathbb{R}^+\times\Omega \tag{1.2}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \Omega \tag{1.3}$$

where $u$ represents a population density evolving in a partial controlled heterogeneous . Here the parameter $\lambda$ is a control related to the practise and $a_1$ represents the region where the control is exerted.

In the literature the characterisation of the positive bounded solutions has been extensively studied for the elliptic equations

$$\mathcal{E}[u] + a_0u + \lambda a_1(x)u - \beta(x)u^p = 0 \quad \text{in} \quad \Omega, \tag{1.4}$$

$$u(x) = 0, \quad \text{in} \quad \partial\Omega. \tag{1.5}$$

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where $\mathcal{E}[u] := a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)$ is uniform elliptic [3, 4, 5, 6, 8, 9, 10, 17, 18, 26]. Nowadays, the structure of the positive bounded solutions $u_\lambda$ to (1.4)–(1.5) is well understood. More precisely, a positive bounded solution $u$ to (1.4)– (1.5) exists **if and only if**

$$\mu_1(\mathcal{E} + a_0 + \lambda a_1, \Omega) < 0 < \mu_1(\mathcal{E} + a_0 + \lambda a_1, \omega), \tag{1.6}$$

where $\omega$ denotes the refuge zone, i.e. $\omega := \{x \in \Omega \,|\, \beta(x) = 0\}$ and $\mu_1(\Omega)$ denotes the first eigenvalue of the spectral problem $\mathcal{E}[\phi] + a_0\phi + \lambda a_1(x)\phi + \mu\phi = 0$, $\phi = 0$ on $\partial\Omega$. Depending on the properties of $\beta$ and $a_1$ a description of the curves $(\lambda, u_\lambda)$ can be found in [16, 17, 18, 26].

For nonlocal equations such as (1.1), less is known and the analysis of the existence, uniqueness and the bifurcation diagram have been only studied in particular situations [2, 12, 14, 15, 19, 22, 25, 31]. A large part of the literature is devoted to the existence of positive solution to (1.1) in situations where no refuge zone exists and for a fixed $\lambda$ [2, 12, 14, 15, 22, 31]. To our knowledge [19] is the first paper which considers a nonlocal logistic equation with a refuge zone and analyses the curves $(\lambda, u_\lambda)$. More precisely, the authors investigate the existence, uniqueness of a positive bounded solution of

$$J \star u - u + \lambda u - \beta(x)u^p = 0 \quad \text{in} \quad \Omega, \tag{1.7}$$

$$u \equiv 0 \quad \text{in} \quad \mathbb{R}^n \setminus \bar{\Omega}, \tag{1.8}$$

where $J$ is a symmetric density of probability. They prove that a positive solution of the above problem exists **if and only if**

$$\mu_1(J \star u - u, \Omega) < \lambda < \mu_1(J \star u - u, \omega).$$

Moreover, they have showed that this solution is unique and have established the following asymptotic behaviours:

$$\lim_{\lambda \to \mu_1(J\star u - u, \Omega)} u_\lambda(x) = 0 \quad \text{for all } x \in \Omega,$$

$$\lim_{\lambda \to \mu_1(J\star u - u, \omega)} u_\lambda(x) = +\infty \quad \text{for all } x \in \Omega.$$

These results have been recently extended to the more general equation (1.1) with a quadratic nonlinearity ($s(a(x) - b(x)s)$) and under some assumptions on the symmetry of the kernel $K$ and some extra conditions on $a$ and $\lambda$, see [25].

Here we address these questions of existence, uniqueness and the description of some bifurcation diagrams for a general kernel $K$ and with no restriction on the coefficients $a_i$, $\lambda$ and $\beta$.

In what follows we will always assume that the functions $a_i$ and $\beta$ satisfy:

$$\begin{cases} a_i, \beta \in C(\Omega), a_1 \geq 0, \beta \geq 0 \\ \Omega \setminus \overline{supp(a_1)} \text{ is a open set of } \mathbb{R}^n \\ K \in C(\mathbb{R}^n \times \mathbb{R}^n), K \geq 0 \end{cases} \tag{1.9}$$

For the dispersal kernel, we will also require that $K$ satisfies:

$$\exists c_0 > 0, \epsilon_0 > 0 \quad \text{such that} \inf_{x\in\Omega} \left( \inf_{y \in B(x,\epsilon_0)} K(x,y) \right) > c_0. \tag{1.10}$$

A typical example of such dispersal kernel is given by

$$K(x,y) = J\left( \frac{x_1 - y_1}{g_1(y)h_1(x)}; \frac{x_2 - y_2}{g_2(y)h_2(x)}; \dots; \frac{x_n - y_n}{g_n(y)h_n(x)} \right),$$

with $J \in C(\mathbb{R}^n)$ continuous, $J(0) > 0$ and $0 < \alpha_i \leq g_i \leq \beta_i$ and $0 \leq h_i \leq \beta_i$. Such type of kernel have been recently introduced in [11] to model a nonlocal heterogeneous dispersal process. To simplify the presentation of our results, we also introduce the notation $\mathcal{L}_\Omega[u]$ for the continuous linear operator

$$\mathcal{L}_\Omega[u] := \int_\Omega K(x,y)u(y)\,dy - k(x)u(x).$$

In [19, 25] the analysis essentially relies on the existence of positive eigenfunction associated with a principal eigenvalue $\mu_1$ and a $L^2$ variational characterisation of $\mu_1$. However, such properties ( existence of a positive eigenfunction and a $L^2$ variational characterisation of $\mu_1$) does not hold for general kernels $K$ and $a_i$ [13] and a new approach and characterisation of the principal eigenvalue has to be developed.

In the past few years, the spectral properties of nonlocal operators such as $\mathcal{L}_{\Omega} + a$ have been intensively studied [2, 12, 13, 14, 15, 20, 22, 23, 24]. In particular a notion of generalized principal eigenvalue $\mu_p$ of a linear operator $\mathcal{L}_{\Omega} + a$ has been introduced in [12, 15] and is defined by

$$\mu_p(\mathcal{L}_{\Omega} + a) := \sup\{\mu \in \mathbb{R} \,|\, \exists \phi \in C(\bar{\Omega}), \phi > 0, \text{ so that } \mathcal{L}_{\Omega}[\phi] + (a + \lambda)\phi \leq 0\}.$$

$\mu_p$ is called a generalized principal eigenvalue because $\mu_p$ is not necessarily associated with a $L^1$ positive eigenfunction [12, 13, 24, 30]. Such notion has been successfully used to derive an optimal criterium for the existence of a unique positive solution of (1.1) in absence of a refuge zone [12, 15].

Equipped with this notion of generalised eigenvalue, we can now state our results. We first present an optimal criterium for the existence of a unique positive bounded solution to (1.1). Namely, we show

**Theorem 1.1.** *Let $K$, $a_i$, $\beta$ satisfy the assumptions* (1.9)–(1.10) *and let $\omega$ be the refuge set*

$$\omega := \{x \in \bar{\Omega} | \beta(x) = 0\}.$$

*Then a positive continuous bounded solution $u$ of* (1.1) *exists* ***if and only if***

$$\mu_p(\mathcal{L}_{\Omega} + a_0 + \lambda a_1) < 0 < \mu_p(\mathcal{L}_{\omega} + a_0 + \lambda a_1),$$

*where we set $\mu_p(\mathcal{L}_{\omega} + a_0 + \lambda a_1) = -\sup_{\omega}(a_0 + \lambda a_1)$ when $\overset{\circ}{\omega} = \emptyset$. Moreover the solution is unique.*

Next we analyse the partially controlled problem (1.1) i.e. we describe the set $\{\lambda, u_\lambda\}$ where $u_\lambda$ is a positive bounded continuous solution to (1.1). We start by describing $\{\lambda, u_\lambda\}$ in a case of the absence of a refuge zone. We prove the following

**Theorem 1.2.** *Assume that $K, a_i$ and $\beta$ satisfy* (1.9)–(1.10)*. Assume further that $\beta > 0$ in $\bar{\Omega}$ then there exists $\lambda^* \in [-\infty; \infty)$, so that for all $\lambda > \lambda^*$ there exists a unique positive continuous solution $u_\lambda$ to* (1.1)*. When $\lambda^* \in \mathbb{R}$, there is no positive solution to* (1.1) *for all $\lambda \leq \lambda^*$. Moreover, we have the following trichotomy:*

- *$\lambda^* = -\infty$ when $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) < 0$,*
- *$\lambda^* \in [-\infty, \infty)$ when $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) = 0$.*
- *$\lambda^* \in \mathbb{R}$ when $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) > 0$.*

*In addition, the map $\lambda \to u_\lambda$ is monotone increasing and we have*

$$\forall\, x \in \bar{\Omega} \quad \lim_{\lambda \to +\infty} u_\lambda(x) = +\infty,$$
$$\forall\, x \in \bar{\Omega} \quad \lim_{\lambda \to \lambda^{*,+}} u_\lambda(x) = u_\infty(x),$$

*where $u_\infty \equiv 0$ on $\Omega_1 := \{x \in \Omega |\, a_1(x) > 0\}$ and $u_\infty$ is a nonnegative solution to*

$$\int_{\Omega\setminus\Omega_1} K(x,y)u(y)dy - k(x)u + a_0(x)u - \beta u^p = 0 \quad \text{in} \quad \Omega \setminus \Omega_1.$$

*Furthermore, $u_\infty$ is non trivial when $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) < 0$.*

Finally, we describe the set $\{\lambda, u_\lambda\}$ in the situation where a refuge zone exists. We prove the following

**Theorem 1.3.** *Assume that $K, a_i$ and $\beta$ satisfy* (1.9)–(1.10)*. Assume further that $\omega \neq \emptyset$, then there exists two quantities $\lambda^*, \lambda^{**} \in [-\infty, +\infty]$ so that we have the following dichotomy :*

- *Either $\lambda^{**} \leq \lambda^*$ and there exists no positive bounded solution to* (1.1)*.*
- *Or $\lambda^* < \lambda^{**}$, and for all $\lambda \in (\lambda^*, \lambda^{**})$ there exists a unique positive bounded continuous solution to* (1.1)*. When $\lambda^*, \lambda^{**} \in \mathbb{R}$, there is no positive bounded solution to* (1.1) *for all $\lambda \leq \lambda^*$ and for all $\lambda \geq \lambda^{**}$. Moreover, the map $\lambda \to u_\lambda$ is monotone increasing and we have*

(i)
$$\lim_{\lambda\to\lambda^{**,-}} \|u_\lambda\|_{\infty,\omega} = +\infty,$$
*where* $\|u_\lambda\|_{\infty,\omega} := \sup_{x\in\omega}|u_\lambda(x)|$.

(ii) *If* $\mu_p(\mathcal{L}_\omega + a_0 + \lambda^{**}a_1)$ *is an eigenvalue in* $L^1(\omega)$ *or* $\lambda^{**} = +\infty$ *then*
$$\forall\, x\in\bar\Omega, \lim_{\lambda\to\lambda^{**,-}} u_\lambda(x) = +\infty.$$

(iii) *For all* $x\in\bar\Omega$ *we have* $\lim_{\lambda\to\lambda^{*,+}} u_\lambda(x) = u_\infty(x)$, *where* $u_\infty$ *is a function satisfying on* $\Omega_1 := \{x\in\Omega|\, a_1(x) > 0\, u_\infty \equiv 0$ *and* $u_\infty$ *is a nonnegative solution to*
$$\int_{\Omega\setminus\Omega_1} K(x,y)u(y)dy - k(x)u + a_0(x)u - \beta u^p = 0 \quad in \quad \Omega\setminus\Omega_1.$$
*Furthermore,* $u_\infty$ *is non trivial when* $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) < 0$.

Before going to the proofs of theses results we would like to make some additional comments. The assumption  can be relaxed and we can get a full description of the curves when $a_1 > 0$ in $\bar\Omega$.

The paper is organised as follows. In a preliminary section we recall some known results on $\mu_p$ and on the positive solution of a KPP equation. Then in Section 3 we prove the existence criterium of Theorem 1.1. The proof of Theorem 1.2 is done in Section 4. Finally, in Section 5 we analyse the bifurcation diagram of (1.1) in the presence of a refuge zone (Theorem 1.3).

## 2. Preliminaries

In this section, we recall some results on the principal eigenvalue of a linear nonlocal operator and some known results about the KPP equation below

$$\mathcal{L}_\Omega[u] + f(x,u) = 0 \quad \text{in} \quad \Omega \tag{2.1}$$

where
$$\mathcal{L}_\Omega[u] := \int_\Omega K(x,y)u(y)\,dy - k(x)u(x)$$
and $f(x,s)$ is satisfying

$$\begin{cases} f\in C(\Omega\times[0,\infty)) \text{ and is differentiable with respect to } s\\ f_u(\cdot,0)\in C(\Omega)\\ f(\cdot,0)\equiv 0 \text{ and } f(x,s)/s \text{ is decreasing with respect to } s\\ \text{there exists } M>0 \text{ such that } f(x,s)\le 0 \text{ for all } s\ge M \text{ and all } x.\end{cases} \tag{2.2}$$

The simplest example of such a nonlinearity is
$$f(x,u) = u(a(x)-u),$$
where $a(x)\in C(\Omega)$.

It has been shown in [2, 12] that the existence of a positive solution of (2.1) is conditioned to the sign of the principal eigenvalue $\mu_p$ of the linear operator $\mathcal{L}_\Omega + f_u(x,0)$ where $\mu_p$ is defined by the formula
$$\mu_p(\mathcal{L}_\Omega + f_u(x,0)) := \sup\{\mu\in\mathbb{R}\,|\,\exists\,\phi\in C(\Omega), \phi>0 \text{ so that } \mathcal{L}_\Omega[\phi] + f_u(x,0)\phi + \mu\phi \le 0\}.$$
That is to say

**Theorem 2.1** ([2, 12])**.** *Let $\Omega$ be a bounded open set anf assume that $K$ and $f$ satisfy respectively* (1.9)–(1.10) *and* (2.2)*. Then there exists a unique positive continuous solution to* (2.1) ***if and only if*** $\mu_p(\mathcal{L}_\Omega + f_u(x,0)) < 0$. *Moreover, if $\mu_p\ge 0$ then any non negative uniformly bounded solution of* (2.1) *is identically zero.*

Also noted in [12] the principal eigenvalue is not always achieved. This means that there is not always a positive continuous eigenfunction associated with $\mu_p$. However as shown in [13], we can always associate a positive measure $d\mu$ with $\mu_p$. More precisely,

**Theorem 2.2** ([13]). *Let $\Omega$ be an open bounded set and assume that $K$ and $f_u(x,0)$ satisfy the assumptions* (1.9) *and* (2.2). *Then there exists a positive measure $d\mu \in \mathcal{M}^+(\Omega)$, so that for any $\phi \in C_c(\Omega)$ we have*

$$\int_\Omega \phi(x)\left(\int_\Omega K(x,y)d\mu(y)\right)dx + \int_\Omega \phi(x)(f_u(x,0) - k(x) + \mu_p)d\mu(x) = 0.$$

*Moreover, there exists a positive function $\phi_p \in L^1(\Omega) \cap C(\Omega \setminus \Sigma)$ so that $\inf_\Omega \phi_p > 0$ and $d\mu(x) = \phi_p(x)dx + d\mu_s(x)$ where $d\mu_s(x)$ is a non negative singular measure with respect to the Lebesgue measure whose support lies in the set $\Sigma := \{y \in \bar\Omega | f_u(y,0) - k(y) = \sup_{x\in\Omega}(f_u(x,0) - k(x))\}$.*

As proved in [12, 24, 30], when $\Omega$ is an open bounded set we can find a condition on the coefficients which guarantees that $d\mu_s(x) \equiv 0$ and the existence of a positive continuous eigenfunction. For example the existence of principal eigenfunction is guaranteed, if we assume that the function $a(x) := f_u(x,0) - \int_\Omega K(y,x)dy$ satisfies

$$\frac{1}{\sup_\Omega a - a(x)} \notin L^1(\Omega') \quad \text{for some open bounded domain} \quad \Omega' \subset \bar\Omega.$$

For the existence of principal eigenfunction as remark in [15] we also have this useful criteria:

**Proposition 2.3.** Let $\Omega$ be a bounded open set, then there exists a positive continuous eigenfunction associated to $\mu_p$ **if and only if** $\mu_p(\mathcal{L}_\Omega + a) < -\sup_\Omega a$.

Next we recall some properties of the principal eigenvalue $\mu_p$ that we will constantly use along this paper:

**Proposition 2.4.**

(i) Assume $\Omega_1 \subset \Omega_2$, then for the two operators

$$\mathcal{L}_{\Omega_1}[u] + a(x)u := \int_{\Omega_1} K(x,y)u(y)\,dy - k(x)u + a(x)u$$
$$\mathcal{L}_{\Omega_2}[u] + a(x)u := \int_{\Omega_2} K(x,y)u(y)\,dy - k(x)u + a(x)u$$

respectively defined on $C(\Omega_1)$ and $C(\Omega_2)$ we have

$$\mu_p(\mathcal{L}_{\Omega_1} + a(x)) \geq \mu_p(\mathcal{L}_{\Omega_2} + a(x)).$$

(ii) Fix $\Omega$ and assume that $a_1(x) \geq a_2(x)$, then

$$\mu_p(\mathcal{L}_\Omega + a_2(x)) \geq \mu_p(\mathcal{L}_\Omega + a_1(x)).$$

Moreover, if $a_1(x) \geq a_2(x) + \delta$ for some $\delta > 0$ then

$$\mu_p(\mathcal{L}_\Omega + a_2(x)) > \mu_p(\mathcal{L}_\Omega + a_1(x)).$$

(iii) $\mu_p(\mathcal{L}_\Omega + a(x))$ is Lipschitz continuous in $a(x)$. More precisely,

$$|\mu_p(\mathcal{L}_\Omega + a(x)) - \mu_p(\mathcal{L}_\Omega + b(x))| \leq \|a(x) - b(x)\|_\infty$$

(iv) Assume $\Omega_1 \subset \Omega_2$, then for the two operators

$$\mathcal{L}_{\Omega_1}[u] + a(x)u := \int_{\Omega_1} K(x,y)u(y)\,dy - k(x)u + a(x)u$$
$$\mathcal{L}_{\Omega_2}[u] + a(x)u := \int_{\Omega_2} K(x,y)u(y)\,dy - k(x)u + a(x)u$$

respectively defined on $C(\Omega_1)$ and $C(\Omega_2)$. Assume that the corresponding principal eigenvalue are associated to a positive continuous principal eigenfunction. Then we have

$$|\mu_p(\mathcal{L}_{\Omega_1} + a(x)) - \mu_p(\mathcal{L}_{\Omega_2} + a(x))| \leq C_0|\Omega_2 \setminus \Omega_1|,$$

where $C_0$ depends on $K$ and $\phi_2$.

(v) We always have the following estimate

$$-\sup_\Omega\left(a(x) + \int_\Omega K(x,y)\,dy\right) \leq \mu_p(\mathcal{L}_\Omega + a) \leq -\sup_\Omega a.$$

**Proof:**

We refer to [12] for the proofs of $(i)-(iii)$ and $(v)$, so we will be concerned only with $(iv)$. Let us introduce the following quantity:

$$\mu_p'(\mathcal{L}_\Omega + a) := \inf\{\mu \in \mathbb{R} \,|\, \exists\, \phi \in C(\bar{\Omega}), \phi > 0 \text{ so that } \mathcal{L}_\Omega[\phi] + a\phi + \mu\phi \geq 0\}.$$

One can check that $\mu_p(\mathcal{L}_\Omega + a) = \mu_p'(\mathcal{L}_\Omega + a)$. Indeed since there is a positive eigenfunction associated with $\mu_p(\mathcal{L}_\Omega + a)$ one has $\mu_p'(\mathcal{L}_\Omega + a) \leq \mu_p(\mathcal{L}_\Omega + a)$ by definition of $\mu_p'(\mathcal{L}_\Omega + a)$. We obtain the equality by arguing as follows. Assume by contradiction that $\mu_p(\mathcal{L}_\Omega + a) > \mu_p'(\mathcal{L}_\Omega + a)$. Then there exists $\mu$ so that $\mu_p'(\mathcal{L}_\Omega + a) < \mu < \mu_p(\mathcal{L}_\Omega + a)$ and from the definition of $\mu_p$ and $\mu_p'$ there exists two positive continuous functions $\psi$ and $\phi$ so that

$$\mathcal{L}_\Omega[\phi] + a(x)\phi + \mu\phi \geq 0,$$
$$\mathcal{L}_\Omega[\psi] + a(x)\psi + \mu\psi < 0.$$

From the last inequalities we deduce that $\psi > 0$ in $\bar{\Omega}$ and by setting $w := \frac{\phi}{\psi}$ it follows that

$$\begin{aligned} 0 \leq \mathcal{L}_\Omega[\phi] + (a(x) + \mu)\phi &= \mathcal{L}_\Omega[\phi] + (a(x) + \phi)\frac{\phi}{\psi}\psi, \\ &\leq \mathcal{L}_\Omega[\phi] - \frac{\phi}{\psi}(x)\mathcal{L}_\Omega[\psi], \\ &\leq \int_\Omega K(x,y)\psi(y)(w(y) - w(x))\,dy. \end{aligned}$$

Thus $w$ cannot achieve a maximum in $\bar{\Omega}$ without being constant. $w$ being continuous in $\bar{\Omega}$, it follows that $\phi = c\psi$ for some positive constant $c$. Thus we get the contradiction

$$0 \leq \mathcal{L}_\Omega[\phi] + (a(x) + \mu)\phi = c\left(\mathcal{L}_\Omega[\psi] + (a(x) + \mu)\psi\right) < 0.$$

We are now in position to prove (iv). Let $\phi_2$ be the eigenfunction associated to$\mu_p(\mathcal{L}_{\Omega_2} + a(x))$ normalized by $\|\phi_2\|_\infty = 1$ and let us set $C_0 := \frac{\|K(\cdot,\cdot)\|_\infty}{\min_{\Omega_2}\phi_2}$.

Now, let us show that $(\phi_2, \mu_p(\mathcal{L}_{\Omega_2} + a) + C_0|\Omega_2 \setminus \Omega_1|)$ is an adequate test function for $\mu_p'(\mathcal{L}_{\Omega_1} + a)$. By a direct computation and by using the normalisation of $\phi_2$ we have

$$\begin{aligned} \mathcal{L}_{\Omega_1}[\phi_2] + (a + \mu_p(\mathcal{L}_{\Omega_2} + a) + C_0|\Omega_2 \setminus \Omega_1|))\phi_2 &= -\int_{\Omega_2\setminus\Omega_1} K(x,y)\phi_2(y)\,dy + C_0|\Omega_2 \setminus \Omega_1|\phi_2, \\ &\geq -\|K(\cdot,\cdot)\|_\infty|\Omega_2 \setminus \Omega_1| + C_0|\Omega_2 \setminus \Omega_1|\phi_2, \\ &\geq \left(\frac{\phi_2}{\min_{\Omega_2}\phi_2} - 1\right)\|K(\cdot,\cdot)\|_\infty|\Omega_2 \setminus \Omega_1|. \end{aligned}$$

Therefore $\mu_p'(\mathcal{L}_{\Omega_1} + a) \leq \mu_p(\mathcal{L}_{\Omega_2} + a) + C_0|\Omega_2 \setminus \Omega_1|$, which combined with $(i)$ and using that $\mu_p'(\mathcal{L}_{\Omega_1} + a) = \mu_p(\mathcal{L}_{\Omega_1} + a)$ leads to

$$|\mu_p(\mathcal{L}_{\Omega_1} + a(x)) - \mu_p(\mathcal{L}_{\Omega_2} + a(x))| \leq C_0|\Omega_2 \setminus \Omega_1|.$$

□

## 3. Optimal existence criterium

In this section we establish an optimal criterium for the existence of a positive continuous bounded solution to

$$\mathcal{L}_\Omega[u] + a(x)u - \beta(x)u^p = 0 \quad \text{in} \quad \Omega, \tag{3.1}$$

when there exists $\omega \subset \Omega$ so that $\beta_{|\omega} \equiv 0$. Note that (3.1) is a particular case of (2.1) with $f(x,s) := a(x)s - \beta(x)s^p$. However, due to the presence of refuge zone (i.e. $\beta_{|\omega} \equiv 0$) the function $f(x,s)$ does not satisfy the assumptions (2.2) and the Theorem 2.1 does not apply. But we still have a complete characterisation of the existence of a bounded positive solution. Namely we can show the following Theorem:

**Theorem 3.1.** *Assume that $K, a$ and $\beta$ satisfy* (1.9)–(1.10). *Assume further that there exists $\omega \subset \Omega$ so that $\beta_{|\omega} \equiv 0$. Then there exists a bounded positive continuous solution to* (3.1) ***if and only if***

$$\mu_p(\mathcal{L}_\omega + a) > 0 > \mu_p(\mathcal{L}_\Omega + a),$$

*where we set $\mu_p(\mathcal{L}_\omega + a) = -\sup_\omega a$ when $\overset{\circ}{\omega} = \emptyset$.*

**Proof:**

First let us assume that $\mu_p(\mathcal{L}_\omega + a) \leq 0$, we will show that there is no positive bounded solution to (3.1). Let us suppose by contradiction that there exists $u$, a positive bounded solution to (3.1). So in $\omega$, $u$ satisfies

$$\mathcal{L}_\Omega[u] + au = 0,$$

which implies that $\max_{\bar{\omega}} a < 0$ and $u$ is continuous on $\bar{\omega}$. Furthermore, we have

$$\mathcal{L}_\omega[u] + au = -\int_{\Omega\setminus\omega} K(x,y)u(y)\,dy \leq 0. \tag{3.2}$$

If $\overset{\circ}{\omega} = \emptyset$ then we obtain easily a contradiction. Indeed in such case, we have $\mu_p(\mathcal{L}_\omega + a) = -\sup_\omega a$ which leads to the contradiction

$$0 < -\sup_\omega a = \mu_p(\mathcal{L}_\omega + a) \leq 0.$$

In the other situations, $\overset{\circ}{\omega} \neq \emptyset$ and to obtain our desired contradiction we argue as follows. Since $\mu_p(\mathcal{L}_\omega + a) \leq 0 < -\max_{\bar{\omega}} a$, by Proposition 2.3 there exists a positive continuous eigenfunction associated with $\mu_p(\mathcal{L}_\omega + a)$. As a consequence there exists also a positive continuous eigenfunction associated with $\mu_p(\mathcal{L}^*_\omega + a)$ where $\mathcal{L}^*_\omega + a$ is defined by

$$\mathcal{L}^*_\omega[\phi] + a\phi := \int_\omega K(y,x)\phi(y)dy - k(x)\phi + a(x)\phi.$$

We can easily check that $\mu_p(\mathcal{L}_\omega + a) = \mu_p(\mathcal{L}^*_\omega + a)$. Let us denote by $\phi^*$ the positive continuous principal eigenfunction associated with $\mu_p(\mathcal{L}^*_\omega + a)$. Now by multiplying (3.2) by $\phi^*$ and then integrating over $\omega$, it follows

$$\int_\omega \phi^*(x)\mathcal{L}_\omega[u](x)dx + au\phi^*(x)dx \leq -c_0\int_\omega \phi^*\left(\int_{\Omega\setminus\omega} K(x,y)\,dy\right).$$

By using Fubini's Theorem in the above inequality we get the contradiction

$$0 \leq -\mu_p(\mathcal{L}^*_\omega + a)\int_\omega \phi^* u \leq -c_0\int_\omega \phi^*\left(\int_{\Omega\setminus\omega} K(x,y)\,dy\right) < 0.$$

Thus in both cases, there is no bounded solution to (3.1) when $\mu_p(\mathcal{L}_\omega + a) \leq 0$.

Next we see that there is no positive bounded solution for (3.1) when $\mu_p(\mathcal{L}_\Omega + a) \geq 0$. In this situation, with some modifications we can reproduce the argumentation developed in [12] (Subsection 6.2). Let us assume that a positive solution of (3.1) exists and let us denote $u$ this solution. We first observe that by following the argument developed in [2] we can see that $u$ is continuous in $\Omega$ and there exists positive constants $\delta$ and $c_0$ so that

$$\begin{cases} \inf_\Omega u \geq c_0, \\ \inf_{x\in\Omega}(k(x) - a(x) + \beta(x)u^{p-1}) \geq \delta. \end{cases}$$

From the monotone behaviour with respect to the $s$ of the function $g(x,s) := (a - \beta(x)s^{p-1})$, we deduce that $a - \beta u^{p-1} \leq a(x) - \beta c_0^{p-1} \leq a$. Now let us denote $\gamma(x) = a(x) - \beta(x)c_0^{p-1}$. By construction, we have $\gamma(x) \leq a(x)$ and we see by $(ii)$ of Proposition 2.4 that

$$\mu_p(\mathcal{L}_\Omega + \gamma(x)) \geq \mu_p(\mathcal{L}_\Omega + a(x)) \geq 0.$$

Moreover, since $u$ is a solution of (3.1), we have

$$\mathcal{L}_\Omega[u] + \gamma u \geq \mathcal{L}_\Omega[u] + au - \beta u^p = 0, \tag{3.3}$$

with a strict inequality for any $x \in \Omega \setminus \omega$.

We claim that

***Claim*** – 3.1. There exists $\delta > 0$ and a positive continuous function $\phi$ so that $\inf_{\Omega} \phi > \delta$ and

$$\mathcal{L}_{\Omega}[\phi] + \gamma\phi \leq 0.$$

Assume for the moment that the Claim holds true then we get our desired contradiction by arguing as follow. Since $\phi > \delta$ we can define the following quantity

$$\tau^* := \inf\{\tau > 0 | u \leq \tau\phi\}.$$

Obviously, by proving that $\tau^* = 0$ we get the contradiction

$$c_0 \leq u \leq 0.$$

Assume by contradiction that $\tau^* > 0$ and let us denote $w := \tau^*\phi - u$. By definition of $\tau^*$, there exists $x_0 \in \bar{\Omega}$ such that $\tau^*\phi(x_0) = u(x_0) > 0$ and from (3.3) we see that $w$ satisfies

$$\mathcal{L}_{\Omega}[w] + \gamma w \leq 0.$$

By evaluating the above expression at $x_0$, since $w \geq 0$ we see that

$$0 \leq \int_{\Omega} K(x_0, y)w(y)\, dy \leq 0.$$

Therefore, since K satisfies (1.10) we must have $w(y) = 0$ for almost every $y \in \bar{\Omega}$. Thus, we end up with $\tau^*\phi \equiv u$ and we get the following contradiction

$$0 < \mathcal{L}_{\Omega}[u] + \gamma u = \mathcal{L}_{\Omega}[\tau^*\phi] + \gamma\tau^*\phi \leq 0 \quad \text{on} \quad \Omega \setminus \omega.$$

Hence $\tau^* = 0$.

**Proof of the Claim:**

When $\mu_p(\mathcal{L}_{\Omega} + \gamma) > 0$ then by definition of the principal eigenvalue for all positive $0 < \mu < \mu_p(\mathcal{L}_{\Omega} + \gamma)$ there exists a positive continuous function $\phi$ such that

$$\mathcal{L}_{\Omega}[\phi] + \gamma\phi \leq -\mu\phi < 0.$$

Observe that $\phi \geq \delta$ for some positive $\delta$ since otherwise there exists $x_0 \in \bar{\Omega}$ so that $\phi(x_0) = 0$ and we get the contradiction

$$0 < \mathcal{L}_{\Omega}[\phi](x_0) + \gamma(x_0)\phi(x_0) \leq 0.$$

When $\mu_p(\mathcal{L}_{\Omega} + \gamma) = 0$ we argue as follows. By construction, $a \geq \gamma$ and on $\bar{\Omega} \setminus \omega$ we have

$$\gamma < a \leq \sup_{\Omega} a \leq -\mu_p(\mathcal{L}_{\Omega} + a) \leq 0. \tag{3.4}$$

And another hand on $\omega$ since $\beta_{|\omega} \equiv 0$, we have

$$\mathcal{L}_{\Omega}[u] + a(x)u = 0,$$

which leads to $\sup_{\bar{\omega}} a < 0$. So on $\bar{\omega}$ we also have

$$\gamma \leq \sup_{\omega} a < 0. \tag{3.5}$$

By combining (3.4) and (3.5) it follows that $\sup_{\bar{\Omega}} \gamma < 0$. Now, since $0 = \mu_p(\mathcal{L}_{\Omega} + \gamma) < -\sup_{\bar{\Omega}} \gamma$ we deduce from Proposition 2.3 that there exists a continuous positive principal eigenfunction $\phi$ associated with $\mu_p(\mathcal{L}_{\Omega} + \gamma)$. As above, we have $\inf_{\Omega} \phi > \delta$ for some positive $\delta$.

□

Lastly, let us construct a positive bounded solution to (3.1) when the condition

$$\mu_p(\mathcal{L}_{\omega} + a) > 0 > \mu_p(\mathcal{L}_{\Omega} + a) \tag{3.6}$$

is satisfied. The uniqueness of this solution follows form a similar argumentation as in [2, 12], so we will omit the proof here.

From the condition $0 > \mu_p(\mathcal{L}_{\Omega} + a)$, by reproducing the argument in [12] we can find a positive bounded subsolution $\phi_0$ of the problem (3.1) so that $\kappa\phi_0$ is still a subsolution for any $\kappa$ small and positive. Here the main difficulty is to find a positive supersolution $\psi$. Indeed, due to the existence of a refuge zone, the large positive constants are not supersolutions of (3.1). We claim

***Claim* – 3.2.** When the condition (3.6) is satisfied, then there exists $\psi > 0, \psi \in C(\bar{\Omega})$ supersolution of (3.1)

Note that by proving the claim we end the construction of the solution to (3.1). Indeed, since for $\kappa$ small we have $\kappa\phi \le \psi$, by the monotone iterative scheme there exists a solution $u$ to (3.1) so that $\kappa\phi \le u \le \psi$.

□

Now, let us turn our attention to the proof of the Claim.

**Proof of the Claim:**

Let us first assume that $\overset{\circ}{\omega} \neq \emptyset$.

In this situation, by following the argument in [12] (Subsection 6.1) we can introduce a regularisation $a_\epsilon \in C(\Omega)$ of $a - k$ so that the following operator

$$\mathcal{L}_{\epsilon,\omega}[u] := \int_\omega K(x,y)u(y)dy + a_\epsilon(x)u$$

has a positive continuous principal eigenfunction. By continuity of $\mu_p(\mathcal{L}_{\epsilon,\omega})$ with respect to $a_\epsilon$ ($(iii)$ of Proposition 2.4) we can find $\epsilon$ small so that

$$\mu_p(\mathcal{L}_{\epsilon,\omega}) - \|a_\epsilon - a + k\|_\infty \ge \frac{\mu_p(\mathcal{L}_\omega + a)}{2}.$$

Let $\epsilon$ be fixed and let us denote $\omega_\delta$ the following set

$$\omega_\delta := \{x \in \Omega \mid d(x;\omega) < \delta\}.$$

By continuity of the function $\sup_{\omega_\delta} a_\epsilon$ with respect to $\delta$, there exists $\delta_0$ so that for all $\delta \le \delta_0$ we have

$$|\sup_{\omega_\delta} a_\epsilon - \sup_{\omega} a_\epsilon| \le \frac{-\mu_p(\mathcal{L}_\omega + a_\epsilon) - \sup_\omega a_\epsilon}{2}.$$

So, by $(i)$ of Proposition 2.4, we deduce from the above inequality that we have for all $\delta \le \delta_0$,

$$\mu_p(\mathcal{L}_{\omega_\delta} + a_\epsilon) \le \mu_p(\mathcal{L}_\omega + a_\epsilon) < -\sup_{\omega_\delta} a_\epsilon.$$

Therefore, thanks to Proposition 2.3 for all $\delta \le \delta_0$ there exists a positive continuous eigenfunction associated with $\mu_p(\mathcal{L}_{\epsilon,\omega_\delta})$.

By continuity of $\mu_p(\mathcal{L}_{\epsilon,\omega_\delta})$ with respect to the domain ((iv) of Proposition 2.4) we achieve for $\delta$ small enough, say $\delta \le \delta_1$,

$$\mu_p(\mathcal{L}_{\epsilon,\omega}) \ge \mu_p(\mathcal{L}_{\epsilon,\omega_\delta}) \ge \mu_p(\mathcal{L}_{\epsilon,\omega_{\delta_1}}) \ge \|a_\epsilon - a + k\|_\infty + \frac{\mu_p(\mathcal{L}_\omega + a)}{8}. \tag{3.7}$$

By construction $\bar{\Omega} \setminus \omega_\delta$ and $\bar{\omega}_{\frac{\delta}{2}}$ are two disjoints bounded closed set, so by the Urysohn Lemma there exists a nonnegative continuous function $\eta_1$ such that $0 \le \eta_1 \le 1, \eta_1(x) = 1$ in $\bar{\Omega}\setminus\omega_\delta, \eta_1(x) = 0$ in $\bar{\omega}_{\frac{\delta}{2}}$.

Let $\psi_1, \psi_2$ be the following continuous functions

$$\psi_1 := \begin{cases} C_1\eta_1 & \text{in} \quad \Omega \setminus \omega_{\frac{\delta}{2}} \\ 0 & \text{elsewhere,} \end{cases} \qquad \psi_2 := \begin{cases} C_2(1-\eta_1)\Psi_\delta & \text{in} \quad \omega_\delta \\ 0 & \text{elsewhere.} \end{cases}$$

where $\Psi_\delta$ denotes the positive continuous eigenfunction associated with $\mu_p(\mathcal{L}_{\epsilon,\omega_\delta})$ normalized by $\|\Psi_\delta\|_\infty = 1$ and $C_1$ and $C_2$ are positive constants to be specified later on. Consider now the function $\psi := \sup(\psi_1, \psi_2)$, we will prove that for well chosen $C_1$ and $C_2$, $\psi$ is a supersolution of (3.1).

On $\Omega \setminus \omega_{\frac{\delta}{2}}$, a short computation shows that for $C_1$ large

$$\begin{aligned}\mathcal{L}_\Omega[\psi] + a\psi - \beta\psi^p &\le C_1^{\frac{p+1}{2}}\left(\int_\Omega K(x,y)\,dy + a - \beta C_1^{\frac{p-1}{2}}\right), \\ &\le C_1^{\frac{p+1}{2}}\left(\int_\Omega K(x,y)\,dy + a - \inf_{\Omega\setminus\omega_{\frac{\delta}{2}}}(\beta) C_1^{\frac{p-1}{2}}\right).\end{aligned}$$

By construction $\inf_{\Omega\setminus\omega_{\frac{\delta}{2}}}(\beta) > 0$ and $\frac{p-1}{2} > 0$, so for $C_1$ large enough we have on $\Omega\setminus\omega_{\frac{\delta}{2}}$

$$\mathcal{L}_\Omega[\psi] + a\psi - \beta\psi^p \leq 0. \tag{3.8}$$

Now, observe that for $C_2 \geq C_1$ large, on $\omega_{\frac{\delta}{2}}$ we have

$$\begin{aligned}\mathcal{L}_\Omega[\psi] + a\psi - \beta\psi^p \leq C_1\int_{\Omega\setminus\omega_\delta} K(x,y)\,dy + C_2\int_{\omega_\delta\setminus\omega_{\frac{\delta}{2}}} K(x,y)\,dy\\ + C_2\left(\int_{\omega_{\frac{\delta}{2}}} K(x,y)\Psi_\delta(y)\,dy - k(x)\Psi_\delta + a(x)\Psi_\delta\right).\end{aligned}$$

Since $\Psi_\delta > 0$ in $\omega_\delta$, we have

$$\begin{aligned}\mathcal{L}_\Omega[\psi] + a\psi - \beta\psi^p \leq C_2\left(\frac{C_1}{C_2}\bar{K}_1 + \bar{K}_2|\omega_\delta\setminus\omega_{\frac{\delta}{2}}|\right)\\ + C_2\left(\int_{\omega_\delta} K(x,y)\Psi_\delta(y)\,dy - k(x)\Psi_\delta + a(x))\Psi_\delta\right),\end{aligned}$$

where $\bar{K}_1 := \sup_{x\in\Omega}\int_\Omega K(x,y)\,dy$ and $\bar{K}_2 := \|K(\cdot,\cdot)\|_\infty$.

Recall that $\Psi_\delta$ is the eigenfunction associated with $\mu_p(\mathcal{L}_{\epsilon,\omega})$, so it follows that

$$\mathcal{L}_\Omega[\psi] + a\psi - \beta\psi^p \leq C_2\left(\frac{C_1}{C_2}\bar{K}_1 + \bar{K}_2|\omega_\delta\setminus\omega_{\frac{\delta}{2}}| + (a - k - a_\epsilon - \mu_p(\mathcal{L}_{\epsilon,\omega_\delta}))\Psi_\delta\right)$$

which combined with (3.7) reduces to

$$\mathcal{L}_\Omega[\psi] + a\psi - \beta\psi^p \leq C_2\left(\frac{C_1}{C_2}\bar{K}_1 + \bar{K}_2|\omega_\delta\setminus\omega_{\frac{\delta}{2}}| - \frac{\mu_p(\mathcal{L}_\omega + a)}{8}\Psi_\delta\right).$$

By using that for all $\delta\in[0,\delta_0]$, the principal eigenfunction $\Psi_\delta$ associated to $\mu_p(\mathcal{L}_{\epsilon,\omega_\delta})$ is positive and continuous, we can see that

$$\inf_{\delta\in[0,\delta_0]}\inf_{\omega_\delta}\Psi_\delta > c,$$

for some positive constant $c$. Moreover we can find $\delta$ small, say $\delta\leq\delta_1$ so that for all $\delta\leq\delta_1$

$$\bar{K}_2|\omega_\delta\setminus\omega_{\frac{\delta}{2}}| - \frac{\mu_p(\mathcal{L}_\omega)}{8}\Psi_\delta \leq -\frac{\mu_p(\mathcal{L}_\omega + a)}{16}c.$$

Thus for $\delta\leq\delta_1$, we achieve on $\omega_{\frac{\delta}{2}}$

$$\mathcal{L}_\Omega[\psi] + a\psi - \beta\psi^p \leq C_2\left(\frac{C_1}{C_2}\bar{K}_1 - \frac{\mu_p(\mathcal{L}_\omega + a)}{16}c\right).$$

Now by choosing $C_2 := C_1^{\frac{p+1}{2}}$ we have $\lim_{C_1\to+\infty}\frac{C_1}{C_2} = 0$ since $p > 1$. So for $C_1$ large enough, say $C_1 \geq C_1^* := \left(\frac{32\bar{K}_1}{\mu_p(\mathcal{L}_\omega + a)c}\right)^{\frac{2}{p-1}}$ we achieve on $\omega_\delta$

$$\mathcal{L}_\Omega[\psi] + a\psi - \beta\psi^p \leq -cC_1^{\frac{p+1}{2}}\frac{\mu_p(\mathcal{L}_\omega + a)}{32} < 0. \tag{3.9}$$

Hence from (3.8) and (3.9) we see that the function $\psi$ is a positive continuous supersolution of (3.1).

Let us now assume that $\overset{\circ}{\omega} = \emptyset$. In this situation, we have $0 < \mu_p(\mathcal{L}_\omega + a) = -\sup_\omega a$. By continuity of $a$ and $K$ there exists $\delta$ small so that

$$\sup_{x\in\omega_\delta}\int_{\omega_\delta} K(x,y)dy \leq \frac{\mu_p(\mathcal{L}_\omega + a)}{2} < -\sup_{\omega_\delta} a,$$

where as above $\omega_\delta := \{x\in\Omega | d(x,\omega) < \delta\}$.

From the above inequality it follows from $(v)$ of Proposition 2.4 that $0 < \mu_p(\mathcal{L}_{\omega_\delta} + a)$. Let us consider $\beta_\delta := \beta\eta_1$, where $\eta_1$ is constructed above, then we have $\beta_\delta \leq \beta$ and $\omega_{\frac{\delta}{2}} = \{x \in$

$\Omega|\beta_\delta(x) = 0\}$. By construction $\overset{\circ}{\omega}_{\frac{\delta}{2}} \neq \emptyset$ and $0 < \mu_p(\mathcal{L}_{\omega_\delta} + a) \leq \mu_p(\mathcal{L}_{\omega_{\frac{\delta}{2}}} + a)$, therefore by using the above arguments there exists a positive continuous supersolution $\psi$ to

$$\mathcal{L}_\Omega[u] + a(x)u - \beta_\delta u^p = 0 \quad \text{in} \quad \Omega.$$

Thanks to $\beta_\delta \leq \beta$, we have

$$\mathcal{L}_\Omega[\psi] + a(x)\psi - \beta\psi^p \leq \mathcal{L}_\Omega[\psi] + a(x)\psi - \beta_\delta\psi^p \leq 0 \quad \text{in} \quad \Omega$$

and $\psi$ is our desired supersolution.

□

## 4. THE PARTIALLY CONTROLLED PROBLEM: THE KPP CASE

In this section we analyse the dependence in $\lambda$ of the positive continuous solutions to (1.1) in absence of a refuge zone and we prove the Theorem 1.2 that we recall below. More precisely, we look for positive continuous solution of the partially controlled problem:

$$\int_\Omega K(x,y)u(y)\,dy - k(x)u(x) + a_0(x)u + \lambda a_1(x)u - \beta u^p = 0 \tag{4.1}$$

when $\beta > 0$ and $\lambda \in \mathbb{R}$.

In absence of a refuge zone, we can show that there exists a critical value $\lambda^*$ characterising completely the existence/non existence of a positive stationary solution. More precisely we have,

**Theorem 4.1.** *Assume that $K, a_i$ and $\beta$ satisfy* (1.9)–(1.10). *Assume further that $\beta > 0$ in $\bar{\Omega}$ then there exists $\lambda^* \in [-\infty;\infty)$, so that for all $\lambda > \lambda^*$ there exists a unique positive continuous solution $u_\lambda$ to* (4.1). *When $\lambda^* \in \mathbb{R}$, there is no positive solution to* (1.1) *for all $\lambda \leq \lambda^*$. Moreover, we have the following trichotomy:*

- *$\lambda^* = -\infty$ when $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) < 0$,*
- *$\lambda^* \in [-\infty,\infty)$ when $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) = 0$.*
- *$\lambda^* \in \mathbb{R}$ when $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) > 0$.*

*In addition, the map $\lambda \to u_\lambda$ is monotone increasing and we have*

$$\forall\, x \in \bar{\Omega} \qquad \lim_{\lambda\to+\infty} u_\lambda(x) = +\infty,$$
$$\forall\, x \in \bar{\Omega} \qquad \lim_{\lambda\to\lambda^{*,+}} u_\lambda(x) = u_\infty(x),$$

*where $u_\infty \equiv 0$ on $\Omega_1 := \{x \in \Omega|\, a_1(x) > 0\}$ and $u_\infty$ is a nonnegative solution to*

$$\int_{\Omega\setminus\Omega_1} K(x,y)u(y)dy - k(x)u + a_0(x)u - \beta u^p = 0 \quad in \quad \Omega\setminus\Omega_1.$$

*Furthermore, $u_\infty$ is non trivial when $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) < 0$.*

**Proof:**

In absence of a refuge zone, we observe that the problem (4.1) is a particular case of the KPP equation (2.1) where the nonlinearity $f$ is given by $f(x,s) := a_0 s + \lambda a_1 s - \beta s^p$. Therefore by the Theorem 2.1, for each $\lambda \in \mathbb{R}$ the existence of a positive solution to (4.1) is conditioned by the sign of $\mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1)$.

First let us observe that for $\lambda > \frac{\|k\|_\infty - \|a_0\|_\infty}{\|a_1\|_\infty}$ we have $sup_{x\in\Omega}(a_0(x) + \lambda a_1(x) - k(x)) > 0$ and by $(v)$ of Proposition 2.4 we have $\mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1) \leq -sup_\Omega(a_0(x) + \lambda a_1(x) - k(x)) < 0$. Therefore by Theorem 2.1, there exists a positive solution to (4.1) for all $\lambda > \frac{\|k\|_\infty - \|a_0\|_\infty}{\|a_1\|_\infty}$. Let us consider the following set $\{\lambda\,|\,\mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1) = 0\}$. When $\{\lambda\,|\,\mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1) = 0\} \neq \emptyset$, by monotonicity of $\mu_p$ with respect to $\lambda$ ($(ii)$ of Proposition 2.4 ), we can see that $\{\lambda\,|\,\mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1) = 0\}$ is bounded from above. Therefore we can define $\lambda^* \in [-\infty, +\infty)$ by the following formula

$$\lambda^* := \sup\{\lambda\,|\,\mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1) = 0\},$$

where we set $\lambda^* = -\infty$ when $\{\lambda\,|\,\mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1) = 0\} = \emptyset$.

By construction, thanks to Theorem 2.1 for all $\lambda > \lambda^*$ there exists a unique positive continuous solution to (4.1) and when $\lambda^* \in \mathbb{R}$, there is no positive solution to (1.1) for all $\lambda \le \lambda^*$.

Before proving the trichotomy, let us look at the asymptotic behaviours with respect to $\lambda$ of the unique solution $u_\lambda$. First let us observe that the map $\lambda \mapsto u_\lambda$ is monotone non decreasing. Indeed, thanks to the nonnegativity of $a_1$, for any $\lambda \ge \lambda'$, the continuous bounded function $u_{\lambda'}$ is a subsolution of the problem

$$\int_\Omega K(x,y)u(y)\,dy - k(x)u(x) + a_0(x)u + \lambda a_1(x)u - \beta u^p = 0. \tag{4.2}$$

Observe that any large constant M is a super-solution of (4.2). Therefore by taking $M$ large enough we have $u_{\lambda'} \le M$ and by the monotone iteration scheme we can construct a positive bounded solution of (4.2) which satisfies $u_{\lambda'} \le u \le M$. We conclude by using the uniqueness of the solution of problem (4.2). Hence, $u_{\lambda'} \le u_\lambda \equiv u$.

The asymptotic behaviour of $u_\lambda$ when $\lambda \to +\infty$ is obtained by establishing a bound from below for the solution $u_\lambda$ when $\lambda \to +\infty$. More precisely we show that for all $x \in \Omega_1$ we have for $\lambda$ large enough

$$u_\lambda(x) \ge \left(\frac{\lambda a_1 + a_0 - k(x)}{\sup_\Omega \beta}\right)^{\frac{1}{p-1}}. \tag{4.3}$$

Indeed from (4.1) using that $u_\lambda$ is non negative we have

$$\beta(x)u_\lambda^p(x) \ge [k(x) + a_0(x) + \lambda a_1(x)]u_\lambda.$$

Thus for $x \in \Omega_1$ (4.3) holds for $\lambda$ large enough. From (4.3) we get trivially that for all $x \in \Omega_1$

$$\lim_{\lambda\to+\infty} u_\lambda(x) \ge \lim_{\lambda\to+\infty} \left(\frac{\lambda a_1 + a_0 - k(x)}{\sup_\Omega \beta}\right)^{\frac{1}{p-1}} = +\infty.$$

So for $x \in \Omega \setminus \Omega_1$ so that $|B_{\epsilon_0}(x) \cap \Omega_1| > 0$ where $\epsilon_0$ is given by (1.10) we conclude that

$$\lim_{\lambda\to+\infty} \int_{B_{\epsilon_0}(x)\cap\Omega_1} u_\lambda(y)\,dy = +\infty.$$

Therefore from (1.10), (4.1) and $u_\lambda \ge 0$ we deduce that

$$\left(\beta(x)u_\lambda^{p-1} + k(x)\right)u_\lambda(x) \ge \int_\Omega K(x,y)u_\lambda(y)\,dy \ge c_0 \int_{B_{\epsilon_0}(x)\cap\Omega_1} u_\lambda(y)\,dy$$

which leads to

$$\lim_{\lambda\to+\infty} u_\lambda \left(\beta(x)u_\lambda^{p-1} + k(x)\right) \ge \lim_{\lambda\to+\infty} c_0 \int_{B_{\epsilon_0}(x)\cap\Omega_1} u_\lambda(y)\,dy = +\infty \quad \text{for all} \quad x \in \bigcup_{z\in\Omega_1} B_{\epsilon_0}(z).$$

The later implies that

$$\lim_{\lambda\to+\infty} u_\lambda(x) = +\infty \quad \text{for all} \quad x \in \bigcup_{z\in\Omega_1} B_{\epsilon_0}(z).$$

By repeating the above argument with $\bigcup_{z\in\Omega_1} B_{\epsilon_0}(z)$ instead $\Omega_1$, we show that

$$\lim_{\lambda\to+\infty} u_\lambda(x) = +\infty \quad \text{for all} \quad x \in \bigcup_{z\in\Omega_1} B_{2\epsilon_0}(z).$$

By a finite iteration of the above argumentation, we get

$$\lim_{\lambda\to+\infty} u_\lambda(x) = +\infty \quad \text{for all} \quad x \in \bar{\Omega}.$$

Let us now deal with the limit of $u_\lambda$ when $\lambda \to \lambda^{*,+}$. First let us assume that $\lambda^{*,+} \in \mathbb{R}$. In this situation by using the positivity of $u_\lambda$ and the monotonicity of $u_\lambda$ with respect to $\lambda$, we deduce that $u_\lambda$ converges pointwise to $u_{\lambda^{*,+}}$ when $\lambda \to \lambda^{*,+}$. Moreover thanks to the Lebesgue dominated convergence Theorem by passing to the limit in (4.1), we see that $u_{\lambda^{*,+}}$ is a non negative solution of (4.1) with $\lambda = \lambda^{*,+}$. Therefore by Theorem 2.1 we deduce that $u_{\lambda^{*,+}} \equiv 0$ since $\mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1) = 0$. Thus in this case

$$\lim_{\lambda \to \lambda^{*,+}} u_\lambda(x) = 0 \quad \text{for all} \quad x \in \bar{\Omega}.$$

Lastly assume that $\lambda^{*,+} = -\infty$. Again by using the positivity of $u_\lambda$ and the monotonicity of $u_\lambda$ with respect to $\lambda$, we deduce that $u_\lambda$ converges pointwise to $u_\infty$ when $\lambda \to -\infty$. Now observe that by the monotonicity of $u_\lambda$, we have for all $\lambda \le 0$, $u_\lambda \le M_0 := \|u_0\|_\infty$ and

$$a_1(x)|\lambda| u_\lambda \le C_0,$$

where

$$C_0 := M_0 \left\| \int_\Omega K(\cdot, y)dy + k + a_0 \right\|_\infty + \|\beta\|_\infty M_0^p.$$

Therefore for $x \in \Omega_1$ , we deduce that

$$0 \le u_\infty(x) = \lim_{\lambda \to -\infty} u_\lambda(x) \le \lim_{\lambda \to -\infty} \frac{C_0}{a_1(x)|\lambda|} = 0.$$

By passing to the limit in the equation (4.1), thanks to the Lebesgue dominated convergence Theorem we see that $u_\infty$ satisfies the equation below

$$\mathcal{L}_{\Omega \setminus \Omega_1}[u] + a_0(x)u + \lambda a_1(x)u - \beta u^p = 0. \tag{4.4}$$

By Theorem 2.1, the existence of a positive solution to the above equation is governed by the sign of $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0)$. Therefore when $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) < 0$ there is a unique positive solution whereas for $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) \ge 0$ there is none. In the later case, we deduce that

$$\lim_{\lambda \to -\infty} u_\lambda(x) = 0 \quad \text{for all} \quad x \in \bar{\Omega}.$$

Now let us look more closely at the properties of $\lambda^*$ and prove the trichotomy

(1) $\lambda^* = -\infty$ when $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) < 0$,
(2) $\lambda^* \in [-\infty, \infty)$ when $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) = 0$,
(3) $\lambda^* \in \mathbb{R}$ when $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) > 0$.

**Case 1: $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) < 0$.** In this situation, observe that by $(i)$ of Proposition 2.4, we have for all $\lambda$

$$0 > \mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) = \mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0 + \lambda a_1) \ge \mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1).$$

Therefore, thanks to Theorem 2.1 there exists a positive non trivial solution to (4.1) for all $\lambda \in \mathbb{R}$ . Thus $\lambda^* = -\infty$.

**Case 2:$\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) = 0$.** In this situation, by monotonicity of $\mu_p$ with respect to $\lambda$ ($(ii)$ of Proposition 2.4 ) and $(i)$ of Proposition 2.4 either $\mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1) < 0$ for all $\lambda \le 0$ or there exists $\lambda_0 \le 0$ so that $\mu_p(\mathcal{L}_\Omega + a_0 + \lambda_0 a_1) = 0$. In the first situation, as above there exists a positive solution to (4.1) for any $\lambda$ and $\lambda^* = -\infty$. In the other case, $\lambda^* \ge \lambda_0$ and $\lambda^* \in \mathbb{R}$.

**Case 3:$\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1} + a_0) > 0$.** In this last situation, we claim that

***Claim*** – 4.1.

$$\liminf_{\lambda \to -\infty} \mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1) > 0.$$

Assume the claim holds true then this implies that $\{\lambda \,|\, \mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1) = 0\}$ is non empty and therefore $\lambda^* \in \mathbb{R}$. Indeed, since $\mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1) < 0$ for any $\lambda > \frac{\|k\|_\infty - \|a_0\|_\infty}{\|a_1\|_\infty}$ and by the claim there exists $\bar{\lambda}$ so that $\mu_p(\mathcal{L}_\Omega + a_0 + \bar{\lambda} a_1) > 0$, by continuity of $\mu_p$ with respect to $\lambda$ there exists a $\bar{\lambda} < \lambda_0 < \frac{\|k\|_\infty - \|a_0\|_\infty}{\|a_1\|_\infty}$ so that $\mu_p(\mathcal{L}_\Omega + a_0 + \lambda_0 a_1) = 0$.

**Proof of the Claim:**

The proof of this claim relies on the construction of an adequate test function. By arguing as in the proof of Claim 3.2 we can introduce a regularisation $a_\epsilon$ of $a_0-k$ so that the following operator

$$\mathcal{L}_{\epsilon,\Omega\setminus\Omega_1}[u]:=\int_{\Omega\setminus\Omega_1}K(x,y)u(y)dy+a_\epsilon(x)u$$

has a positive continuous principal eigenfunction. By continuity of $\mu_p(\mathcal{L}_{\epsilon,\Omega\setminus\Omega_1})$ with respect to $a_\epsilon$ ($(iii)$ of Proposition 2.4) we can find $\epsilon$ small so that

$$\mu_p(\mathcal{L}_{\epsilon,\Omega\setminus\Omega_1})-\|a_\epsilon-a_0+k\|_\infty\geq\frac{\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1}+a)}{2}.$$

Let $\epsilon$ be fixed and let $\Omega_\delta$ be the set

$$\Omega_\delta:=\{x\in\Omega_1\mid d(x;\partial\Omega_1)>\delta\}.$$

As in the proof of the Claim 3.2, by continuity of $\sup_{\Omega\setminus\Omega_\delta}$ and $\mu_p(\mathcal{L}_{\epsilon,\Omega\setminus\Omega_\delta})$ with respect to the domain we achieve for $\delta$ small enough, say $\delta\leq\delta_0$,

$$\mu_p(\mathcal{L}_{\epsilon,\Omega\setminus\Omega_1})\geq\mu_p(\mathcal{L}_{\epsilon,\Omega\setminus\Omega_\delta})\geq\mu_p(\mathcal{L}_{\epsilon,\Omega\setminus\Omega_{\delta_0}})\geq\|a_\epsilon-a_0+k\|_\infty+\frac{\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1}+a_0)}{8}. \tag{4.5}$$

By construction $\bar\Omega\setminus\Omega_{\frac{\delta}{2}}$ and $\bar\Omega_\delta$ are two disjoints bounded closed set, so by the Urysohn Lemma there exists a nonnegative continuous function $\eta_1$ such that $0\leq\eta_1\leq1,\eta_1(x)=1$ in $\bar\Omega\setminus\Omega_{\frac{\delta}{2}},\eta_1(x)=0$ in $\bar\Omega_\delta$.

Let $\psi_1,\psi_2$ be the following continuous functions

$$\psi_1:=\begin{cases}\Psi_\delta\eta_1 & \text{in}\quad \Omega\setminus\Omega_\delta\\ 0 & \text{elsewhere,}\end{cases}\qquad \psi_2:=\begin{cases}c_0\eta_2 & \text{in}\quad \Omega_1\\ 0 & \text{elsewhere.}\end{cases}$$

where $\Psi_\delta$ is the positive continuous eigenfunction associated with $\mu_p(\mathcal{L}_{\epsilon,\Omega\setminus\Omega_\delta})$ normalized by $\|\Psi_\delta\|_\infty=1$ and $c_0$ is positive constant to be specified later on. Consider now the function $\psi:=\sup(\psi_1,\psi_2)$ and let $\gamma$ be a positive constant to be fixed later on. We will prove that for $\gamma,\delta,\lambda$ and $c_0$ well chosen the function $\psi$ is an adequate test function for $\mathcal{L}_\Omega+a_0+\lambda a_1+\gamma$. So let us compute $\mathcal{L}_\Omega[\psi]+a\psi+\lambda a_1\psi+\gamma\psi$.

On $\Omega\setminus\Omega_{\frac{\delta}{2}}$, by construction we have

$$\begin{aligned}\mathcal{L}_\Omega[\psi]+a_0\psi+\lambda a_1\psi+\gamma\psi\leq\int_{\Omega\setminus\Omega_{\frac{\delta}{2}}}K(x,y)\Psi_\delta(y)\,dy+\|K(\cdot,\cdot)\|_\infty\left(|\Omega_{\frac{\delta}{2}}\setminus\Omega_\delta|+c_0|\Omega_\delta|\right)\\ +a_\epsilon\Psi_\delta+\|a_\epsilon+k-a_0\|_\infty\Psi_\delta+\lambda a_1\Psi_\delta+\gamma\Psi_\delta.\end{aligned}$$

Therefore by (4.5) we see that

$$\mathcal{L}_\Omega[\psi]+a_0\psi+\lambda a_1\psi+\gamma\psi\leq\left(-\frac{\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1}+a_0)}{8}+\gamma\right)\Psi_\delta+\|K(\cdot,\cdot)\|_\infty\left(|\Omega_{\frac{\delta}{2}}\setminus\Omega_\delta|+c_0|\Omega_\delta|\right). \tag{4.6}$$

Let $m_0$ be the constant

$$m_0:=\inf_{\delta\in[0,\delta_0]}\left(\inf_{x\in\Omega\setminus\Omega_\delta}\Psi_\delta(x)\right).$$

We have $m_0>0$ since for all $\delta\in[0,\delta_0]$ $\Psi_\delta$ is positive and continuous in $\bar\Omega\setminus\Omega_\delta$.

Now let us fix $\gamma:=\frac{\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1}+a_0)}{16}$ and choose $\delta$ and $c_0$ such that

$$|\Omega_{\frac{\delta}{2}}\setminus\Omega_\delta|\leq\frac{m_0\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1}+a_0)}{64\|K(\cdot,\cdot)\|_\infty}, \tag{4.7}$$

$$c_0\leq\frac{m_0\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1}+a_0)}{64\|K(\cdot,\cdot)\|_\infty|\Omega_1|}. \tag{4.8}$$

By combining (4.6), (4.7) and (4.8) we see that

$$\left(-\frac{\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1}+a_0)}{8}+\gamma\right)\Psi_\delta+\|K(\cdot,\cdot)\|_\infty\left(|\Omega_{\frac{\delta}{2}}\setminus\Omega_\delta|+c_0|\Omega_\delta|\right)\leq 0.$$

Therefore on $\Omega\setminus\Omega_{\frac{\delta}{2}}$, we achieve for all $\lambda\leq 0$

$$\mathcal{L}_\Omega[\psi]+(a_0+\lambda a_1+\gamma)\psi\leq 0. \tag{4.9}$$

Now on $\Omega_{\frac{\delta}{2}}$ we have by construction

$$\mathcal{L}_\Omega[\psi]+(a_0+\lambda a_1+\gamma)\psi\leq\|K(\cdot,\cdot)\|_\infty|\Omega|+\|a_0\|_\infty+\gamma+\lambda c_0\left(\inf_{\Omega_{\frac{\delta}{2}}}a_1\right).$$

Since $a_1>0$ in $\bar{\Omega}_{\frac{\delta}{2}}$, by choosing $\lambda\leq-\frac{\|K(\cdot,\cdot)\|_\infty|\Omega|+\|a_0\|_\infty+\gamma}{c_0\left(\inf_{\Omega_{\frac{\delta}{2}}}a_1\right)}$ we get

$$\mathcal{L}_\Omega[\psi]+(a_0+\lambda a_1+\gamma)\psi\leq 0\quad\text{in}\quad\Omega_{\frac{\delta}{2}}. \tag{4.10}$$

Hence from (4.9) and (4.10) we see that for $\lambda$ negative enough, the function $(\psi,\gamma)$ is an adequate test function for the operator $\mathcal{L}_\Omega+a_0+\lambda a_1$. That is: $\psi$ is a positive continuous function on $\Omega$ which satisfies

$$\mathcal{L}_\Omega[\psi]+(a_0+\lambda a_1+\gamma)\psi\leq 0.$$

So by definition of $\mu_p(\mathcal{L}_\Omega+a_0+\lambda a_1)$ we deduce that for $\lambda$ negative enough we have $\mu_p(\mathcal{L}_\Omega+a_0+\lambda a_1)\geq\gamma>0$.

□

## 5. The partially controlled problem: The refuge case

In this Section, we analyse (1.1) in the presence of a refuge zone, i.e. when there exists $\omega\subset\Omega$ so that $\beta_{|\omega}\equiv 0$. In a presence of a refuge zone, the analysis of (1.1) is more involved and the characterisation of the existence/non-existence of a positive solution of (1.1) cannot always be summarised to a single critical value $\lambda^*$. In this situation, we prove the Theorem 1.3 that we recall below:

**Theorem 5.1.** *Assume that $K, a_i$ and $\beta$ satisfy* (1.9)–(1.10). *Assume further that $\omega\neq\emptyset$, then there exists two quantities $\lambda^*,\lambda^{**}\in[-\infty,+\infty]$ so that we have the following dichotomy :*

- *Either $\lambda^{**}\leq\lambda^*$ and there exists no positive bounded solution to* (1.1).
- *Or $\lambda^*<\lambda^{**}$, and for all $\lambda\in(\lambda^*,\lambda^{**})$ there exists a unique positive bounded continuous solution to* (1.1). *When $\lambda^*,\lambda^{**}\in\mathbb{R}$, there is no positive bounded solution to* (1.1) *for all $\lambda\leq\lambda^*$ and for all $\lambda\geq\lambda^{**}$. Moreover, the map $\lambda\to u_\lambda$ is monotone increasing and we have*
  (i)
  $$\lim_{\lambda\to\lambda^{**,-}}\|u_\lambda\|_{\infty,\omega}=+\infty,$$
  *where $\|u_\lambda\|_{\infty,\omega}:=\sup_{x\in\omega}|u_\lambda(x)|$.*
  (ii) *If $\mu_p(\mathcal{L}_\omega+a_0+\lambda^{**}a_1)$ is an eigenvalue in $L^1(\omega)$ or $\lambda^{**}=+\infty$ then*
  $$\forall\, x\in\bar{\Omega},\ \lim_{\lambda\to\lambda^{**,-}}u_\lambda(x)=+\infty.$$
  (iii) *For all $x\in\bar{\Omega}$ we have $\lim_{\lambda\to\lambda^{*,+}}u_\lambda(x)=u_\infty(x)$, where $u_\infty$ is a function satisfying on $\Omega_1:=\{x\in\Omega|\,a_1(x)>0\,u_\infty\equiv 0$ and $u_\infty$ is a nonnegative solution to*
  $$\int_{\Omega\setminus\Omega_1}K(x,y)u(y)dy-k(x)u+a_0(x)u-\beta u^p=0\quad\text{in}\quad\Omega\setminus\Omega_1.$$
  *Furthermore, $u_\infty$ is non trivial when $\mu_p(\mathcal{L}_{\Omega\setminus\Omega_1}+a_0)<0$.*

**Proof:**

Thanks to Theorem 3.1 the existence of a positive unique bounded solution to (1.1) in presence of a refuge zone is conditioned by the following inequality

$$\mu_p(\mathcal{L}_\omega + a_0 + \lambda a_1) > 0 > \mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1). \tag{5.1}$$

Let us introduce the following quantities:

$$\lambda^* := \sup\{\lambda \,|\, \mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1) \geq 0\},$$
$$\lambda^{**} := \inf\{\lambda \,|\, \mu_p(\mathcal{L}_\omega + a_0 + \lambda a_1) \leq 0\}.$$

We can see that the description of the set of positive bounded solutions of (1.1) is then equivalent to show whether or not we have $\lambda^* < \lambda^{**}$. To answer this question, we analyse separately the three different situations :

(1) $\omega \subset \Omega \setminus \Omega_1$,
(2) $\omega \subset\subset \Omega_1$,
(3) $\omega \not\subset \Omega \setminus \Omega_1$ and $\omega \not\subset \Omega_1$.

Let us start with the analysis the first situation.

**Case 1:** $\omega \subset \Omega \backslash \Omega_1$**.** In this situation, since $\omega \subset \Omega\backslash\Omega_1$, we have $\mu_p(\mathcal{L}_\omega + a_0 + \lambda a_1) = \mu_p(\mathcal{L}_\omega + a_0)$. So from (5.1) we see that for all $\lambda$ there is no bounded solution to (4.1) when $\mu_p(\mathcal{L}_\omega + a_0) \leq 0$ whereas the existence of a bounded solution will be conditioned only by the sign of $\mu_p(\mathcal{L}_\Omega + a_0 + \lambda a_1)$ when $\mu_p(\mathcal{L}_\omega + a_0) > 0$. In the later case, the analysis of the Section 4 can be reproduced, so we get $\lambda^* \in [-\infty, \infty) < \lambda^{**} = +\infty$.

**Case 2:** $\omega \subset\subset \Omega_1$**.** In this situation, since $\omega \subset \Omega_1$, by $(v)$ of Proposition 2.4 we can see that for some positive constant $C$

$$-\lambda \sup_\omega a_1 - C \leq \mu_p(\mathcal{L}_\omega + a_0 + \lambda a_1) \leq C - \lambda \sup_\omega a_1.$$

Therefore, we see that $\lambda^{**} \in \mathbb{R}$ and by definition of $\lambda^{**}$ and $(i)$ of Proposition 2.4 we have

$$\mu_p(\mathcal{L}_\Omega + a_0 + \lambda^{**} a_1) \leq \mu_p(\mathcal{L}_\omega + a_0 + \lambda^{**} a_1) = 0.$$

From the above inequality, we get the following dichotomy:

- Either $\mu_p(\mathcal{L}_\Omega + a_0 + \lambda^{**} a_1) < 0$ and by $(ii)$ of Proposition 2.4, we deduce that $\lambda^* < \lambda^{**}$.
- Or $\mu_p(\mathcal{L}_\Omega + a_0 + \lambda^{**} a_1) = 0$ and by definition of $\lambda^*$ we have $\lambda^* \geq \lambda^{**}$ and for all $\lambda$ there is no positive bounded solution to (1.1).

**Case 3:** $\omega \not\subset \Omega \setminus \Omega_1$ **and** $\omega \not\subset \Omega_1$**.** In this situation, since $\omega \cap \Omega_1 \neq \emptyset$, by $(v)$ of Proposition 2.4 we can see that for some positive constant $C$

$$\mu_p(\mathcal{L}_\omega + a_0 + \lambda a_1) \leq C - \lambda \sup_{\omega \cap \Omega_1} a_1.$$

Therefore $\lambda^{**} \in [-\infty, +\infty)$. Now let us observe that in this situation we also have for all $\lambda$

$$\mu_p(\mathcal{L}_\omega + a_0 + \lambda a_1) \leq \mu_p(\mathcal{L}_{\omega \cap (\Omega \setminus \Omega_1)} + a_0 + \lambda a_1) = \mu_p(\mathcal{L}_{\omega \cap (\Omega \setminus \Omega_1)} + a_0).$$

We then have two case to analyse:

(*i*) $\mu_p(\mathcal{L}_{\omega \cap (\Omega \setminus \Omega_1)} + a_0) \leq 0$ :
In this situation, from the above inequality, we can already conclude that $\lambda^{**} = -\infty$. Hence in this situation, for all $\lambda$ there is no positive bounded solution of (1.1).

(*ii*) $\mu_p(\mathcal{L}_{\omega \cap (\Omega \setminus \Omega_1)} + a_0) > 0$ :
In this situation, by working as in Section 4 we see that there exists $\lambda << -1$ so that $\mu_p(\mathcal{L}_\omega + a_0 + \lambda a_1) > 0$. Therefore, $\mu^{**} \in \mathbb{R}$ and we can argue as in the Case 2.

Let us now look at the asymptotic behaviour of $u_\lambda$ with respect to $\lambda$. The monotone behaviour of $u_\lambda$ and its limit as $\lambda \to \lambda^*$ (i.e (iii)) can be obtained by following the arguments in Section 4, so we drop the proof here and prove only $(i)$ and $(ii)$ i.e. we analyse the limits of $u_\lambda$ as $\lambda \to \lambda^{**}$.

When $\lambda^{**} = +\infty$, the behaviour of $u_\lambda$ can be obtained by reproducing the arguments of Section 4 and we get for all $x \in \bar\Omega$

$$\lim_{\lambda\to+\infty} u_\lambda(x) = +\infty.$$

Now, let us assume that $\lambda^{**} \in \mathbb{R}$. By definition of $\lambda^{**}$, we must have

$$\mu_p(\mathcal{L}_\omega + a_0 + \lambda^{**}a_1) = 0.$$

As a consequence we also have $\mu_p(\mathcal{L}^*_\omega + a_0 + \lambda^{**}a_1) = 0$ where $\mathcal{L}^*_\omega + a_0 + \lambda^{**}a_1$ is defined by

$$\mathcal{L}^*_\omega[\phi] + a_0\phi + \lambda^{**}a_1\phi := \int_\omega K(y,x)\phi(y)dy - k(x)\phi + a_0\phi + \lambda^{**}a_1\phi.$$

Let us start with the proof of $(i)$. First assume that $\overset{\circ}{\omega} \neq \emptyset$, then by Theorem 2.2 there exists a positive measure $d\mu^*$ associated with $\mu_p(\mathcal{L}^*_\omega + a_0 + \lambda^{**}a_1)$. Moreover, $\phi^*_p(x)dx$ the regular part of $d\mu^*$ satisfies $\inf_\omega \phi^*_p > 0$. By integrating the equation (1.1) over $\omega$ with respect to the measure $d\mu^*$, we get for any $\lambda^* < \lambda < \lambda^{**}$

$$\int_\omega \left(\int_\Omega K(x,y)u_\lambda(y)\,dy\right) d\mu^* + \int_\omega (-k(x) + a_0 + \lambda a_1)u_\lambda\, d\mu^* = 0.$$

By definition of $\mu_p(\mathcal{L}^*_\omega + a_0 + \lambda^{**})$, it follows from the above equality

$$\int_\omega \left(\int_{\Omega\setminus\omega} K(x,y)u_\lambda(y)\,dy\right) d\mu^* = (\lambda^{**} - \lambda)\int_\omega a_1 u_\lambda\, d\mu^*.$$

Therefore, by using the monotonicity of the map $u_\lambda$ we have for $\lambda_0 < \lambda < \lambda^{**}$

$$\frac{1}{\lambda^{**}-\lambda}\int_\omega \left(\int_{\Omega\setminus\omega} K(x,y)u_{\lambda_0}(y)\,dy\right) d\mu^* \le \|a_1\|_\infty \|u_\lambda\|_{\infty,\omega}\int_\omega d\mu^*,$$

which enforces

$$\lim_{\lambda\to\lambda^{**}} \|u_\lambda\|_{\infty,\omega} = +\infty.$$

Assume now that $\overset{\circ}{\omega} = \emptyset$. In this situation, we have $\mu_p(\mathcal{L}_\omega + a_0 + \lambda^{**}a_1) = -\sup_\omega(-k + a_0 + \lambda^{**}a_1)$. Since $\omega$ is a compact set there exists $x_0 \in \omega$ so that $-k(x_0)+a_0(x_0)+\lambda^{**}a_1(x_0) = 0$. We can check that $x_0 \in \Omega_1$, otherwise we have $\sup_{\omega\cap(\Omega\setminus\Omega_1)}(-k+a_0) = \sup_{\omega\cap(\Omega\setminus\Omega_1)}(-k+a_0+\lambda^{**}a_1) = 0$ and $\mu_p(\mathcal{L}_{\omega\cap(\Omega\setminus\Omega_1)} + a_0) = 0$. The latter equality leads to the contradiction $-\infty < \lambda^{**} = -\infty$, since for all $\lambda$ we have

$$\mu_p(\mathcal{L}_\omega + a_0 + \lambda a_1) \le \mu_p(\mathcal{L}_{\omega\cap(\Omega\setminus\Omega_1)} + a_0) = 0.$$

Now at $x_0$, we have

$$\int_\Omega K(x_0,y)u_\lambda(y)\,dy = (\lambda^{**} - \lambda)a_1(x_0)u_\lambda(x_0).$$

By using that $u_\lambda$ is monotone with respect to $\lambda$ we get for all $\lambda_0 \le \lambda < \lambda^{**}$

$$\frac{1}{(\lambda^{**}-\lambda)a_1(x_0)}\int_\Omega K(x_0,y)u_{\lambda_0}(y)\,dy = u_\lambda(x_0),$$

which implies

$$\lim_{\lambda\to\lambda^{**}} u_\lambda(x_0) = +\infty.$$

Let us now prove $(ii)$. When $\mu_p(\mathcal{L}_\omega + a_0 + \lambda^{**}a_1)$ is associated with a positive $L^1(\omega)$ eigenfunction we claim that

***Claim*** – 5.1.

$$\lim_{\lambda\to\lambda^{**}} \int_\Omega u_\lambda(x)\,dx = +\infty.$$

Assume for the moment that the claim holds then we get $(ii)$ by arguing as follows. Since $\bar\Omega$ is compact, in view of the claim there exists $\bar x \in \bar\Omega$ so that

$$\lim_{\lambda\to\lambda^{**}} \int_{B(\bar x, \frac{\epsilon_0}{4})\cap\Omega} u_\lambda(x)\,dx = +\infty.$$

From the equation (1.1) and by the assumption (1.10) we always have

$$c_0 \int_{B(x,\epsilon_0)\cap\Omega} u_\lambda(x)\,dx \le (k(x) - a_0(x) - \lambda a_1(x))u_\lambda(x) + \beta(x)u_\lambda^p(x).$$

Therefore for all $x \in B(\bar x, \frac{\epsilon_0}{2})$, we have $B(\bar x, \frac{\epsilon_0}{4}) \subset B(x, \epsilon_0)$ which combined with the above inequalities implies that

$$\lim_{\lambda\to\lambda^{**}} (k(x) - a_0(x) - \lambda a_1(x))u_\lambda(x) + \beta(x)u_\lambda^p(x) = +\infty.$$

Thus

$$\lim_{\lambda\to\lambda^{**}} u_\lambda(x) = +\infty \quad \text{for all} \quad x \in B(\bar x, \frac{\epsilon_0}{2}) \cap \bar\Omega.$$

In the above arguments by replacing $\bar x$ by any $x \in B(\bar x, \frac{\epsilon_0}{4}) \cap \Omega$, we achieve

$$\lim_{\lambda\to\lambda^{**}} u_\lambda(x) = +\infty \quad \text{for all} \quad x \in \bigcup_{x\in B(\bar x, \frac{\epsilon_0}{4})} B(x, \frac{\epsilon_0}{2}) \cap \Omega.$$

Since $\Omega$ is compact we achieve $\lim_{\lambda\to\lambda^{**}} u_\lambda(x) = +\infty$ for all $x \in \Omega$ after a finite iteration of this argument.

□

**Proof of the Claim**

Assume by contradiction that $\sup_\lambda \int_\Omega u_\lambda(x)\,dx < +\infty$. Since $u_\lambda$ is monotone, by Lebesgue monotone convergence Theorem we have $u_\lambda \to \bar u$ in $L^1(\Omega)$ when $\lambda \to \lambda^{**}$ and $\bar u > u_{\lambda_0} > 0$ satisfies the equation

$$\mathcal{L}_\Omega[\bar u](x) + (a_0 + \lambda^{**}a_1)\bar u(x) - \beta(x)\bar u^p(x) = 0 \quad \text{for almost every } x \text{ in } \Omega.$$

Therefore we have

$$\mathcal{L}_\Omega[\bar u](x) + (a_0 + \lambda^{**}a_1)\bar u(x) = 0 \quad \text{for almost every } x \text{ in } \omega. \tag{5.2}$$

By assumption there exists a positive $L^1$ eigenfunction $\phi_p$ associated with $\mu_p(\mathcal{L}_\omega + a_0 + \lambda^{**}a_1)$. Moreover the positive function $\frac{1}{k(x)-a_0(x)-\lambda^{**}a_1(x)} \in L^1(\omega)$ and the compact operator $\mathcal{K}$:

$$\begin{array}{rcl} C(\omega) & \to & C(\omega) \\ v & \mapsto & \mathcal{K}[v] := \int_\omega K(x,y)\frac{v(y)dy}{k(y)-a_0(y)-\lambda^{**}a_1(y)}. \end{array}$$

is well defined. By construction, we can check that $\mu_p(\mathcal{K}) = -1$. Indeed, let $v_p := (k(x) - a_0(x) - \lambda^{**}a_1(x))\phi_p$ then we can see that $v_p$ is positive and continuous, since by assumption we have

$$\mathcal{L}_\omega[\phi_p] - v_p = 0.$$

Moreover, $v_p$ satisfies $\mathcal{K}[v_p] = v_p$. Thus by the Krein-Rutman theory, we have $\mu_p(\mathcal{K}) = -1$ and $\psi_1 := v_p$ where $\psi_1$ is the principal positive continuous eigenfunction associated with $\mu_p(\mathcal{K})$.

Let us now consider $\mathcal{K}^*$ the following compact operator

$$\begin{array}{rcl} L^1(\omega) & \to & L^1(\omega) \\ v & \mapsto & \mathcal{K}^*[v] := \frac{1}{k(x)-a_0(x)-\lambda^{**}a_1(x)}\int_\omega K(y,x)v(y)dy. \end{array}$$

By the Krein-Rutman Theory there exists an eigenvalue $\nu_1$ associated with a positive $L^1(\omega)$ function $\phi^*$. Furthermore we can check that $\nu_1 = -1$. Indeed, since $\phi^*$ is associated with $\nu_1$ we have

$$\mathcal{K}^*[\phi_p^*] = -\nu_1\phi_p^*.$$

By multiplying the above equation by $v_p$ and then integrating over $\omega$ it follows that

$$\nu_1 \int_o v_p(x)\phi^*(x)dx = -\int_\omega \frac{v_p(x)}{k(x)-a_0(x)-\lambda^{**}a_1(x)}\left(\int_\omega K(y,x)\phi^*(y)dy\right)dx$$
$$= -\int_\omega v_p(y)\phi^*(y)dy,$$

which implies that $\nu_1 = -1$.

Let $\bar{v}$ be the $L^1(\omega)$ function $\bar{v} := (k(x) - a_0(x) - \lambda^{**})\bar{u}$ then we get by (5.2)

$$\mathcal{K}[\bar{v}] - \bar{v} = -\int_{\Omega\setminus\omega} K(x,y)\bar{u}(y)\,dy \quad \text{for almost every } x \text{ in } \omega. \tag{5.3}$$

Since $\mathcal{K}[\bar{v}]$ is continuous, we deduce from (5.3) that $\bar{v}$ is continuous in $\omega$. So by multiplying (5.2) by $\phi^*$ and then integrating over $\omega$ we get

$$\int_\omega \phi_1^*(x)\left(\int_{\Omega\setminus\omega} K(x,y)\bar{u}(y)\,dy\right)dx = \int_\omega \phi_1^*(x)[\mathcal{K}[\bar{v}]-\bar{v}]dx$$

Since $\nu_1 = -1$ and $\bar{u} > 0$ we end up with the contradiction

$$0 < c_0 \leq \int_\omega \phi_1^*(x)\int_{\Omega\setminus\omega} K(x,y)\bar{u}(y)\,dydx = 0.$$

□

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