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A B S T R A C T

In this paper we employ the direct approach to the theory of rods and beams, which is based on the deformable curve model with a triad of rotating directors attached to each point. We show that this model (also called directed curve) is an efficient approach for analyzing the deformation of elastic beams with a complex material structure. Thus, we consider non homogeneous, composite and functionally graded beams made of isotropic or orthotropic materials and we determine the effective stiffness properties in terms of the three dimensional elasticity constants. We present general analytical expressions of the effective stiffness coefficients, valid for beams of arbitrary cross section shape. Finally, we apply this method for FGM beams made of metal foams and compare our analytical results with the numerical results obtained by a finite element analysis.

1. Introduction

Nowadays, the rod and beam structures made of functionally graded or non homogeneous materials are widely used in engineering applications. One of the convenient ways to describe the mechanical properties of composite rods is to use the direct approach, which is based on the deformable curve model.

The classical way to derive theories of beams and rods is to use the thinness hypothesis and to perform an approximate analysis of the stress strain state of three dimensional bodies for which two dimensions are much smaller in comparison with the third one. Thus, the derivation of a set of one dimensional approximate equations can be achieved by the application of kinematical or stress hypotheses, and the use of mathematical techniques like series expansions and asymptotic analysis [1 5]. Theory, finite element analysis, and various applications of FGM rods and beams are presented in many works, see e.g. [6 12].

As an alternative to these methods, one can follow the direct approach by considering a deformable curve endowed with a certain microstructure, as a model for rods. The direct approach, which is well known since Euler, was summarized for the first time in the monograph of the Cosserat brothers [13] presenting the kinematical model of a continuum with material points which behave like rigid bodies (having 6 degrees of freedom, instead of 3 in the classical continuum mechanics). Later, this idea has been developed by Ericksen and Truesdell [14], Green and Naghdi [15,16] within the so called theory of Cosserat curves [17,18]. A different direct approach for shells and rods has been presented by Zhilin [19 21], who elaborated the so called theory of directed curves and surfaces. This theory follows the original idea of Cosserat and considers deformable continua (surfaces or curves) endowed with a triad of rigidly rotating orthonormal vectors connected to each point. This kinematical model was supplemented with appropriate constitutive equations [20 22], thus making the theory applicable to solve practical problems for rods.

In this paper we show that the model of directed curves is an efficient approach for analyzing elastic beams and rods with a complex internal structure (functionally graded, composite, non homogeneous, etc.). In order to be able to describe the complex mechanical behavior of functionally graded or composite beams and rods, we need to employ a quite general set of constitutive equations, which allows for the coupling of the extensional shear and bending torsion deformations. The structure of the constitutive tensors and the form of the constitutive equations are presented in Section 2.

The main difficulty in any direct approach is the determination of the effective stiffness coefficients appearing in the one dimensional constitutive equations in terms of the three dimensional elasticity constants. The determination of effective stiffness coefficients is important because it allows to reduce the treatment of

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three dimensional problems to much simpler one dimensional problems. To identify these mechanical properties for general non homogeneous rods, we compare the solutions of extension, bending and torsion problems in the direct approach with the corresponding results from the three dimensional theory [23,24]. Thus, we obtain the effective bending stiffness, extensional stiffness, torsional rigidity and other coupling coefficients. Also, to determine the effective shear stiffness, we compare the shear vibrations of rectangular beams in the two approaches (direct and three dimensional). These results are presented in Sections 3 and 4 in the case of isotropic non homogeneous beams with arbitrary cross section shape. In Sections 5 and 6 we consider beams composed of two different non homogeneous materials, either orthotropic or isotropic, and we derive general formulas for the effective stiffness coefficients. These formulas are expressed in terms of the solutions to some auxiliary plane strain boundary value problems defined on the cross section domain. In general, the solutions of these auxiliary boundary value problems are not easy to find in a closed form, but we present in Section 7 some special cases for the geometry/material parameters in which we can obtain the results in closed form. In Section 8 we employ our analytical modeling to analyze the deformation of FGM beams made of metal foams. The mass density distribution of the cellular material in the beam is given by a power law function of the cross section coordinate, while the Young's modulus is expressed by the Gibson Ashby formula for closed cell aluminum foams [25]. Finally, we verify our analytical modeling by comparing the results obtained in the direct approach of FGM beams with the numerical solution of various bending problems obtained by a finite element analysis using ABAQUS.

The close agreement between the analytical and numerical solutions indicates that the direct approach to rods, together with the formulas for the effective stiffness coefficients derived in this paper, represent an efficient tool for the analysis of the deformation of functionally graded rods.

2. Equations for curved rods in the direct approach

2.1. Material independent equations

In this expository section we present the basic non linear equations for beams and rods, obtained by the direct approach in [20,21]. In this approach, the thin body is modeled as a deformable curve endowed with a triad of rigidly rotating vectors attached to each point.

We denote by $\mathcal{C}_0$ the deformable curve in its reference (initial) configuration and by s the material coordinate along $\mathcal{C}_0$, which is also the arclength parameter. The position of the directed curve is described by the position vector $\boldsymbol{r}(s)$ and the attached vectors $\boldsymbol{d}_i(s)$, $i = 1,\ 2,\ 3$, also called directors. The unit vectors $\boldsymbol{d}_i(s)$ are mutually orthogonal and they are chosen such that $\boldsymbol{d}_3$ coincides with the unit tangent $\boldsymbol{t} \equiv \boldsymbol{r}'$, and $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ belong to the normal plane to the curve $\mathcal{C}_0$. The rotations of the attached triad of directors describe the rotations of the rod's cross sections during deformation.

Let $\mathcal{C}$ be the deformed configuration of the rod at time t, which is characterized by the vector fields (see Fig. 1)

$$\boldsymbol{R} \quad \boldsymbol{R}(s,t), \quad \boldsymbol{D}_i \quad \boldsymbol{D}_i(s,t), \quad i \quad 1,2,3, \tag{1}$$

where $\boldsymbol{R}$ is the position vector and $\boldsymbol{D}_i$ are the directors after deformation. We have $\boldsymbol{D}_i \ \boldsymbol{D}_j = \delta_{ij}$ (the Kronecker symbol), but $\boldsymbol{D}_3$ is not tangent to the curve $\mathcal{C}$, i.e. the initial cross sections are no longer normal to the middle curve after deformation. In this model it is assumed that the cross sections of the beam do not deform, but they only rotate with respect to the middle curve.

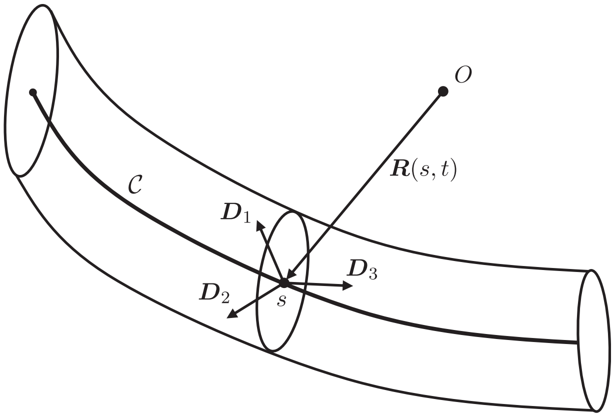


Fig. 1. The deformed configuration of the rod.

Let $\boldsymbol{P}(s,t) = \boldsymbol{D}_k(s,t) \otimes \boldsymbol{d}_k(s)$ be the rotation tensor. We employ throughout the Einstein's summation convention and the direct tensor notation in the sense of [26,27]. Greek indices range over the set $\{1,2\}$, while Latin indices take the values $\{1,2,3\}$. Denote by a superposed dot the material time derivative and by $()' \quad \frac{\mathrm{d}}{\mathrm{d}s}$.

The velocity vector is $\boldsymbol{V}(s,t) \quad \dot{\boldsymbol{R}}(s,t)$, and the angular velocity vector $\boldsymbol{\omega}(s,t)$ is determined by the relation $\dot{\boldsymbol{P}} \quad \boldsymbol{\omega} \times \boldsymbol{P}$, i.e. $\boldsymbol{\omega}$ is the axial vector of the antisymmetric tensor $\dot{\boldsymbol{P}} \ \ \boldsymbol{P}^T$. The equations of motion for rods are

$$\begin{aligned}
&\boldsymbol{N}'(s,t) + \rho_0 \boldsymbol{\mathcal{F}} \quad \rho_0(\boldsymbol{V} + \boldsymbol{\Theta}_1 \cdot \boldsymbol{\omega})^{\bullet},\\
&\boldsymbol{M}'(s,t) + \boldsymbol{R}' \times \boldsymbol{N}(s,t) + \rho_0 \boldsymbol{\mathcal{L}} \quad \rho_0[\boldsymbol{V} \times \boldsymbol{\Theta}_1 \cdot \boldsymbol{\omega} + (\boldsymbol{V} \cdot \boldsymbol{\Theta}_1 + \boldsymbol{\Theta}_2 \cdot \boldsymbol{\omega})^{\bullet}],
\end{aligned} \tag{2}$$

where $\boldsymbol{N}$ is the force vector, $\boldsymbol{M}$ is the moment vector, $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{L}}$ are the external body force and moment per unit mass, ρ_0 is the mass density per unit length of $\mathcal{C}_0$, while the second order tensors $\boldsymbol{\Theta}_1(s,t)$ and $\boldsymbol{\Theta}_2(s,t)$ are the inertia tensors per unit mass. According to [21], the tensors $\boldsymbol{\Theta}_\alpha$ are expressed by $\boldsymbol{\Theta}_\alpha(s,t) \quad \boldsymbol{P}(s,t) \ \ \boldsymbol{\Theta}^0_\alpha(s)$ $\boldsymbol{P}^{\mathrm{T}}(s,t)$, where $\boldsymbol{\Theta}^0_\alpha(s)$ are the inertia tensors in the reference configuration, which are given by

$$\begin{aligned}
&\rho_0 \boldsymbol{\Theta}^0_1 \qquad \int_\Sigma (\boldsymbol{1} \times \boldsymbol{a})\rho^* \ \tilde{\mu}\ \mathrm{d}x_1 \mathrm{d}x_2,\\
&\rho_0 \boldsymbol{\Theta}^0_2 \qquad \int_\Sigma [(\boldsymbol{a} \cdot \boldsymbol{a})\boldsymbol{1} \quad \boldsymbol{a} \otimes \boldsymbol{a}]\ \rho^* \ \tilde{\mu}\ \mathrm{d}x_1 \mathrm{d}x_2.
\end{aligned} \tag{3}$$

Here ρ^* is the mass density in the three dimensional rod, $\boldsymbol{1}$ is the second order unit tensor, Σ is the domain of the cross section in the normal plane, $\boldsymbol{a} = x_1\boldsymbol{d}_1 + x_2\boldsymbol{d}_2$ and $\mu \equiv 1 + \frac{\boldsymbol{a}\cdot\boldsymbol{n}}{R_c}$, where R_c is the radius of curvature of the curve $\mathcal{C}_0$ and $\boldsymbol{n}$ is the principal normal unit vector. In the case of straight rods, we have clearly $\mu \quad 1$. In the general case of curved rods, since the diameter of the rod is much smaller than R_c, we have $\frac{|\boldsymbol{a}\cdot\boldsymbol{n}|}{R_c} \ll 1$, and thus $\mu > 0$ and μ has a value close to 1.

We note that $\boldsymbol{\Theta}_1$ is antisymmetric, $\boldsymbol{\Theta}_2$ is symmetric. The fields $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{L}}$ account also for the loads acting on the lateral surface of three dimensional rods.

The vectors of deformation are defined as follows: the vector of extension shear $\boldsymbol{\varepsilon} \quad \boldsymbol{R}' \quad \boldsymbol{P}\ \ \boldsymbol{t}$, and the vector of bending torsion $\boldsymbol{\Phi}$ is given by $\boldsymbol{P}' = \boldsymbol{\Phi} \times \boldsymbol{P}$, i.e. $\boldsymbol{\Phi}$ is the axial vector of the antisymmetric tensor $\boldsymbol{P}' \ \ \boldsymbol{P}^{\mathrm{T}}$. We also introduce the energetic vectors of deformation $\boldsymbol{\varepsilon}_*$ and $\boldsymbol{\Phi}_*$ defined by $\boldsymbol{\varepsilon}_* \quad \boldsymbol{P}^T \ \ \boldsymbol{\varepsilon}$ and $\boldsymbol{\Phi}_* = \boldsymbol{P}^T \ \ \boldsymbol{\Phi}$ [20,21].

For general elastic beams, the constitutive assumptions imply that the internal energy density $\mathcal{U}$ is a function of the following arguments $\{\boldsymbol{\varepsilon}_*, \boldsymbol{\Phi}_*\}$. In our work we consider that the internal energy is a quadratic function of its arguments. Thus, we have the following constitutive equations

$$\rho_0\mathcal{U} \quad \mathcal{U}_0 + \boldsymbol{N}_0 \cdot \boldsymbol{\mathcal{E}}_* + \boldsymbol{M}_0 \cdot \boldsymbol{\Phi}_* + \tfrac{1}{2}\boldsymbol{\mathcal{E}}_* \cdot \boldsymbol{A} \cdot \boldsymbol{\mathcal{E}}_* + \boldsymbol{\mathcal{E}}_* \cdot \boldsymbol{B} \cdot \boldsymbol{\Phi}_* + \tfrac{1}{2}\boldsymbol{\Phi}_* \cdot \boldsymbol{C} \cdot \boldsymbol{\Phi}_*,$$
$$\boldsymbol{N} \quad \frac{\partial(\rho_0\mathcal{U})}{\partial \boldsymbol{\mathcal{E}}_*} \cdot \boldsymbol{P}^T, \qquad \boldsymbol{M} \quad \frac{\partial(\rho_0\mathcal{U})}{\partial \boldsymbol{\Phi}_*} \cdot \boldsymbol{P}^T, \tag{4}$$

where $\mathcal{U}_0$ is a scalar, $\boldsymbol{N}_0$, $\boldsymbol{M}_0$ are vectors, and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are second order tensors, defined on the reference configuration. The structure and significance of the elasticity tensors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ have been discussed in [20,21].

2.2. Structure of constitutive tensors

In our study we are interested to determine the structure of constitutive tensors for beams and rods made of functionally graded materials. We assume that the material properties do not vary along the length of the beam, but only across the cross sec tions. In other words, they depend on (x_1,x_2), but not on s. In each cross section we chose the directors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ along the principal axes of inertia. Thus, we have

$$\langle \rho^* x_1 \rangle \quad \langle \rho^* x_2 \rangle \quad 0, \quad \langle \rho^* x_1 x_2 \rangle \quad 0, \tag{5}$$

where we denote by $\langle f \rangle \quad \int_\Sigma f \, dx_1 \; dx_2$ for any field f.

The structure of the constitutive tensors can be determined using the generalized theory of tensor symmetry [21,28]. In the general case of curved rods, the constitutive tensors depend on the geometry of the rod through the Darboux vector $\boldsymbol{\tau}$ of the curve $\mathcal{C}_0$, and through the angle of natural twisting $\sigma = \sphericalangle(\boldsymbol{d}_1, \boldsymbol{n})$. The expressions of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ for homogeneous curved rods are pre sented in [20,21]. If we restrict for simplicity to straight rods with out natural twisting, then we have $\boldsymbol{\tau} = \boldsymbol{0}$ and $\sigma = 0$. Imposing that the orthogonal tensor $\boldsymbol{1} \quad 2\boldsymbol{t} \otimes \boldsymbol{t}$ belongs to the symmetry group of any constitutive tensor, we find that for non homogeneous rods $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ have the following structures

$$\begin{aligned}
\boldsymbol{A} &\quad A_1\boldsymbol{d}_1 \otimes \boldsymbol{d}_1 + A_2\boldsymbol{d}_2 \otimes \boldsymbol{d}_2 + A_3\boldsymbol{t} \otimes \boldsymbol{t} + A_{12}(\boldsymbol{d}_1 \otimes \boldsymbol{d}_2 + \boldsymbol{d}_2 \otimes \boldsymbol{d}_1),\\
\boldsymbol{B} &\quad B_{13}\boldsymbol{d}_1 \otimes \boldsymbol{t} + B_{31}\boldsymbol{t} \otimes \boldsymbol{d}_1 + B_{23}\boldsymbol{d}_2 \otimes \boldsymbol{t} + B_{32}\boldsymbol{t} \otimes \boldsymbol{d}_2,\\
\boldsymbol{C} &\quad C_1\boldsymbol{d}_1 \otimes \boldsymbol{d}_1 + C_2\boldsymbol{d}_2 \otimes \boldsymbol{d}_2 + C_3\boldsymbol{t} \otimes \boldsymbol{t} + C_{12}(\boldsymbol{d}_1 \otimes \boldsymbol{d}_2 + \boldsymbol{d}_2 \otimes \boldsymbol{d}_1).
\end{aligned} \tag{6}$$

Remark. The structure of the constitutive tensors can be derived also in the more general case of rods with natural twisting. In this case, the constitutive coefficients depend also on the angle of natural twist $\sigma(s)$, and the expressions corresponding to (6) have to be supplemented with additional terms. □

Our aim is to determine the constitutive coefficients A_i, C_i, A_{12}, C_{12}, $B_{\alpha 3}$ and $B_{3\alpha}$ for functionally graded beams and rods, in terms of the three dimensional elastic properties. These coefficients de scribe the effective stiffness properties of thin beams and rods. Since the constitutive coefficients do not depend on the deforma tion, their expressions can be derived by comparison of exact solu tions for directed curves with the results from three dimensional elasticity in the framework of linear theory.

In order to realize such comparison of exact solutions, we re strict ourselves to the linear theory. Let us note that in the the ory of beams and rods there is long tradition of using linear elasticity to derive one dimensional beams and rods theories including some non linear effects. This tradition is based on the fact that one can calculate stiffness parameters of beam or rod on the base of linear elasticity and then use the stiffness moduli in geometrically non linear theory of beams and rods. In deed, the coefficients of the strain energy density considered as the quadratic function of strain measures coincide for linear and for geometrically non linear theories of beams and rods. This fact is used for example in [3,20,21] where different approaches are applied. In the paper the geometrically non linear approach with physically linear constitutive relations is considered. Such a theory can be applied for standard material. Exception is, for example, a rubber like material for which the quadratic form of the strain energy density is not valid in the case of large deformations, in general. Some recent attempts to apply non lin ear elasticity to construction of one dimensional theories of beams and rods are given for example in [29 37].

3. Linearized equations for directed curves

3.1. Geometrical linearization

In the linear setting, the displacement $\boldsymbol{u}(s,t) = \boldsymbol{R}(s,t) \quad \boldsymbol{r}(s)$ is as sumed to be infinitesimal. Also, the rotation tensor can be repre sented as $\boldsymbol{P} = \boldsymbol{1} + \boldsymbol{\psi} \times \boldsymbol{1}$, where $\boldsymbol{\psi}(s,t)$ is the vector of small rotations. The field $\boldsymbol{\psi}$, which is assumed to be infinitesimal, satis fies the relations $\dot{\boldsymbol{\psi}} \quad \boldsymbol{\omega}$ and $\boldsymbol{\psi}' = \boldsymbol{\Phi}$. The vectors of deformation are denoted in the linear case by $\boldsymbol{e}$ and $\boldsymbol{\kappa}$, and they are given by

$$\boldsymbol{e} \equiv \boldsymbol{u}' + \boldsymbol{t} \times \boldsymbol{\psi} \quad \boldsymbol{\mathcal{E}} \quad \boldsymbol{\mathcal{E}}_*, \quad \boldsymbol{\kappa} \equiv \boldsymbol{\psi}' \quad \boldsymbol{\Phi} \quad \boldsymbol{\Phi}_*. \tag{7}$$

The constitutive Eq. (4) reduce to

$$\rho_0\mathcal{U}(\boldsymbol{e},\boldsymbol{\kappa}) \quad \frac{1}{2}\boldsymbol{e} \cdot \boldsymbol{A} \cdot \boldsymbol{e} + \boldsymbol{e} \cdot \boldsymbol{B} \cdot \boldsymbol{\kappa} + \frac{1}{2}\boldsymbol{\kappa} \cdot \boldsymbol{C} \cdot \boldsymbol{\kappa},$$
$$\boldsymbol{N} \quad \frac{\partial(\rho_0\mathcal{U})}{\partial \boldsymbol{e}}, \quad \boldsymbol{M} \quad \frac{\partial(\rho_0\mathcal{U})}{\partial \boldsymbol{\kappa}}. \tag{8}$$

The equations of motion (2) simplify to the forms

$$\begin{aligned}
\boldsymbol{N}' + \rho_0\boldsymbol{\mathcal{F}} &\quad \rho_0(\ddot{\boldsymbol{u}} + \boldsymbol{\Theta}_1^0 \cdot \ddot{\boldsymbol{\psi}}), \quad \boldsymbol{M}' + \boldsymbol{t} \times \boldsymbol{N} + \rho_0\boldsymbol{\mathcal{L}}\\
&\quad \rho_0(\ddot{\boldsymbol{u}} \cdot \boldsymbol{\Theta}_1^0 + \boldsymbol{\Theta}_2^0 \cdot \ddot{\boldsymbol{\psi}}).
\end{aligned} \tag{9}$$

To the governing field Eqs. (7) (9) we adjoin boundary conditions and initial conditions. Let l be the length of the rod, so that the arc length parameter range over the interval $s \in [0,l]$. We denote the two endpoints by $\bar{s}_1 \quad 0$ and $\bar{s}_2 \quad l$ for convenience, and we consider boundary conditions of the type

$$\begin{aligned}
&\boldsymbol{u}(s_\gamma,t) \quad \boldsymbol{u}^{(\gamma)}(t) \quad \text{or} \quad \boldsymbol{N}(s_\gamma,t) \quad \boldsymbol{N}^{(\gamma)}(t), \quad \text{for}\ \gamma \quad 1,2,\\
&\boldsymbol{\psi}(s_\gamma,t) \quad \boldsymbol{\psi}^{(\gamma)}(t) \quad \text{or} \quad \boldsymbol{M}(s_\gamma,t) \quad \boldsymbol{M}^{(\gamma)}(t), \quad \text{for}\ \gamma \quad 1,2.
\end{aligned}$$

The initial conditions are

$$\begin{aligned}
&\boldsymbol{u}(s,0) \quad \boldsymbol{u}_0(s), \quad \dot{\boldsymbol{u}}(s,0) \quad \boldsymbol{v}_0(s), \quad \boldsymbol{\psi}(s,0) \quad \boldsymbol{\psi}_0(s),\\
&\dot{\boldsymbol{\psi}}(s,0) \quad \boldsymbol{\omega}_0(s),
\end{aligned}$$

where the functions $\boldsymbol{u}_0$, $\boldsymbol{\psi}_0$, $\boldsymbol{v}_0$, $\boldsymbol{\omega}_0$, as well as $\boldsymbol{u}^{(\gamma)}$, $\boldsymbol{\psi}^{(\gamma)}$, $\boldsymbol{N}^{(\gamma)}$, $\boldsymbol{M}^{(\gamma)}$ are prescribed.

The correspondence between the displacement and rotation fields $\{\boldsymbol{u},\boldsymbol{\psi}\}$ for directed curves and the displacement vector $\boldsymbol{u}^*$ for three dimensional rods is established by the following relations [21]

$$\begin{aligned}
\rho_0(\boldsymbol{u} + \boldsymbol{\Theta}_1^0 \cdot \boldsymbol{\psi}) &\quad \langle \rho^*\boldsymbol{u}^*\tilde{\mu} \rangle, \quad \rho_0(\boldsymbol{u} \cdot \boldsymbol{\Theta}_1^0 + \boldsymbol{\Theta}_2^0 \cdot \boldsymbol{\psi})\\
&\quad \langle \rho^*(\boldsymbol{a} \times \boldsymbol{u}^*)\tilde{\mu} \rangle.
\end{aligned} \tag{10}$$

Also, the relations between the fields $\{\boldsymbol{N},\boldsymbol{M}\}$ and the Cauchy stress tensor $\boldsymbol{T}^*$ from three dimensional theory are given by

$$\boldsymbol{N} \quad \langle \boldsymbol{t} \cdot \boldsymbol{T}^* \rangle, \quad \boldsymbol{M} \quad \langle \boldsymbol{a} \times (\boldsymbol{t} \cdot \boldsymbol{T}^*) \rangle. \tag{11}$$

These relations are useful when comparing the solutions of some problems in the two different approaches.

3.2. Straight rods

In what follows we restrict our attention to straight rods without natural twisting. In this case, we can chose the Cartesian coordinate frame $Ox_1x_2x_3$ such that the curve $\mathcal{C}_0$ is situated on the axis Ox_3, between the limits $x_3 = 0, l$, and we have

$$\boldsymbol{t} \quad \boldsymbol{d}_3 \quad \boldsymbol{e}_3, \ \boldsymbol{n} \quad \boldsymbol{d}_1 \quad \boldsymbol{e}_1, \ \boldsymbol{d}_2 \quad \boldsymbol{e}_2, \ s \quad x_3, \ \mu \quad 1, \ \boldsymbol{\Theta}_1^0 \quad \boldsymbol{0},$$
$$\rho_0\boldsymbol{\Theta}_2^0 \quad I_1\boldsymbol{e}_1\otimes\boldsymbol{e}_1 + I_2\boldsymbol{e}_2\otimes\boldsymbol{e}_2 + (I_1+I_2)\boldsymbol{e}_3\otimes\boldsymbol{e}_3, \ I_1 \quad \langle\rho^* x_2^2\rangle, \ I_2 \quad \langle\rho^* x_1^2\rangle,$$

where $\boldsymbol{e}_i$ denote the unit vectors along Ox_i.

To distinguish between the extensional, torsional, bending, and shear deformation, we decompose the vectors $\boldsymbol{u}$, $\boldsymbol{\psi}$, $\boldsymbol{e}$, $\boldsymbol{\kappa}$, $\boldsymbol{N}$, $\boldsymbol{M}$, $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{L}}$ by the tangent direction $\boldsymbol{t}$ and the normal plane ($\boldsymbol{e}_1$, $\boldsymbol{e}_2$):

$$\begin{aligned} &\boldsymbol{u} \quad u\boldsymbol{t}+\boldsymbol{w}, \ \boldsymbol{\psi} \quad \psi\boldsymbol{t}+\boldsymbol{t}\times\boldsymbol{\vartheta}, \ \boldsymbol{e} \quad u'\boldsymbol{t}+\boldsymbol{\gamma}, \ \boldsymbol{\kappa} \quad \psi'\boldsymbol{t}+\boldsymbol{t}\times\boldsymbol{\vartheta}', \\ &\boldsymbol{N} \quad F\boldsymbol{t}+\boldsymbol{Q}, \ \boldsymbol{M} \quad H\boldsymbol{t}+\boldsymbol{t}\times\boldsymbol{L}, \ \boldsymbol{\mathcal{F}} \quad \mathcal{F}_t\boldsymbol{t}+\boldsymbol{\mathcal{F}}_n, \ \boldsymbol{\mathcal{L}} \quad \mathcal{L}_t\boldsymbol{t}+\boldsymbol{\mathcal{L}}_n, \end{aligned} \tag{12}$$

with $\boldsymbol{\gamma} = \boldsymbol{w}' \quad \boldsymbol{\vartheta}$. The vectors $\boldsymbol{w}$, $\boldsymbol{\vartheta}$, $\boldsymbol{\gamma}$, $\boldsymbol{Q}$, $\boldsymbol{L}$, $\boldsymbol{\mathcal{F}}_n$ and $\boldsymbol{\mathcal{L}}_n$ are orthog onal to $\boldsymbol{t}$. Here $\boldsymbol{\gamma}$ is the transverse shear vector, u is the longitudinal displacement, $\boldsymbol{w} = w_\alpha\boldsymbol{e}_\alpha$ is the vector of transversal displacement, ψ is the torsion, $\boldsymbol{\vartheta}' \quad \vartheta'_\alpha\boldsymbol{e}_\alpha$ is the vector of bending deformation, F is the longitudinal force, $\boldsymbol{Q} = Q_\alpha\boldsymbol{e}_\alpha$ is the vector of transversal force, H is the torsion moment and $\boldsymbol{L} = L_\alpha\boldsymbol{e}_\alpha$ is the vector of bending mo ment. Using the decompositions (12) and the structure of constitu tive tensors (6), we remark that the constitutive Eq. (8) can be written in component form as

$$\begin{aligned} &Q_1 \quad A_1(w_1' \quad \vartheta_1)+A_{12}(w_2' \quad \vartheta_2)+B_{13}\psi', \\ &Q_2 \quad A_{12}(w_1' \quad \vartheta_1)+A_2(w_2' \quad \vartheta_2)+B_{23}\psi', \\ &F \quad A_3u' \quad B_{31}\vartheta_2'+B_{32}\vartheta_1', \ H \quad C_3\psi'+B_{13}(w_1' \quad \vartheta_1)+B_{23}(w_2' \quad \vartheta_2), \\ &L_1 \quad C_2\vartheta_1' \quad C_{12}\vartheta_2'+B_{32}u', \ L_2 \qquad C_{12}\vartheta_1'+C_1\vartheta_2' \quad B_{31}u'. \end{aligned} \tag{13}$$

The constitutive coefficients are constants, since we consider rods made of non homogeneous materials which properties do not de pend on the axial coordinate s.

We observe that the general boundary initial value problem for non homogeneous rods does not decouple into sub problems. Note that in the case of homogeneous materials the general prob lem decouples into the extension torsion problem and the bend ing shear problem, see [21]. The relations of identification (10) and (11), written for straight rods, become

$$\begin{aligned} &\rho_0 w_\alpha \quad \langle\rho^* u_\alpha^*\rangle, \quad \rho_0 u \quad \langle\rho^* u_3^*\rangle, \quad \rho_0 \quad \langle\rho^*\rangle, \\ &\vartheta_1 \qquad \frac{\langle\rho^* x_1 u_3^*\rangle}{I_2}, \quad \vartheta_2 \qquad \frac{\langle\rho^* x_2 u_3^*\rangle}{I_1}, \quad \psi \quad \frac{\langle\rho^*(x_1u_2^* \quad x_2u_1^*)\rangle}{I_1+I_2}, \\ &Q_\alpha \quad \langle t_{3\alpha}^*\rangle, \quad F \quad \langle t_{33}^*\rangle, \quad L_\alpha \qquad \langle x_\alpha t_{33}^*\rangle, \quad H \quad \langle x_1t_{32}^* \quad x_2t_{31}^*\rangle, \end{aligned} \tag{14}$$

where u_i^* and t_{ij}^* are the components of $\boldsymbol{u}^*$ and $\boldsymbol{T}^*$, respectively. The relations (14) will be used to identify the corresponding fields in the two approaches (directed curves and three dimensional).

3.3. Extension, bending and torsion in the direct approach

Let us find the exact solution of the problem of extension, bend ing and torsion of directed curves. We mention that this solution is exact up to rigid body displacement and rotation fields. In the lin ear theory the rigid body fields have the general form $\boldsymbol{u} \quad \boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{r}, \boldsymbol{\psi} \quad \boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary constant vectors.

Let us determine the equilibrium of a straight rod subjected to an axial force $\overline{F}$, a torsion moment $\overline{H}$, and bending moments $\overline{L}_\alpha$ ap plied to both ends. The body forces and moments are absent. In our case, the equilibrium equations corresponding to (9) are

$$Q_\alpha'(s) \quad 0, \quad F'(s) \quad 0, \quad L_\alpha'(s)+Q_\alpha(s) \quad 0, \quad H'(s) \quad 0, \quad s\in(0,l), \tag{15}$$

while the boundary conditions on the ends of the rods are

$$\begin{aligned} &Q_\alpha(0) \quad Q_\alpha(l) \quad 0, \quad F(0) \quad F(l) \quad \overline{F}, \\ &L_\alpha(0) \quad L_\alpha(l) \quad \overline{L}_\alpha, \quad H(0) \quad H(l) \quad \overline{H}. \end{aligned} \tag{16}$$

Using the constitutive Eq. (13) we obtain a system of ordinary dif ferential equations which yields the solution

$$w_\alpha(s) \qquad \frac{1}{2}a_\alpha s^2+b_\alpha s, \quad u(s) \quad a_3s, \quad \vartheta_\alpha(s) \qquad a_\alpha s, \quad \psi(s) \quad b_3s, \tag{17}$$

where the constants a_i and b_i are determined by the algebraic linear systems

$$\begin{bmatrix} C_2 & C_{12} & B_{32} \\ C_{12} & C_1 & B_{31} \\ B_{32} & B_{31} & A_3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \begin{bmatrix} \overline{L}_1 \\ \overline{L}_2 \\ \overline{F} \end{bmatrix}, \quad \begin{bmatrix} A_1 & A_{12} & B_{13} \\ A_{12} & A_2 & B_{23} \\ B_{13} & B_{23} & C_3 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ \overline{H} \end{bmatrix} \tag{18}$$

The force and moment vector fields corresponding to this solution are given by

$$\boldsymbol{N} \quad \overline{F}\boldsymbol{e}_3, \quad \boldsymbol{M} \qquad \overline{L}_2\boldsymbol{e}_1+\overline{L}_1\boldsymbol{e}_2+\overline{H}\boldsymbol{e}_3. \tag{19}$$

This solution will be used later for comparison with three dimen sional solutions, in order to identify the effective stiffness coeffi cients for non homogeneous thin rods.

4. Determination of constitutive coefficients for isotropic rods

4.1. Deformation of non homogeneous three dimensional rods

Let us consider a three dimensional rod which occupies the do main $\mathcal{B} \quad \{(x_1,x_2,x_3)|(x_1,x_2)\in\Sigma, x_3\in[0,l]\}$. The cross section Σ is arbitrary and the symmetry relations (5) are satisfied. The body $\mathcal{B}$ is made of an isotropic and non homogenous material such that the mass density ρ^* and the Lamé moduli λ, μ are independent of the axial coordinate, i.e. we have

$$\rho^* \quad \rho^*(x_1,x_2), \quad \lambda \quad \lambda(x_1,x_2), \quad \mu \quad \mu(x_1,x_2).$$

We consider the deformation of such cylinders under the action of terminal forces and moments.

We assume that the body $\mathcal{B}$ is in equilibrium, in the absence of external body loads and tractions on the lateral surfaces. On the two ends of the cylinder act a resultant axial force and a resultant moment. We consider the same problem as in Section 3.4, but for mulated in the three dimensional setting. In view of the relations $(14)_{7\ \ 10}$ we take the boundary conditions

$$\begin{aligned} &\langle t_{3\alpha}^*\rangle \quad 0, \quad \langle t_{33}^*\rangle \quad \overline{F}, \quad \langle x_\alpha t_{33}^*\rangle \qquad \overline{L}_\alpha, \\ &\langle x_1t_{32}^* \quad x_2t_{31}^*\rangle \quad \overline{H} \text{ for } x_3 \quad 0, l. \end{aligned} \tag{20}$$

The solution of this three dimensional problem for non homoge neous rods is presented in [23] Section 3.3 and Section 3.4, where it is expressed in terms of the solutions to some auxiliary plane strain problems. For the sake of completeness and for later reference we present these three dimensional results.

We denote by $u_\alpha^{(1)}$, $u_\alpha^{(2)}$ and $u_\alpha^{(3)}$ the solutions of the 3 plane strain problems $\mathcal{D}^{(1)}$, $\mathcal{D}^{(2)}$ and $\mathcal{D}^{(3)}$ respectively, defined on the do main Σ by

$$\begin{aligned} &\mathcal{D}^{(\gamma)}: \quad t_{\beta\alpha,\beta}^{(\gamma)}+(\lambda x_\gamma)_{,\alpha} \quad 0 \ \text{ in } \Sigma, \quad t_{\beta\alpha}^{(\gamma)}n_\beta^* \qquad \lambda x_\gamma n_\alpha^* \ \text{ on } \partial\Sigma, \\ &\mathcal{D}^{(3)}: \quad t_{\beta\alpha,\beta}^{(3)}+\lambda_{,\alpha} \quad 0 \ \text{ in } \Sigma, \quad t_{\beta\alpha}^{(3)}n_\beta^* \qquad \lambda n_\alpha^* \ \text{ on } \partial\Sigma, \end{aligned} \tag{21}$$

where obviously $t_{\alpha\beta}^{(k)} \quad \lambda u_{\rho,\rho}^{(k)}\delta_{\alpha\beta}+\mu\left(u_{\alpha,\beta}^{(k)}+u_{\beta,\alpha}^{(k)}\right)$, ($k = 1, 2, 3$ and $\alpha, \beta = 1, 2$) and $\boldsymbol{n}^* \quad n_\alpha^*\boldsymbol{e}_\alpha$ is the outward unit normal to $\partial\Sigma$. Let $\varphi(x_1,x_2)$ be the solution of the Neumann type boundary value problem

$$(\mu\varphi_{,\alpha})_{,\alpha} \quad \mu_{,1}x_2 \quad \mu_{,2}x_1 \quad \text{in } \Sigma, \quad \frac{\partial\varphi}{\partial n^*} \quad x_2n_1^* \quad x_1n_2^* \quad \text{on } \partial\Sigma. \tag{22}$$

The existence of solutions to the above boundary value problems (21) and (22) is proved in [23], Sections 3.2 and 3.4. Then, the solution of our three dimensional problem for the loads (20) is given by

$$u_1 \quad \frac{1}{2}\hat{a}_1 x_3^2 \quad \tau x_2 x_3 + \sum_{k\ 1}^{3} \hat{a}_k u_1^{(k)}(x_1,x_2),$$
$$u_2 \quad \frac{1}{2}\hat{a}_2 x_3^2 + \tau x_1 x_3 + \sum_{k\ 1}^{3} \hat{a}_k u_2^{(k)}(x_1,x_2), \tag{23}$$
$$u_3 \quad (\hat{a}_1 x_1 + \hat{a}_2 x_2 + \hat{a}_3) x_3 + \tau\varphi(x_1,x_2),$$

where the constants τ and $\hat{a}_i$ are given by the relations

$$\tau \quad \frac{\overline{H}}{D_*} \quad \text{and} \quad D_{\alpha j}\hat{a}_j \quad \overline{L}_\alpha, \quad D_{3j}\hat{a}_j \quad \overline{F}. \tag{24}$$

Here the torsional rigidity D_* is expressed by

$$D_* \quad \langle \mu[x_1(x_1+\varphi_{,2}) + x_2(x_2 \quad \varphi_{,1})]\rangle, \tag{25}$$

while the coefficients D_{ij} are given by

$$D_{\alpha\beta} \quad \langle(\lambda+2\mu)x_\alpha x_\beta + \lambda x_\alpha u^{(\beta)}_{\gamma,\gamma}\rangle, \quad D_{33} \quad \langle(\lambda+2\mu)+\lambda u^{(3)}_{\gamma,\gamma}\rangle,$$
$$D_{\alpha 3} \quad \langle(\lambda+2\mu)x_\alpha + \lambda x_\alpha u^{(3)}_{\gamma,\gamma}\rangle, \quad D_{3\alpha} \quad \langle(\lambda+2\mu)x_\alpha + \lambda u^{(\alpha)}_{\gamma,\gamma}\rangle. \tag{26}$$

In [23], Section 3.3, it is shown that $D_{ij} = D_{ji}$ and $\det(D_{ij})_{3\times 3} \neq 0$, so that we can determine the constants $\hat{a}_i$ from the system $(24)_{2,3}$.

Remark. If we introduce the stress function $\chi(x_1,x_2)$ by the relations

$$\chi_{,1} \quad \mu(\varphi_{,2}+x_1), \quad \chi_{,2} \quad \mu(\varphi_{,1} \quad x_2),$$

then the torsional rigidity is given by

$$D_* \quad 2\langle\chi(x_1,x_2)\rangle. \tag{27}$$

The stress function χ can be obtained from the boundary value problem

$$\left(\frac{1}{\mu}\chi_{,\alpha}\right)_{,\alpha} \quad 2 \quad \text{in } \Sigma, \quad \chi \quad 0 \quad \text{on } \partial\Sigma,$$

provided that the domain Σ is simply connected. In the case of multiply connected cross sections Σ, the torsion problem has been studied in, e.g., [38,39]. □

Let us compare now the three dimensional solution (23) with the solution (17) obtained in the direct approach to rods, taking into account the relations (5) and (14). By comparison, it follows that we have to identify the constants

$$C_3 \quad D_*, \quad A_3 \quad D_{33}, \quad C_1 \quad D_{22}, \quad C_2 \quad D_{11}, \quad C_{12} \quad D_{12},$$
$$B_{31} \quad D_{23}, \quad B_{32} \quad D_{13}, \quad B_{13} \quad B_{23} \quad 0. \tag{28}$$

Thus, from (26) (28) we obtain the following expressions for the constitutive coefficients

$$C_3 \quad 2\langle\chi(x_1,x_2)\rangle, \ A_3 \quad \langle(\lambda+2\mu)+\lambda u^{(3)}_{\gamma,\gamma}\rangle, \ C_1 \quad \langle(\lambda+2\mu)x_2^2+\lambda x_2 u^{(2)}_{\gamma,\gamma}\rangle,$$
$$C_2 \quad \langle(\lambda+2\mu)x_1^2+\lambda x_1 u^{(1)}_{\gamma,\gamma}\rangle, \ C_{12} \quad \langle(\lambda+2\mu)x_1x_2+\lambda x_1 u^{(2)}_{\gamma,\gamma}\rangle, \ B_{\alpha 3} \quad 0,$$
$$B_{31} \quad \langle(\lambda+2\mu)x_2+\lambda x_2 u^{(3)}_{\gamma,\gamma}\rangle, \ B_{32} \quad \langle(\lambda+2\mu)x_1+\lambda x_1 u^{(3)}_{\gamma,\gamma}\rangle. \tag{29}$$

By virtue of the identifications (14) and (29) we can verify that the fields u, w_α, ψ, $\boldsymbol{N}$ and $\boldsymbol{M}$ calculated for the solutions in the two different approaches coincide.

Remark. For the fields ϑ_α corresponding to the three dimensional solution (23) we obtain from $(14)_{1,2}$ and (5) the expressions

$$\vartheta_\alpha \quad \hat{a}_\alpha x_3 \quad \tau\frac{\langle\rho^* x_\alpha \varphi\rangle}{\langle\rho^* x_\alpha^2\rangle}, \quad \alpha \quad 1,2, \text{not summed}.$$

Comparing this relation with the field $\vartheta_\alpha$from the solution $(17)_3$ for directed curves, we see that we have to approximate

$$\frac{\langle\rho^* x_\alpha \varphi\rangle}{\langle\rho^* x_\alpha^2\rangle} \simeq 0, \quad \alpha \quad 1,2, \text{not summed},$$

where $\varphi(x_1,x_2)$ is the torsion function given by (22). For example, in the case when Σ is an elliptical domain $\Sigma \quad \left\{(x_1,x_2)|\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}<1\right\}$ and μ is constant, then we have $\varphi(x_1,x_2) \quad \frac{b^2 \ a^2}{a^2+b^2}x_1x_2$ so that the above approximation is justified. □

We remark that, due to the shear bending coupling in the case of static problems, the effective shear stiffness coefficients A_1, A_2 and A_{12} cannot be obtained by analyzing static shear problems and using the same procedure as above. (For *thin* beams, the coefficients A_1, A_2, A_{12} will not enter in the leading order terms of the solutions.) For this reason, we determine the effective shear stiffness coefficients by solving a free vibration problem.[1] The necessity of considering free vibration problems for the determination of effective shear stiffness properties is also discussed in details in [20] Section 6, and in [21, pp. 34 38].

4.2. Shear vibrations of rectangular rods

Consider a three dimensional rod which occupies the domain $\mathcal{R} \quad \{(x_1,x_2,x_3)|x_1\in(\ \frac{a}{2},\frac{a}{2}),x_2\in(\ \frac{b}{2},\frac{b}{2}),x_3\in(0,l)\}$, made of a non homogeneous isotropic material. The material parameters λ, μ and ρ^* are given functions of (x_1,x_2). Assume that the mass density ρ^* has a symmetrical distribution across the thickness: $\rho^*(x_1,x_2) = \rho^*(\ \ x_1,x_2)$.

The body loads are zero, the lateral surfaces $x_1 \quad \pm\frac{a}{2}$ and $x_2 \quad \pm\frac{b}{2}$ are traction free, and the end boundary conditions are given by

$$u_1^* \quad u_2^* \quad 0 \quad \text{and} \quad t_{33}^* \quad 0 \quad \text{for} \quad x_3 \quad 0,l. \tag{30}$$

To determine the shear vibrations of this rod, we search for solutions $\boldsymbol{u}^*$ of the form

$$\boldsymbol{u}^* \quad W\cos(\omega t)\sin\left(\frac{\pi}{a}x_1\right)\boldsymbol{e}_3, \tag{31}$$

where W is a constant and ω is the lowest natural frequency. We observe that all the boundary conditions are satisfied by the field (31), and the equations of motion reduce to $t^*_{13,1} \quad \rho^*\ddot{u}_3^*$, which by integration with respect to x_1 gives

$$t_{13}^* \quad W\omega^2\cos(\omega t)\int_{\ a/2}^{x_1}\rho^*(x_1,x_2)\sin\left(\frac{\pi}{a}x_1\right)dx_1.$$

Using the constitutive equation for t^*_{13} we get

$$\mu(x_1,x_2)\frac{\pi}{a}\cos\left(\frac{\pi}{a}x_1\right) \quad \omega^2\int_{\ a/2}^{x_1}\rho^*(x_1,x_2)\sin\left(\frac{\pi}{a}x_1\right)dx_1 \tag{32}$$

We apply the mean value theorem for the integral in (32) and we deduce that there exists a point $\alpha \quad \alpha(x_1,x_2)\in(\ \frac{a}{2},x_1)$ such that

$$\int_{\ a/2}^{x_1}\rho^*(x_1,x_2)\sin\left(\frac{\pi}{a}x_1\right)dx_1 \quad \rho^*(\alpha,x_2)\int_{\ a/2}^{x_1}\sin\left(\frac{\pi}{a}x_1\right)dx_1. \tag{33}$$

Substituting (33) into (32) and integrating over Σ we obtain

$$\omega^2 \quad \left(\frac{\pi}{a}\right)^2\frac{\langle\mu(x_1,x_2)\rangle}{\langle\rho^*(\alpha,x_2)\rangle}. \tag{34}$$

Let us treat the same problem using the approach of directed curves. We consider a straight rod along the Ox_3 axis for which the arclength parameter $s\in(0,l)$. The external body loads $\boldsymbol{F}$ and $\boldsymbol{L}$ are zero. According to (14) and (30) we have the following boundary conditions on the rod ends

[1] Note that the use of static and dynamic problems for identification purposes must result in the same effective stiffness properties. The type of the problem (static or dynamic) should not influence the final results [19].

$$w_\alpha \quad 0, \quad F \quad 0, \quad \psi \quad 0, \quad L_\alpha \quad 0 \quad (\alpha \quad 1,2), \quad \text{for } s \quad 0, l. \tag{35}$$

In order to study the shear vibrations, we search for solutions of the Eqs. (9), (13) of the form

$$\vartheta_1 \quad \overline{W}\cos(\omega t), \quad \vartheta_2 \quad \psi \quad 0, \quad u \quad w_\alpha \quad 0, \tag{36}$$

where $\overline{W}$ is a constant and $\bar{\omega}$ is the natural frequency of the rod. In view of the constitutive Eq. (13), we see that the boundary condi tions (35) are satisfied. Imposing that the fields (36) verify the equa tions of motion (9) we find

$$\omega^2 \quad \frac{A_1}{I_2} \quad \text{and} \quad A_{12} \quad 0. \tag{37}$$

We identify the natural frequencies ω and $\bar{\omega}$ from (34) and (36)$_1$, and we obtain the expression of the constitutive coefficient A_1 as follows:

$$A_1 \quad k\frac{\langle\mu\rangle\langle\rho^* x_1^2\rangle \mathrm{Area}(\Sigma)}{\langle\rho^*(\alpha, x_2)\rangle\langle x_1^2\rangle} \quad \text{with} \quad k \quad \frac{\pi^2}{12}, \tag{38}$$

where the factor k is similar to the shear correction factor intro duced first by Timoshenko [40] in the theory of beams (note that in the original contribution of Timoshenko the value is 2/3). One can proceed analogously for the x_2 direction and find a similar expression for A_2. These relations express the transverse shear stiff ness coefficients for non homogeneous rectangular rods. The value given by (38) will be verified in Section 8, where we consider the bending of cantilever functionally graded beams and make a com parison with numerical results.

Remarks.

1. In the case of homogeneous rods, μ and ρ^* are constant, and from (38) we get the well known formulas [20]

$$A_1 \quad A_2 \quad k\mu\mathrm{Area}(\Sigma), \quad A_{12} \quad 0. \tag{39}$$

The value of the factor k in relation (39) has been discussed in [21].

2. In the case of thin rods, when ρ^* has a smooth variation across the thickness, we can employ the approximation

$$\langle\rho^*(\alpha, x_2)\rangle \simeq \langle\rho^*(x_1, x_2)\rangle. \tag{40}$$

Then, we substitute (40) into (38) and find

$$A_\gamma \quad k\langle\mu\rangle\frac{\langle\rho^* x_\gamma^2\rangle \mathrm{Area}(\Sigma)}{\langle\rho^*\rangle\langle x_\gamma^2\rangle} \quad (\gamma \quad 1, 2 \text{ not summed}), \quad A_{12} \quad 0. \tag{41}$$

The simplified (approximate) formulas (41) can be used to estimate the transverse shear stiffness for arbitrary non homogeneous rods (not necessarily rectangular or symmetrical) in most cases.

5. Beams composed of two different materials

In this section we consider beams and rods made of two isotro pic and non homogeneous materials. The body $\mathcal{B}$ is decomposed in two regions $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{B}_\rho \quad \{(x_1, x_2, x_3) | (x_1, x_2) \in S_\rho, x_3 \in (0, l)\}$. Thus, the cross section Σ is decomposed in two domains S_1 and S_2 with $S_1 \cap S_2 = \emptyset$, see Fig. 2a. We denote by Γ_0 the curve of separation between the domains S_1 and S_2 and by Γ_1, Γ_2 the complementary subsets of $\partial\Sigma$ such that $\partial \;\; S_\rho = \Gamma_0 \cap \Gamma_\rho$. Let $\Pi_0 = \{(x_1, x_2, x_3) | \; (x_1, x_2) \in \Gamma_0, x_3 \in (0, l)\}$ be the surface of separation of the two materials. We assume that the two materials are welded together along Π_0 and there is no separation of material along Π_0, so we have the conditions

$$[\boldsymbol{u}^*]_1 \quad [\boldsymbol{u}^*]_2, \quad [\boldsymbol{T}^*]_1 \cdot \boldsymbol{n}^0 \quad [\boldsymbol{T}^*]_2 \cdot \boldsymbol{n}^0 \quad \text{on} \quad \Pi_0, \tag{42}$$

where $\boldsymbol{n}^0 \quad n_\alpha^0 \boldsymbol{e}_\alpha$ is the unit normal of Π_0, outward to $\mathcal{B}_1$. The nota tions $[f]_1$ and $[f]_2$ represent the values of any field f on Π_0, calculated

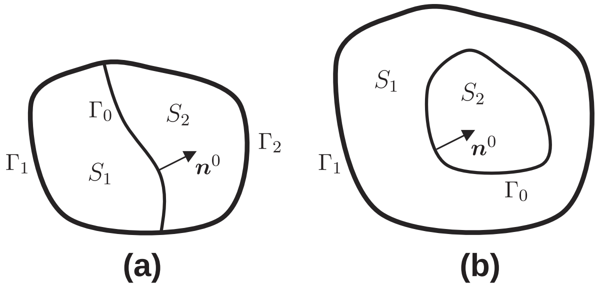


Fig. 2. The cross-section of rods composed of two materials.

as the limits of the values from the domains $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Let us denote the Lamé moduli of the material occupying the do main $\mathcal{B}_\rho$ by $\lambda^{(\rho)}(x_1, x_2)$ and $\mu^{(\rho)}(x_1, x_2)$, with $(x_1, x_2) \in S_\rho$, $\rho = 1, 2$.

Consider the problem of extension, bending and torsion of such a compound three dimensional beam, under the resultant forces and moments (20) acting on the ends. This problem has been trea ted in [23, Section 3.6], and the exact solution is expressed in terms of the solutions to some auxiliary plane strain problems. Let us de note by $u_\alpha^{(1)}$, $u_\alpha^{(2)}$ and $u_\alpha^{(3)}$ the solutions of the 3 plane strain prob lems $\mathcal{P}^{(1)}$, $\mathcal{P}^{(2)}$ and $\mathcal{P}^{(3)}$ respectively, formulated on the domain $\Sigma = S_1 \cup S_2 \cup \Gamma_0$ by

$$\begin{aligned}
\mathcal{P}^{(\gamma)}:\; & t_{\beta\alpha,\beta}^{(\gamma)} + (\lambda^{(\rho)} x_\gamma)_{,\alpha} \quad 0 \;\text{ in } S_\rho, \quad t_{\beta\alpha}^{(\gamma)} n_\beta^* \quad \lambda^{(\rho)} x_\gamma n_\alpha^* \;\text{ on } \Gamma_\rho, \\
& [u_\alpha^{(\gamma)}]_1 \quad [u_\alpha^{(\gamma)}]_2, \quad \left[t_{\beta\alpha}^{(\gamma)}\right]_1 n_\beta^0 \quad \left[t_{\beta\alpha}^{(\gamma)}\right]_2 n_\beta^0 + (\lambda^{(2)} \quad \lambda^{(1)}) x_\gamma n_\alpha^0 \;\text{ on } \Gamma_0, \\
\mathcal{P}^{(3)}:\; & t_{\beta\alpha,\beta}^{(3)} + \lambda_{,\alpha}^{(\rho)} \quad 0 \;\text{ in } S_\rho, \quad t_{\beta\alpha}^{(3)} n_\beta^* \quad \lambda^{(\rho)} n_\alpha^* \;\text{ on } \Gamma_\rho, \\
& [u_\alpha^{(3)}]_1 \quad [u_\alpha^{(3)}]_2, \quad \left[t_{\beta\alpha}^{(3)}\right]_1 n_\beta^0 \quad \left[t_{\beta\alpha}^{(3)}\right]_2 n_\beta^0 + (\lambda^{(2)} \quad \lambda^{(1)}) n_\alpha^0 \;\text{ on } \Gamma_0.
\end{aligned} \tag{43}$$

We also introduce the function $\varphi(x_1, x_2)$ which is the solution of the boundary value problem

$$\begin{aligned}
& (\mu^{(\rho)}\varphi_{,\alpha})_{,\alpha} \quad \mu_{,1}^{(\rho)} x_2 \quad \mu_{,2}^{(\rho)} x_1 \;\text{ in } S_\rho, \quad \frac{\partial\varphi}{\partial n^*} \quad x_2 n_1^* \quad x_1 n_2^* \;\text{ on } \Gamma_\rho, \\
& [\varphi]_1 \quad [\varphi]_2, \quad \mu^{(1)}\left[\frac{\partial\varphi}{\partial n^0}\right]_1 \quad \mu^{(2)}\left[\frac{\partial\varphi}{\partial n^0}\right]_2 + (\mu^{(1)} \quad \mu^{(2)})(x_2 n_1^0 \quad x_1 n_2^0) \;\text{ on } \Gamma_0.
\end{aligned} \tag{44}$$

Comparing the solution of the extension bending torsion problem in the direct approach given in Section 3.2 with the solution of the corresponding three dimensional problem presented in [23, Section 3.6], we deduce (in the same manner as in Section 4.1) the following expressions for the constitutive coefficients

$$\begin{aligned}
A_3 \quad & \sum_{\rho \quad 1}^{2} \int_{S_\rho} (\lambda^{(\rho)} + 2\mu^{(\rho)} + \lambda^{(\rho)} u_{\gamma,\gamma}^{(3)}) \mathrm{d}x_1 \mathrm{d}x_2, \quad B_{13} \quad B_{23} \quad 0, \\
B_{31} \quad & \sum_{\rho \quad 1}^{2} \int_{S_\rho} x_2 \left(\lambda^{(\rho)} + 2\mu^{(\rho)} + \lambda^{(\rho)} u_{\gamma,\gamma}^{(3)}\right) \mathrm{d}x_1 \mathrm{d}x_2, \\
B_{32} \quad & \sum_{\rho \quad 1}^{2} \int_{S_\rho} x_1 \left(\lambda^{(\rho)} + 2\mu^{(\rho)} + \lambda^{(\rho)} u_{\gamma,\gamma}^{(3)}\right) \mathrm{d}x_1 \mathrm{d}x_2, \\
C_1 \quad & \sum_{\rho \quad 1}^{2} \int_{S_\rho} x_2 \left[(\lambda^{(\rho)} + 2\mu^{(\rho)}) x_2 + \lambda^{(\rho)} u_{\gamma,\gamma}^{(2)}\right] \mathrm{d}x_1 \mathrm{d}x_2, \\
C_2 \quad & \sum_{\rho \quad 1}^{2} \int_{S_\rho} x_1 \left[(\lambda^{(\rho)} + 2\mu^{(\rho)}) x_1 + \lambda^{(\rho)} u_{\gamma,\gamma}^{(1)}\right] \mathrm{d}x_1 \mathrm{d}x_2, \\
C_{12} \quad & \sum_{\rho \quad 1}^{2} \int_{S_\rho} x_1 \left[(\lambda^{(\rho)} + 2\mu^{(\rho)}) x_2 + \lambda^{(\rho)} u_{\gamma,\gamma}^{(2)}\right] \mathrm{d}x_1 \mathrm{d}x_2, \\
C_3 \quad & \sum_{\rho \quad 1}^{2} \int_{S_\rho} \mu^{(\rho)} \left[x_1(x_1 + \varphi_{,2}) + x_2(x_2 \quad \varphi_{,1})\right] \mathrm{d}x_1 \mathrm{d}x_2,
\end{aligned} \tag{45}$$

where the functions $u_\alpha^{(k)}(x_1, x_2)$ are determined by (43) and $\varphi(x_1, x_2)$ is given by (44).

Remarks

1. The above results (45) also hold when the distribution of the material in the beam is such that the separation curve Γ_0 is a closed curve included in Σ, see Fig. 2b. In this case we have $\Gamma_1 = \partial\Sigma$, $\Gamma_2 = \emptyset$, $\partial S_1 = \Gamma_1 \cup \Gamma_0$, $\partial S_2 = \Gamma_0$, and the boundary value problems (43), (44) keep the same forms.
2. The results of this section can be extended to the case when the beam $\mathcal{B}$ is composed of n $(n \geqslant 2)$ non homogeneous and isotro pic materials with different mechanical properties.

6. Orthotropic and non-homogeneous materials

Let us consider next beams and rods made of orthotropic and non homogeneous materials. The three dimensional constitutive equations for such materials are

$$\begin{aligned} &t^*_{11} = c_{11}e^*_{11} + c_{12}e^*_{22} + c_{13}e^*_{33}, \quad t^*_{22} = c_{12}e^*_{11} + c_{22}e^*_{22} + c_{23}e^*_{33},\\ &t^*_{33} = c_{13}e^*_{11} + c_{23}e^*_{22} + c_{33}e^*_{33}, \quad t^*_{23} = 2c_{44}e^*_{23}, \quad t^*_{31} = 2c_{55}e^*_{31},\\ &t^*_{12} = 2c_{66}e^*_{12}, \end{aligned} \tag{46}$$

where the constitutive coefficients c_{ij} depend on $(x_1,x_2) \in \Sigma$.

Our aim is to determine the effective stiffness coefficients from the direct approach in terms of $c_{ij}(x_1,x_2)$. In this purpose, we con sider the extension, bending and torsion of the beam $\mathcal{B}$ due to the terminal loads (20).This three dimensional problem has been solved in [23, Section 4.11], with the help of some auxiliary plane strain problems defined on the domain Σ, which are recorded be low. We designate by $u^{(k)}_\alpha(x_1,x_2)$ the solutions of the plane strain problems $\mathcal{Q}^{(k)}, k = 1,2,3,$ given by

$$\begin{aligned} &\mathcal{Q}^{(\gamma)}: \quad t^{(\gamma)}_{\beta\alpha,\beta} + (c_{\alpha 3}x_\gamma)_{,\alpha} = 0 \quad \text{in } \Sigma, \quad t^{(\gamma)}_{\beta\alpha}n^*_\beta = -c_{\alpha 3}x_\gamma n^*_\alpha \quad \text{on } \partial\Sigma,\\ &\mathcal{Q}^{(3)}: \quad t^{(3)}_{\beta\alpha,\beta} + c_{\alpha 3,\alpha} = 0 \quad \text{in } \Sigma, \quad t^{(3)}_{\beta\alpha}n^*_\beta = -c_{\alpha 3}n^*_\alpha \quad \text{on } \partial\Sigma. \end{aligned} \tag{47}$$

The subscript $\alpha = 1, 2$ is not summed in the relations (47). The tor sion function $\varphi(x_1,x_2)$ is determined by the boundary value problem

$$\begin{aligned} &(c_{55}\varphi_{,1})_{,1} + (c_{44}\varphi_{,2})_{,2} = c_{55,1}x_2 - c_{44,2}x_1 \quad \text{in} \quad \Sigma,\\ &c_{55}\varphi_{,1}n^*_1 + c_{44}\varphi_{,2}n^*_2 = c_{55}x_2n^*_1 - c_{44}x_1n^*_2 \quad \text{on} \quad \partial\Sigma. \end{aligned} \tag{48}$$

By identification of the three dimensional solution from [23, Sec tion 4.11], with the solution (17) (19) in the direct approach we get the following effective stiffness coefficients

$$\begin{aligned} &A_3 = \langle c_{33} + c_{13}u^{(3)}_{1,1} + c_{23}u^{(3)}_{2,2}\rangle,\\ &B_{13} = B_{23} = 0,\\ &B_{31} = \langle x_2(c_{33} + c_{13}u^{(3)}_{1,1} + c_{23}u^{(3)}_{2,2})\rangle,\\ &B_{32} = -\langle x_1(c_{33} + c_{13}u^{(3)}_{1,1} + c_{23}u^{(3)}_{2,2})\rangle,\\ &C_1 = \langle x_2(c_{33}x_2 + c_{13}u^{(2)}_{1,1} + c_{23}u^{(2)}_{2,2})\rangle,\\ &C_2 = \langle x_1\left(c_{33}x_1 + c_{13}u^{(1)}_{1,1} + c_{23}u^{(1)}_{2,2}\right)\rangle,\\ &C_{12} = -\langle x_1\left(c_{33}x_2 + c_{13}u^{(2)}_{1,1} + c_{23}u^{(2)}_{2,2}\right)\rangle,\\ &C_3 = \langle c_{44}x_1(x_1 + \varphi_{,2}) + c_{55}x_2(x_2 - \varphi_{,1})\rangle. \end{aligned} \tag{49}$$

In view of the identifications (49) one can show that the fields u, w_α, ψ, $\boldsymbol{N}$ and $\boldsymbol{M}$ corresponding to the solutions in the two approaches coincide.

Remark. This method can be applied also for beams composed of two different orthotropic materials. Using the notations introduced in the beginning of Section 5, we assume that the non homogeneous orthotropic material which occupies the domain $\mathcal{B}_\rho$ has the consti tutive coefficients $c^{(\rho)}_{ij}(x_1,x_2)$. If we employ the same procedure as in Section 5 and compare with the results of [23, Section 4.11], then we obtain the following expressions for the effective stiffness coefficients

$$\begin{aligned} &A_3 = \sum_{\rho=1}^{2}\int_{S_\rho}\left(c^{(\rho)}_{33} + c^{(\rho)}_{13}u^{(3)}_{1,1} + c^{(\rho)}_{23}u^{(3)}_{2,2}\right)dx_1dx_2, \quad B_{13} = B_{23} = 0,\\ &B_{31} = \sum_{\rho=1}^{2}\int_{S_\rho}x_2\left(c^{(\rho)}_{33} + c^{(\rho)}_{13}u^{(3)}_{1,1} + c^{(\rho)}_{23}u^{(3)}_{2,2}\right)dx_1dx_2,\\ &B_{32} = -\sum_{\rho=1}^{2}\int_{S_\rho}x_1\left(c^{(\rho)}_{33} + c^{(\rho)}_{13}u^{(3)}_{1,1} + c^{(\rho)}_{23}u^{(3)}_{2,2}\right)dx_1dx_2,\\ &C_1 = \sum_{\rho=1}^{2}\int_{S_\rho}x_2\left(c^{(\rho)}_{33}x_2 + c^{(\rho)}_{13}u^{(2)}_{1,1} + c^{(\rho)}_{23}u^{(2)}_{2,2}\right)dx_1dx_2,\\ &C_2 = \sum_{\rho=1}^{2}\int_{S_\rho}x_1(c^{(\rho)}_{33}x_1 + c^{(\rho)}_{13}u^{(1)}_{1,1} + c^{(\rho)}_{23}u^{(1)}_{2,2})dx_1dx_2,\\ &C_{12} = -\sum_{\rho=1}^{2}\int_{S_\rho}x_1\left(c^{(\rho)}_{33}x_2 + c^{(\rho)}_{13}u^{(2)}_{1,1} + c^{(\rho)}_{23}u^{(2)}_{2,2}\right)dx_1dx_2,\\ &C_3 = \sum_{\rho=1}^{2}\int_{S_\rho}\left[c^{(\rho)}_{44}x_1(x_1 + \varphi_{,2}) + c^{(\rho)}_{55}x_2(x_2 - \varphi_{,1})\right]dx_1dx_2, \end{aligned} \tag{50}$$

where $u^{(k)}_\alpha(x_1,x_2), k = 1,2,3,$ are the solutions of the three plane strain problems

$$\begin{aligned} &t^{(\gamma)}_{\beta\alpha,\beta} + \left(c^{(\rho)}_{\alpha 3}x_\gamma\right)_{,\alpha} = 0 \quad \text{in } S_\rho, \quad t^{(\gamma)}_{\beta\alpha}n^*_\beta = -c^{(\rho)}_{\alpha 3}x_\gamma n^*_\alpha \quad \text{on } \Gamma_\rho,\\ &[u^{(\gamma)}_\alpha]_1 = [u^{(\gamma)}_\alpha]_2, \quad \left[t^{(\gamma)}_{\beta\alpha}\right]_1 n^0_\beta = \left[t^{(\gamma)}_{\beta\alpha}\right]_2 n^0_\beta + \left(c^{(2)}_{\alpha 3} - c^{(1)}_{\alpha 3}\right)x_\gamma n^0_\alpha \quad \text{on } \Gamma_0, \end{aligned} \tag{51}$$

$$\begin{aligned} &t^{(3)}_{\beta\alpha,\beta} + c^{(\rho)}_{\alpha 3,\alpha} = 0 \quad \text{in } S_\rho, \quad t^{(3)}_{\beta\alpha}n^*_\beta = -c^{(\rho)}_{\alpha 3}n^*_\alpha \quad \text{on } \Gamma_\rho,\\ &[u^{(3)}_\alpha]_1 = [u^{(3)}_\alpha]_2, \quad \left[t^{(3)}_{\beta\alpha}\right]_1 n^0_\beta = \left[t^{(3)}_{\beta\alpha}\right]_2 n^0_\beta + \left(c^{(2)}_{\alpha 3} - c^{(1)}_{\alpha 3}\right)n^0_\alpha \quad \text{on } \Gamma_0. \end{aligned} \tag{52}$$

In the relations (51) and (52) the subscript $\alpha = 1, 2$ is not summed. The torsion function $\varphi(x_1,x_2)$ appearing in (50) is the solution of the following boundary value problem

$$\begin{aligned} &\left(c^{(\rho)}_{55}\varphi_{,1}\right)_{,1} + \left(c^{(\rho)}_{44}\varphi_{,2}\right)_{,2} = c^{(\rho)}_{55,1}x_2 - c^{(\rho)}_{44,2}x_1 \quad \text{in} \quad S_\rho,\\ &c^{(\rho)}_{55}\varphi_{,1}n^*_1 + c^{(\rho)}_{44}\varphi_{,2}n^*_2 = c^{(\rho)}_{55}x_2n^*_1 - c^{(\rho)}_{44}x_1n^*_2 \quad \text{on} \quad \Gamma_\rho, \quad [\varphi]_1 = [\varphi]_2 \text{on } \Gamma_0,\\ &\left[c^{(1)}_{55}\varphi_{,1}n^0_1 + c^{(1)}_{44}\varphi_{,2}n^0_2\right]_1 = \left[c^{(2)}_{55}\varphi_{,1}n^0_1 + c^{(2)}_{44}\varphi_{,2}n^0_2\right]_2\\ &\qquad + \left(c^{(1)}_{55} - c^{(2)}_{55}\right)x_2n^0_1 - \left(c^{(1)}_{44} - c^{(2)}_{44}\right)x_1n^0_2 \text{ on } \Gamma_0. \end{aligned} \tag{53}$$

The relations (50) for the constitutive coefficients are valid also in the case when Γ_0 is a closed curve included in Σ. Moreover, these formulas can be extended to the case of beams composed of n dif ferent orthotropic materials $(n \geqslant 2)$.

6.1. Transverse shear stiffness

To determine the transverse shear stiffness coefficients A_1, A_2 and A_{12} for orthotropic non homogeneous rods, we consider the problem of shear vibrations of rectangular rods formulated in Sec tion 4.2. Assume that ρ^* has a symmetrical distribution in the x_1 direction: $\rho^*(x_1,x_2) = \rho^*(-x_1,x_2)$.

We search for a solution in the form (31). Then the boundary conditions (30) are satisfied and the equations of motion reduce to

$$c_{55}(x_1,x_2)\frac{\pi}{a}\cos\left(\frac{\pi}{a}x_1\right) \qquad -\omega^2\int_{-a/2}^{x_1}\rho^*(x_1,x_2)\sin\left(\frac{\pi}{a}x_1\right)dx_1$$

Inserting here the relation (33) and integrating over Σ we find the lowest natural frequency

$$\omega^2 = \left(\frac{\pi}{a}\right)^2 \frac{\langle c_{55}(x_1,x_2)\rangle}{\langle \rho^*(\alpha,x_2)\rangle}. \tag{54}$$

On the other hand, we solve the same problem by the direct ap proach and we find the rod's natural frequency $\bar{\omega}$ given by (37). We identify $\omega = \bar{\omega}$ and from relations (37) and (54) we obtain

$$A_1 = k\frac{\langle c_{55}\rangle\langle \rho^* x_1^2\rangle \mathrm{Area}(\Sigma)}{\langle \rho^*(\alpha,x_2)\rangle\langle x_1^2\rangle}, \quad A_{12} = 0. \tag{55}$$

To determine A_2, one can proceed analogously in the x_2 direction.

Remarks

1. If we admit the approximation (40) then we deduce

$$A_1 = k\langle c_{55}\rangle \frac{\langle \rho^* x_1^2\rangle \mathrm{Area}(\Sigma)}{\langle \rho^*\rangle\langle x_1^2\rangle},$$
$$A_2 = k\langle c_{44}\rangle \frac{\langle \rho^* x_2^2\rangle \mathrm{Area}(\Sigma)}{\langle \rho^*\rangle\langle x_2^2\rangle}. \tag{56}$$

where $\frac{\pi^2}{12}$ stands for the value of the factor k. In most cases, these formulas are applicable for orthotropic non homogeneous rods with arbitrary cross section properties (not necessarily rectan gular or symmetrical).

2. Consider the case of non homogeneous rods composed of two different orthotropic materials: in the region $\mathcal{B}_\gamma$ of the body we have the mass density $\rho^{(\gamma)}(x_1,x_2)$ and the constitutive coef ficients $c_{ij}^{(\gamma)}(x_1,x_2), \gamma = 1,2$. Eqs. (55) and (56) for transverse shear stiffness coefficients remain valid also in this case, with the specifications

$$\langle c_{ij}\rangle = \sum_{\gamma=1}^{2}\int_{S_\gamma} c_{ij}^{(\gamma)}\mathrm{d}x_1\mathrm{d}x_2, \quad \langle \rho^*\rangle = \sum_{\gamma=1}^{2}\int_{S_\gamma}\rho^{(\gamma)}\mathrm{d}x_1\mathrm{d}x_2,$$
$$\langle \rho^* x_\alpha^2\rangle = \sum_{\gamma=1}^{2}\int_{S_\gamma}\rho^{(\gamma)}x_\alpha^2\mathrm{d}x_1\mathrm{d}x_2. \tag{57}$$

The extension of formulas (56) and (57) to the case of rods com posed of n orthotropic materials is also possible.

7. Special cases and examples

7.1. Non homogeneous rods with constant Poisson ratio

Let us consider the case when the rod is made of an isotropic material with constant Poisson ratio ν. The Young's modulus E is an arbitrary function of (x_1,x_2) and the shape of cross section Σ is arbitrary. This type of material is of practical interest and it has been studied in many works, see e.g. [41]. In this case the solu tions $u_\alpha^{(k)}(x_1,x_2)$ of the problems $\mathcal{D}^{(k)}$, $k = 1,2,3$, defined by (21) have a simple form

$$u_1^{(1)} = -\frac{1}{2}\nu(x_1^2 - x_2^2), \quad u_2^{(1)} = -\nu x_1x_2;$$
$$u_1^{(2)} = -\nu x_1x_2, \quad u_2^{(2)} = \frac{1}{2}\nu(x_1^2 - x_2^2); \quad u_1^{(3)} = -\nu x_1, \quad u_2^{(3)} = -\nu x_2. \tag{58}$$

Then, from (29) we obtain the following expressions for the effec tive stiffness coefficients

$$A_3 = \langle E(x_1,x_2)\rangle, \; C_{12} = -\langle x_1x_2E(x_1,x_2)\rangle, \; C_1 = \langle x_2^2E(x_1,x_2)\rangle,$$
$$C_2 = \langle x_1^2E(x_1,x_2)\rangle, \; B_{31} = \langle x_2E(x_1,x_2)\rangle, \; B_{32} = -\langle x_1E(x_1,x_2)\rangle, \; B_{\alpha3} = 0. \tag{59}$$

The constitutive coefficients C_3, A_1, A_2 and A_{12} keep the same form as in the general case, given by $(29)_1$ and (38).

Remark. In the case of a homogeneous isotropic rod, i.e. when E is also constant, from (59) and (5) we obtain the well known formulas

$$A_3 = E\mathrm{Area}(\Sigma), \quad C_1 = E\langle x_2^2\rangle, \quad C_2 = E\langle x_1^2\rangle, \quad C_{12} = 0,$$
$$B_{\alpha3} = B_{3\alpha} = 0.$$

In view of $(29)_1$ the torsional rigidity C_3 for simply connected cross sections is given by

$$C_3 = 2\mu\langle\phi(x_1,x_2)\rangle \quad \text{with} \quad \Delta\phi = -2 \text{ in } \Sigma, \quad \phi = 0 \text{ on } \partial\Sigma.$$

The effective transverse shear coefficients are given by (39). The above expressions of the effective stiffness coefficients for homoge neous and isotropic directed curves have been presented in [20,21].

7.2. Circular rod composed of two materials

For rods composed of two different isotropic and non homoge neous materials we use the notations and developments of Section 5. The cross section of the rod is decomposed as $\Sigma = S_1 \cup \bar{S}_2$, where $S_1 = \left\{(x_1,x_2)|a^2 < x_1^2 + x_2^2 < b^2\right\}$ and $S_2 = \{(x_1,x_2)|x_1^2 + x_2^2 < a^2\}$. The first material occupies the region $S_1 \times (0,l)$ and has the Lamé moduli

$$\lambda^{(1)}(x_1,x_2) = \lambda_0 r^{-m}, \quad \mu^{(1)}(x_1,x_2) = \mu_0 r^{-m},$$
$$r = \sqrt{x_1^2 + x_2^2}, \; (x_1,x_2) \in S_1, \tag{60}$$

where $m > 0$, λ_0 and μ_0 are constants. This kind of inhomogeneity has been investigated in many works, e.g. [41,42]. We denote by $\nu_0 = \frac{\lambda_0}{2(\lambda_0+\mu_0)}$ and $E_0 = \frac{\mu_0(3\lambda_0+2\mu_0)}{\lambda_0+\mu_0}$. The second material occupies the re gion $S_2 \times (0,l)$ and its elastic properties are described by

$$E^{(2)}(x_1,x_2) = E(r), \quad \nu^{(2)}(x_1,x_2) = \nu_0(\text{constant}),$$
$$(x_1,x_2) \in S_2, \tag{61}$$

where $E(r)$ is an arbitrary given function of r.

In order to use the results presented in Section 5 we have to solve the plane strain problems $\mathcal{P}^{(k)}$ given by (43) and the bound ary value problem (44) for the torsion function. In our case, we ob serve that these problems admit the following solutions

$$u_1^{(1)} = -u_2^{(2)} = \frac{1}{2}\nu_0(x_2^2 - x_1^2), \quad u_2^{(1)} = u_1^{(2)} = -\nu_0 x_1x_2,$$
$$u_\alpha^{(3)} = -\nu_0 x_\alpha. \tag{62}$$

Inserting these functions into the general results (45) we find the effective stiffness coefficients for this compound rod

$$C_3 = \frac{\pi}{1+\nu_0}\int_0^a r^3E(r)\mathrm{d}r + 2\pi\mu_0 d_m, \quad A_3 = 2\pi\left(\int_0^a rE(r)\mathrm{d}r + E_0c_m\right),$$
$$C_1 = C_2 = \pi\left(\int_0^a r^3E(r)\mathrm{d}r + E_0d_m\right), \quad C_{12} = 0, \quad B_{3\alpha} = B_{\alpha3} = 0, \tag{63}$$

where we have denoted by c_m and d_m the expressions

$$c_m = \begin{cases} \frac{a^{2-m} - b^{2-m}}{m-2} & \text{for } m \neq 2 \\ \log(b/a) & \text{for } m = 2 \end{cases}, \quad d_m = \begin{cases} \frac{a^{4-m} - b^{4-m}}{m-4} & \text{for } m \neq 4 \\ \log(b/a) & \text{for } m = 4 \end{cases}. \tag{64}$$

Let us find also the transverse shear stiffness coefficients A_1 and A_2. Assume that the mass density function $\rho^*(x_1,x_2)$ is given by

$$\rho^*(x_1,x_2) = \begin{cases} \rho_0^* r^{-m} & \text{for } (x_1,x_2) \in S_1 \\ \rho(r) & \text{for } (x_1,x_2) \in S_2 \end{cases}, \tag{65}$$

where $\rho_0^* > 0$ is a constant and $\rho(r)$ is an arbitrary function. Then, using the results (56), (57) specialized for isotropic materials we find the expressions

$$A_1 \quad A_2 \quad \frac{\pi^3}{6b^2}\left(\int_0^a \frac{rE(r)}{1+\nu_0}\mathrm{d}r + 2\mu_0 c_m\right)\left(\int_0^a r^3\rho(r)\mathrm{d}r + \rho_0^* d_m\right)$$
$$\times\left(\int_0^a r\rho(r)\mathrm{d}r + \rho_0^* c_m\right)^{\ 1}. \tag{66}$$

7.3. Orthotropic circular rod

Let us consider an orthotropic rod with cross section $\Sigma \quad \{(x_1,x_2)|x_1^2+x_2^2<a^2\}$. We assume that the constitutive coeffi cients satisfy

$$c_{ij} \quad c_{ij}^* e^{\ \sigma r}, \quad r \quad \sqrt{x_1^2+x_2^2}, \tag{67}$$

where $\sigma > 0$ and c_{ij}^* are constants. Let us introduce the notations

$$E_0 \quad E_0^* e^{\ \sigma r}, \quad E_0^* \quad c_{33}^* \quad c_{13}^*\nu_1^* \quad c_{23}^*\nu_2^*,$$
$$\nu_1^* \quad \frac{c_{13}^*c_{22}^* \quad c_{23}^*c_{12}^*}{\delta_1^*}, \quad \nu_2^* \quad \frac{c_{23}^*c_{11}^* \quad c_{13}^*c_{12}^*}{\delta_1^*}, \quad \delta_1^* \quad c_{11}^*c_{22}^* \quad (c_{12}^*)^2. \tag{68}$$

The solutions $u_\alpha^{(k)}(x_1,x_2)$ of the problems $\mathcal{Q}^{(k)}$ given by (47) are in this case

$$\begin{aligned} &u_1^{(1)} \quad \frac{1}{2}(\nu_1^*x_1^2 \quad \nu_2^*x_2^2), \quad u_2^{(1)} \quad \nu_2^*x_1x_2;\\ &u_1^{(2)} \quad \nu_1^*x_1x_2, \quad u_2^{(2)} \quad \frac{1}{2}(\nu_1^*x_1^2 \quad \nu_2^*x_2^2);\\ &u_1^{(3)} \quad \nu_1^*x_1, \quad u_2^{(3)} \quad \nu_2^*x_2, \end{aligned} \tag{69}$$

while the torsion function $\varphi(x_1,x_2)$ which solves the boundary va lue problem (48) is

$$\varphi(x_1,x_2) \quad \frac{c_{55}^* \quad c_{44}^*}{c_{44}^*+c_{55}^*}x_1x_2. \tag{70}$$

Thus, in view of (49) and (69) and (70) we find the effective stiffness coefficients

$$\begin{aligned} &C_3 \quad \frac{4\pi c_{44}^*c_{55}^*}{c_{44}^*+c_{55}^*}\int_0^a r^3e^{\ \sigma r}\mathrm{d}r, \qquad A_3 \quad 2\pi E_0^*\int_0^a re^{\ \sigma r}\mathrm{d}r,\\ &C_1 \quad C_2 \quad \pi E_0^*\int_0^a r^3e^{\ \sigma r}\mathrm{d}r, \quad C_{12} \quad 0, \quad B_{3\alpha} \quad B_{\alpha 3} \quad 0. \end{aligned} \tag{71}$$

Assume that the mass density of the rod is of the form $\rho^*(x_1,x_2) \quad \rho_0^*e^{\ \sigma r}$, where $\rho_0^*>0$ is constant. Then, from relations (56) we obtain the effective transverse shear stiffness coefficients

$$A_1 \quad \frac{\pi^3}{3}\frac{c_{55}^*}{a^2}\int_0^a r^3e^{\ \sigma r}\mathrm{d}r, \qquad A_2 \quad \frac{\pi^3}{3}\frac{c_{44}^*}{a^2}\int_0^a r^3e^{\ \sigma r}\mathrm{d}r. \tag{72}$$

8. Functionally graded beams made of metal foams

8.1. Distribution of the material properties

The mechanical properties of cellular solids have been pre sented in the books [25,43]. In this section we analyze rectangular beams made of metal foams.

The cross section domain is given by $\Sigma \quad \{(x_1,x_2)|x_1\in(\ \frac{h}{2},\frac{h}{2}), x_2\in(\ \frac{b}{2},\frac{b}{2})\}$. We consider that the porous material is functionally graded in the x_1 direction, such that the mass density ρ of the foam is given as a function of x_1 by the power law

$$\rho(x_1) \quad \rho_m + (\rho_s \quad \rho_m)\left(\frac{2|x_1|}{h}\right)^N, \tag{73}$$

where ρ_s is the density of the bulk (matrix) material, ρ_m is the min imum value of the density of the foam, and N is an exponent. This type of functionally graded porous materials has been studied in the case of plates in [43,44]. To express the Young modulus E of the foam we use the formula indicated by Gibson and Ashby [25]

$$E(x_1) \quad E_s\left(\frac{\rho(x_1)}{\rho_s}\right)^\kappa, \tag{74}$$

where E_s is the Young modulus of the bulk material. In what follows, we consider closed cell aluminum foams, for which the exponent κ is given by $\kappa = 2$, and the Poisson ratio is assumed to be constant with the value $\nu = 0.3$ [25]. Let us denote by $G_s \quad \frac{E_s}{2(1+\nu)}$ the shear modulus of the bulk material. The variations of ρ and E as functions of x_1, as given in (73) and (74), are depicted in Fig. 3 for several val ues of the exponent N.

Let us calculate the effective stiffness coefficients for this func tionally graded porous beam. Since the Poisson ratio is assumed constant, we can use the relation (59), in conjunction with (73) and (74), to derive the extensional and bending stiffness coefficients

$$\begin{aligned} &A_3 \quad bhE_s\left[r^2+\frac{2}{N+1}r(1 \quad r)+\frac{1}{2N+1}(1 \quad r)^2\right], \quad B_{3\alpha} \quad B_{\alpha 3} \quad 0,\\ &C_1 \quad \frac{b^3h}{12}E_s\left[r^2+\frac{2}{N+1}r(1 \quad r)+\frac{1}{2N+1}(1 \quad r)^2\right], \quad C_{12} \quad 0,\\ &C_2 \quad \frac{bh^3}{12}E_s\left[r^2+\frac{6}{N+3}r(1 \quad r)+\frac{3}{2N+3}(1 \quad r)^2\right], \end{aligned} \tag{75}$$

where we denote by r the ratio $r \quad \frac{\rho_m}{\rho_s}$. The effective shear stiffness can be calculated from the relations (38). We insert the expression for ρ from (73) into (38) and obtain

$$\begin{aligned} &A_2 \quad kbhG_s\left[r^2+\frac{2}{N+1}r(1 \quad r)+\frac{1}{2N+1}(1 \quad r)^2\right],\\ &A_1 \quad kbhG_s\frac{(bh)\rho_s}{\langle\rho(\alpha(x_1))\rangle}\left[r+\frac{3}{N+3}(1 \quad r)\right]\\ &\qquad\times\left[r^2+\frac{2}{N+1}r(1 \quad r)+\frac{1}{2N+1}(1 \quad r)^2\right], \end{aligned} \tag{76}$$

where, according to (33), $\langle\rho(\alpha(x_1))\rangle$ is given by

$$\langle\rho(\alpha(x_1))\rangle \quad \frac{\pi b}{h}\int_{\ \frac{h}{2}}^{\frac{h}{2}}\left(\cos\frac{\pi x_1}{h}\right)^{\ 1}\int_{\ \frac{h}{2}}^{x_1}\rho(\zeta)\sin\frac{\pi\zeta}{h}\mathrm{d}\zeta\mathrm{d}x_1. \tag{77}$$

Using the expression (73) in (77) and making some mathematical calculations, we get

$$\langle\rho(\alpha(x_1))\rangle \quad bh\rho_s\left[r+\frac{1 \quad r}{N+1}+(1 \quad r)J_N\right], \tag{78}$$

where we have denoted by

$$J_N \quad \left(\frac{2}{\pi}\right)^{N+1}\left[p_N(0) \quad p_N\left(\frac{\pi}{2}\right)+\int_0^{\frac{\pi}{2}}(\cos x)^{\ 1}\left(p_N\left(\frac{\pi}{2}\right) \quad p_N(x)\sin x\right)\mathrm{d}x\right].$$

In the last relation, the polynomial function $p_N(x)$ is given by

$$\begin{aligned} p_N(x) \quad &Nx^{N\ 1} \quad N(N \quad 1)(N \quad 2)x^{N\ 3}+N(N \quad 1) \quad (N \quad 4)x^{N\ 5}\\ &\sum_{i\ 0}^{m}(\ 1)^iN(N \quad 1) \quad (N \quad 2i)x^{N\ 2i\ 1}, \quad p_1(x) \quad 1, \ p_2(x) \quad 2x, \end{aligned}$$

where $m \quad [\frac{N\ 1}{2}]$ is the greatest integer not exceeding $\frac{N\ 1}{2}$.

Finally, if we substitute (78) into $(76)_2$ we find

$$A_1 \quad kbhG_s\frac{r+\frac{3}{N+3}(1 \quad r)}{r+\left(\frac{1}{N+1}+J_N\right)(1 \quad r)}\left[r^2+\frac{2}{N+1}r(1 \quad r)+\frac{1}{2N+1}(1 \quad r)^2\right]. \tag{79}$$

The formula (79) represent the 'exact' expression for the effective shear stiffness, calculated on the basis of (38). On the other hand, if we employ the 'approximate' relation (41) instead of (38), then we deduce the following simplified (approximate) expression for A_1

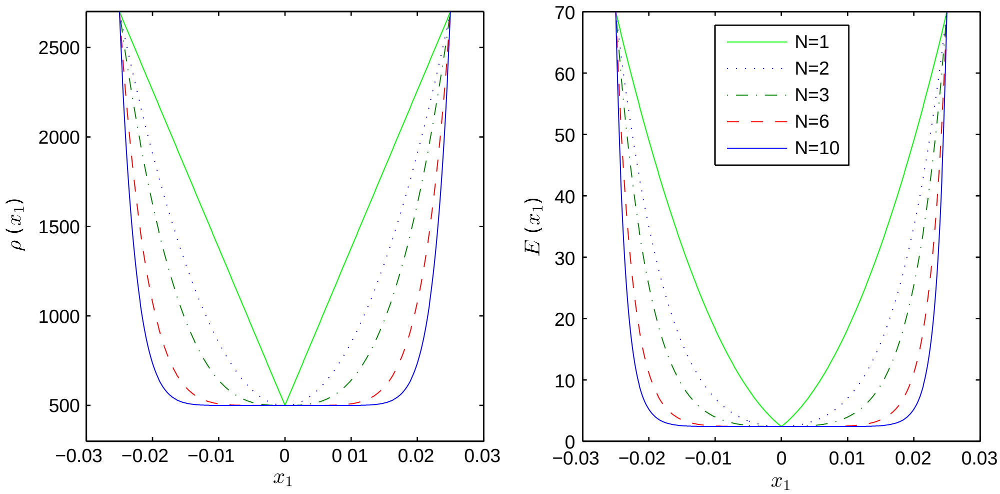


Fig. 3. The distributions of density ρ and Young modulus E for different values of N. For the aluminum foam we take $\rho_m = 500$ kg m^{-3}, $\rho_s = 2700$ kg m^{-3}, $E_s = 70$ GPa, and the thickness $h = 0.05$ m.

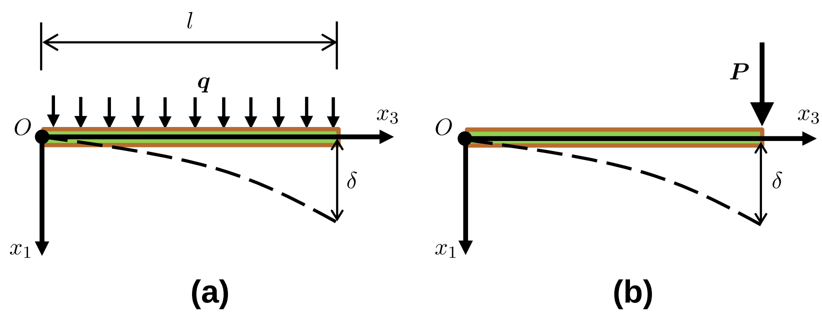


Fig. 4. (a) Cantilever beam with uniform distributed load q. (b) Cantilever beam with concentrated end force P.

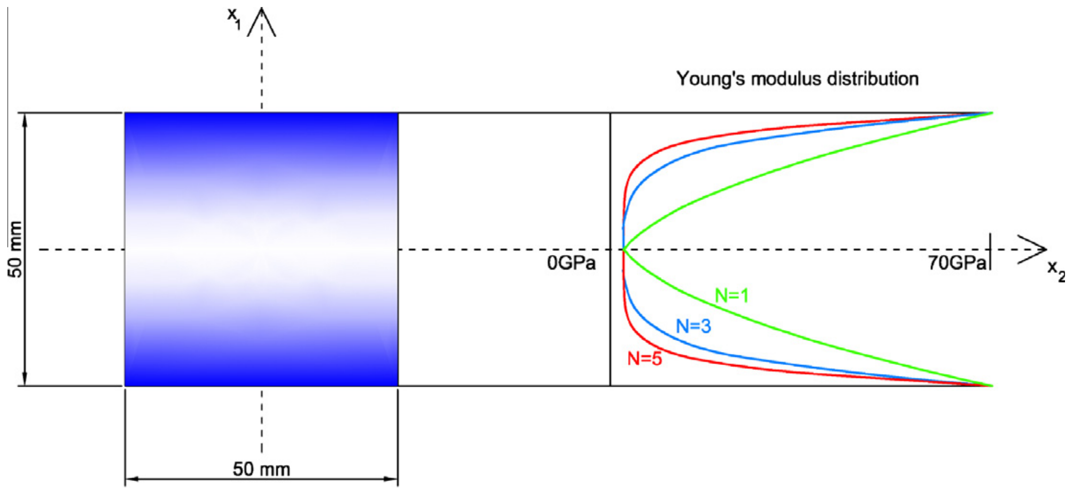


Fig. 5. Cross-section of the FGM beam and distribution of Young's modulus.

$$\widetilde{A}_1 = kbhG_s \times \frac{r + \frac{3}{N+3}(1-r)}{r + \frac{1}{N+1}(1-r)}\left[r^2 + \frac{2}{N+1}r(1-r) + \frac{1}{2N+1}(1-r)^2\right]. \tag{80}$$

Let us use the effective stiffness coefficients for FGM porous beams determined previously to solve some bending problems and compare the analytical solutions with the results obtained by a finite element analysis.

8.2. Cantilever beams

Consider a cantilever beam made of functionally graded closed cell aluminum foam subject to bending and shear under

Table 1
Comparison of results for cantilever FGM beam with uniform load.

N	1	2	3	4	5	6	7	8	9	10
δ_{FEM} (mm)	26.316	34.825	42.818	50.406	57.649	64.581	71.223	77.591	83.699	89.557
δ_{exact} (mm)	26.129	34.471	42.410	50.025	57.358	64.435	71.276	77.896	84.308	90.523
δ_{approx} (mm)	26.097	34.407	42.317	49.907	57.218	64.276	71.100	77.705	84.105	90.309
Δ (%)	0.716	1.027	0.962	0.762	0.507	0.227	−0.074	−0.393	−0.728	−1.079

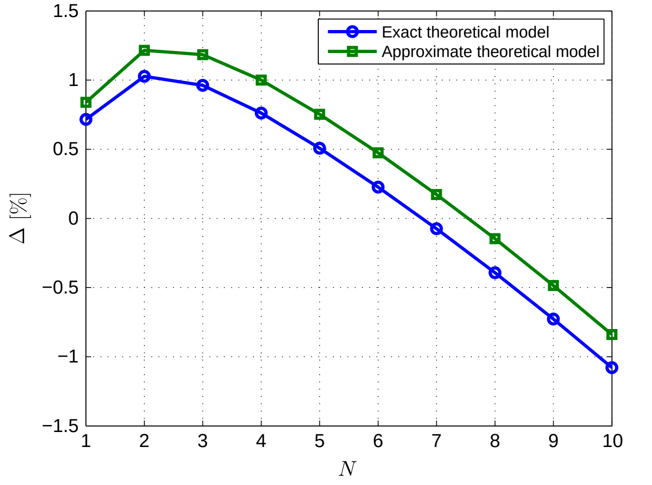


Fig. 6. Error Δ in terms of the exponent N, for the maximum deflection of a cantilever FGM beam with uniform load.

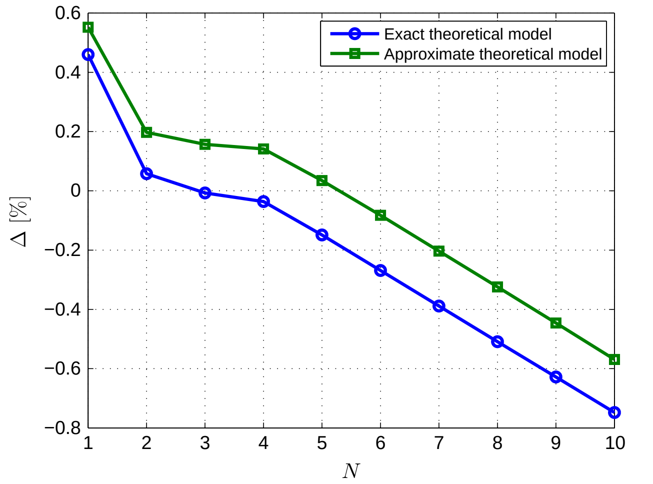


Fig. 7. Error Δ in terms of the exponent N, for the maximum deflection of a cantilever FGM beam with end load.

the following loads: (a) uniformly distributed force q acting in the x_1 direction; or (b) concentrated end force P acting in the x_1 direction. We denote by l the length of the beam (see Fig. 4).

The analytical solutions of these problems can easily be derived from the one dimensional governing differential equations of directed rods presented in Section 3. For the maximum deflection δ of the beam we obtain the well known relations

$$\delta = \frac{ql^2}{2}\left(\frac{1}{A_1}+\frac{l^2}{4C_2}\right) \quad \text{for uniformly distributed force } q,$$
$$\delta = Pl\left(\frac{1}{A_1}+\frac{l^2}{3C_2}\right) \quad \text{for concentrated end force } P, \tag{81}$$

where the values of the effective shear stiffness A_1 and bending stiffness C_2 for FGM porous beams are given by (79) (or the approximate form (80)) and (75), respectively. The theoretical predictions (81) will be compared with numerical solutions obtained by the finite element method.

The cross section of the beam has the dimensions $h = 50$ mm and $b = 50$ mm (see Fig. 5), the length is $l = 1$ m, and the closed cell aluminum foam is characterized by the material parameters $\rho_m = 500$ kg m^{-3}, $\rho_s = 2700$ kg m^{-3}, $E_s = 70$ GPa. We have calculated the maximum deflection of the beam numerically, using the software ABAQUS. To describe its functionally graded structure, the beam domain has been divided into layers orthogonal to the x_1 direction. Each layer is assumed to have constant material parameters E and ρ, which satisfy the power laws (73) and (74) stepwise.

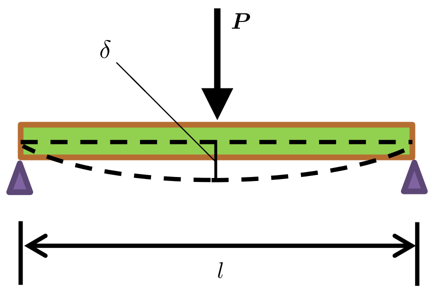


Fig. 8. Three-point bending of a FGM beam.

For the problems presented here a number of 64 or 128 layers is sufficient. The calculation has been performed using 3D shell elements and very dense mesh. The finite elements have been taken square, with one element per layer thickness.

We denote by δ_{FEM} the maximum deflection calculated by finite element analysis, let δ_{exact} be the theoretical value of the maximum deflection given by (81) with the exact formula (79) for A_1, and δ_{approx} be the theoretical value given by (81) with the approximate formula (80). We calculate the relative error Δ by the relation

$$\Delta = \frac{\delta_{\mathrm{FEM}} - \delta_{\mathrm{exact}}}{\min(\delta_{\mathrm{FEM}};\delta_{\mathrm{exact}})}.$$

Table 2
Comparison of results for cantilever FGM beam with concentrated end load.

N	1	2	3	4	5	6	7	8	9	10
δ_{FEM} (mm)	69.920	91.856	112.927	133.158	152.499	171.107	189.044	206.355	223.076	239.238
δ_{exact} (mm)	69.600	91.803	112.935	133.206	152.726	171.567	189.779	207.405	224.477	241.027
δ_{approx} (mm)	69.536	91.675	112.750	132.970	152.446	171.248	189.428	207.024	224.071	240.599
Δ (%)	0.460	0.058	−0.007	−0.036	−0.149	−0.269	−0.389	−0.509	−0.628	−0.748

Table 3
Comparison of results for FGM beam in three-point bending.

N	1	2	3	4	5	6	7	8	9	10
δ_{FEM} (mm)	4.476	5.972	7.356	8.660	9.900	11.088	12.230	13.329	14.389	15.412
δ_{exact} (mm)	4.393	5.804	7.147	8.435	9.674	10.870	12.024	13.141	14.223	15.270
δ_{approx} (mm)	4.377	5.773	7.101	8.376	9.604	10.790	11.937	13.046	14.121	15.163
Δ (%)	1.889	2.895	2.924	2.667	2.336	2.006	1.713	1.431	1.167	0.930

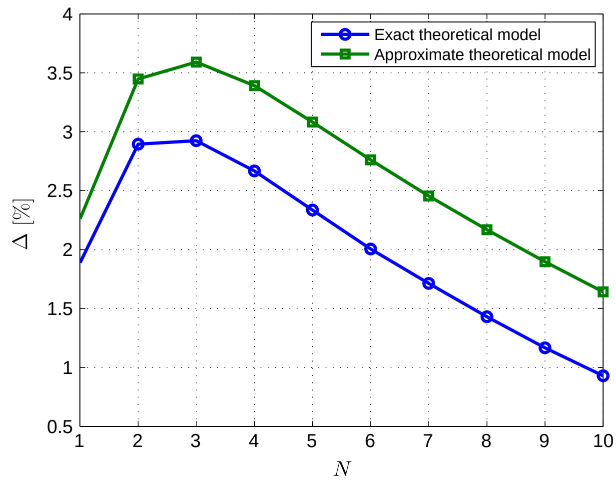


Fig. 9. Error Δ in terms of the exponent N, for the maximum deflection of a FGM beam in three-point bending.

(a) For the bending of cantilever beam by uniformly distributed force q 5 kN m 1, we have employed 64 layers. The com parison of the results is presented in Table 1, for the values of the exponent $N = 1, 2, \ldots, 10$. We can observe a very good agreement between the analytical and the numerical results, since the errors range between 1% to 1%. The percentage of relative error Δ is plotted in Fig. 6, in terms of the exponent N.

(b) For the bending of the beam by a concentrated end force $P = 5$ kN, we have employed 128 layers. The concentrated force has been divided into equal parts acting in the nodes along the whole edge of the beam. This procedure reduces the concentration of stress in the numerical solution. The comparison between the analytical and the finite element solutions is shown in Table 2. The errors Δ are very small: between 0.7% and 0.5%. Fig. 7 presents the percentage of relative error, for the exact and approximate solutions, with respect to the numerical one.

From Figs. 6 and 7 we notice that the exact theoretical model gi ven by (79) is slightly better than the approximate one (in the least square sense). Moreover, we see that the approximate theoretical model (80) yields results in good agreement with the numerical and exact solutions, and it has the advantage of simplicity.

8.3. Three point bending of functionally graded beam

Let us consider the functionally graded beam described previ ously in relations (73) (80) subjected to three point bending. A concentrated central force $P = 5$ kN acts at the mid span of the beam ($x_3 = l/2$) in the x_1 direction, and the end edges $x_3 = 0$, l are simply supported, see Fig. 8. The analytical solution of this bending problem can be derived from the equations given in Section 3. For the maximum deflection δ of the beam, we get

$$\delta_{\text{exact}} \quad \frac{Pl}{4}\left(\frac{1}{A_1} + \frac{l^2}{12C_2}\right), \qquad \delta_{\text{approx}} \quad \frac{Pl}{4}\left(\frac{1}{\widetilde{A}_1} + \frac{l^2}{12C_2}\right), \tag{82}$$

where the effective bending stiffness C_2 is given by (75), while the effective shear stiffness A_1 has the exact expression (79), and $\widetilde{A}_1$ is the approximate form (80).

To obtain the maximum deflection δ_{FEM} by a finite element analysis, we use 128 layers to divide the beam domain. Table 3 shows the comparison of the theoretical and numerical solutions, together with the relative error Δ. In Fig. 9 we plot the relative er ror with respect to the numerical solution, for $N = 1, \ldots, 10$. We ob serve that the errors range between 0.9% and 2.9%, depending on the value of N.

The shape of the beam in the deformed configuration is de picted in Fig. 10 for $N = 1, 5, 10$, in both numerical and theoretical approaches. The results are in very good agreement, so that the curves for the analytical and numerical solutions are very close in Fig. 10. Indeed, according to Table 3, the relative errors for the

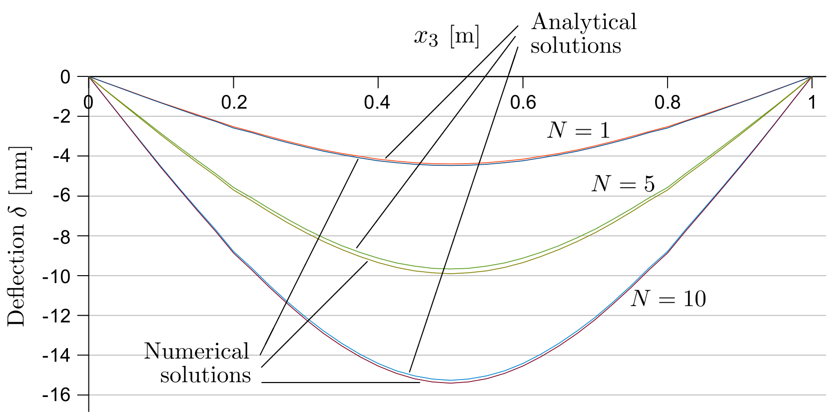
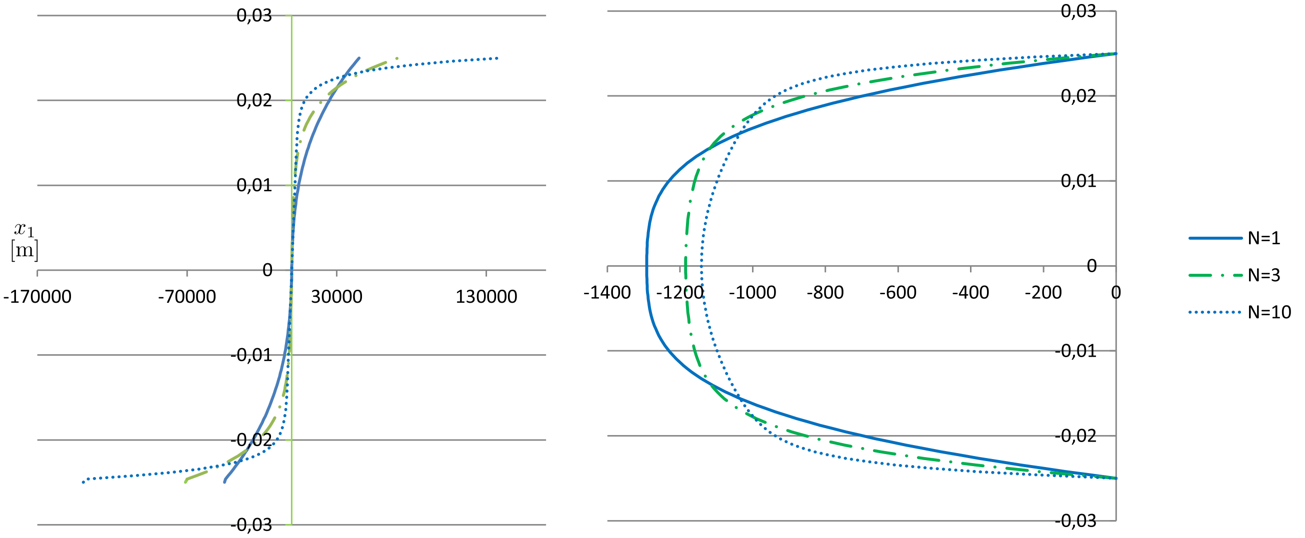


Fig. 10. Deflection of the FGM beam under three-point bending: numerical and analytical results, for the values of the exponent $N = 1, 5, 10$.



(a) Normal stress [kPa] (b) Shear stress [kPa]

Fig. 11. Distribution of normal and shear stresses in the cross-section of the FGM beam, for $x_3 = l/4$.

maximum deflections δ for N=1, 5, 10, are 1.8%, 2.3%, and respec tively 0.9%.

Let us present some results about the stress state in the FGM beam. For the cross section of the beam characterized by the axial coordinate $x_3 = l/4$, the distributions of the normal stress t^*_{33} and shear stress t^*_{31} versus the thickness coordinate x_1 are obtained by the finite element analysis and depicted in Fig. 11.

On the other hand, the analytical solution of this three point bending problem in the direct approach yields the following trans versal force Q_1 and bending moment L_1, calculated at the axial coordinate $x_3 = l/4$:

$$Q_1 \quad \frac{P}{2}, \quad L_1 \quad \frac{Pl}{8}. \tag{83}$$

According to $(14)_{7,9}$, the correspondence between Q_1, L_1 and the three dimensional stress state is given by

$$Q_1 \quad b\int_{\frac{h}{2}}^{\frac{h}{2}} t^*_{31}\,\mathrm{d}x_1, \quad L_1 \quad b\int_{\frac{h}{2}}^{\frac{h}{2}} x_1 t^*_{33}\,\mathrm{d}x_1. \tag{84}$$

Then, we can compare the theoretical predictions (83) with the numerical solution in the form of the resultants (84). As expected, the agreement between the two approaches is very good: for the bending moment L_1 the relative error is in the range 0.005 0.007%; for the transversal force Q_1 the relative error is about 0.00003% (for every exponent $N = 1, \ldots, 10$).

9. Conclusions

In this paper we have employed the theory of directed curves to investigate the mechanical behavior of non homogeneous, com posite, and functionally graded beams. The structure of the consti tutive tensors and the form of the linear constitutive equations have been established in Sections 2, 3, and are presented in the relations (6) and (13). We determine the effective stiffness coeffi cients via comparison with three dimensional elasticity static and free vibration solutions in Sections 4 6. Thus, for non homo geneous isotropic beams we find the formulas (29) and (38), while for composite beams made of two different materials we have the effective stiffness properties (45). For orthotropic non homoge neous beams, the effective shear stiffness is expressed by (55), (56), and the effective bending stiffness, extensional stiffness, tor sional rigidity and coupling coefficients are given by (50).

In Section 7 we apply these general formulas to determine the effective stiffness properties of some special functionally graded beams, such as orthotropic beams with exponential distribution law, or composite circular beams with power law distribution of material properties.

In Section 8 we consider rectangular functionally graded beams made of metal foams. Using the Gibson Ashby formula (74) for the Young modulus of closed cell aluminum foams, combined with the power law distribution of mass density (73), we find the effective stiffness coefficients in the form (75) and (79). In view of these re sults, we deduce the analytical beam like solutions for the bending of a FGM cantilever beam subjected to uniform and end loadings in Section 8.2, and for a FGM beam in three point bending in Section 8.3. The theoretical predictions are in good agreement with numer ical results obtained by a finite element analysis.

This comparison with finite element solutions represents a val idation of our analytical modeling concerning the effective stiffness properties of FGM beams. Nevertheless, our approach is much more general and it can be used to analyze the mechanical proper ties of various functionally graded rods, with different geometrical and material characteristics.

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