Ten-parameters deformations of the sixth order Peregrine breather solutions of the NLS equation.

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Abstract

In this paper, we construct new deformations of the Peregrine breather of order 6 with 10 real parameters. We obtain new families of quasi-rational solutions of the NLS equation. With this method, we construct new patterns of different types of rogue waves. We get as already found for the lower order, the triangular configurations and rings isolated. Moreover, one sees for certain values of the parameters the appearance of new configurations of concentric rings.

1 Introduction

Since fundamental work of Zakharov and Shabat in 1972, and the first expressions of the quasi-rational solutions given by Peregrine in 1983, a considerable number of studies were carried out. Eleonski, Akhmediev and Kulagin obtained the first higher order analogue of the Peregrine breather[3] in 1986. Akhmediev et al. [1, 4] , constructed other families of higher order, using Darboux transformations.
In 2010, rational solutions of the NLS equation have been written as a quotient of two Wronskians in [8]. In 2011, an other representation of the solutions of the NLS equation has been constructed in [10], also in terms of a ratio of two Wronskians determinants of order $2N$.

In 2012, an other representation of the solutions of the focusing NLS equation, as a ratio of two determinants has been given in [13] using generalized Darboux transform.
Ohta and Yang [17] have given a new approach where solutions of the focusing NLS equation by means of a determinant representation, obtained from Hirota bilinear method.
A the beginning of the year 2012, one obtained a representation in terms of determinants which does not involve limits [12].
These first two formulations given in [10, 12] did depend in fact only on two parameters. Then we found for the order N (for determinants of order 2N), solutions depending on $2N-2$ real parameters.
The purpose of this study is to present new solutions depending this time on strictly more than two parameters, to get all the possible patterns for the solutions of NLS equation. We construct solutions depending on 10 parameters which give the Peregrine breather as particular case when all the parameters are equal to 0 : it is the reason why we will call these solutions, 10 parameters deformations of the Peregrine of order 6.
We first recall the expressions of solutions of the two dimensional focusing nonlinear Schrödinger equation [10] in terms of wronskians. Then, we construct new quasi rational solutions depending a priori on $2N-2$ parameters at the order N. After, one builds various drawings to illustrate the evolution of the solutions according to the parameters.
One obtains at the same time triangular configurations and ring structures with a maximum of 21 peaks. The complete analytical expression of the solutions depending on 10 parameters is found, but is too monstrous to be published. These deformations are completely new and gives by new patterns a best understanding of the NLS equation.

2 Determinant representation of solutions of NLS equation

We recall the results obtained in [10] and [12]. We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0. \tag{1}$$

In the following, we consider $2N$ parameters λ_ν, $\nu = 1, \ldots, 2N$ satisfying the relations

$$0 < \lambda_j < 1, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \leq j \leq N. \tag{2}$$

We define the terms κ_ν, δ_ν, γ_ν by the following equations,

$$\kappa_\nu = 2\sqrt{1-\lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}}, \tag{3}$$

and

$$\kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \quad j = 1 \ldots N. \tag{4}$$

The terms $x_{r,\nu}$ $(r = 3, 1)$ are defined by

$$x_{r,\nu} = (r-1)\ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \leq j \leq 2N. \tag{5}$$

The parameters e_ν are defined by

$$e_j = ia_j - b_j, \quad e_{N+j} = ia_j + b_j, \quad 1 \leq j \leq N, \tag{6}$$

where a_j and b_j, for $1 \leq j \leq N$ are arbitrary real numbers.
We use the following notations :

$$A_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2,$$

$$B_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2,$$

for $1 \leq \nu \leq 2N$, with κ_ν, δ_ν, $x_{r,\nu}$ defined in (3), (4) and (5).
The parameters e_ν are defined by (6).
Here, the parameters a_j and b_j, for $1 \leq N$ are chosen in the form

$$a_j = \sum_{k=1}^{N-1} \tilde{a_k} \epsilon^{2k+1} j^{2k+1}, \quad b_j = \sum_{k=1}^{N-1} \tilde{b_k} \epsilon^{2k+1} j^{2k+1}, \quad 1 \leq j \leq N. \tag{7}$$

We consider the following functions :

$$\begin{gathered} f_{4j+1,k} = \gamma_k^{4j-1} \sin A_k, \quad f_{4j+2,k} = \gamma_k^{4j} \cos A_k, \\ f_{4j+3,k} = -\gamma_k^{4j+1} \sin A_k, \quad f_{4j+4,k} = -\gamma_k^{4j+2} \cos A_k, \end{gathered} \tag{8}$$

for $1 \leq k \leq N$, and

$$f_{4j+1,k} = \gamma_k^{2N-4j-2} \cos A_k, \quad f_{4j+2,k} = -\gamma_k^{2N-4j-3} \sin A_k,$$
$$f_{4j+3,k} = -\gamma_k^{2N-4j-4} \cos A_k, \quad f_{4j+4,k} = \gamma_k^{2N-4j-5} \sin A_k, \tag{9}$$

for $N+1 \leq k \leq 2N$.
We define the functions $g_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way, we replace only the term A_k by B_k.

$$g_{4j+1,k} = \gamma_k^{4j-1} \sin B_k, \quad g_{4j+2,k} = \gamma_k^{4j} \cos B_k,$$
$$g_{4j+3,k} = -\gamma_k^{4j+1} \sin B_k, \quad g_{4j+4,k} = -\gamma_k^{4j+2} \cos B_k, \tag{10}$$

for $1 \leq k \leq N$, and

$$g_{4j+1,k} = \gamma_k^{2N-4j-2} \cos B_k, \quad g_{4j+2,k} = -\gamma_k^{2N-4j-3} \sin B_k,$$
$$g_{4j+3,k} = -\gamma_k^{2N-4j-4} \cos B_k, \quad g_{4j+4,k} = \gamma_k^{2N-4j-5} \sin B_k, \tag{11}$$

for $N+1 \leq k \leq 2N$.
Then we get the following result :

Theorem 2.1 *The function v defined by*

$$v(x,t) = \exp(2it - i\varphi) \times \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} \tag{12}$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

depending on $2N-2$ parameters $\tilde{a}_j$, $\tilde{a}_j$, $1 \leq j \leq N$, where

$n_{j1} = f_{j,1}(x,t,0)$, $1 \leq j \leq 2N$ $\quad n_{jk} = \dfrac{\partial^{2k-2} f_{j,1}}{\partial \epsilon^{2k-2}}(x,t,0)$, $2 \leq k \leq N$, $1 \leq j \leq 2N$

$n_{jN+1} = f_{j,N+1}(x,t,0)$, $1 \leq j \leq 2N$ $\quad n_{jN+k} = \dfrac{\partial^{2k-2} f_{j,N+1}}{\partial \epsilon^{2k-2}}(x,t,0)$, $2 \leq k \leq N$, $1 \leq j \leq 2N$

$d_{j1} = g_{j,1}(x,t,0)$, $1 \leq j \leq 2N$ $\quad d_{jk} = \dfrac{\partial^{2k-2} g_{j,1}}{\partial \epsilon^{2k-2}}(x,t,0)$, $2 \leq k \leq N$, $1 \leq j \leq 2N$

$d_{jN+1} = g_{j,N+1}(x,t,0)$, $1 \leq j \leq 2N$ $\quad d_{jN+k} = \dfrac{\partial^{2k-2} g_{j,N+1}}{\partial \epsilon^{2k-2}}(x,t,0)$, $2 \leq k \leq N$, $1 \leq j \leq 2N$

The functions f and g are defined in (8),(9), (10), (11).

We will give the proof in a forthcoming paper.
The solutions of the NLS equation can also be written in the form :

$$v(x,t) = \exp(2it - i\varphi) \times Q(x,t)$$

where $Q(x,t)$ is defined by :

$$Q(x,t) := \frac{\begin{vmatrix} f_{1,1}[0] & \dots & f_{1,1}[N-1] & f_{1,N+1}[0] & \dots & f_{1,N+1}[N-1] \\ f_{2,1}[0] & \dots & f_{2,1}[N-1] & f_{2,N+1}[0] & \dots & f_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{2N,1}[0] & \dots & f_{2N,1}[N-1] & f_{2N,N+1}[0] & \dots & f_{2N,N+1}[N-1] \end{vmatrix}}{\begin{vmatrix} g_{1,1}[0] & \dots & g_{1,1}[N-1] & g_{1,N+1}[0] & \dots & g_{1,N+1}[N-1] \\ g_{2,1}[0] & \dots & g_{2,1}[N-1] & g_{2,N+1}[0] & \dots & g_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{2N,1}[0] & \dots & g_{2N,1}[N-1] & g_{2N,N+1}[0] & \dots & g_{2N,N+1}[N-1] \end{vmatrix}} \quad (13)$$

3 Quasi-rational solutions of order 6 with ten parameters

Wa have already constructed in [10] solutions for the cases from $N = 1$ until $N = 6$, and in [12] with two parameters.
Because of the length of the expression v of the solution of NLS equation with eight parameters, we can't give here. We only construct figures to show deformations of the fifth Peregrine breathers.
Conversely to the study with two parameters given in preceding works [10, 12], we get other type of symmetries in the plots in the (x,t) plane. We give some examples of this fact in the following.

3.1 Peregrine breather of order 6

If we choose $\tilde{a}_1 = \tilde{b}_1 = \tilde{a}_2 = \tilde{b}_2 = \tilde{a}_3 = \tilde{b}_3 = \tilde{a}_4 = \tilde{b}_4 = \tilde{a}_5 = \tilde{b}_5 = 0$, we obtain the classical Peregrine breather :
Figure 1

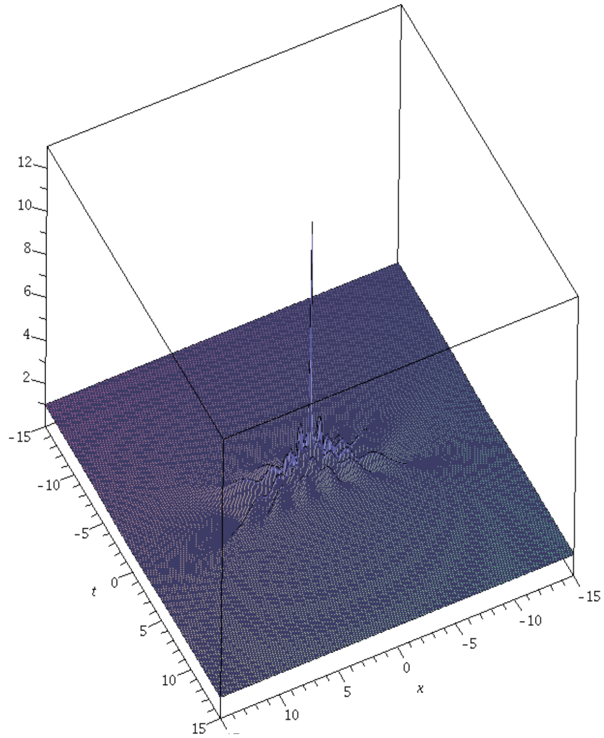

Figure 1: Solution of NLS, N=6, all parameters equal to 0 , Peregrine breather P_6.

With other choices of parameters, we obtain all types of configurations : triangles, overlapping triangular-circular and circular configurations with a maximum of 21 peaks.

3.2 Variation of parameters

In the case of the variation of one parameter, we obtain different types of configuration with a maximum of 21 peaks.
In the cases $a_1 \neq 0$ or $b_1 \neq 0$ we obtain triangles; for $a_2 \neq 0$ or $b_2 \neq 0$, we have 3 concentric rings with two of them with 5 peaks and an another with 10 peaks with a central peak. For $a_3 \neq 0$ or $b_3 \neq 0$, we obtain 3 concentric rings with 7 peaks on each of them without a central peak. For $a_4 \neq 0$ or $b_4 \neq 0$, we have 2 concentric rings with 9 peaks with inside (for large values of parameters) the apparition of the Peregrine breather of order 2 with 3

peaks. For $a_5 \neq 0$ or $b_5 \neq 0$, we have only one ring with $2N-1=11$ peaks with inside (for large values of parameters) the apparition of the Peregrine breather of order $N-2=4$ with 10 peaks.
Figure 2 $a_1 \neq 0$

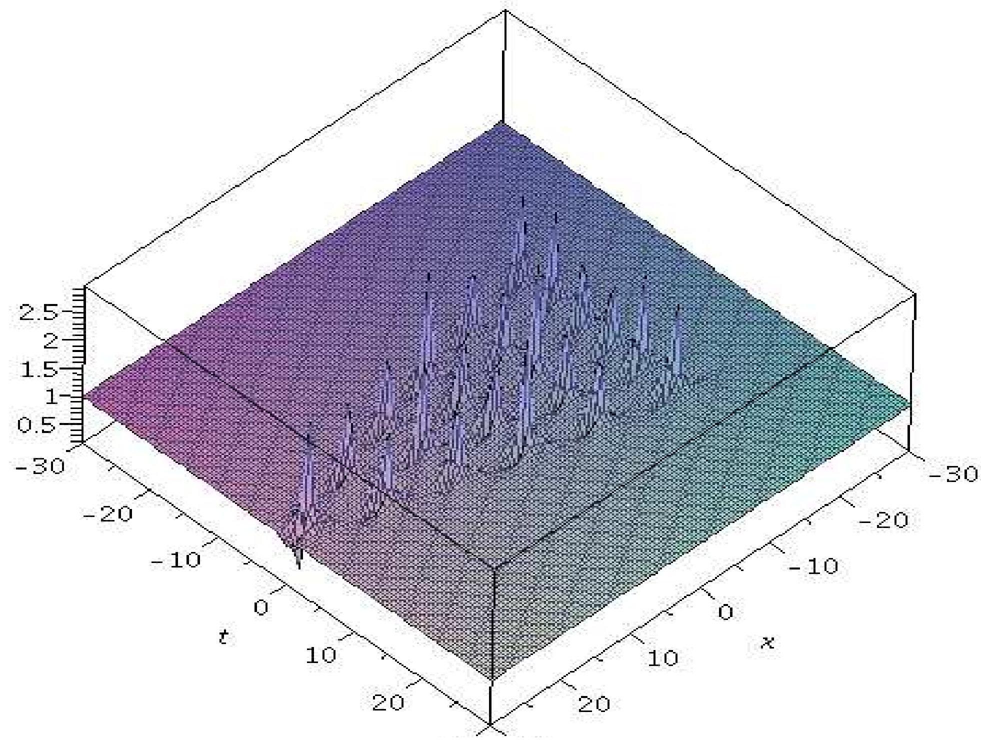

Figure 2: Solution of NLS, N=6, $\tilde{a}_1 = 10^3$, triangle with 21 peaks.

Figure 3 $b_1 \neq 0$

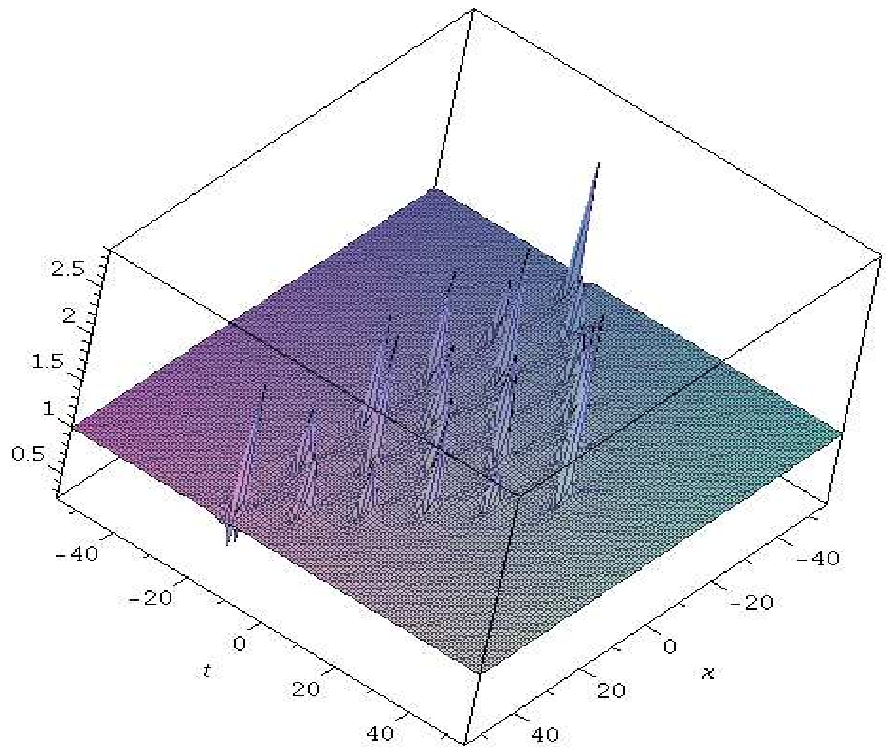

Figure 3: Solution of NLS, N=6, $\tilde{b}_1 = 10^4$, triangle with 21 peaks.

Figure 4 $a_2 \neq 0$

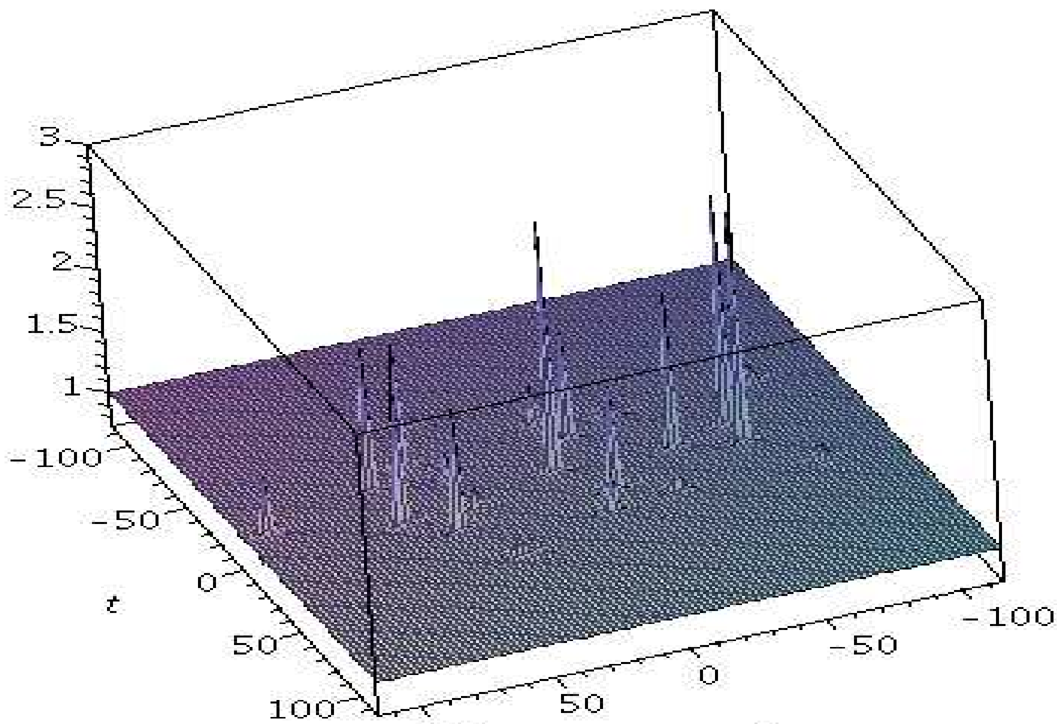

Figure 4: Solution of NLS, N=6, $\tilde{a}_2 = 10^8$, 3 rings with respectively 5, 10, 5 peaks with in the center one peak.

Figure 5 $b_2 \neq 0$

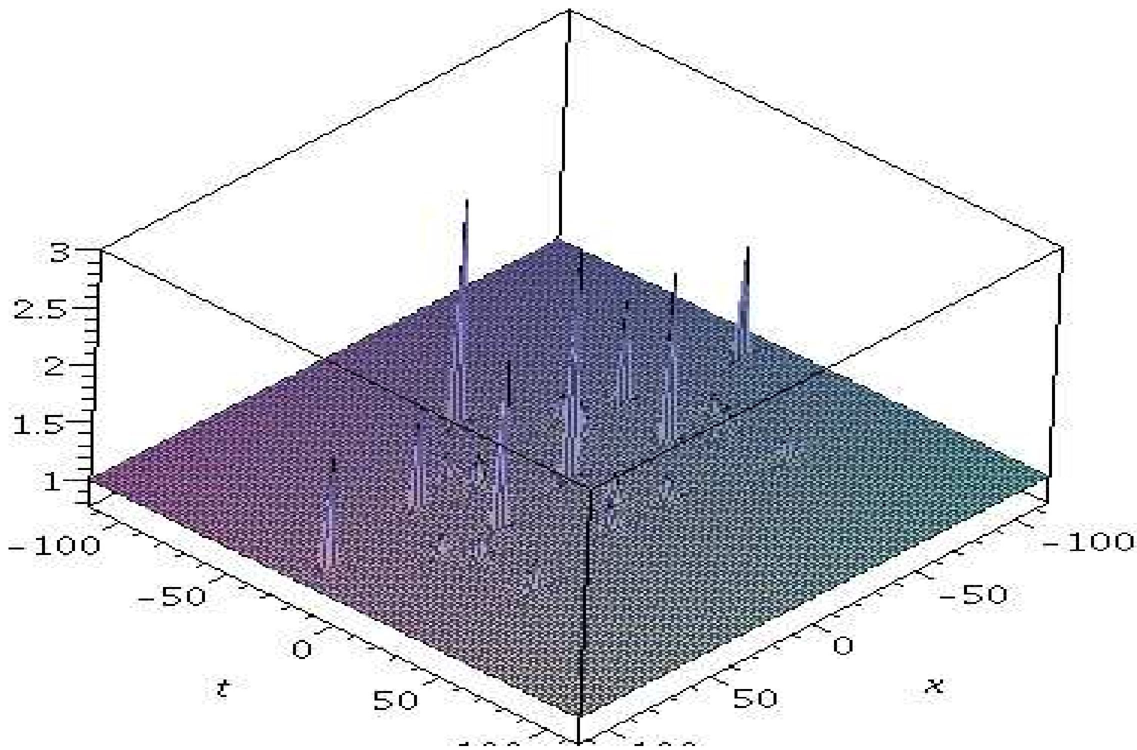

Figure 5: Solution of NLS, N=6, $\tilde{b}_2 = 10^8$, 3 rings with respectively 5, 10, 5 peaks with in the center one peak.

Figure 6 $a_3 \neq 0$

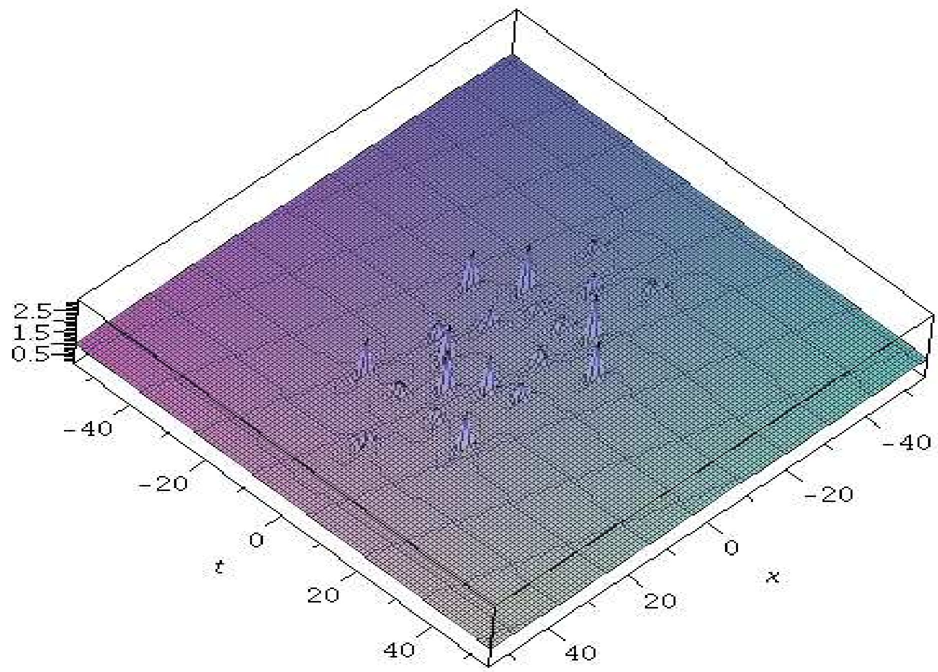

Figure 6: Solution of NLS, N=6, $\tilde{a}_3 = 10^8$, 3 rings with 7 peaks on each of them without central peak.

Figure 7 $b_3 \neq 0$

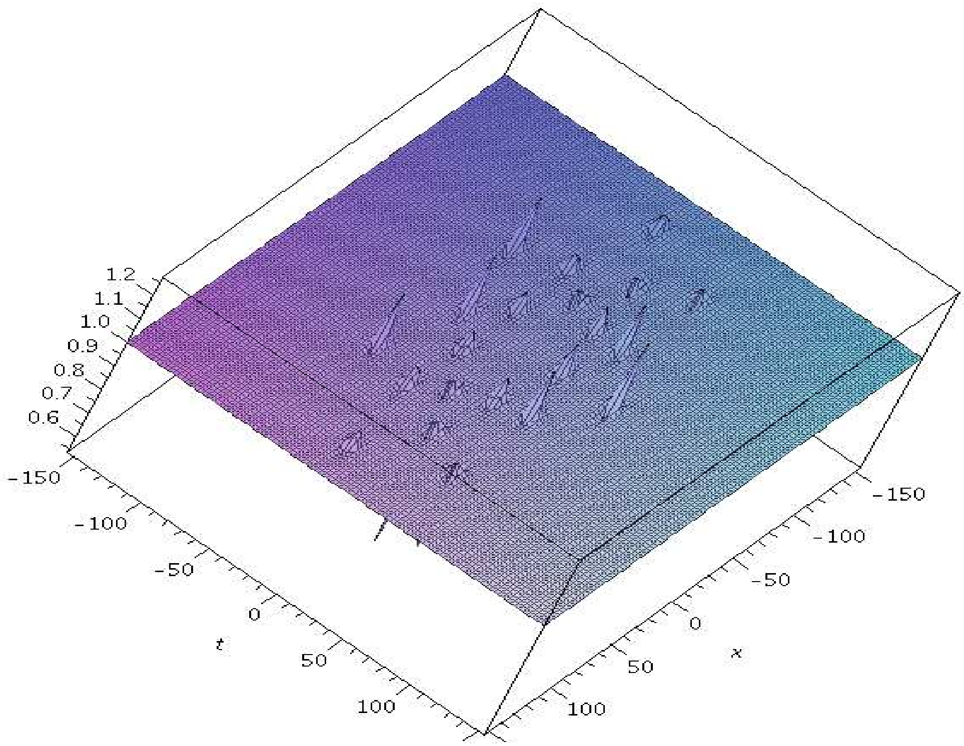

Figure 7: Solution of NLS, N=6, $\tilde{b}_3 = 10^{12}$, 3 rings with 7 peaks on each of them without central peak.

Figure 8 $a_4 \neq 0$

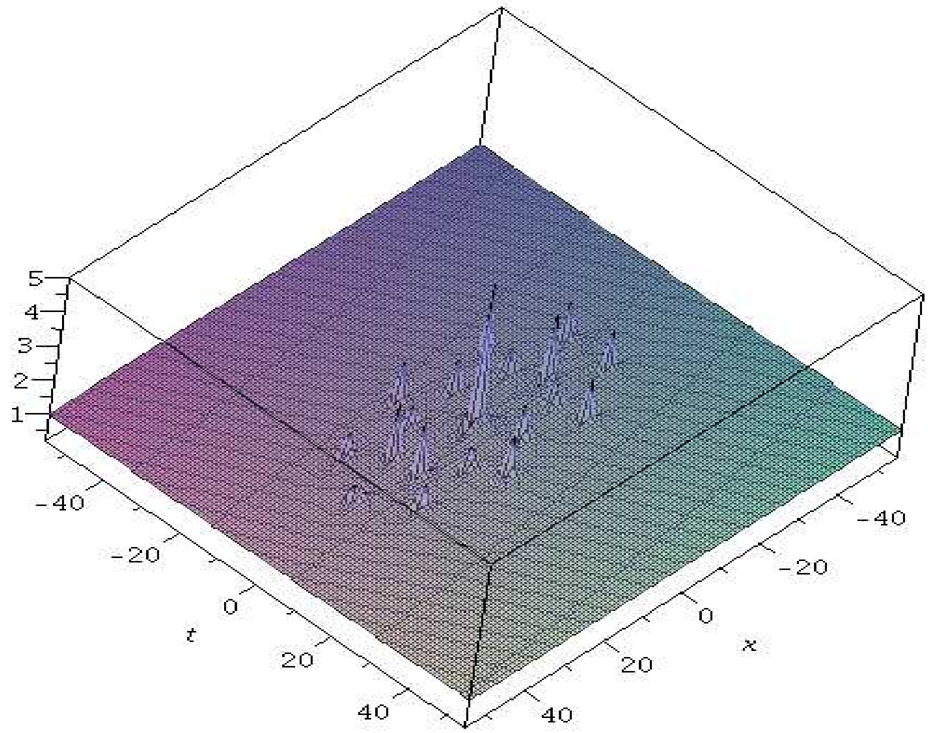

Figure 8: Solution of NLS, N=6, $\tilde{a}_4 = 10^{10}$, 2 rings with 9 peaks with in the center the Peregrine P_2.

Figure 9 $b_4 \neq 0$

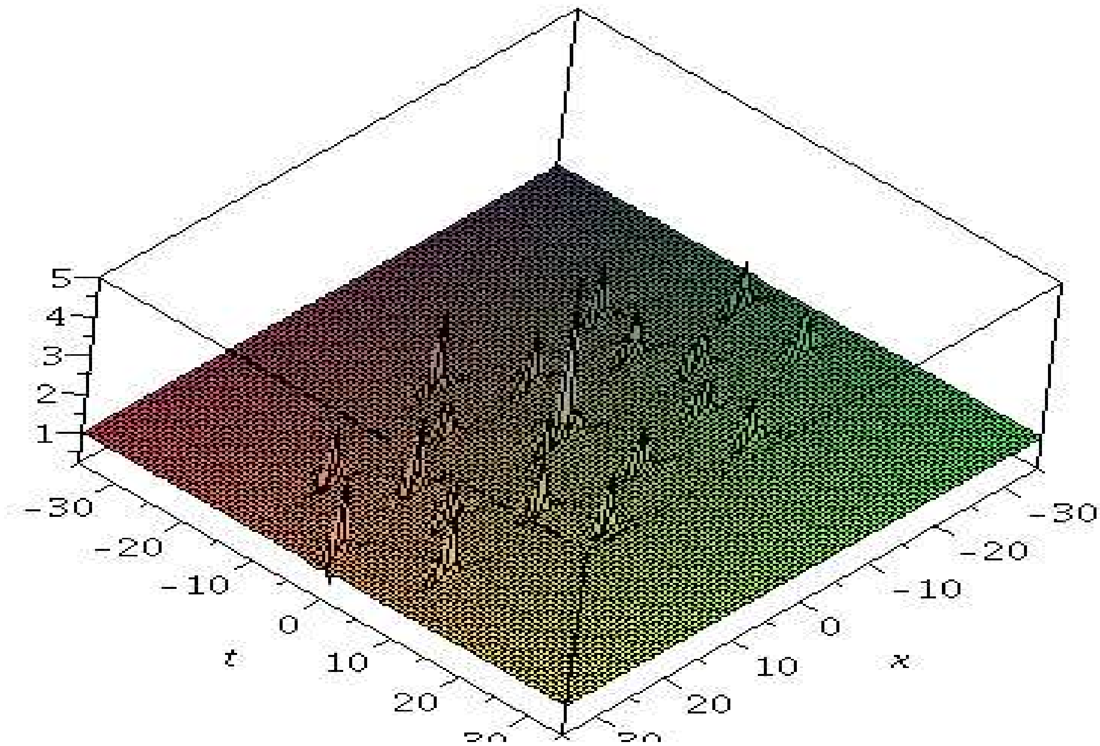

Figure 9: Solution of NLS, N=6, $\tilde{b}_4 = 10^{10}$, 2 rings with 9 peaks with in the center the Peregrine P_2.

Figure 10 $a_5 \neq 0$

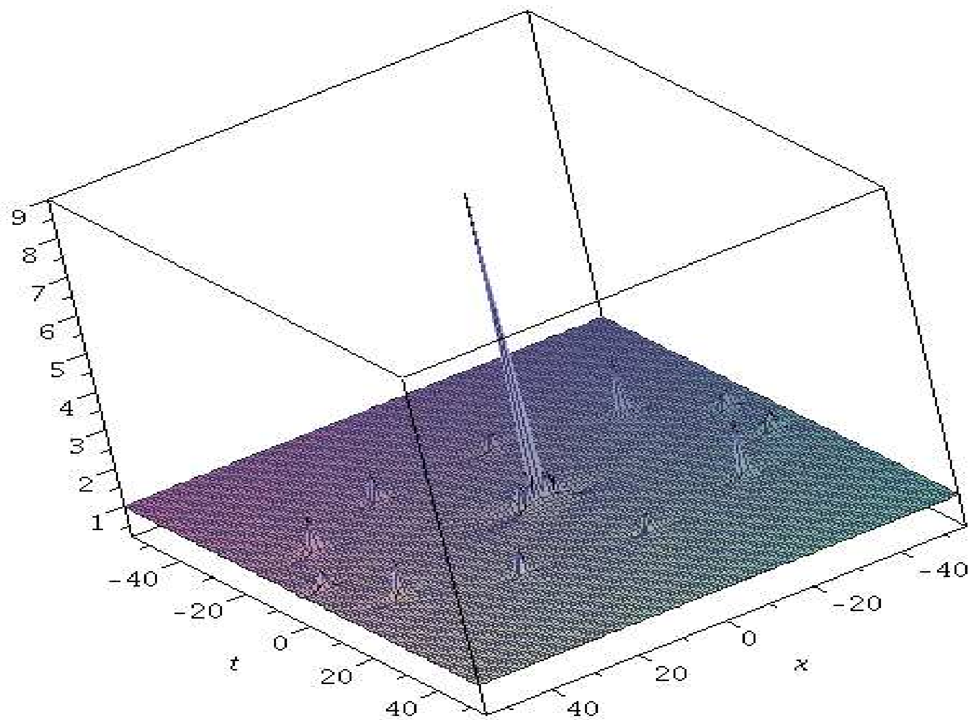

Figure 10: Solution of NLS, N=6, $\tilde{a}_5 = 10^5$, a ring of 11 peaks with in the center the Peregrine of order 4.

Figure 11 $b_5 \neq 0$

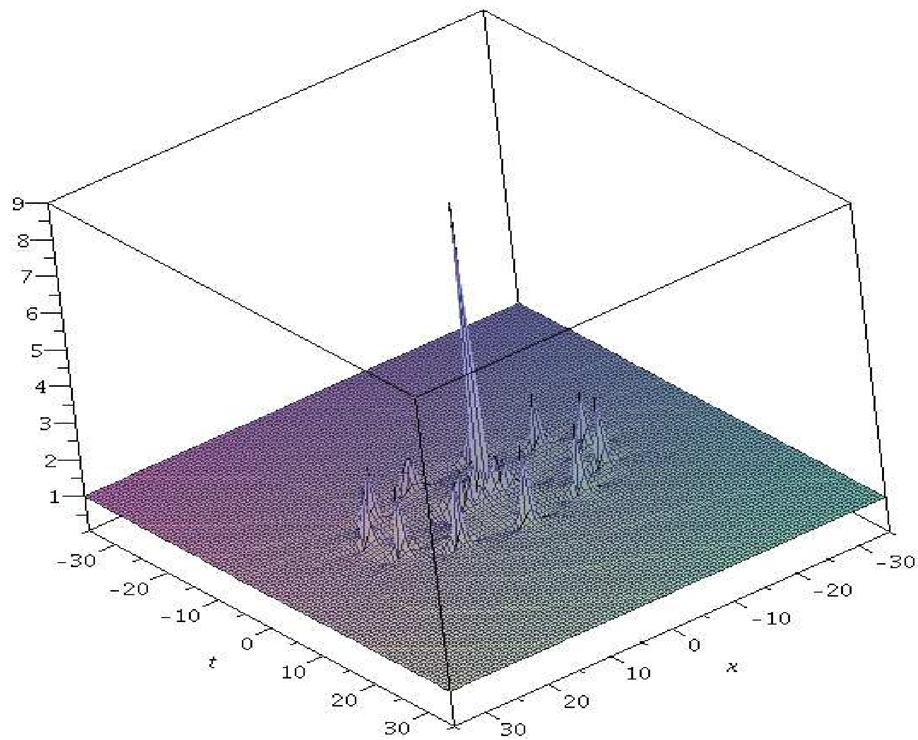

Figure 11: Solution of NLS, N=6, $\tilde{b}_5 = 10^{10}$, a ring of 11 peaks with in the center the Peregrine of order 4.

4 Conclusion

In the present paper we construct explicitly solutions of the NLS equation of order N with $2N-2$ real parameters. The explicit expression in terms of polynomials of x and t is too monstrous to be published.
By different choices of these parameters, we obtained new patterns in the $(x;t)$ plane; we recognized rings as already observed in the case of deformations depending on two parameters [10, 12]. We get news triangular shapes and multiple concentric rings configurations.
All conjectures are verified : the maximum of the modulus of the Peregrine breather of order $N=6$ is equal to $2N-1=11$; in the case of one ring, there is $2N-1$ peaks on the ring; we obtain polynomials in x and t of degree $N(N+1)=42$.
We hope to continue this study for the higher orders in order to give a better understanding of the phenomenon of rogue waves.

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