The vertex-colouring a,b-edge-weighting problem is NP-complete for every pair of weights

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Abstract

Let G be a graph. From an edge-weighting $w : E(G) \to \{a, b\}$ of G such that a and b are two distinct real numbers, one obtains a vertex-colouring χ_w of G defined as $\chi_w(u) = \sum_{v \in N(u)} w(uv)$ for every $u \in V(G)$. If χ_w is a proper colouring of G, i.e. two adjacent vertices of G receive distinct colours by χ_w, then we say that w is vertex-colouring. We investigate the complexity of the problem of deciding whether a graph admits a vertex-colouring edge-weighting taking values among a given pair $\{a, b\}$, which is already known to be NP-complete when $\{a, b\}$ is either $\{0, 1\}$ or $\{1, 2\}$. We show this problem to be NP-complete for every pair of real weights.

Keywords: vertex-colouring edge-weighting of graphs, two weights, complexity, 1-2-3 Conjecture

1 Introduction

Let G be a graph. An *edge-weighting* $w : E(G) \to W$ of G is obtained by associating a *weight* from W with each edge of G, where $W \subset \mathbb{R}$. To make the values assigned by w to the edges of G visible, we also say that w is a W*-edge-weighting* of G. In the case where $W = \{1, 2, ..., k\}$ for some integer k, we call w a k*-edge-weighting* of G.

Assume w is an edge-weighting of G. From w, one deduces a vertex-colouring χ_w of G where $\chi_w(u) = \sum_{v \in N(u)} w(uv)$ for every vertex u of G. In other words, the colour of u by χ_w is the sum of its incident colours by w. We call $\chi_w(u)$ the *weighted degree* of u (by w). If χ_w is proper, i.e. for every pair $\{u, v\}$ of adjacent vertices in G we have $\chi_w(u) \neq \chi_w(v)$, then we say that w is *vertex-colouring*.

In 2004, Karoński, Łuczak and Thomason raised the following famous conjecture on vertex-colouring edge-weightings of graphs [6].

1-2-3 Conjecture. *Every graph with no isolated edge admits a vertex-colouring 3-edge-weighting.*

As a first result towards the 1-2-3 Conjecture, Karoński, Łuczak and Thomason proved that there is a set W of 183 real weights such that any graph with no isolated edge admits a vertex-colouring W-edge-weighting [6]. Since then, many refinements of this result have been introduced. The result which is the closest from the 1-2-3 Conjecture so far is due to Kalkowski, Karoński and Pfender, who proved that every graph with no isolated edge admits a vertex-colouring 5-edge-weighting [5]. The interested reader may refer to [7] for an up-to-date survey on vertex-colouring edge-weightings of graphs.

Consider now the following problem arising from the definitions above.

Vertex-Colouring $\{a, b\}$-Edge-Weighting - $\{a, b\}$-VCEW
Instance: A graph G.
Question: Does G admit a vertex-colouring $\{a, b\}$-edge-weighting?

The problem $\{a, b\}$-VCEW is clearly in NP no matter what are a and b. It was proved by Dudek and Wajc that $0, 1$-VCEW and $1 - 2$-VCEW are NP-complete [3]. They additionally suggested that their hardness reduction could be generalized to prove that $\{a, b\}$-VCEW is NP-complete whenever a and b are two distinct rational numbers. However, they did not give any formal proof of this statement. The problem $\{1, 2\}$-VCEW is also known to be NP-complete when restricted to cubic graphs [1].

In this paper, we prove that $\{a, b\}$-VCEW is NP-complete for every pair $\{a, b\}$ of real weights. We proceed as follows. We first introduce, in Section 3, a hardness reduction framework for showing that some edge-colouring problems are NP-complete by reduction from the well-known SAT problem. A first implementation of this framework with specific gadgets leads, in Section 4, to a proof that $\{a, b\}$-VCEW is NP-complete whenever $0 \notin \{a, b\}$ and $b \neq -a$. We then give two other implementations of our reduction framework in Sections 5 and 6 for showing that $\{a, 0\}$-VCEW and $\{a, -a\}$-VCEW are NP-complete, respectively.

2 Terminology and preliminary results

Before introducing the reduction framework, we first introduce some terminology and results that are used in further sections.

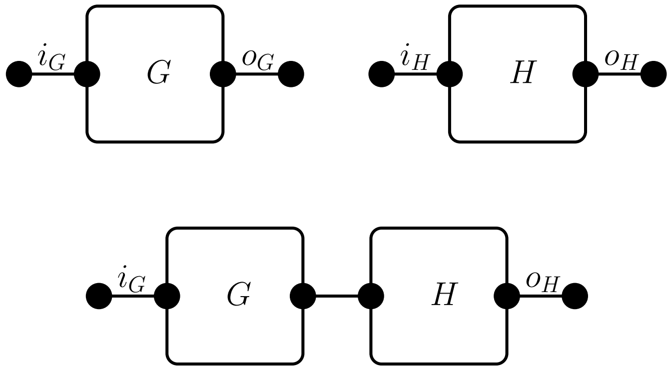


Figure 1: Two graphs G and H with input i_G and i_H, respectively, and output o_G and o_H, respectively, and their connection along o_G and i_H.

Let G be a graph. Given a vertex u of G, we denote by $d(u)$ the *degree* of u, i.e. the number of its neighbours in G. An *input* (resp. *output*) *of* G is an edge $i = uv$ (resp. $o = vu$) such that $d(u) = 1$. Assuming G has x (resp. y) inputs (resp. outputs) ordered arbitrarily, we sometimes refer to these inputs (resp. outputs) as $i_1(G), ..., i_x(G)$ (resp. $o_1(G), ..., o_y(G)$). Consider now two graphs G and H such that o and i are an output of G and an input of H, respectively. The *connection of* G *and* H *along* o *and* i is the graph obtained by taking the disjoint union of G and H, and then identifying the edges o and i. Assuming we are given a one-to-one correspondence ϕ from a set $\{o_1, ..., o_x\}$ of x outputs of G to a set $\{i_1, ..., i_x\}$ of x inputs of H, one can similarly define the connection of G and H along x inputs of G and x outputs of H where the resulting graph is obtained by identifying o_j and $\phi(o_j)$ for every $j \in \{1, ..., x\}$. In this situation we say that G and H are connected along $(o_1, ..., o_x)$ and $(\phi(o_1), ..., \phi(o_x))$. The inputs and outputs of any graph resulting from the connection of G and H are those of G and H which have not been used for the connection. This construction is depicted in Figure 1.

Denote by G' the graph obtained by connecting G and H along $(o_1, ..., o_x)$ and $(i_1, ..., i_x)$. Given a vertex-colouring W-edge-weighting w of G, an *extension* of w from G to G' is a vertex-colouring W-edge-weighting w' of G' such that we have $w'(e) = w(e)$ for every edge e that originally belonged to G. Note in particular that if e is an edge resulting from the identification of o_j and i_j for some $j \in \{1, ..., x\}$, then we have $w'(e) = w(o_j)$.

Assume uv is an edge of G. Now consider a graph H such that H has two inputs $u''z$ and zv''. According to the definitions above, we have $d(u'') = d(v'') = 1$. Besides, we have $d(z) \geq 2$. By *H-subdividing* uv, we mean that we "replace" the edge uv with H. More precisely, we first remove the edge uv from G, then attach a new vertex u' to u and one new vertex v' to v (so that $d(u') = d(v') = 1$), and finally connect G and H along (uu', vv')

Figure 2: Two possible vertex-colouring $\{a, b\}$-edge-weightings of P_4.

and $(u''z, v''z)$.

We now give several properties of vertex-colouring $\{a, b\}$-edge weightings.

Observation 2.1. *Let P_4 denote the path $u_1u_2u_3u_4$ of length 3. If w is a vertex-colouring $\{a, b\}$-edge-weighting of P_4, then $w(u_1u_2) \neq w(u_3u_4)$.*

Proof. Suppose $w(u_1u_2) = a$ without loss of generality. Then u_2u_3 can be either coloured a or b. In the first case, we have $w(u_3u_4) = b$ since otherwise we would have $\chi_w(u_2) = \chi_w(u_3) = 2a$. In the second case, we have $w(u_3u_4) = b$ since otherwise we would have $\chi_w(u_2) = \chi_w(u_3) = a + b$ $\square$

Suppose we have $w(o) = a$ for an output o of some graph G. Then, by connecting G and $u_1u_2u_3u_4$ along o and u_1u_2 (which is similar to P_4-subdividing o), we get that any extension of w from G to the resulting graph is such that $w(u_3u_4) = b$ by Observation 2.1. Therefore, P_4-subdividing an output is an operation that can be used to "invert" the colour at some output of a graph by a vertex-colouring $\{a, b\}$-edge-weighting.

In the next result, we denote by $m_w(u)$ the multiset of colours incident with a vertex u by w. In other words, if x (resp. y) of the edges incident with u are coloured a (resp. b) by w, i.e. we have $\chi_w(u) = xa + yb$, then $m_w(u)$ contains the value a (resp. b) exactly x (resp. y) times.

Lemma 2.2. *Let u and v be two adjacent vertices of some graph G such that $d(u) = d(v)$, and w be an $\{a, b\}$-edge-weighting of G. If $\chi_w(u) = \chi_w(v)$, then $m_w(u) = m_w(v)$.*

Proof. We clearly have $\chi_w(u) = \chi_w(v)$ when $m_w(u) = m_w(v)$. Now suppose that $m_w(u) \neq m_w(v)$. Then, u and v are incident to x and x' edges coloured a by w, respectively, and y and y' edges coloured b, respectively. Besides, we have $x \neq x'$ and $y \neq y'$ since $m_w(u) \neq m_w(v)$. Because $d(u) = d(v)$, we have $x + y = x' + y'$. Since $\chi_w(u) = \chi_w(v)$, we have $xa + yb = x'a + y'b$, and, because $x - x' \neq 0$, we get that $a = b$ which is impossible by definition of an $\{a, b\}$-edge-weighting. $\square$

By Lemma 2.2, it follows that if u and v are two adjacent vertices of G such that $d(u) = d(v)$, then we only have to check whether $m_w(u) \neq m_w(v)$ while checking whether an edge-weighting w of G is vertex-colouring.

We now introduce the notion of *replacement gadget*. A replacement gadget R for a pair $\{a, b\}$ of real weights is a graph with the following structural and colouring properties:

1. the graph R has two inputs $i_1(R) = uz$ and $i_2(R) = zv$, and no output,
2. we have $w(i_1(R)) = w(i_2(R))$ for every vertex-colouring $\{a, b\}$-edge-weighting w of R,
3. there exist vertex-colouring $\{a, b\}$-edge-weightings of R such that $i_1(R)$ is coloured a,
4. there exist vertex-colouring $\{a, b\}$-edge-weightings of R such that $i_1(R)$ is coloured b,
5. the weighted degree of z is x by every $\{a, b\}$-edge-weighting w of R such that $w(i_1(R)) = a$, where x is some real number,
6. the weighted degree of z is y by every $\{a, b\}$-edge-weighting w of R such that $w(i_1(R)) = b$, where y is some real number.

We refer to the vertex z as $r(R)$ for convenience. To make the weighted degree of $r(R)$ by all vertex-colouring $\{a, b\}$-edge-weightings of R apparent, we call R a (x, y)*-replacement gadget*.

Replacement gadgets may be used to reduce the number of conflicts by a non-vertex-colouring $\{a, b\}$-edge-weighting w of some graph G. Indeed, suppose that for an edge uv of G we have $w(uv) = a$ and $\chi_w(u) = \chi_w(v)$. To solve this local conflict, one way to proceed is to "replace" uv by a (x_1, y_1)-replacement gadget R for $\{a, b\}$, i.e. to R-subdivide uv, and then extend w to the resulting graph, i.e. to the edges of R, in such a way that $w(i_1(R)) = w(i_2(R)) = a$. Such an extension exists by the definition of a replacement gadget.

Clearly, the weighted degree of both u and v is not altered by the extension of w. Therefore, the only new possible conflicts which may arise are $\chi_w(u) = x_1$ or $\chi_w(v) = x_1$. If such a situation occurs, then we reveal what is the weighted degree of one of u and v by w. By then repeating the procedure above but with a (x_2, y_2)-replacement gadget for $\{a, b\}$ such that $x_1 \neq x_2$, the possibilities for getting another conflict when extending w to the resulting graph are reduced. We catch this observation within the next lemma.

Lemma 2.3. *Assume w is an $\{a, b\}$-edge-weighting of some graph G, and suppose we are given one (x_1, y_1)-, one (x_2, y_2)- and one (x_3, y_3)-replacement gadget R_1, R_2 and R_3 for $\{a, b\}$, respectively, where the x_i's are distinct and the y_i's are distinct. If $\chi_w(u) = \chi_w(v)$ for some edge uv of G, then there* *is a value of $i \in \{1,2,3\}$ for which there is an extension of w to the graph resulting from the R_i-subdivision of uv such that $\chi_w(u) \neq \chi_w(r(R_i))$ and $\chi_w(v) \neq \chi_w(r(R_i))$.*

Proof. Suppose $w(uv) = a$. Start by R_1-subdividing uv, and then extend w to the resulting graph in such a way that $w(i_1(R_1)) = w(i_2(R_1)) = a$. Such an extension exists by definition. If the claim is not verified, then $\chi_w(u) = \chi_w(r(R_1)) = x_1$ without loss of generality. Start over from the original graph and colouring. Now, R_2-subdivide uv before extending w to the resulting graph in such a way that $w(i_1(R_2)) = w(i_2(R_2)) = a$. Clearly we cannot have $\chi_w(u) = \chi_w(r(R_2))$ since $x_1 \neq x_2$. Thus, if the claim is still not verified, then $\chi_w(v) = x_2$. In this situation, repeat the same procedure a third time but with a R_3-subdivision of uv. Now the claim has to be true since other we would have either $\chi_w(u) = \chi_w(r(R_3))$ or $\chi_w(v) = \chi_w(r(R_3))$, which is impossible since $\chi_w(r(R_3)) = x_3$, $\chi_w(u) = x_1$, $\chi_w(v) = x_2$, and x_1, x_2 and x_3 are distinct. □

Hence, assuming we are provided three sufficiently different replacement gadgets for $\{a,b\}$, by repeating the procedure used in the proof of Lemma 2.3 for every conflicting edge of G, i.e. every edge uv such that $\chi_w(u) = \chi_w(v)$, we can deduce a graph which looks like G and which admits a vertex-colouring $\{a,b\}$-edge-weighting which looks like w.

Corollary 2.4. *Assume w is an $\{a,b\}$-edge-weighting of some graph G on n vertices, and suppose we are given one (x_1,y_1)-, one (x_2,y_2)-, and one (x_3,y_3)-replacement gadget R_1, R_2 and R_3 for $\{a,b\}$, respectively, where the x_i's are distinct and the y_i's are distinct. Then there is a combination of subdivisions of the edges of G involving the R_i's such that the resulting graph admits an extension of w which is vertex-colouring.*

Proof. Suppose there are $x < n^2$ edges $e_1, ..., e_x$ of G whose incident vertices have the same weighted degree by w. Consider each such edge $e_i = uv$. Suppose $w(e_i) = a$ without loss of generality. By Lemma 2.3, there is one replacement gadget R of R_1, R_2 and R_3 for which we can R-subdivide e_i and then extend w to the resulting graph in such a way that $w(i_1(R)) = w(i_2(R)) = a$, $\chi_w(u) \neq \chi_w(r(R))$ and $\chi_w(v) \neq \chi_w(r(R))$. Since this operation does not alter the weighted degree of both u and v, no new pair of vertices with the same weighted degree appeared after the modification. Hence, after having repeated the same procedure for each of $e_1, ..., e_x$, there are no two neighbouring vertices with the same weighted degree. □

3 The hardness reduction framework

3.1 Overview of the framework

All the hardness reductions performed in this paper are from the following classical NP-complete problem.

3SAT
Instance: A 3CNF formula F over variables $\{x_1, ..., x_n\}$ and clauses $\{C_1, ..., C_m\}$.
Question: Is F satisfiable?

Note that if F has a clause C of the form $(x_i \vee x_i \vee x_i)$ (resp. $(\overline{x_i} \vee \overline{x_i} \vee \overline{x_i})$), then x_i is set to true (resp. false) by any satisfying truth assignment of F. In this situation, we thus say that x_i is *forced* to true (resp. false) by C. Besides, we can suppose that every possible literal appears in F. Indeed, if x_i does not appear in any clause of F, then the 3CNF formula $F \wedge (x_i \vee x_i \vee \overline{x_i})$ is satisfiable if and only if F is satisfiable too. A formula equivalent to F but involving every possible literal over its variables can then be obtained in polynomial time.

The reduction framework used here has been frequently used in the literature to prove the hardness of some graph edge-colouring problems (see, e.g. [2, 4]). From F, we produce a graph G_F such that, given a pair $\{a, b\}$ of real weights, F is satisfiable if and only if G_F admits a vertex-colouring $\{a, b\}$-edge-weighting w_F.

We use a simple analogy to describe our reduction scheme. The reduced graph G_F has to be thought of as an *electrical circuit* made up of *gadgets*, i.e. recurrent subgraphs, connected in a specific way. These gadgets are interconnected along several *inputs* and *outputs* that permit two *signals*, the *positive* and the *negative* ones, to be propagated along G_F. This propagation fulfils properties which are inspired by the propagation of a vertex-colouring $\{a, b\}$-edge-weighting in a graph, where the positive and negative signals may be assimilated with the colours a and b, respectively. The structure of G_F is representative of the structure of F in the sense that the propagation of the positive signal through the gadgets is representative of the consequences on F of setting such or such variable of F to true. In this way, we get a straight analogy between spreading the positive signal through G_F and satisfying F.

The gadgets of G_F are the following. The graph G_F is composed of one *generator gadget* $G_F(S)$, m *clause gadgets* $G_F(C_1), ..., G_F(C_m)$, and $2n$ *literal gadgets* $G_F(\ell_1), ..., G_F(\ell_{2n})$. Each clause C_i in F is associated with the clause gadget $G_F(C_i)$, and similarly for each literal ℓ_i of F and the literal gadget $G_F(\ell_i)$. The graph G_F is obtained by originally considering $G_F(S)$, and then successively connecting gadgets to it. The generator gadget is first connected to all of the m clause gadgets, which are each connected to some of the literal gadgets. In particular, if we denote by $c_i \in \{1, 2, 3\}$

the number of distinct literals in C_i, and by $\ell_{i_1}, ..., \ell_{i_{c_i}}$ these literals, then $G_F(C_i)$ is connected to $G_F(\ell_{i_1}), ..., G_F(\ell_{i_{c_i}})$. Each literal gadget $G_F(\ell_i)$ of G_F thus has n_i inputs, where $n_i \geq 1$ is the number of distinct clauses in F that contain the literal ℓ_i. Finally, the outputs of two literal gadgets $G_F(\ell_i)$ and $G_F(\overline{\ell_i})$ are connected in a specific way.

Assuming each clause gadget $G_F(C_i)$ is supplied with the positive signal by the generator gadget, the main property of $G_F(C_i)$ is that it propagates the positive signal through at least one of its outputs, i.e. to at least one literal gadget $G_F(\ell_j)$ such that $\ell_j \in C_i$. The main property of a literal gadget $G_F(\ell_i)$ is that it outputs a signal if and only if the same signal comes in from all of its n_i inputs. Moreover, if a given signal comes in from the n_i inputs of $G_F(\ell_i)$, then $G_F(\ell_i)$ outputs the same signal, which must be different from the one outputted by $G_F(\overline{\ell_i})$.

Hence, we have an equivalence between satisfying F and propagating the positive signal through G_F from the generator gadget:

- each clause C_i in F must have at least one true literal and $G_F(C_i)$ must spread the positive signal to at least one literal gadget it is connected to,
- every literal ℓ_i must have the same truth value in all clauses it appears in and all the inputs of $G_F(\ell_i)$ must spread the same signal in,
- a variable x_i and its negation $\overline{x_i}$ must have distinct truth values and the outputs of $G_F(x_i)$ and $G_F(\overline{x_i})$ must spread different signals out.

Now consider the graph theory point of view. The graph G_F, which is associated with the electrical circuit, is obtained by successively connecting graphs to the generator gadget $G_F(S)$. These graphs, i.e. the clause and literal gadgets, must fulfil the structural and colouring properties summed up above. In particular, assuming the colour a (resp. b) of the vertex-colouring $\{a, b\}$-edge-weighting w_F of G_F is associated with the positive (resp. negative) signal, the propagation of the positive and negative signals may be seen as successive extensions of w_F to the graphs successively obtained after the connections. We then get an analogy between satisfying F and finding a vertex-colouring $\{a, b\}$-edge-weighting of G_F.

3.2 The reduction framework into details

In this section, we go into the details of the reduction framework by pointing out the properties that must fulfil the gadgets mentioned in Section 3.1, and how these are connected exactly to form G_F. Any implementation of the framework reduction in further sections will thus only consists in exhibiting gadgets and showing that these have the properties exhibited throughout this section.

3.2.1 Spreading gadget $G^{\curlywedge}$ and generator gadget $G_F(S)$

A generator gadget $G_F(S)$ for a given pair $\{a, b\}$ is obtained by connecting several *spreading gadgets* $G^{\curlywedge}$. A spreading gadget $G^{\curlywedge}$ for $\{a, b\}$ has one input, two outputs, and the following colouring property.

Property 1. *Assume w is a vertex-colouring $\{a, b\}$-edge-weighting of $G^{\curlywedge}$. Then we have $w(i_1(G^{\curlywedge})) = w(o_1(G^{\curlywedge})) = w(o_2(G^{\curlywedge}))$.*

Remark that by connecting two copies G_1 and G_2 of $G^{\curlywedge}$ along $o_1(G_1)$ and $i_1(G_2)$, we obtain a graph G' whose input and three outputs all receive the same colour by any vertex-colouring $\{a, b\}$-edge-weighting. Indeed, suppose w is a vertex-colouring $\{a, b\}$-edge-colouring of G' initiated with G_1, and that we have $w(i_1(G_1)) = a$ without loss of generality. Then, we have $w(o_1(G_1)) = w(o_2(G_1)) = a$ by Property 1. Note then that in any extension of w from G_1 to G', we have $w(i_1(G')) = w(o_1(G')) = w(o_2(G')) = w(o_3(G'))$ since G_2 satisfies Property 1, and $o_1(G_1)$ and $i_1(G_2)$ refer to the same edge of G'.

Note that by repeating this construction several times, one obtains a graph with one input and arbitrarily many outputs such that all of these input and outputs necessarily receive the same colour by a vertex-colouring $\{a, b\}$-edge-weighting. Now denote by P_4 the path $u_1u_2u_3u_4$ with length 3, let $i_1(P_4) = u_1u_2$ and $o_1(P_4) = u_3u_4$, and suppose w is a vertex-colouring $\{a, b\}$-edge-weighting of $G^{\curlywedge}$. Note thus that if G' is the graph obtained by connecting $G^{\curlywedge}$ and P_4 along $o_1(G^{\curlywedge})$ and $i_1(P_4)$ (in other words, the graph G' results from a P_4-subdivision of $o_1(G^{\curlywedge})$), then, assuming $w(o_1(G^{\curlywedge})) = a$, we have $w(o_1(G')) = b$ in any extension of w from $G^{\curlywedge}$ to G' by Observation 2.1, where $o_1(G') = u_3u_4$. Regarding the electric circuit analogy, this means that we are able to *invert* some signal. Consequently, by connecting arbitrarily many copies of $G^{\curlywedge}$ and then inverting some outputs, we are able to propagate both the colours a and b towards an arbitrary number of directions assuming the input colour is known.

The generator gadget $G_F(S)$ is obtained in this way, i.e. by connecting several copies of $G^{\curlywedge}$ and then inverting some outputs. The number of necessary connections is not clarified in this paper, but one can easily check that this number in polynomial regarding the size of F.

From now on, we suppose that w_F is a vertex-colouring $\{a, b\}$-edge-weighting of G_F initiated with $G_F(S)$, and extended progressively as $G_F(S)$ is connected to other gadgets to form G_F. Suppose $w(i_1(G_F(S))) = a$ without loss of generality. Then, according to the remarks above, we know which outputs of $G_F(S)$ receive colour a (resp. b) by w_F. We say that these outputs are *positive* (resp. *negative*).

3.2.2 Clause gadget $G_F(C_i)$

Recall that $c_i \in \{1, 2, 3\}$ denotes the number of distinct literals in C_i for every $i \in \{1, ..., m\}$. The structure of the clause gadget $G_F(C_i)$ for $\{a, b\}$ depends on the value of c_i. If $c_i = 1$, then a clause gadget is not necessary. In any other situation, i.e. $c_i = 2$ or $c_i = 3$, the gadget $G_F(C_i)$ has a constant number of inputs and c_i outputs, and the following colouring property.

Property 2. *Assume w is a vertex-colouring $\{a, b\}$-edge-weighting of $G_F(C_i)$ such that a fixed number of inputs of $G_F(C_i)$ are coloured a by w, while its other inputs are coloured b. Then, up to c_i outputs, but at least one, are coloured a by w.*

Now connect $G_F(S)$ and every $G_F(C_i)$ in such a way that each input of $G_F(C_i)$ which is supposed to be coloured a (resp. b) is identified with one distinct positive (resp. negative) output of $G_F(S)$. Then, by Property 2, we know that arbitrarily many, but at least one, outputs of $G_F(C_i)$ can be assigned colour a in any extension of w_F from $G_F(S)$ to G'_F, where G'_F is the graph resulting from the connections.

3.2.3 Collecting gadget $G^{\curlyvee}$ and literal gadget $G_F(\ell_i)$

The structure of a literal gadget $G_F(\ell_i)$ for $\{a, b\}$ depends on the value of n_i, where n_i is the number of distinct clauses of F that contain the literal ℓ_i for every $i \in \{1, ..., 2n\}$. Once again, if $n_i = 1$, then there is no need for a literal gadget. In any other case, i.e. whenever $n_i \geq 2$, a literal gadget is obtained by connecting exactly $n_i - 1$ *collecting gadgets* $G^{\curlyvee}$. A collecting gadget for $\{a, b\}$ has two "regular" inputs $i_1(G^{\curlyvee})$ and $i_2(G^{\curlyvee})$, and one output. It also has some "forcing" inputs which are supposed to be connected with positive or negative outputs of $G_F(S)$ so that the following property is fulfilled.

Property 3. *Assume w is a vertex-colouring $\{a, b\}$-edge-weighting of $G^{\curlyvee}$ such that a fixed number of forcing inputs of $G^{\curlyvee}$ are coloured a by w, while its other forcing inputs are coloured b. Then we have $w(i_1(G^{\curlyvee})) = w(i_2(G^{\curlyvee})) = w(o_1(G^{\curlyvee}))$.*

Now consider every literal ℓ_i. For every distinct clause C_j that contains ℓ_i, associate a distinct output of G'_F with $G_F(\ell_i)$ as follows:

- if ℓ_i is forced to true by C_j, then consider a positive output of $G_F(S)$,
- if ℓ_i is forced to false by C_j, then consider a negative output of $G_F(S)$,
- otherwise, consider one output of $G_F(C_j)$.

This association has to be done in such a way that any chosen output of G'_F is associated with exactly one literal gadget. The two first items above depicts the fact that if a clause contains only one distinct literal, then this literal is forced to some truth value by this clause. The third item reflects the fact that the truth value of a clause by a truth assignment of F depends on the truth values of its literals.

The literal gadget $G_F(\ell_i)$ is obtained as follows. First define an arbitrary ordering $(o_1, ..., o_{n_i})$ over the outputs of G'_F chosen above for $G_F(\ell_i)$. Now consider $n_i - 1$ copies $G_1, ..., G_{n_i-1}$ of $G^{\curlyvee}$, and connect these with G'_F as follows. Start by connecting G'_F and G_1 along (o_1, o_2) and $(i_1(G_1), i_2(G_1))$. Then connect the resulting graph and G_2 along $(o_1(G_1), o_3)$ and $(i_1(G_2), i_2(G_2))$. Next, connect the obtained graph and G_3 along $(o_1(G_2), o_4)$ and $(i_1(G_3), i_2(G_3))$. And so on. Denote by G''_F the resulting graph.

Note that if any two of the n_i outputs chosen above for $G_F(\ell_i)$ do not share the same colour by w_F, then there is no extension of w_F from G'_F to G''_F by Property 3. Thus, all the outputs of G'_F connected to $G_F(\ell_i)$ must receive the same colour by w_F.

3.2.4 Connecting the literal gadgets

The reduced graph G_F is finally obtained by adding some edges to G''_F. Consider every pair $\{\ell_i, \ell_j\}$ of F such that $\ell_j = \overline{\ell_i}$. Now consider the respective output o_i and o_j of G''_F of the literal gadgets $G_F(\ell_i)$ and $G_F(\ell_j)$. More precisely, if $n_i = 1$ and ℓ_i is forced to true (resp. false) by the only clause of F that contains ℓ_i, then o_i is a positive (resp. negative) output of $G_F(S)$. If $n_i = 1$ and ℓ_i is not forced to some truth value by the only clause C_j that contains it, then consider one distinct output of C_j. Otherwise, i.e. $n_i \geq 2$, the output o_i is $o_1(G_F(\ell_i))$. Now, if u and v are the vertices with degree 1 of o_i and o_j, respectively, then let uv be an edge in G_F.

Now suppose any two such outputs o_i and o_j receive the same colour in an extension of w_F from G'_F to G''_F. Then note that w_F cannot be extended from G''_F to G_F since the edges o_i, uv and o_j induce a path on 4 vertices whose end edges have the same colour (Observation 2.1). On the contrary, if we have $w_F(o_i) \neq w_F(o_j)$, then assigning any colour to uv is correct. This simulates the fact that in a truth assignment of the variables of F, a variable and its negation must be assigned distinct truth values.

3.3 Final details

Checking whether any implementation of our reduction framework is correct is quite tedious since, for every two neighbouring vertices u and v of G_F, one has to check whether $\chi_{w_F}(u) \neq \chi_{w_F}(v)$. In the three implementations described in this paper, we only show this to hold for neighbouring vertices of the spreading, clause, and collecting gadgets we exhibit. But this is not sufficient since some conflicts may arise for particular values of $\{a, b\}$ when connecting two of these gadgets.

As an illustration, suppose e.g. that uz is an output of some graph G, and $z'v$ is an input of some graph H. Suppose we are also given vertex-colouring $\{a, b\}$-edge-weightings w_G and w_H of G and H, respectively, such that $w_G(uz) = w_H(z'v) = a$. Now let G' be the graph resulting from the connection of G and H along uz and $z'v$, and consider the vertex-colouring $\{a, b\}$-edge-weighting $w_{G'}$ of G' where any edge e that originally belonged to G is coloured following w_G, and following w_H otherwise. Clearly the only possible conflict is $\chi_{w_{G'}}(u) = \chi_{w_{G'}}(v)$, which may occur for some particular values of a and b.

Thanks to Lemma 2.3, we actually do not need to deeply study every possible connection between two gadgets of a given implementation, i.e. for a given value of $\{a, b\}$, in order to find out the conflicts which may arise when extending an $\{a, b\}$-edge-weighting. Assuming we are given one (x_1, y_1)-, one (x_2, y_2)-, and one (x_3, y_3)-replacement gadget for $\{a, b\}$ such that the x_i's are distinct and the y_i's are distinct, we could "replace" a conflicting edge by one of these gadgets so that we solve the conflict locally, and this without altering the colouring properties of G_F, i.e. without providing new ways for colouring G_F. In this way, the equivalence between satisfying F and colouring G_F would be preserved.

Hence, even if we do not know exactly which edges are conflicting in a vertex-colouring $\{a, b\}$-edge-weighting of G_F for a specific value of $\{a, b\}$, we know that an implementation of our reduction framework can be adapted for $\{a, b\}$ by simply replacing some edges of G_F by a convenient replacement gadget among a triplet of three replacement gadgets for $\{a, b\}$. Each of our framework implementations, i.e. for a given of $\{a, b\}$, is thus provided with a *replacement triplet*, i.e. a triplet (R_1, R_2, R_3) where each R_i is a (x_i, y_i)-replacement gadget for $\{a, b\}$, and such that the x_i's are distinct and the y_i's are distinct.

4 First implementation: $0 \notin \{a, b\}$ and $b \neq -a$

In this section, we give a first implementation of our reduction framework for showing that $\{a, b\}$-VCEW is NP-complete whenever $0 \notin \{a, b\}$ and $b \neq -a$. For this purpose, we introduce several graphs and show that they are spreading, clause, collecting and replacement gadgets, respectively, for $\{a, b\}$. Throughout this section, the thick (resp. thin) edges of our figures represent edges coloured a (resp. b) by a vertex-colouring $\{a, b\}$-edge-weighting.

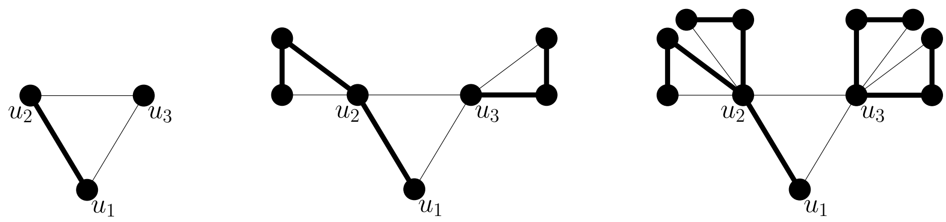


Figure 3: The graphs T_0, T_1 and T_2, and vertex-colouring $\{a, b\}$-edge-weightings of T_0, T_1 and T_2.

4.1 Auxiliary gadget T_k and replacement triplet for $\{a, b\}$

We define the graphs T_k, where $k \geq 0$, which are used in further gadgets to "force" the propagation of a vertex-colouring $\{a, b\}$-edge-weighting. For a given value of $k \geq 0$, the graph T_k is obtained as follows. First consider a triangle $u_1u_2u_3$. If $k = 0$, then we are done. Now, if $k \geq 1$, then identify u_2 with one arbitrary vertex of each of k new triangles. Finally, repeat the last step but with u_3 instead of u_2. We refer to the vertex u_1 as the *root* of T_k. This construction is depicted in Figure 3. In the figures of the next sections, any triangle marked "T_k" indicates that a vertex is identified with the root of a graph T_k.

Any graph T_k with $k \geq 0$ has the following colouring properties.

Lemma 4.1. *Assume w is an $\{a, b\}$-edge-weighting of T_k for some $k \geq 0$. If $\chi_w(u_2) \neq \chi_w(u_3)$, then one of u_2 and u_3 has weighted degree $(k+1)(a+b)$, while the other vertex has weighted degree $(k+2)a+kb$ or $ka+(k+2)b$. Besides, we have $\{w(u_1u_2), w(u_1u_3)\} = \{a, b\}$.*

Proof. Note that for any triangle $v_1v_2u_iv_1$ different from $u_1u_2u_3u_1$ where $i \in \{2, 3\}$, we have $w(u_iv_1) \neq w(u_iv_2)$ since otherwise we would have $\chi_w(v_1) = \chi_w(v_2)$. Therefore, the colouring of the triangles attached to v_2 and v_3 provide $k(a+b)$ in the weighted degree of both u_2 and u_3. Since $\chi_w(u_2) \neq \chi_w(u_3)$, we necessarily have $w(u_2u_1) \neq w(u_3u_1)$. Depending on whether $w(u_2u_3) = a$ or $w(u_2u_3) = b$, one of u_2 and u_3 has weighted degree $(k+2)a+kb$ or $ka+(k+2)b$, respectively. The other vertex has weighted degree $(k+1)(a+b)$. $\square$

Lemma 4.2. *Let $k \geq 0$ be fixed. There is a vertex-colouring $\{a, b\}$-edge-weighting w of T_k, unless $0 \in \{a, b\}$, or $b = -a$, or $\chi_w(u_1) \in \{\chi_w(u_2), \chi_w(u_3)\}$.*

Proof. Recall that for two adjacent vertices of T_k which have the same degree, we only have to make sure that their multisets of colours by w are different according to Lemma 2.2. Note next that, for any triangle $v_1v_2u_iv_1$ different from $u_1u_2u_3u_1$ where $i \in \{2, 3\}$, one of v_1 and v_2 has weighted degree $a+b$, while the other vertex has weighted degree either $2a$ or $2b$ depending on how $v_1v_2u_iv_1$ is coloured (but we can "choose" this weighted degree thanks to local recolouring). Besides, one of u_2 and u_3 has weighted degree $(k+1)(a+b)$ by w, while the other vertex has weighted degree either $(k+2)a+kb$ or $ka+(k+2)b$ according to Lemma 4.1. Once again, this last weighted degree can be "chosen" freely by recolouring the edge u_2u_3.

Suppose $\chi_w(u_2) = (k+1)(a+b)$, and $\chi_w(u_3)$ is either $(k+2)a+kb$ or $ka+(k+2)b$ without loss of generality. On the one hand, consider u_2 and any triangle attached to it. Note first that if we have $\chi_w(u_2) = a+b$, i.e. $(k+1)(a+b) = a+b$, then either $k=0$ or $b=-a$. Now, observe that we cannot both have $\chi_w(u_2) = 2a$ and $\chi_w(u_2) = 2b$, unless $a=b$ which is impossible. On the other hand, consider u_3. Firstly, if we have both $(k+2)a+kb = a+b$ and $ka+(k+2)b = a+b$, then $a=b$. Secondly, if both $\chi_w(u_3) = 2a$ and $\chi_w(u_3) = 2b$ hold, then $a=b$ once again. Hence, the only possible conflict by w under our assumptions on a and b is $\chi_w(u_1) = \chi_w(u_2)$ or $\chi_w(u_1) = \chi_w(u_3)$. □

Let $k \geq 1$ be fixed, and assume $v_1v_2v_3$ is the path of length 2. As T'_k, we refer to the graph obtained by identifying v_2 and the root of each of k graphs T_k. The two inputs of T'_k are $i_1(T'_k) = v_1v_2$ and $i_2(T'_k) = v_2v_3$. We show that T'_k is a replacement gadget for $\{a,b\}$ under our assumptions on a and b.

Lemma 4.3. *Let $k \geq 1$ be fixed. The graph T'_k is a $((k+2)a+kb, ka+(k+2)b)$-replacement gadget for $\{a,b\}$ when $0 \notin \{a,b\}$ and $b \neq -a$.*

Proof. Assume w is a vertex-colouring $\{a,b\}$-edge-weighting of T'_k. Suppose $w(i_1(T'_k)) = a$ without loss of generality. For each of the graphs T_k attached to v_2, one of the two edges incident with v_2 is coloured a by w, while the one is coloured b according to Lemma 4.1. Hence, the graphs T_k attached to v_2 provide $k(a+b)$ in the weighted degree of v_2. Besides, in each graph T_k there is a vertex neighbouring v_2 which has weighted degree $(k+1)(a+b)$, while we may suppose that the other vertex neighbouring v_2 has weighted degree $ka+(k+2)b$.

Note then that if $w(v_2v_3) = b$, then we get $\chi_w(v_2) = (k+1)(a+b)$ and v_2 has the same weighted degree as some of its neighbours. Therefore, we have $w(v_2v_3) = a$. In this situation we have $\chi_w(v_2) = (k+2)a+kb$ while the vertices from the graphs T_k neighbouring v_2 have weighted degree $(k+1)(a+b)$ and $ka+(k+2)b$, respectively. Since v_2 and these vertices have the same degree, these weighted degrees are distinct by Observation 2.1. □

Corollary 4.4. *Any triplet (T'_i, T'_j, T'_k) is a replacement triplet for every pair $\{a,b\}$, such that $0 \notin \{a,b\}$ and $b \neq -a$, when i, j and k are distinct.*

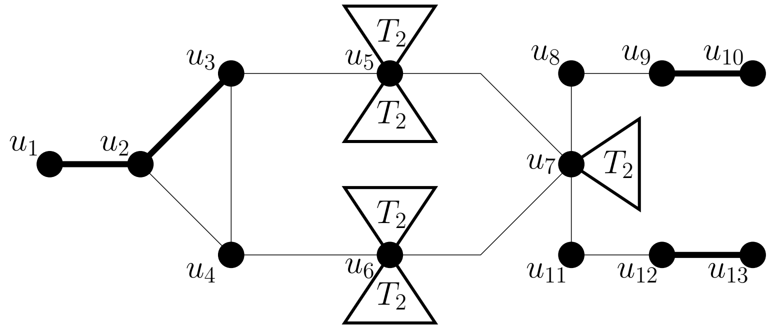


Figure 4: The spreading gadget $G^{\curlywedge}$ for the main implementation of the reduction framework, and a vertex-colouring $\{a,b\}$-edge-weighting of $G^{\curlywedge}$.

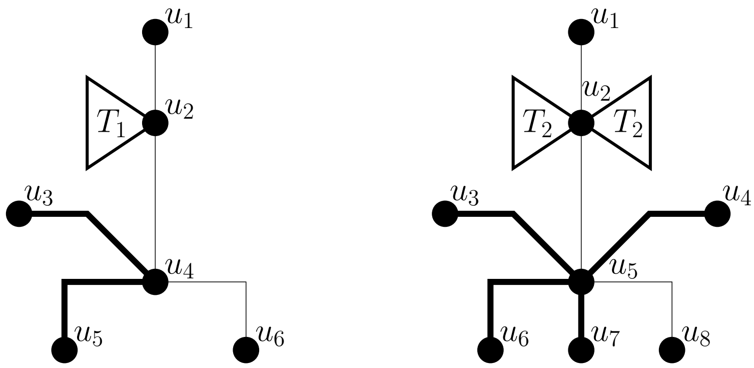
4.2 Spreading gadget $G^{\curlywedge}$ for $\{a,b\}$

Now consider the graph $G^{\curlywedge}$ depicted in Figure 4, whose input is u_1u_2, and whose two outputs are u_9u_{10} and $u_{12}u_{13}$. We show that $G^{\curlywedge}$ is a spreading gadget for $\{a,b\}$, i.e. that $G^{\curlywedge}$ satisfies Property 1, under our assumptions on a and b.

Proposition 4.5. *The graph $G^{\curlywedge}$ satisfies Property 1 for $\{a,b\}$ under our assumptions on a and b.*

Proof. Assume w is a vertex-colouring $\{a,b\}$-edge-weighting w of $G^{\curlywedge}$. Note that we cannot have $w(u_3u_5) \neq w(u_4u_6)$. Indeed, suppose e.g. that $w(u_3u_5) = a$ and $w(u_4u_6) = b$. Because u_5 and u_6 are both attached to two graphs T_2, which form a graph T_2', then we have $w(u_5u_7) = a$ and $w(u_6u_7) = b$ by Lemma 4.3. Besides, we have $\chi_w(u_5) = 4a + 2b$ and $\chi_w(u_6) = 2a + 4b$. We also know that a neighbour of u_7 from the graph T_2 attached to it has weighted degree $3a + 3b$, and that this graph T_2 provides $a + b$ in the weighted degree of u_7 according to Lemma 4.1. Then, the vertex u_7 has weighted degree at least $2a + 2b$, and the two edges u_7u_8 and u_7u_{11} are coloured in such a way that the weighted degree of u_7 does not meet any value in $\{2a + 4b, 3a + 3b, 4a + 2b\}$, but this is impossible.

On the contrary, if $w(u_3u_5) = w(u_4u_6) = a$ without loss of generality, then w can be vertex-colouring. Because of the arguments above, we have $w(u_5u_7) = w(u_6u_7) = a$ and $\chi_w(u_5) = \chi_w(u_6) = 4a+2b$. Recall that we may assume that the colouring of the graph T_2 attached to u_7 is such that the two vertices that are adjacent with u_7 have weighted degree $3a+3b$ and $4a+2b$. Besides, the colouring of this graph T_2 provides $a+b$ in the weighted degree of u_7. Thus, the weighted degree of u_7 is at least $3a+b$, and the edges u_7u_8 and u_7u_{11} are coloured in such a way that the weighted degree of u_7 is not $3a + 3b$ or $4a + 2b$. The only possibility is to have $w(u_7u_8) = w(u_7u_{11}) = a$ since, in this situation, we get $\chi_w(u_7) = 5a + b$.



(a) Case $c_i = 2$.

(b) Case $c_i = 3$.

Figure 5: The two forms of the clause gadget $G_F(C_i)$ for the main implementation of the reduction framework, and vertex-colouring $\{a, b\}$-edge-weightings of $G_F(C_i)$.

Now suppose $w(u_1u_2) = a$, and consider the edges u_2u_3 and u_2u_4. First, if $w(u_2u_3) = w(u_2u_4)$, then note that w cannot be vertex-colouring according to the arguments above since we would necessarily have $w(u_3u_5) \neq w(u_4u_6)$ so that $\chi_w(u_3) \neq \chi_w(u_4)$. Thus, $w(u_2u_3) = a$ and $w(u_2u_4) = b$ without loss of generality, and $\chi_w(u_2) = 2a + b$. Now note that if $w(u_3u_4) = a$, then we necessarily get that $\chi_w(u_3)$ or $\chi_w(u_4)$ is equal to $\chi_w(u_2)$ since we need $w(u_3u_5) = w(u_4u_6)$. Thus $w(u_3u_4) = b$. We then have $w(u_3u_5) = b$ so that $\chi_w(u_3) \neq \chi_w(u_2)$, and also $w(u_4u_6) = b$ so that $\chi_w(u_4) \neq \chi_w(u_3)$.

According to the arguments above, we have $w(u_3u_5) = w(u_4u_6) = b$ and $w(u_7u_8) = w(u_7u_{11}) = b$ under the assumption $w(u_1u_2) = a$. By Observation 2.1, we have $w(u_9u_{10}) = w(u_{12}u_{13}) = a$. $\square$

4.3 Clause gadgets $G_F(C_i)$ for $\{a, b\}$

We distinguish two forms for $G_F(C_i)$, depending on whether $c_i = 2$ or $c_i = 3$. These two forms are depicted in Figure 5. In the first case, i.e. $c_i = 2$, the inputs of $G_F(C_i)$ are u_3u_4, which is supposed to be coloured a, and u_1u_2, which is supposed to be coloured b, while the two outputs of $G_F(C_i)$ are u_4u_5 and u_4u_6. In the second case, i.e. $c_i = 3$, the three inputs of $G_F(C_i)$ are u_1u_2, which is supposed to be coloured b, and u_3u_5 and u_4u_5 which are supposed to be coloured a. The three outputs of $G_F(C_i)$ are u_5u_6, u_5u_7 and u_5u_8 in this case.

We prove that these two types of gadgets satisfy Property 2 under our assumptions on a and b.

Proposition 4.6. *The graph $G_F(C_i)$ for $c_i = 2$ satisfies Property 2 for $\{a, b\}$ under our assumptions on a and b.*

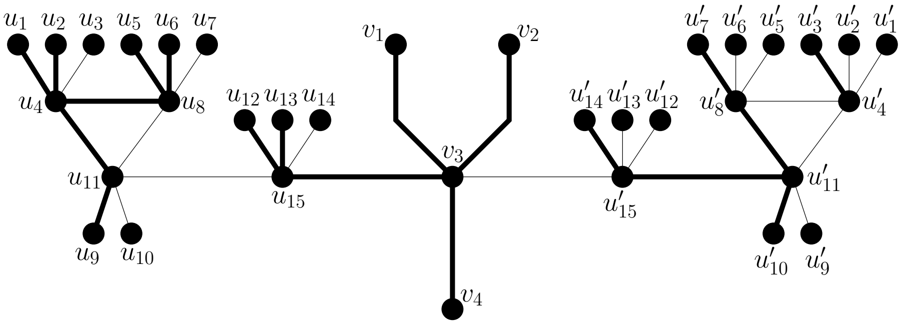


Figure 6: The collecting gadget $G^{\curlyvee}$ for the main implementation of the reduction framework, and a vertex-colouring $\{a, b\}$-edge-weighting of $G^{\curlyvee}$.

Proof. Assume w is a vertex-colouring $\{a, b\}$-edge-weighting of $G_F(C_i)$ such that $w(u_1u_2) = b$ and $w(u_3u_4) = a$. By Lemma 4.3, we have $w(u_2u_4) = b$, and $\chi_w(u_2) = a+3b$. Note then that we cannot have $w(u_4u_5) = w(u_4u_6) = b$ since otherwise we would get $\chi_w(u_4) = a + 3b = \chi_w(u_2)$. Hence, we have either $\{w(u_4u_5), w(u_4u_6)\} = \{a, a\}$ or $\{w(u_4u_5), w(u_4u_6)\} = \{a, b\}$. □

Proposition 4.7. *The graph $G_F(C_i)$ for $c_i = 3$ satisfies Property 2 for $\{a, b\}$ under our assumptions on a and b.*

Proof. Assume similarly that w is a vertex-colouring $\{a, b\}$-edge-weighting of $G_F(C_i)$ such that $w(u_1u_2) = b$ and $w(u_3u_5) = w(u_4u_5) = a$. Then we have $w(u_2u_5) = b$ and $\chi_w(u_2) = 2a + 4b$ according to Lemma 4.3 since the two graphs T_2 attached to u_2 form a graph T'_2. Now note that if $w(u_5u_6) = w(u_5u_7) = w(u_5u_8) = b$, then $\chi_w(u_5) = 2a + 4b = \chi_w(u_2)$. Thus, at least one of u_5u_6, u_5u_7 and u_5u_8 has colour a by w. □

4.4 Collecting gadget $G^{\curlyvee}$ for $\{a, b\}$

The collecting gadget $G^{\curlyvee}$ for this main implementation is depicted in Figure 6. The two regular inputs of $G^{\curlyvee}$ are v_1v_3 and v_2v_3, and its output is v_3v_4. The edges u_1u_4, u_2u_4, u_5u_8, u_6u_8, u_9u_{11}, $u_{12}u_{15}$, $u_{13}u_{15}$, $u'_3u'_4$, $u'_7u'_8$, $u'_{10}u'_{11}$ and $u'_{14}u'_{15}$ of $G^{\curlyvee}$ are forcing inputs which are supposed to be coloured a. The edges u_3u_4, u_7u_8, $u_{10}u_{11}$, $u_{14}u_{15}$, $u'_1u'_4$, $u'_2u'_4$, $u'_5u'_8$, $u'_6u'_8$, $u'_9u'_{11}$, $u'_{12}u'_{15}$ and $u'_{13}u'_{15}$ are forcing inputs supposed to be coloured b.

Under our assumptions on a and b, we prove that $G^{\curlyvee}$ is a collecting gadget, i.e. it satisfies Property 3.

Proposition 4.8. *The graph $G^{\curlyvee}$ satisfies Property 3 for $\{a, b\}$ under our assumptions on a and b.*

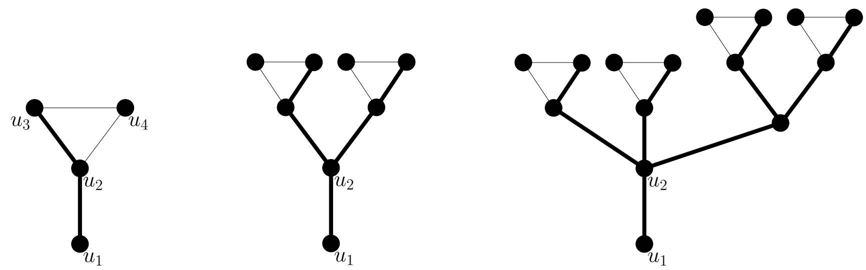


Figure 7: The graphs Y_1, Y_2 and Y_3, and vertex-colouring $\{a, b\}$-edge-weightings of Y_1, Y_2 and Y_3.

Proof. Assume w is a vertex-colouring $\{a, b\}$-edge-weighting of $G^{\curlyvee}$ such that the forcing inputs of $G^{\curlyvee}$ are coloured as requested. Consider first the left side of $G^{\curlyvee}$, i.e. the subgraph of $G^{\curlyvee}$ induced by the u_i's. Note first that we have $w(u_4u_{11}) \neq w(u_8u_{11})$ so that $\chi_w(u_4) \neq \chi_w(u_8)$. Now note that we cannot have $w(u_4u_8) = b$, since otherwise we would have $\{\chi_w(u_4), \chi_w(u_8)\} = \{2a+3b, 3a+2b\}$, and we would necessarily get $\chi_w(u_{11}) \in \{\chi_w(u_4), \chi_w(u_8)\}$ whatever is $w(u_{11}u_{15})$. Thus $w(u_4u_8) = a$, and $\{\chi_w(u_4), \chi_w(u_8)\} = \{4a + b, 3a + 2b\}$. Note now that we have $w(u_{11}u_{15}) = b$ since otherwise we would have $\chi_w(u_{11}) = 3a + 2b$. It follows that $\chi_w(u_{11}) = 2a + 3b$, and we have $w(u_{15}v_3) = a$ since otherwise we would have $\chi_w(u_{15}) = 2a + 3b = \chi_w(u_{11})$. Besides, $\chi_w(u_{15}) = 3a + 2b$.

Thanks to the symmetric structure of $G^{\curlyvee}$, we can deduce similar facts regarding the right side of $G^{\curlyvee}$, i.e. the subgraph of $G^{\curlyvee}$ induced by the u_i''s. In particular, we have $w(v_3u_{15}') = b$ and $\chi_w(u_{15}') = 2a + 3b$. Now observe that we cannot have $w(v_1v_3) \neq w(v_2v_3)$ since otherwise by having $w(v_3v_4) = a$ or $w(v_3v_4) = b$ we would get $\chi_w(v_3) = 3a + 2b = \chi_w(u_{15})$ or $\chi_w(v_3) = 2a + 3b = \chi_w(u_{15}')$, respectively. Therefore, we have $w(v_1v_3) = w(v_2v_3)$, and also $w(v_3v_4) = w(v_1v_3)$ since otherwise we would get $\chi_w(v_3) \in \{\chi_w(u_{15}), \chi_w(u_{15}')\}$. $\square$

5 Second implementation: $b = 0$

The second implementation of our reduction framework is dedicated to the case where one of the two weights from $\{a, b\}$ is 0. We assume throughout this section that $b = 0$. The thick (resp. thin) edges in our figures represent edges coloured a (resp. 0) by a vertex-colouring $\{a, 0\}$-edge-weighting.

5.1 Auxiliary gadget Y_k and replacement triplet for $\{a, 0\}$

Similarly as in the first implementation, we first give an auxiliary graph which is used in our gadgets to "force" the propagation of a vertex-colouring

$\{a, 0\}$-edge-weighting. The graphs Y_k with $k \geq 1$ are defined inductively. By construction, any graph Y_k has only one vertex with degree 1, called the *root* of Y_k. Start with an edge u_1u_2. To build Y_1, just identify u_2 and one vertex from a triangle. Now for the general case, i.e. $k \geq 2$, start over from the edge uv, and identify u_2 and the root of each of $k-1$ copies of Y_1 and one copy of Y_{k-1}. This construction is depicted in Figure 7. In the figures of the following sections, any pendant triangle marked "Y_k" indicates that a vertex is identified with the root of a graph Y_k.

Any graph Y_k has the following colouring property.

Lemma 5.1. *Assume w is a vertex-colouring $\{a, 0\}$-edge-weighting of Y_k for some $k \geq 1$. Then $w(u_1u_2) = a$ and $\chi_w(u_2) = (k+1)a$.*

Proof. We prove this lemma by induction on k. Consider Y_1 first, and denote the vertices of the triangle attached to u_2 by $u_2u_3u_4u_2$. Note that if $w(u_2u_3) = w(u_3u_4)$, then $\chi_w(u_3) = \chi_w(u_4)$. Then $w(u_2u_3) = a$ and $w(u_2u_4) = 0$ without loss of generality, and $w(u_3u_4) = 0$ since otherwise we would have either $\chi_w(u_2) = \chi_w(u_3)$ or $\chi_w(u_2) = \chi_w(u_4)$ by setting $w(u_1u_2) = a$ or $w(u_1u_2) = 0$, respectively. Then $\{\chi_w(u_3), \chi_w(u_4)\} = \{0, a\}$, and we have $w(u_1u_2) = a$ since otherwise we would have $\chi_w(u_2) = a$. In particular, we have $\chi_w(u_2) = 2a$.

Now suppose the claim is true for every k up to some i, and consider $k = i+1$. The graph Y_k is made of $k-1$ copies of Y_1 and one copy of Y_{k-1} whose roots are identified with u_2. By the induction hypothesis, these copies are coloured by w in such a way that their respective edge incident with u_2 is coloured a, and the vertex from Y_{k-1} neighbouring u_2 has weighted degree ka. Thus, these copies provide ka in the weighted degree of u_2. Finally, we have $w(u_1u_2) = a$ so that $\chi_w(u_2) \neq ka$, and we get $\chi_w(u_2) = (k+1)a$. □

Let $k \geq 1$ be fixed, and let $v_1v_2v_3$ denote the vertices of a path with length 2. As L_k, we refer to the graph obtained by identifying v_2 and the roots of k copies of the graphY_k. The two inputs of L_k are the edges v_1v_2 and v_2v_3. We show that L_k is a replacement gadget for $\{a, 0\}$.

Lemma 5.2. *Let $k \geq 1$ be fixed. The graph L_k is a $((k+2)a, ka)$-replacement gadget for $\{a, 0\}$.*

Proof. Assume w is a vertex-colouring $\{a, 0\}$-edge-weighting of L_k. By Lemma 5.1, the k copies of Y_k attached to v_2 provide ka to the weighted degree of v_2, and v_2 is adjacent to vertices with weighted degree $(k+1)a$. Note then that if $\{w(v_1v_2), w(v_2v_3)\} = \{a, 0\}$, then the weighted degree of v_2 is $(k+1)a$, and w is not vertex-colouring. On the contrary, if $w(v_1v_2) = w(v_2v_3) = a$ or $w(v_1v_2) = w(v_2v_3) = 0$, then the weighted degree of v_2 is $(k+2)a$ or ka, respectively. □

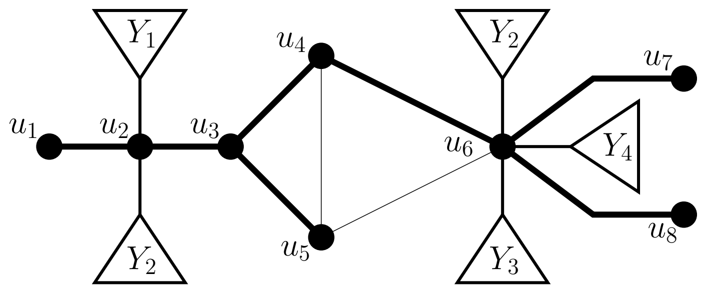


Figure 8: The spreading gadget $G^{\curlywedge}$ for the second implementation of the reduction framework, and a vertex-colouring $\{0, a\}$-edge-weighting of $G^{\curlywedge}$.

Corollary 5.3. *Any triplet (L_i, L_j, L_k) is a replacement triplet for $\{a, 0\}$ when i, j and k are distinct.*

5.2 Spreading gadget $G^{\curlywedge}$ for $\{a, 0\}$

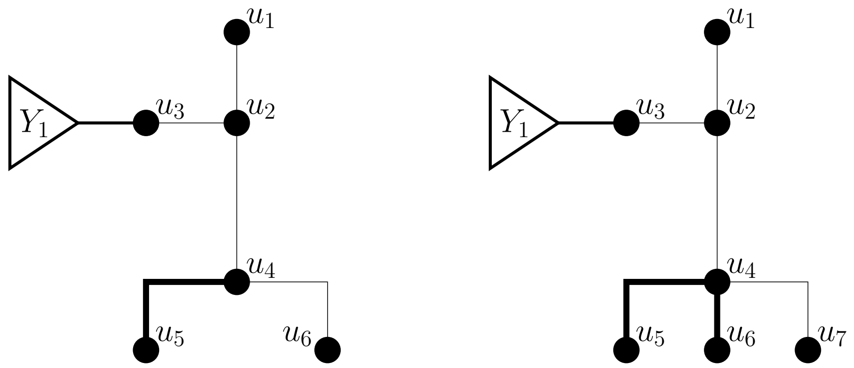
Consider, as $G^{\curlywedge}$, the graph depicted in Figure 8, whose input is u_1u_2, and whose two outputs are u_6u_7 and u_6u_8. We show that $G^{\curlywedge}$ is a spreading gadget for $\{a, 0\}$.

Proposition 5.4. *The graph $G^{\curlywedge}$ satisfies Property 1 for $\{a, 0\}$.*

Proof. Assume w is a vertex-colouring $\{a, 0\}$-edge-weighting of $G^{\curlywedge}$. Recall that the graphs Y_2, Y_3 and Y_4 attached to u_6 provide $3a$ in the weighted degree of u_6, and that u_6 is adjacent to vertices with weighted degree $3a$, $4a$, and $5a$ according to Lemma 5.1. Then we cannot have $\{w(u_4u_6), w(u_5u_6)\} = \{0, 0\}$ since otherwise we would have $\chi_w(u_6) \in \{3a, 4a, 5a\}$ whatever are $w(u_6u_7)$ and $w(u_6u_8)$. Observe also that if $\{w(u_4u_6), w(u_5u_6)\} = \{0, a\}$, then we have $w(u_6u_7) = w(u_6u_8) = a$. In this situation, we have $\chi_w(u_6) = 6a$.

Consider the edge u_1u_2. By Lemma 5.1, the weighted degree of u_2 is at least $2a$, and u_2 is adjacent with vertices whose weighted degrees are $2a$ and $3a$. Then we have $w(u_1u_2) = w(u_2u_3) = a$ since otherwise we would have $\chi_w(u_2) \in \{2a, 3a\}$. In particular, we have $\chi_w(u_2) = 4a$. Now note that we cannot have $w(u_3u_4) = w(u_3u_5) = 0$ since one of u_4 or u_6 would have weighted degree $\chi_w(u_3) = a$. Indeed, no matter what is the colour of u_4u_5, we have $\{w(u_4u_6), w(u_5u_6)\} = \{0, a\}$ so that $\chi_w(u_4) \neq \chi_w(u_5)$. But then, one of u_4 or u_6 necessarily gets weighted degree a, which is $\chi_w(u_3)$.

Suppose now $w(u_3u_4) = w(u_3u_5) = a$. In this situation, we have $\chi_w(u_3) = 3a$. Note that if $w(u_4u_5) = a$, then we have $\{w(u_4u_6), w(u_5u_6)\} = \{0, a\}$ so that u_4 and u_5 have distinct weighted degrees. But then, one of these two vertices has weighted degree $3a$. So $w(u_4u_5) = 0$. Once again,



(a) Case $c_i = 2$.　　(b) Case $c_i = 3$.

Figure 9: The two forms of the clause gadget $G_F(C_i)$ for the second implementation of the reduction framework, and vertex-colouring $\{0, a\}$-edge-weightings of $G_F(C_i)$.

we have $\{w(u_4u_6), w(u_5u_6)\} = \{0, a\}$ so that u_4 and u_5 are distinguished. According to the remarks above, we then have $w(u_6u_7) = w(u_6u_8) = a$, as requested.

Suppose finally that $w(u_3u_4) = a$ and $w(u_3u_5) = 0$ without loss of generality. Then $\chi_w(u_3) = 2a$. Note that we cannot have $w(u_4u_5) = 0$ since otherwise we would have $w(u_4u_6) = 0$ so that $\chi_w(u_4) \neq \chi_w(u_3)$, and $w(u_5u_6) = 0$ so that $\chi_w(u_4) \neq \chi_w(u_5)$. But then $\{w(u_4u_6), w(u_5u_6)\} = \{0, 0\}$, and w is not vertex-colouring. Thus, $w(u_4u_5) = a$. Because $\chi_w(u_4) \neq \chi_w(u_3)$ and $\chi_w(u_5) \neq \chi_w(u_3)$, we have both $w(u_4u_6) = a$ and $w(u_5u_6) = 0$. According to the arguments above, we have $w(u_6u_7) = w(u_6u_8) = a$ once again. □

5.3 Clause gadgets $G_F(C_i)$ for $\{a, 0\}$

The two forms of $G_F(C_i)$ for $\{a, 0\}$, i.e. for the cases $c_i = 2$ and $c_i = 3$, are depicted in Figure 9. In both cases, the input of $G_F(C_i)$ is u_1u_2 and is supposed to be coloured 0. The outputs of $G_F(C_i)$ are u_4u_5, u_4u_6, and also u_4u_7 when $c_i = 3$. We show that $G_F(C_i)$ satisfies Property 2 in any of the two cases.

Proposition 5.5. *The graph $G_F(C_i)$ satisfies Property 2 for $\{a, 0\}$ whatever is the value of c_i.*

Proof. Assume w is a vertex-colouring $\{a, 0\}$-edge-weighting of $G_F(C_i)$ such that $w(u_1u_2) = 0$. Recall that the edge of the graph Y_1 incident with u_3 has colour a, and that the vertex of the graph Y_1 adjacent with u_3 has weighted degree $2a$ (Lemma 5.1). Therefore, we have $w(u_3u_2) = 0$ so that $\chi_w(u_3) =$

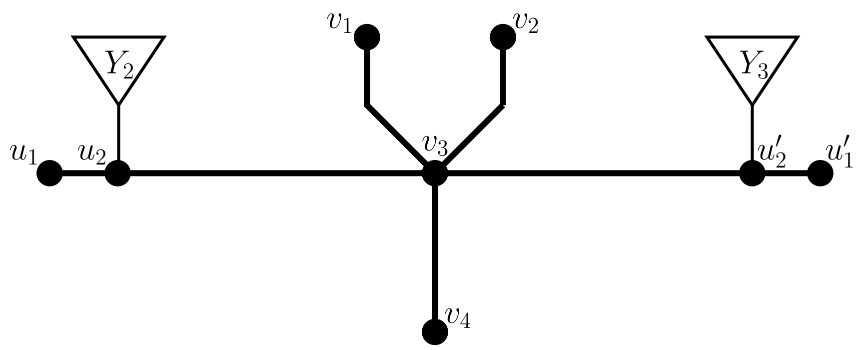


Figure 10: The collecting gadget $G^{\curlyvee}$ for the second implementation of the reduction framework, and a vertex-colouring $\{a,b\}$-edge-weighting of $G^{\curlyvee}$.

$a \neq 2a$, and $w(u_2u_4) = 0$ so that $\chi_w(u_2) \neq \chi_w(u_3)$. In particular, we get $\chi_w(u_2) = 0$. Then note that at least one of the outputs of $G_F(C_i)$ receives colour a by w since otherwise we would have $\chi_w(u_4) = 0 = \chi_w(u_2)$. $\square$

5.4 Collecting gadget $G^{\curlyvee}$ for $\{a, 0\}$

Now consider the graph depicted in Figure 10 as $G^{\curlyvee}$. The two regular inputs of $G^{\curlyvee}$ are v_1v_3 and v_2v_3, and its output is v_3v_4. The forcing inputs of $G^{\curlyvee}$ are u_1u_2 and $u'_1u'_2$, which are supposed to be coloured a. We prove that $G^{\curlyvee}$ satisfies Property 3 for $\{a, 0\}$.

Proposition 5.6. *The graph $G^{\curlyvee}$ satisfies Property 3 for $\{a, 0\}$.*

Proof. Suppose w is a vertex-colouring $\{a,0\}$-edge-weighting of $G_F(G^{\curlyvee})$ such that $w(u_1u_2) = w(u'_1u'_2) = a$. We have $w(u_2v_3) = w(u'_2v_3) = a$ according to Lemma 5.2. Plus, we have $\chi_w(u_2) = 3a$ and $\chi_w(u'_2) = 4a$. Under these assumptions, we cannot have $w(v_1v_3) \neq w(v_2v_3)$. Indeed, in such a situation, by having $w(v_3v_4) = a$ or $w(v_3v_4) = 0$, we would get $\chi_w(v_3) = 4a$ or $\chi_w(v_3) = 3a$, respectively.

Now suppose $w(v_1v_3) = w(v_2v_3)$. On the one hand, if $w(v_1v_3) = w(v_2v_3) = a$, then we have $w(v_3v_4) = a$ since otherwise we would get $\chi_w(v_3) = 4a = \chi_w(u'_2)$. In this situation, we get $\chi_w(v_3) = 5a$. On the other hand, suppose $w(v_1v_3) = w(v_2v_3) = 0$. Note that if $w(v_3v_4) = a$, then $\chi_w(v_3) = 3a = \chi_w(u_2)$. On the contrary, we have $\chi_w(v_3) = 2a$ when $w(v_3v_4) = 0$. $\square$

6 Third implementation: $b = -a$

In this section, we give the gadgets for implementing our reduction framework in the case where $\{a, b\} = \{a, -a\}$. In all the figures of this section, the thick (resp. thin) edges represent edges coloured a (resp. $-a$) by a vertex-colouring $\{a,-a\}$-edge-weighting.

6.1 Auxiliary gadget T and replacement triplet for $\{a,-a\}$

Once again, we use a graph to force the propagation of a vertex-colouring $\{a,-a\}$-edge-weighting in a graph. This graph, denoted T, is just a triangle $u_1u_2u_3u_1$ whose vertex u_1 is the *root* of T. Hence, every triangle marked "T" in our figures of further sections refers to the graph T. This graph T has some interesting properties when dealing with vertex-colouring $\{a,-a\}$-edge-weightings.

Lemma 6.1. *Assume w is an $\{a,-a\}$-edge-weighting of T. If $\chi_w(u_2) \neq \chi_w(u_3)$, then one of u_2 and u_3 has weighted degree 0, while the other vertex has weighted degree $2a$ or $-2a$. Besides, we have $\{w(u_1u_2), w(u_1u_3)\} = \{a,-a\}$.*

Proof. The proof is similar to the one of Lemma 4.1 since T is isomorphic to T_0. Because $\chi_w(u_2) \neq \chi_w(u_3)$, we have $w(u_1u_2) \neq w(u_1u_3)$. Suppose e.g. $w(u_1u_2) = a$ and $w(u_1u_3) = -a$ without loss of generality. Now, by setting either $w(u_2u_3) = a$ or $w(u_2u_3) = -a$, we get $\chi_w(u_3) = 0$ or $\chi_w(u_2) = 0$, respectively. Besides, we have $\chi_w(u_2) = 2a$ or $\chi_w(u_3) = -2a$, respectively. □

We now introduce the replacement gadgets for $\{a,-a\}$. The first replacement gadget R_1 is obtained by identifying the root of T and v_2, where v_2 denotes the inner vertex of some path $v_1v_2v_3$ with length 2. The two inputs of R_1 then are v_1v_2 and v_2v_3.

Lemma 6.2. *The graph R_1 is a $(2a,-2a)$-replacement gadget for $\{a,-a\}$.*

Proof. Assume w is a vertex-colouring $\{a,-a\}$-edge-weighting of R_1. By Lemma 6.1, the weighted degrees of the vertices adjacent with v_2 which belong to the graph T are 0, and either $2a$ or $-2a$, where this last weighted degree can be "chosen" thanks to local recolouring of T. Besides, the colouring of T provides $a + (-a) = 0$ in the weighted degree of v_2. Note then that if $w(v_1v_2) \neq w(v_2v_3)$, then we have $\chi_w(v_2) = 0$. Hence, we have $w(v_1v_2) = w(v_2v_3)$, and $\chi_w(v_2) = 2a$ or $\chi_w(v_2) = -2a$ depending on $w(v_1v_2) = a$ or $w(v_1v_2) = -a$, respectively. □

The second replacement gadget R_2 for $\{a,-a\}$ is obtained as follows. As for R_1, start from a path $v_1v_2v_3$ with length 2, and identify v_2 and u_1, where $u_1u_2u_3u_4u_5u_1$ is a cycle with length 5. The inputs of R_2 are v_1v_2 and v_2v_3.

Lemma 6.3. *The graph R_2 is a $(4a, -4a)$-replacement gadget for $\{a, -a\}$.*

Proof. Assume w is a vertex-colouring $\{a, -a\}$-edge-weighting of R_2. Note first that $w(u_1u_2) = w(u_5u_1)$ according to Observation 2.1. Besides, we have $\{\chi_w(u_2), \chi_w(u_5)\} = \{0, 2 \cdot w(u_1u_2)\}$ and the cycle attached to v_2 provides $2 \cdot w(u_1u_2)$ in the weighted degree of v_2. Suppose now $w(u_1u_2) = w(u_5u_1) = a$. Then note that if $w(v_1v_2) = w(v_2v_3) = -a$ or $w(v_1v_2) \neq w(v_2v_3)$, then we have $\chi_w(v_2) = 0$ or $\chi_w(v_2) = 2a$, and w is not vertex-colouring. Hence $w(v_1v_2) = w(v_2v_3) = a$ and $\chi_w(v_2) = 4a$ in this situation. The proof follows similarly from the assumption $w(u_1u_2) = w(u_5u_1) = -a$. $\square$

The other replacement gadgets for $\{a, -a\}$ are defined inductively. To obtain the graph R_k with $k \geq 3$, start from a path $v_1v_2v_3$ with length 2. Next identify v_2 and the root of each of $k-1$ copies of the graph T. For every i^{th} resulting copy $v_2u_2u_3v_2$ of T, with $i \in \{1, ..., k-1\}$, now R_i-subdivide each of the edges v_2u_2 and v_2u_3. This results in a cycle $s_1s_2s_3s_4s_5s_1$ with length 5 such that $s_1 = v_2$, and the edges s_1s_2 and s_2s_3, and s_1s_5 and s_5s_4 are the inputs of two replacement gadgets R_i. To finish the construction of R_k, identify v_2 and one vertex of each of $k-1$ cycles with length 5. The inputs of R_k are v_1v_2 and v_2v_3.

Lemma 6.4. *Let $k \geq 3$ be fixed. The graph R_k is a $(2ka, -2ka)$-replacement gadget for $\{a, -a\}$.*

Proof. Assume the claim is true for every k up to some value of i, and consider $k = i+1$. Let w be a vertex-colouring $\{a, -a\}$-edge-weighting of R_k. Consider first every cycle $v_2s_2s_3s_4s_5v_2$ with length 5 such that the edges v_2s_2 and s_2s_3, and s_4s_5 and s_5v_2 are the inputs of two graphs $R_{k'}$, with $k' < k$. Note that we cannot have $w(s_2s_3) = w(s_4s_5)$ since otherwise we would have $\chi_w(s_3) = \chi_w(s_4)$ (Observation 2.1). Thus we have $w(s_2s_3) = a$ and $w(s_4s_5) = -a$ without loss of generality, and $w(v_2s_2) = a$ and $w(v_2s_5) = -a$ according to the induction hypothesis. Besides, $\chi_w(s_2) = 2k'a$ and $\chi_w(s_5) = -2k'a$. Hence, the $k-1$ cycles of this form attached to v_2 provide $(k-1)(a + (-a)) = 0$ in the weighted degree of v_2, and v_2 is adjacent with vertices whose weighted degrees lie in $\{-2(k-1)a, -2(k-2)a, ..., -2a, 2a, ..., 2(k-2)a, 2(k-1)a\}$.

Now consider any "regular" cycle $v_2s_2s_3s_4s_5v_2$ with length 5 attached to v_2. For the same reasons as those given in the proof of Lemma 6.3, we have $w(v_2s_2) = w(v_2s_5)$, and $0 \in \{\chi_w(s_2), \chi_w(s_5)\}$. Hence, each regular cycle provides either $2a$ or $-2a$ in the weighted degree of v_2. Is is then easy to check that the only way for w to be vertex-colouring is to have each of the $k-1$ regular cycles providing $2a$ (resp. $-2a$) to the weighted degree of v_2, and $w(v_1v_2) = w(v_2v_3) = a$ (resp. $w(v_1v_2) = w(v_2v_3) = -a$). In this situation, we get $\chi_w(v_2) = 2ka$ (resp. $\chi_w(v_2) = -2ka$). For every other

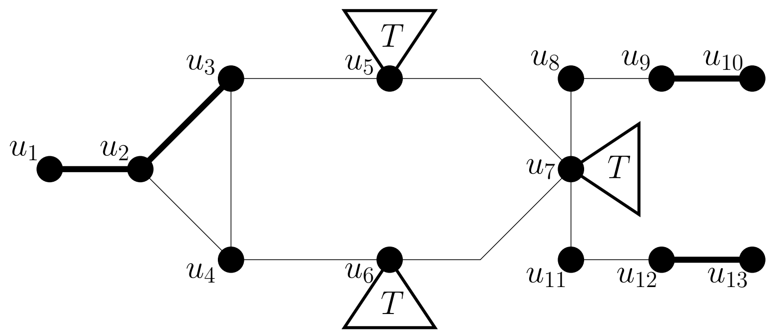


Figure 11: The spreading gadget $G^{\curlywedge}$ for the third implementation of the reduction framework, and a vertex-colouring $\{a, -a\}$-edge-weighting of $G^{\curlywedge}$.

possible colouring, we necessarily get that $\chi_w(v_2)$ lies in $\{-2(k-1)a, -2(k-2)a, ..., -2a, 0, 2a, ..., 2(k-2)a, 2(k-1)a\}$. $\square$

Corollary 6.5. *Any triplet (R_i, R_j, R_k) is a replacement triplet for $\{a, -a\}$ when i, j and k are distinct.*

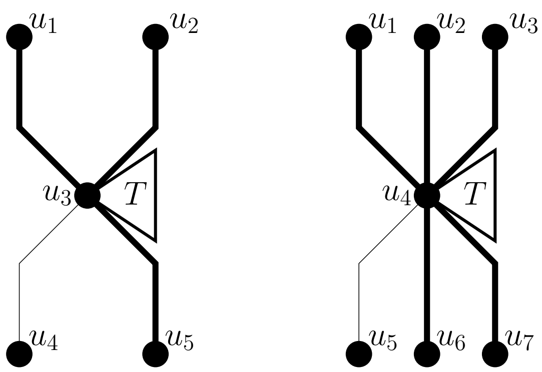
6.2 Spreading gadget $G^{\curlywedge}$ for $\{a, -a\}$

The spreading gadget $G^{\curlywedge}$ for $\{a, -a\}$ is depicted in Figure 11. The input of $G^{\curlywedge}$ is u_1u_2, while its outputs are u_9u_{10} and $u_{12}u_{13}$. We prove that $G^{\curlywedge}$ satisfies the spreading gadget property.

Proposition 6.6. *The graph $G^{\curlywedge}$ satisfies Property 1 for $\{a, -a\}$.*

Proof. Suppose w is a vertex-colouring $\{a, -a\}$-edge-weighting of $G^{\curlywedge}$. Note first that we cannot have $w(u_3u_5) \neq w(u_4u_6)$. Indeed, suppose e.g. that $w(u_3u_5) = a$ and $w(u_4u_6) = -a$. Then $w(u_5u_7) = a$ and $w(u_6u_7) = -a$ according to Lemma 6.2. Besides, $\chi_w(u_5) = 2a$ and $\chi_w(u_6) = -2a$. Note further that the colouring of the T graph attached to u_7 provides $a+(-a) = 0$ in the weighted degree of u_7, and that u_7 has a neighbour with weighted degree 0 (Lemma 6.1). Then note that for any value of $\{w(u_7u_8), w(u_7u_{11})\}$, i.e. $\{a, a\}$, $\{a, -a\}$ or $\{-a, -a\}$, we get that $\chi_w(u_7)$ is either $2a$, 0 or $-2a$, respectively. Hence w is not vertex-colouring under the assumption $w(u_3u_5) \neq w(u_4u_6)$.

On the contrary, note that if $w(u_3u_5) = w(u_4u_6) = -a$, then w can be vertex-colouring. Note first that we have $w(u_5u_7) = w(u_6u_7) = -a$ according to Lemma 6.2. Besides, $\chi_w(u_5) = \chi_w(u_6) = -2a$. Recall that u_7 has a neighbour with weighted degree 0, and that the graph T attached to u_7 provides 0 to the weighted degree of u_7. Now note that if $\{w(u_7u_8), w(u_7u_{11})\}$ is $\{a, a\}$ or $\{a, -a\}$, then we have $\chi_w(u_7) = 0$ or $\chi_w(u_7) = -2a$, respectively.



(a) Case $c_i = 2$. (b) Case $c_i = 3$.

Figure 12: The two forms of the clause gadget $G_F(C_i)$ for the third implementation of the reduction framework, and vertex-colouring $\{a, -a\}$-edge-weightings of $G_F(C_i)$.

On the contrary, if $w(u_7u_8) = w(u_7u_{11}) = -a$, then we get $\chi_w(u_7) = -4a$. Besides, we have $w(u_9u_{10}) = w(u_{12}u_{13}) = a$ by Observation 2.1.

Now assume $w(u_1u_2) = a$. First, note that we cannot have $w(u_2u_3) = w(u_2u_4)$. Indeed, in this situation, we would have $w(u_3u_5) \neq w(u_4u_6)$ so that u_3 and u_4 have distinct weighted degree, and this whatever is $w(u_3u_4)$. According to the arguments above, w is not vertex-colouring under this assumption. Then, $w(u_2u_3) = a$ and $w(u_2u_4) = -a$ without loss of generality. In this situation, $\chi_w(u_2) = a$. On the one hand, if $w(u_3u_4) = a$, then w cannot be vertex-colouring. Indeed, we would have $w(u_3u_5) = a$ so that $\chi_w(u_3) \neq \chi_w(u_2)$, and $w(u_4u_6) = -a$ so that $\chi_w(u_4) \neq \chi_w(u_2)$. But then $w(u_3u_5) \neq w(u_4u_6)$, and w is not vertex-colouring, again according to the arguments above. On the other hand, i.e. $w(u_3u_4) = -a$, then we have $w(u_3u_5) = -a$ so that $\chi_w(u_2) \neq \chi_w(u_3)$, and $w(u_4u_6) = -a$ so that $\chi_w(u_3) \neq \chi_w(u_4)$. As pointed out above, we have $w(u_9u_{10}) = w(u_{12}u_{13}) = a$ as requested. □

6.3 Clause gadgets $G_F(C_i)$ for $\{a, -a\}$

Consider, as $G_F(C_i)$, the graphs depicted in Figure 12. In the first (resp. second) form, i.e. for $c_i = 2$ (resp. $c_i = 3$), the inputs of $G_F(C_i)$ are u_1u_3 and u_2u_3 (resp u_1u_4, u_2u_4 and u_3u_4) and are supposed to be coloured a. The outputs of $G_F(C_i)$ are u_3u_4 and u_3u_5 (resp. u_4u_5, u_4u_6 and u_4u_7). We show that the two forms of $G_F(C_i)$ satisfy the clause gadget property.

Proposition 6.7. *The graph $G_F(C_i)$ satisfies Property 2 for $\{a, -a\}$ whatever is the value of c_i.*

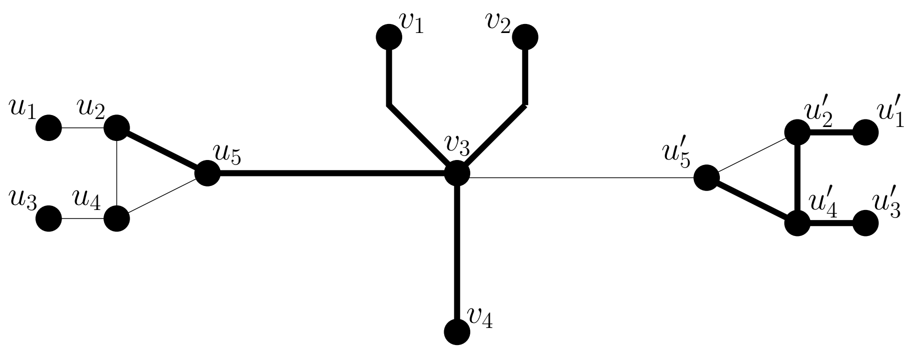


Figure 13: The collecting gadget $G^{\curlyvee}$ for the third implementation of the reduction framework, and a vertex-colouring $\{a, -a\}$-edge-weighting of $G^{\curlyvee}$.

Proof. Assume w is a vertex-colouring $\{a, -a\}$-edge-weighting of $G_F(C_i)$ such that all the inputs of $G_F(C_i)$ are coloured a. We show the claim to be true when $c_i = 2$, but the proof is similar for the case $c_i = 3$. Recall that the graph T attached to u_3 provides $a + (-a) = 0$ in the weighted degree of u_3, and that one of its vertices has weighted degree 0 (Lemma 6.1). Note then that if $w(u_3u_4) = w(u_3u_5) = -a$, then $\chi_w(u_3) = 0$. Therefore, at least one output of $G_F(C_i)$ receives colour a by w. $\square$

6.4 Collecting gadget $G^{\curlyvee}$ for $\{a, -a\}$

Let $G^{\curlyvee}$ be the graph depicted in Figure 13. The two regular inputs of $G^{\curlyvee}$ are v_1v_3 and v_2v_3, while its output is v_3v_4. The forcing inputs of $G^{\curlyvee}$ are u_1u_2 and u_3u_4, which are supposed to be coloured $-a$, and $u'_1u'_2$ and $u'_3u'_4$ which are supposed to be coloured a. We show that $G^{\curlyvee}$ is a collecting gadget for $\{a, -a\}$.

Proposition 6.8. *The graph $G^{\curlyvee}$ satisfies Property 3 for $\{a, -a\}$.*

Proof. Suppose w is a vertex-colouring $\{a, -a\}$-edge-weighting of $G^{\curlyvee}$ such that $w(u_1u_2) = w(u_3u_4) = -a$ and $w(u'_1u'_2) = w(u'_3u'_4) = a$. Note that we cannot have $w(u_2u_4) = a$. Indeed, in this situation, we would have $w(u_2u_5) \neq w(u_4u_5)$ so that $\chi_w(u_2) \neq \chi_w(u_4)$. But then we would get $\{\chi_w(u_2), \chi_w(u_4)\} = \{a, -a\}$ and we would have $\chi_w(u_5) \in \{a, -a\}$ no matter what is $w(u_5v_3)$. Therefore $w(u_2u_4) = -a$. For the same reasons, we have $w(u_2u_5) = a$ and $w(u_4u_5) = -a$ without loss of generality. Then, $\chi_w(u_2) = -a$ and $\chi_w(u_4) = -3a$, and we have $w(u_5v_3) = a$ since otherwise we would have $\chi_w(u_5) = -a$. Besides, we have $\chi_w(u_5) = a$.

Repeating the same arguments for the graph induced by $\{u'_1, u'_2, u'_3, u'_4, u'_5, v_3\}$, we get that $w(u'_5v_3) = -a$ and $\chi_w(u'_5) = -a$. Therefore, the edges u_5v_3 and u'_5v_3 provide $a + (-a) = 0$ in the weighted degree of v_3, and v_3 is adjacent to vertices with respective weighted degree a and $-a$. Now observe that we cannot have $w(v_1v_3) \neq w(v_2v_3)$. Indeed, by then having $w(v_3v_4) = a$ or $w(v_3v_4) = -a$, we would get $\chi_w(v_3) = a$ or $\chi_w(v_3) = -a$, respectively. On the contrary, if $w(v_1v_3) = w(v_2v_3) = a$ (resp. $w(v_1v_3) = w(v_2v_3) = -a$), then we have $w(v_3v_4) = a$ (resp. $w(v_3v_4) = -a$) since otherwise we would have $\chi_w(v_3) = a$ (resp. $\chi_w(v_3) = -a$). In particular, we get $\chi_w(v_3) = 3a$ (resp. $\chi_w(v_3) = -3a$). $\square$

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