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A ONE-PARAMETER DEFORMATION OF THE FARAHAT-HIGMAN ALGEBRA

JEAN-PAUL BULTEL

Abstract. We show, by introducing an appropriate basis, that a one-parameter family of Hopf algebras introduced by Foissy [Adv. Math. 218 (2008) 136-162] interpolates between the Faà di Bruno algebra and the Farahat-Higman algebra. Its structure constants in this basis are deformation of the top connection coefficients, for which we obtain analogues of Macdonald’s formulas.

1. Introduction

The center \( Z_n \) of the group algebra \( \mathbb{Z}[S_n] \) of the symmetric group is spanned by conjugacy classes \( C_\mu \), which are parametrized by partitions \( \mu \) of \( n \). The connection coefficients \( a^\lambda_{\mu\nu} \) are the structure constants

\[
C_\mu C_\nu = \sum_{\lambda \vdash n} a^\lambda_{\mu\nu} C_\lambda
\]

of \( Z_n \). These coefficients, whose calculation is in general very hard, have important applications to various enumerative problems or to the calculation of certain matrix integrals [7].

For a partition \( \mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r > 0) \), define

\[
\bar{\mu} = (\mu_1 - 1, \mu_2 - 1, \ldots, \mu_r - 1),
\]

the reduced cycle type of any permutation of cycle type \( \mu \). Denoting by \( c_\rho(n) \) the conjugacy class \( C_\rho \) of \( S_n \) such that \( \bar{\rho} = \rho \), we can rewrite (1) in the form

\[
c_\mu(n)c_\nu(n) = \sum_\lambda a^\lambda_{\mu\nu}(n)c_\lambda(n).
\]

It has been proved by Farahat and Higman [4] that the connection coefficients \( a^\lambda_{\mu\nu}(n) \) are polynomial functions of \( n \), and are independent of \( n \) if \( |\lambda| = |\mu| + |\nu| \). These are called the top connection coefficients.

One may use the top connection coefficients to define an algebra \( R \), spanned by formal symbols \( c_\mu \) indexed by all partitions, with multiplication rule

\[
c_\mu c_\nu = \sum_{|\lambda| = |\mu| + |\nu|} a^\lambda_{\mu\nu} c_\lambda.
\]

This is the Farahat-Higman algebra ([4], see also [10, ex. 24 p 131]). It is also proved in [4] that \( R \) is isomorphic to the algebra of symmetric functions \( \Lambda \). The construction of an explicit isomorphism \( \varphi : \Lambda \to R \) is more recent, and due to Macdonald [10, ex.

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(another proof has been given by Goulden and Jackson [6]). More precisely, Macdonald constructed a basis \( (g_\mu) \) of \( \Lambda \) such that
\[
g_\mu g_\nu = \sum_{|\lambda|=|\mu|+|\nu|} a_{\mu \nu}^\lambda g_\lambda,
\]
and obtained a recursive formula for the calculation of \( a_{\mu \nu}^\lambda \).

Murray [12] explicited the images in \( \Lambda \) of the projections of symmetric functions of Jucys-Murphy elements in \( R \) under this isomorphism. In Section 3, we give a new derivation of his result, and new proofs of various results of Biane [1] and Matsumoto-Novak [11].

It is well-known that the algebra of symmetric functions is a Hopf algebra. Its standard coproduct, denoted here by \( \Delta_0 \), comes from its interpretation as the algebra of polynomial functions on the multiplicative group \( G_0 \) of formal power series with constant term 1. It has, however, another Hopf algebra structure, known as the Faà di Bruno algebra, coming from its interpretation as the algebra of polynomial functions on the group \( G_1 = \{ ta(t) | a \in G_0 \} \) of formal diffeomorphisms of the line under composition (see, e.g., [3]). In fact, Macdonald’s basis \( g_\mu \) is the dual of the image \( h_\mu = S_1(h_\mu) \) of the basis of complete homogeneous functions \( h_\mu \) by the antipode of the Faà di Bruno algebra \( F \).

This suggests to interpret \( g_\mu \) as living in the dual \( F^* \) of \( F \). However, contrary to \( R \), this dual is not commutative, being the universal enveloping algebra of the Lie algebra \( g_1 \) of \( G_1 \). Thus, such an interpretation does not make sense a priori.

To clarify this situation, we make use of a one-parameter deformation \( F_\gamma \) of \( F \) recently discovered by Foissy [5] in his investigation of combinatorial Dyson-Schwinger equations in the Connes-Kreimer algebra. We then obtain for the structure constants of \( F_\gamma \) in the dual basis of \( S_1(h_\mu) \) a one-parameter deformation of Macdonald’s formulas which are recovered for \( \gamma = 0 \).

We follow the conventions of [10]. For the convenience of the reader, the most essential ones are recalled in Section 2.

2. Notations and Background

2.1. Partitions. Let \( n \) be a positive integer. A finite sequence of strictly positive integers \( (\lambda_1, \lambda_2, \ldots) \) is called a partition of \( n \) if \( \lambda_1 \geq \lambda_2 \geq \ldots \) and \( \lambda_1 + \lambda_2 + \ldots = n \). We then write \( \lambda \vdash n \). The \( \lambda_i \) are the parts of \( \lambda \), \( |\lambda| = n \) is the weight of \( \lambda \) and the number \( l(\lambda) \) of parts in \( \lambda \) is the length of \( \lambda \). The multiplicity \( m_\lambda(k) \) of \( k \) in \( \lambda \) is the number of parts in \( \lambda \) equal to \( k \). We set
\[
z_\lambda = \prod_{k \geq 1} k^{m_\lambda(k)} m_\lambda(k)!
\]

2.2. The algebra of symmetric functions. We denote by \( \Lambda \) the algebra of symmetric functions. The bases \( (m_\lambda), (e_\lambda), (h_\lambda), (p_\lambda) \) and \( (s_\lambda) \) are respectively the monomial, elementary, complete, power sum and Schur symmetric functions. These bases of \( \Lambda \) are parametrised by partitions of all integers. For any basis \( (b_\lambda) \), we denote by \( b_n \)
the symmetric function $b_n$. Denote by $\langle \cdot, \cdot \rangle$ the usual scalar product on $\Lambda$, for which $(s_{\lambda})$ is an orthonormal basis. For this scalar product, $(p_{\lambda})$ is an orthogonal basis, and one has

$$\langle p_{\lambda}, p_{\mu} \rangle = \delta_{\lambda \mu} z_{\mu}$$

(7) The bases $(m_{\lambda})$ and $(h_{\lambda})$ are dual to each other.

$$\langle h_{\lambda}, m_{\mu} \rangle = \delta_{\lambda \mu}$$

(8) $(\delta$ is the Kronecker symbol). We denote by $p_n^\perp$ the adjoint of the multiplication by $p_n$, in the sense

$$\langle p_n f, g \rangle = \langle f, p_n^\perp g \rangle$$

(9) where $f$ and $g$ are any two symmetric functions. This operator is a derivation. More precisely,

$$p_n^\perp = n \frac{\partial}{\partial p_n}$$

(10) This operator acts on the $h_n$ as follows

$$p_n^\perp h_N = h_{N-n}$$

(11) for any $N > n$, so that

$$p_n^\perp = \sum_{r \geq 0} h_r \frac{\partial}{\partial h_{n+r}}$$

(12) Now, let $X$ and $Y$ be two alphabets. For any symmetric function $f$ and any scalar $k$, we identify $f$ with $f(X)$, and we define the algebra morphisms $f \rightarrow f(X + Y)$, $f \rightarrow f(kX)$ and $f \rightarrow f(k)$ by

$$p_n(X + Y) = p_n(X) + p_n(Y)$$

(13) $$p_n(kX) = kp_n(X)$$

(14) $$p_n(k) = k$$

(15) The standard Hopf algebra structure of $\Lambda$ is defined by the coproduct $\Delta_0$

$$\Delta_0(f) = f(X + Y)$$

(16) where we identify $f(X + Y)$ with an element of $\Lambda \otimes \Lambda$ by identifying $f \otimes g$ with $f(X)g(Y)$. The counit $\epsilon$ and the antipode $S_0$ are given by

$$\epsilon(f) = f(0)$$

(17) and

$$S_0(f) = f(-X)$$

(18) Denote by $H_0$ this Hopf algebra.

$$H_0 = (\Lambda, \cdot, 1, \Delta_0, \epsilon, S_0)$$

(19) One can give another interpretation of $H_0$. Let

$$G_0 = \{ a \mid a(t) = 1 + a_1 t + a_2 t^2 + \ldots \} = 1 + t \mathbb{C}[t]$$

(20)
be the multiplicative group of formal power series with constant term 1, and let \( H \) be the algebra of polynomial functions on \( G_0 \). Let \( k_n \in H \) be the map defined by
\[
(21) \quad k_n(a) = a^n
\]
The standard coproduct for functions on a group is
\[
(22) \quad \Delta f(a, b) = f(ab)
\]
where \( \Delta f \) is identified with an element of \( H \otimes H \). Then, \( h_n \mapsto k_n \) is an isomorphism of Hopf algebras \( H_0 \to H \).

2.3. Other bases of the algebra of symmetric functions. The formal series
\[
(23) \quad u = tH(t) = t + h_1 t^2 + h_2 t^3 + \ldots
\]
has an inverse for the composition, so that we can rewrite \( t \) in terms of \( u \), as follows.
\[
(24) \quad t = u + h_1^* u^2 + h_2^* u^3 + \ldots
\]
The \( h_k^* \) are homogeneous symmetric functions of degree \( k \), and the algebra morphism \( \psi \) from \( \Lambda \) to \( \Lambda \) defined by \( \psi(h_k) = h_k^* \) is an involution:
\[
(25) \quad \psi^2 = Id_\Lambda
\]
The Lagrange inversion formula shows that \((n + 1)h_n^*\) is the coefficient of \( t^n \) in \( H(t)^{(n+1)} \), so that
\[
(26) \quad h_n^* = \frac{h_n(-(n + 1)X)}{n + 1}
\]
We can define a multiplicative \( Z \)-basis \( (h_\lambda^*) \) of \( \Lambda \), by
\[
(27) \quad h_\lambda^* = h_\lambda^*_1 h_\lambda^*_2 \ldots = \psi(h_\lambda)
\]
In [10, ex. 24 p 35], Macdonald shows that
\[
(28) \quad (n + 1)h_n^* = \sum_{\lambda \vdash n} (-1)^{l(\lambda)} \binom{n + l(\lambda)}{n} u_\lambda h_\lambda
\]
where
\[
(29) \quad u_\lambda = \frac{l(\lambda)}{\prod_{i \geq 1} m_i(\lambda)!}
\]
Let \( (g_\lambda) \) be the adjoint basis of \( (h_\lambda^*) \), that is
\[
(30) \quad \langle g_\lambda, h_\mu^* \rangle = \delta_{\lambda\mu}
\]
One has
\[
(31) \quad g_n = -m_n = -p_n
\]
The algebra morphism \( \psi^* \) (adjoint of \( \psi \)) is also an involution, and it maps \( g_\lambda \) to \( m_\lambda \). The matrix \( \Psi \) of \( \psi \) is strictly upper triangular in the basis \( (h_\lambda) \) and one has
\[
(32) \quad \Psi_{\lambda\lambda} = (-1)^{l(\lambda)}
Theorem 2.1 (Macdonald [10]). The linear map

\[ \Phi: \begin{cases} \mathbb{R} \to \Lambda \to \lambda \mapsto g_\lambda \end{cases} \]

is an isomorphism of algebras.

2.4. Top connection coefficients. Now, let us recall some properties of the top connection coefficients \( a^\nu_{\lambda \mu} \), which are the structure constants of the Farahat-Higman algebra \( R \). By Theorem 2.1,

\[ g_\lambda g_\mu = \sum_{|\nu|=|\lambda|+|\mu|} a^\nu_{\lambda \mu} g_\nu \]

When \( \mu \) and \( \nu \) are partitions of length 1 (\( \mu = (r) \) and \( \nu = (m) \)), Macdonald gives in [10, ex. 24 p 131] an explicit formula for \( a^m_{\lambda(r)} \), \( m = |\lambda| + r \), that is

\[ a^m_{\lambda(r)} = \begin{cases} \frac{(m+1)r!}{(r+1-l(\lambda)) \prod_{i=0}^{l(\lambda)} m_i(\lambda)!} & \text{if } l(\lambda) \leq r + 1 \\ 0 & \text{otherwise} \end{cases} \]

Moreover, Macdonald gives a recurrence formula, for any partition \( \nu \) with \( |\nu| = |\lambda| + r \)

\[ a^\nu_{\lambda(r)} = \sum_{(i, \mu) / \mu \cup \nu = \lambda(r)} a^{(\nu)}_{\mu(r)} \]

One can deduce from these formulas that the \( a^\nu_{\lambda(r)} \) are zero except if \( \nu \geq \lambda \cup (r) \), and that \( a^\nu_{\lambda(r)} > 0 \). The multiplicative structure of the Farahat-Higman algebra is uniquely determined by (35) and (36).

2.5. Jucys-Murphy elements. Let \( n \) and \( i \) be two integers, \( i \leq n \). Define

\[ \xi_i = \sum_{j<i} (j, i) \in \mathbb{C}[S_n], \]

called the \( i \)th Jucys-Murphy element. It is the sum of all transpositions \( (i, j) \) with \( j < i \). The Jucys-Murphy elements do not belong to the center \( Z_n \) of \( \mathbb{C}[S_n] \), but they generate a maximal commutative subalgebra of \( \mathbb{C}[S_n] \). Let

\[ \Xi_n = \{ \xi_1, \xi_2, \ldots, \xi_n \} \]

be the alphabet of Jucys-Murphy elements. It is known that the algebra of symmetric functions in \( \Xi_n \) is exactly \( Z_n \) [8]. Hence, since \( (C_\mu)_{\mu \vdash n} \) is a basis of \( Z_n \), one can rewrite \( f(\Xi_n) \) as

\[ f(\Xi_n) = \sum_{\mu \vdash n} k_{f, \mu(n)} C_\mu \]

for any homogeneous symmetric function \( f \), and this decomposition is unique. We can rewrite this formula in terms of the reduced cycle types

\[ f(\Xi_n) = \sum k_{f, \mu(n)} c_\mu(n). \]
Let \( r \) be the degree of \( f \). When \( |\mu| = r \), \( k_{f,\mu}(n) \) does not depend on \( n \). Let us rewrite it as \( k_{f,\mu} \) and define a new element \( f(\Xi) \) of the Farahat-Higman algebra \( R \), by considering only maximal terms in the expansion of \( f(\Xi_n) \) for \( n \) large enough.

\[
f(\Xi) = \sum_{\mu \vdash r} k_{f,\mu} c_{\mu}
\]

Murray [12] has shown that for any \( f \), the isomorphism \( \Phi \) between \( R \) and \( \Lambda \) maps \( f(\Xi) \) to \( f(-X) = S_0(f) \).

\[
\Phi(f(\Xi)) = f(-X)
\]

Since \( \Phi(c_{\mu}) = g_{\mu} \), using the duality between the bases \( (h_\lambda^*) \) and \( (g_\lambda) \), one has

\[
k_{f,\mu} = \langle f(-X), h_\mu^* \rangle
\]

3. More about Murray’s result


\[
p_m(\Xi_n) = \sum_{k=1}^{m+1} \sum_{\nu \vdash k, l(\nu) \leq m-k+2} \phi_{\kappa,m} a_{\kappa,n}
\]

where \( a_{\kappa,n} \) is defined by

\[
a_{\kappa,n} = \frac{1}{(n-k)!} z_\kappa \cup_1 n \cup_2^{n-k} C_\kappa \cup_1 \cup_2^{n-k}
\]

The \( C_\lambda \) are the classical conjugacy classes, and the \( \phi_{\kappa,m} \) are defined from

\[
\phi_\kappa(t) = \frac{(1 - q^{-1})^{k-1} p_\kappa(q-1)}{k! z_k} \bigg|_{q = e^t}
\]

by

\[
\phi_\kappa(t) = \sum_{m \geq |\kappa|+l(\kappa)-2} \phi_{\kappa,m} \frac{t^m}{m!}.
\]

From (44), \( C_{(m+1)\cup_1\cup_2^{n-k}} \) is the only class of modified weight \( m \) which can give a contribution to \( p_m(\Xi_n) \). Indeed, for \(|\kappa| - l(\kappa) = m \), if we had \( l(\kappa) > 1 \) we would have \(|\kappa| > m + 1 \), but the \( \kappa \) satisfying this inequality do not occur in the sum (44), so that the formula

\[
p_m(\Xi) = \sum_{\mu \vdash m} k_{\mu} c_\mu
\]

becomes

\[
p_m(\Xi) = k_m c_m
\]

Now we only have to determine the coefficient \( k_m \). In order to do that, suppose that \( n = m + 1 \). In this case we have

\[
p_m(\Xi_{m+1}) = \xi_1^m + \xi_2^m + \ldots + \xi_{m+1}^m.
\]
that is,
\begin{equation}
    p_m(\Xi_{m+1}) = 0^m + (1, 2)^m + ((1, 3) + (2, 3))^m + \ldots
    + ((1, m+1) + (2, m+1) + \ldots + (m, m+1) )^m
\end{equation}

Only the last term can give a contribution to \( C_{m+1} \). This term can be rewritten
\begin{equation}
    \sum (i_1, m+1)(i_2, m+1)\ldots(i_m, m+1)
\end{equation}

where the sum is over the \((i_1, i_2, \ldots, i_m)\) where the \(i_k\) are integers such that \(1 \leq i_k \leq m\). In this new sum, only terms with the \(i_k\) all distinct can give a contribution to \( C_{m+1} \). The number of these terms is \( m! \), that is, the cardinality of \( C_{m+1} \), so that the coefficient of \( C_{m+1} \) in \( p_m(\Xi_{m+1}) \) is 1. This coefficient does not depend on \( n \) since \((m+1)\) has a maximal modified weight, so that the coefficient of \( c_m \) in \( p_m(\Xi) \), that is \( k_m \), is again 1, so that
\begin{equation}
    p_m(\Xi) = c_m.
\end{equation}

Hence, \( \Phi(p_m(\Xi)) = \Phi(c_m) = g_m = -p_m \), and
\begin{equation}
    \Phi(p_\lambda(\Xi)) = \Phi(\prod_i p_{\lambda_i}(\Xi)) = \prod_i \Phi(p_{\lambda_i}(\Xi)).
\end{equation}

From that we have \( \Phi(p_\lambda(\Xi)) = \prod_i (-p_{\lambda_i}) = (-1)^{l(\lambda)}p_\lambda = p_\lambda(-X) \), and since \( \Lambda \) is spanned by the \( p_\mu \), this implies (42).

3.2. Examples. Using his result and simple calculations in \( \Lambda \), Murray [12] computes coefficients in the expansion of certain symmetric functions of the Jucys-Murphy elements over the \( c_\lambda \). For example, he shows the following formulas
\begin{equation}
    \langle e_k(-X), h_\lambda^* \rangle = 1
\end{equation}
\begin{equation}
    \langle h_k(-X), h_\lambda^* \rangle = \prod_i \text{Cat}_{\lambda, i-1}
\end{equation}

(where \( \text{Cat}_i \) is the \( i \)th Catalan number) giving the top coefficients in the expansion of \( e_k(\Xi) \) and \( h_k(\Xi) \). In the same vein, let us give a new proof of a result of Matsumoto and Novak for the monomial functions. Let \( k \) be an integer, \( \lambda \vdash k \) and \( \mu \vdash k \). We denote by \( L_\mu^\lambda \) the coefficient defined by
\begin{equation}
    m_\lambda(\Xi) = \sum_{|\mu|=|\lambda|} L_\mu^\lambda c_\mu
\end{equation}

Matsumoto and Novak [11] show that
\begin{equation}
    L_\mu^\lambda = \sum_{(\lambda^{(1)}, \lambda^{(2)}, \ldots) \in \mathfrak{M}(\lambda, \mu)} RC(\lambda^{(1)})RC(\lambda^{(2)})\ldots
\end{equation}

where
\begin{equation}
    \mathfrak{M}(\lambda, \mu) = \{(\lambda^{(1)}, \lambda^{(2)}, \ldots)/\forall i, \lambda^{(i)} \vdash \mu_i \text{ and } \lambda = \lambda^{(1)} \cup \lambda^{(2)} \cup \ldots\}
\end{equation}
and for any partition $\lambda$ of $N$,
\begin{equation}
RC(\lambda) = \frac{1}{N+1} \sum_{\lambda} m_\lambda (N+1) = \frac{|\lambda|!}{(|\lambda|-l(\lambda)+1)!} \prod_{i \geq 1} m_\lambda(i)!
\end{equation}
Matsumoto and Novak also give a combinatorial interpretation for $RC(\lambda)$. Let us show how to derive (58) from (43). We have
\begin{equation}
L^\lambda_\mu = \langle m_\lambda(-X), h^*_\mu \rangle
\end{equation}
\begin{align*}
&= \langle m_\lambda, h^*_\mu(-X) \rangle \\
&= \left\langle \frac{h_{\mu_1}((\mu_1 + 1)X)h_{\mu_2}((\mu_2 + 1)X)\ldots}{(\mu_1 + 1)(\mu_2 + 1)\ldots} \right\rangle \\
&= \frac{1}{\prod_i (\mu_i + 1)} \left\langle m_\lambda, \prod_i \sum_{\lambda^{(i)} \in \mathcal{R}(\lambda, \mu)} m_\lambda^{(i)}(\mu_i + 1)h_{\lambda^{(i)}} \right\rangle
\end{align*}
Expanding the right factor of the scalar product, we obtain nonzero terms only if $h_{\lambda^{(1)}}h_{\lambda^{(2)}}\ldots = h_\lambda$, since otherwise one has $\langle m_\lambda, h_{\lambda^{(1)}}h_{\lambda^{(2)}}\ldots \rangle = 0$. Hence,
\begin{equation}
L^\lambda_\mu = \frac{1}{\prod_i (\mu_i + 1)} \sum (m_\lambda^{(i)}(\mu_1 + 1))(m_\lambda^{(i)}(\mu_2 + 1))\ldots (m_\lambda, h_\lambda)
\end{equation}
The sum is over the $(\lambda^{(1)}, \lambda^{(2)}, \ldots)$ such that $\lambda^{(i)} \vdash \mu_i$ for all $i$ and $\bigcup_i \lambda^{(i)} = \lambda$, that is, over the set $\mathcal{R}(\lambda, \mu)$. Moreover, one has $\langle m_\lambda, h_\lambda \rangle = 1$, so that
\begin{equation}
L^\lambda_\mu = \sum_{(\lambda^{(1)}, \lambda^{(2)}, \ldots) \in \mathcal{R}(\lambda, \mu)} \frac{m_\lambda^{(1)}(\mu_1 + 1) m_\lambda^{(2)}(\mu_2 + 1)}{\mu_1 + 1} \frac{m_\lambda^{(3)}(\mu_3 + 1)}{\mu_2 + 1} \ldots
\end{equation}
This is the result of Matsumoto and Novak.

3.3. Coefficient of the cycle with maximal length in $p_\lambda(\Xi_n)$. We call modified cycle type of a product of cycles the sequence $(l_1 - 1, l_2 - 1, \ldots)$, where the $l_i$ are the lengths of the factors. For example, the product $(13)(234)$ has modified cycle type $(2 - 1, 3 - 1) = (1, 2)$. Biane [1] obtains an explicit formula for the number $\alpha_\lambda$ of factorisations with modified cycle type $\lambda$ for a cycle of length $n + 1$, where $n = |\lambda| = \sum_i \lambda_i$, that is
\begin{equation}
\alpha_\lambda = (n + 1)^{|\lambda| - 1}
\end{equation}
Let us prove this with symmetric functions. We consider in the Farahat-Higman algebra $R$ the product
\begin{equation}
c^\lambda = c_{\lambda_1}c_{\lambda_2}\ldots
\end{equation}
Since $R$ is isomorphic to $\Lambda$, it is commutative, so that the order of the elements of $\lambda$ has no importance. Hence, we can assume that $\lambda$ is a partition. One has
\begin{equation}
\Phi(c^\lambda) = g_{\lambda_1}g_{\lambda_2}\ldots = (-p_{\lambda_1})(-p_{\lambda_2})\ldots = (-1)^{|\lambda|} p_\lambda = p_\lambda(-X)
\end{equation}
Hence, from (42),
\begin{equation}
c^\lambda = p_\lambda(\Xi)
\end{equation}
Moreover, $c^{\lambda}$ is the sum of all cycle products of modified type $\lambda$, and $c_n$ is the sum of all the cycles of length $n + 1$. Hence, the total number of cycle products of modified type $\lambda$ is $k^{\lambda}_n \text{card}(c_n)$, where $k^{\lambda}_n$ is the coefficient of $c_n$ in $c^{\lambda}$. The number $\alpha_\lambda$ of these factorisations does not depend on the choice of the cycle, so that $\alpha_\lambda = k^{\lambda}_n \text{card}(c_n) = k^{\lambda}_n$, which is also from (67) the coefficient of $c_n$ in $p_\lambda(\Xi)$. Hence, one has

$$\alpha_\lambda = (-1)^{(\lambda)} \langle p_\lambda, h_n^* \rangle,$$

so that

$$\alpha_\lambda = \frac{1}{n + 1} (n + 1)^{(\lambda)} \langle p_\lambda, h_n \rangle = (n + 1)^{(\lambda) - 1} \langle p_\lambda, h_n \rangle$$

(since the bases $(m_\lambda)$ and $(h_\lambda)$ are dual to each other, $\langle p_\lambda, h_n \rangle$ is the coefficient of $m_n$ in $p_\lambda$, that is, 1).

3.4. **Generalization.** Here, we give a combinatorial interpretation for the coefficient of $c_\mu$ in $p_\lambda(\Xi)$. Now we have

$$\langle p_\lambda, h_n^* \rangle = (-1)^{(\lambda)} (n + 1)^{(\lambda) - 1}$$

On another hand, since $p_\lambda(\Xi) = c_{\lambda_1} c_{\lambda_2} \ldots$ in $R$,

$$c_{\lambda_1} c_{\lambda_2} \ldots = \sum_{|\lambda| = |\mu|} \langle p_\lambda(-X), h_\mu^* \rangle c_\mu,$$

so that $\langle p_\lambda(-X), h_\mu^* \rangle$ corresponds to the total number of decompositions in an arbitrary product of modified type $\lambda$ of a product of disjoint cycles with modified type $\mu$. In such a decomposition, each $c_\mu$ comes from certain $\prod_i c_{\lambda(i)}$, where $\lambda(i) \vdash \mu_i$ is a subpartition of $\lambda$, with $\lambda(1) \cup \lambda(2) \cup \ldots = \lambda$, that is, $(\lambda(1), \lambda(2), \ldots) \in \mathcal{R}(\lambda, \mu)$. Moreover, each $(\lambda(1), \lambda(2), \ldots)$ must be counted $m(\lambda(1), \lambda(2), \ldots)$ times, where

$$m(\lambda(1), \lambda(2), \ldots) = \prod_{j \geq 1} \frac{m_j(\lambda)!}{m_j(\lambda(1))! m_j(\lambda(2))! \ldots}$$

For a given $c_\mu$, the number of decompositions of $c_\mu$ of a certain type $\prod_i c_{\lambda(i)}$ corresponds to the coefficient of $c_\mu$ in $p_{\lambda(i)}(\Xi)$, that is $\langle p_{\lambda(i)}(-X), h_\mu^* \rangle$, so that we have from (70)

$$\langle p_\lambda(-X), h_\mu^* \rangle = \sum_{(\lambda(1), \lambda(2), \ldots) \in \mathcal{R}(\lambda, \mu)} m(\lambda(1), \lambda(2), \ldots) \prod_i (\mu_i + 1)^{(\lambda(i)} - 1$$

Note that it is nonzero only if $\lambda$ is a refinement of $\mu$. Finally, one has

$$p_\lambda(\Xi) = \sum_{|\lambda| = |\mu|} \left( \sum_{(\lambda(1), \lambda(2), \ldots) \in \mathcal{R}(\lambda, \mu)} \left( \prod_j \frac{m_j(\lambda)!}{m_j(\lambda(1))! m_j(\lambda(2))! \ldots} \right) \left( \prod_i (\mu_i + 1)^{(\lambda(i)} - 1 \right) \right) c_\mu$$

Note that from (70) we have

$$h_n^* = \sum_{\mu \vdash n} (-1)^{(\mu)} (n + 1)^{(\mu) - 1} \frac{p_\mu}{z_\mu}$$
Expanding $h^*_\mu$ by means of this expression, since we have for $\bigcup i, \lambda(i) = \lambda$

\begin{equation}
\prod_{j \geq 1} \frac{m_j(\lambda)!}{m_j(\lambda(1))!m_j(\lambda(2))! \ldots} = \frac{z_\lambda}{z_{\lambda(1)}z_{\lambda(2)} \ldots}
\end{equation}

we see that (73) can also be obtained by simple calculations in $\Lambda$.

4. The Faà di Bruno algebra and its deformation

4.1. The Faà di Bruno algebra. There is another coproduct on $\Lambda$, denoted here by $\Delta_1$ and defined by

\begin{equation}
\Delta_1 h_n = \sum_{k=0}^{n} h_k(X) \otimes h_{n-k}((k + 1)X)
\end{equation}

or equivalently

\begin{equation}
\Delta_1 h_n = \sum_{k=0}^{n} \sum_{\mu \vdash n-k} m_{\mu}(k + 1) h_k \otimes h_\mu
\end{equation}

This coproduct defines a Hopf algebra with the counit $\epsilon$ as in $H_0$, and the antipode $S_1 = \psi$, that is, the involution mapping $h_\lambda$ to $h^*_\lambda$. The Hopf algebra

\begin{equation}
H_1 = (\Lambda, 1, \Delta_1, \epsilon, S_1)
\end{equation}

is called the Faà di Bruno algebra. It has the following interpretation. Let

\begin{equation}
G_1 = \{\alpha \mid \exists a \in G_0, \forall t, \alpha(t) = ta(t)\} = tG_0 = t + t^2C[[t]]
\end{equation}

be the group of formal diffeomorphisms of the line tangent to the identity, and let again $k_n$ be the linear form

\begin{equation}
k_n : \alpha(t) = t + a_1t^2 + a_2t^3 + \ldots \mapsto a_n
\end{equation}

Let $\Delta$ be the coproduct such that $\Delta k_n(\alpha, \beta)$ is the coefficient of $t^{n+1}$ in

\begin{equation}
(\alpha \circ \beta)(t) = \alpha(\beta(t))
\end{equation}

The bialgebra $F$ defined by this coproduct is isomorphic to $H_1$ under the correspondence $k_n \mapsto h_n$.

4.2. A deformation of the Faà di Bruno algebra. Let $\gamma$ be a real parameter in $[0, 1]$, and $\Delta_\gamma$ be the coproduct on $\Lambda$ defined by

\begin{equation}
\Delta_\gamma(h_n) = \sum_{k=0}^{n} h_k \otimes h_{n-k}((k\gamma + 1)X)
\end{equation}

or equivalently

\begin{equation}
\Delta_\gamma(h_n) = \sum_{k=0}^{n} \sum_{\mu \vdash n-k} m_{\mu}(k\gamma + 1) h_k \otimes h_\mu
\end{equation}

Foissy obtains this coproduct in [5] in his investigation of formal Dyson-Schwinger equations in the Connes-Kreimer Hopf algebra, and shows that for $\gamma \in [0, 1]$, the
resulting bialgebras $\mathcal{H}_\gamma$ are Hopf algebras, isomorphic to the Faà di Bruno algebra. Obviously, $\Delta_0$ corresponds to $\mathcal{H}_0$.

4.3. **Graded dual of the deformed Faà di Bruno algebra.** Consider now the Hopf algebra $\mathcal{H}'_\gamma$, the graded dual of the deformation $\mathcal{H}_\gamma$ considered in section 3. Denote by $\ast_\gamma$ the product on $\mathcal{H}'_\gamma$. For any $f \in \mathcal{H}'_\gamma$ and any symmetric function $g$ in $\mathcal{H}_\gamma$, denote by $\langle f, g \rangle$ the action of $f$ on $g$. One has for any $f \in \mathcal{H}'_\gamma$, $g \in \mathcal{H}'_\gamma$ and $h \in \Lambda$, by definition of a dual Hopf algebra,

$$\langle f \ast_\gamma g, h \rangle = \langle f \otimes g, \Delta_\gamma(h) \rangle \tag{85}$$

where

$$\langle a \otimes b, c \otimes d \rangle = \langle a, c \rangle \langle b, d \rangle \tag{86}$$

Denote by $(d_\lambda)$, $(q_\lambda)$ and $(b_\lambda)$ the bases respectively dual to $(h_\lambda)$, $(p_\lambda z_\lambda)$ and $h'_\lambda$ in the sense of $\mathcal{H}'_\gamma$, that is, for example

$$\langle d_\lambda, h_\mu \rangle = \delta_{\lambda\mu} \tag{87}$$

For any $\gamma$ in $[0, 1]$,

$$q_n = d_n = -b_n \tag{88}$$

Note that these $b_n$ generate $\mathcal{H}'_\gamma$. $\mathcal{H}_0$ is self-dual, so that in the case $\gamma = 0$,

$$d_\lambda = m_\lambda, \quad q_\lambda = p_\lambda, \quad b_\lambda = g_\lambda \tag{89}$$

When $\gamma \neq 0$, $\mathcal{H}_\gamma$ is not self-dual and not commutative, since the coproduct $\Delta_\gamma$ on $\Lambda$ is not cocommutative.

4.4. **Multiplicative structure of $\mathcal{H}'_\gamma$.** Now, let $f$ and $g$ be two elements of $\mathcal{H}'_\gamma$. One has for any partition $\mu$,

$$\langle f \ast_\gamma g, h_\mu \rangle = \langle f \otimes g, \Delta_\gamma(h_\mu) \rangle$$

$$= \langle f \otimes g, \Delta_\gamma(h_{\mu_1})\Delta_\gamma(h_{\mu_2}) \ldots \rangle$$

$$= \left\langle f \otimes g, \prod_i \sum_{k_i+I_i=\mu_i} h_{k_i} \otimes h_{I_i}(\gamma k_i + 1)X) \right\rangle$$

$$= \left\langle f \otimes g, \sum_{k_i+I_i=\mu_i} \prod_i h_{k_i} \otimes h_{I_i}(\gamma k_i + 1)X \right\rangle$$

$$= \sum \langle f \otimes g, \prod_i h_{k_i} \otimes h_{I_i}(\gamma k_i + 1)X \rangle \tag{90}$$

where the parameters of the sum are the same as above. Hence,

$$\langle f \ast_\gamma g, h_\mu \rangle = \sum \langle f, h_{(k_1,k_2,\ldots)} \rangle \langle g, \prod_i h_{I_i}(\gamma k_i + 1)X \rangle \tag{91}$$

$$= \sum \langle f, h_{(k_1,k_2,\ldots)} \rangle \langle g, \prod_i \sum_{\rho+I_i} m_\rho(\gamma k_i + 1)h_\rho \rangle$$
Now let us change the parameters of the sum, considering the partitions \( \rho_i \), with \( |\rho_i| = l_i \) and \( k_i = \mu_i - |\rho_i| \). The sum is now over the \((\rho_1,\rho_2,\ldots,\rho)\), such that for all \( i \), \( |\rho_i| \leq \mu_i \), and \( \rho \) is the union of the \( \rho_i \). We obtain

\[
\langle f \ast \gamma g, h_\mu \rangle = \sum \langle f, h_{(\mu_1-|\rho_1|,\mu_2-|\rho_2|,\ldots)} \rangle \langle g, \prod_i m_{\rho_i} (\gamma \mu_i - |\rho_i| + 1) h_{\rho_i} \rangle,
\]

and finally,

\[
\langle f \ast \gamma g, h_\mu \rangle = \sum \prod_i m_{\rho_i} (\gamma \mu_i - |\rho_i| + 1) \langle f, h_{(\mu_1-|\rho_1|,\mu_2-|\rho_2|,\ldots)} \rangle \langle g, h_\rho \rangle.
\]

One can use this formula to expand \( f \ast \gamma g \) on the \( d_\mu \), where \( f \) and \( g \) are any two elements in \( H'_{\gamma} \).

### 4.5. Action of \( q_n^\perp \) on the \( h_\mu \)

We define an operator \( q_n^\perp \) on \( \Lambda \) as follows, for \( f \in H'_{\gamma} \) and \( g \in H_{\gamma} \).

\[
\langle f \ast \gamma q_n, g \rangle = \langle f, q_n^\perp g \rangle.
\]

The operator \( q_n^\perp \) is a \textit{derivation}, because it is the adjoint of the right multiplication by a \textit{primitive element}:

\[
q_n^\perp (fg) = f q_n^\perp (g) + q_n^\perp (f) g.
\]

We shall need its action on the \( h_\mu \). Let \( n > 0 \) and \( \mu \) be a partition. We can write from (93):\n
\[
\langle f \ast \gamma q_n, h_\mu \rangle = \sum \prod_i m_{\rho_i} (\gamma \mu_i - |\rho_i| + 1) \langle f, h_{(\mu_1-|\rho_1|,\mu_2-|\rho_2|,\ldots)} \rangle \langle q_n, h_\rho \rangle.
\]

Since \( q_n = d_n \) for all \( n \), one has \( \langle q_n, h_\rho \rangle = \delta_{n\rho} \), so that a term in this sum gives a nonzero contribution only if

\[
\rho = (n)
\]

In this case we have also \( \langle q_n, h_\rho \rangle = 1 \), and

\[
\langle f \ast \gamma q_n, h_\mu \rangle = \sum_{i/\mu_i \geq n} m_n (\gamma \mu_i - \gamma n + 1) \langle f, h_{(\mu_1-\mu_i,\mu_i,\ldots)} \rangle
\]

Since \( m_n (\gamma \mu_i - \gamma n + 1) = \gamma (\mu_i - n) + 1 \), one can deduce

\[
q_n^\perp h_\mu = \sum_{i/\mu_i \geq n} (\gamma \mu_i - \gamma n + 1) h_{(\mu_1-\mu_i,\mu_i,\ldots)}
\]

When \( \mu \) consists only in one part \( N \), this formula can be rewritten

\[
q_n^\perp h_N = (\gamma N - \gamma n + 1) h_{N-n}
\]

Since \( q_n^\perp \) is a derivation, one has its action on the multiplicative basis \((h_\mu)\), and one can use (100) to fully explicit it. We have \( q_n^\perp = D_n + E_n \), where \( D_n \) and \( E_n \) are the derivations defined by

\[
D_n h_n = h_{N-n}
\]

and

\[
E_n h_n = h_{N-n}
\]
and
\begin{equation}
E_n h_n = \gamma (N - n) h_{N-n},
\end{equation}
that is,
\begin{equation}
D_n = \sum_{r \geq 0} h_r \frac{\partial}{\partial h_{n+r}}
\end{equation}
and
\begin{equation}
E_n = \gamma \sum_{r \geq 0} r h_r \frac{\partial}{\partial h_{n+r}}
\end{equation}
so that
\begin{equation}
q_n^\perp = \sum_{r \geq 0} (1 + \gamma r) h_r \frac{\partial}{\partial h_{n+r}}
\end{equation}
We can see that this is valid also in the degenerate case \( \gamma = 0 \).

4.6. **Action of** \( q_n^\perp \) **on the** \( p_\mu \). From (100), we have
\begin{equation}
q_n^\perp h_N = p_n^\perp h_N + \gamma (N - n) h_{N-n}
\end{equation}
On another hand,
\begin{equation}
(N - n) h_{N-n} = p_1 h_{N-n-1} + p_2 h_{N-n-2} + p_3 h_{N-n-3} + \ldots
\end{equation}
Then,
\begin{equation}
q_n^\perp h_N = p_n^\perp h_N + \gamma p_1 h_{N-n-1} + \gamma p_2 h_{N-n-2} + \ldots
= p_n^\perp h_N + \gamma (p_1 h_{n+1} + p_2 (p_{n+2} h_N) + \ldots
= p_n^\perp h_N + \gamma \sum_{r > n} p_{r-n} p_r^\perp h_N
\end{equation}
so that
\begin{equation}
q_n^\perp = p_n^\perp + \gamma \sum_{r > n} p_{r-n} p_r^\perp
\end{equation}

5. **A deformation of the Farahat-Higman algebra**

5.1. **Recurrences for the structure constants of** \( \mathcal{H}'_\gamma \). Denote by \( a_{\lambda,\mu}^\kappa (\gamma) \) the structure constants in the basis \( b_\mu \).
\begin{equation}
b_\lambda \star_\gamma b_\mu = \sum_\nu a_{\lambda,\mu}^\kappa (\gamma) b_\nu
\end{equation}
When \( \gamma = 0 \), the \( b_\mu \) are identified with the \( g_\mu \), and then \( a_{\lambda,\mu}^\kappa (0) \) coincides with \( a_{\lambda,\mu}^\kappa \), the top connection coefficient.

Since \( b_n = -q_n \), one has
\begin{equation}
a_{\lambda,(n)}^\kappa (\gamma) = \langle b_\lambda \star_\gamma b_n, h_\nu^* \rangle = -\langle b_\lambda \star_\gamma q_n, h_\nu^* \rangle
\end{equation}
Hence,
\begin{equation}
(112) \quad a_{\lambda,(n)}^\nu(\gamma) = -\langle b_\lambda, q_n^+ h_n^* \rangle
\end{equation}
Since $q_n^+$ is a derivation, one can rewrite this as
\begin{equation}
(113) \quad a_{\lambda,(n)}^\nu(\gamma) = -\sum_i \langle b_\lambda, h_{\nu \setminus \nu_i}^* q_n^+ (h_{\nu_i}^*) \rangle
\end{equation}
The $i$th term in this sum corresponds to the coefficient of $h_{\nu_i}^*$ in
\begin{equation}
(114) \quad h_{\nu \setminus \nu_i}^* q_n^+ (h_{\nu_i}^*)
\end{equation}
Hence this term gives a nonegative contribution only if there exists a partition $\mu$ such that
\begin{equation}
(115) \quad (\nu \setminus (\nu_i)) \cup \mu = \lambda,
\end{equation}
that is,
\begin{equation}
(116) \quad \mu \cup \nu = \lambda \cup (\nu_i)
\end{equation}
Hence, the $i$th term is also equal to the coefficient of $h_{\mu}^*$ in $q_n^+ (h_{\nu_i}^*)$, that is, $a_{\mu,(n)}^\nu(\gamma)$.

Summarizing, we have proved:

**Theorem 5.1.** The structure constants $a_{\lambda,(n)}^\nu(\gamma)$ satisfy the recursion
\begin{equation}
(117) \quad a_{\lambda,(n)}^\nu(\gamma) = \sum a_{\mu,(n)}^\nu_i(\gamma)
\end{equation}
where the sum is over the $(i, \mu)$ such that
\begin{equation}
(118) \quad \mu \cup \nu = \lambda \cup (\nu_i)
\end{equation}
This formula is a generalization of (36), which is recovered for $\gamma = 0$.

**5.2. Multiplicative structure of the deformed Farahat-Higman algebra.** In the case where $\nu$ has only one part, there is a closed formula for $a_{\lambda,(r)}^\nu(\gamma)$.

**Theorem 5.2.** For $r \in \mathbb{N}$, $N \in \mathbb{N}$ and $\mu$ a partition,
\begin{equation}
(119) \quad a_{\lambda,(r)}^{(N)}(\gamma) = \left(1 - \frac{N - r}{N + 1}\right) a_{\lambda,(r)}^{(N)}(0)
\end{equation}
$a_{\lambda,(r)}^{(N)}(0)$ corresponds to $a_{\lambda,(r)}^{(N)}$ in formula (35), where $m$ is replaced by $N$.

Together with the recurrence formula (117), this determines completely the multiplicative structure of $\mathcal{H}_\gamma^r$, since it is generated by the $b_n$. In order to derive (119), we shall need the following lemmas.

**Lemma 5.3.** Let $r$ and $n$ be two positive integers, with $r < n$. Then,
\begin{equation}
(120) \quad p_r^+ h_n^* = \sum_{\rho \vdash n - r} (-1)^{|\rho|-1}(n + 1)^{|\rho|} \frac{p_\rho}{z_\rho}
\end{equation}
Proof. From (75) we have

(121) \[ p^\bot_r h_n^* = \sum_{\mu \vdash n} \frac{(-1)^l(\mu)(n+1)^{l(\mu)-1}}{z_\mu} p^\bot_r(p_\mu) \]

The action of \( p^\bot_r \) on \( p_\mu \) is

(122) \[ p^\bot_r(p_\mu) = r \frac{\partial p_\mu}{\partial p_r} = rm_r(\mu)p_{\mu \setminus (r)} \]

On another hand,

(123) \[ z_{\mu \setminus (r)} = \frac{z_\mu m_r(\mu \setminus (r))}{rm_r(\mu)!} \]

with \( m_r(\mu \setminus (r)) = m_r(\mu) - 1 \), hence \( \frac{z_\mu}{rm_r(\mu)} = z_{\mu \setminus (r)} \), so that

(124) \[ \frac{z_\mu}{rm_r(\mu)} = z_{\mu \setminus (r)} \]

Hence,

(125) \[ \frac{p^\bot_r(p_\mu)}{z_\mu} = \frac{p_{\mu \setminus (r)}}{z_{\mu \setminus (r)}} \]

and

(126) \[ p^\bot_r h_n^* = \sum_{\mu \vdash n} (-1)^l(\mu)(n+1)^{l(\mu)-1} \frac{p_{\mu \setminus (r)}}{z_{\mu \setminus (r)}} p_\mu \]

or, equivalently,

(127) \[ p^\bot_r h_n^* = \sum_{\rho \vdash n-r} (-1)^l(\rho)(n+1)^{l(\rho)-1} \frac{P_\rho}{z_\rho} \]

Since \( l(\rho \cup (r)) = l(\rho) - 1 \), we deduce the required formula.

Lemma 5.4. Let \( r \) and \( N \) be two positive integers, with \( N > r \). Then,

(128) \[ \sum_{n > r} p_{n-r}(p^\bot_r h_N^*) = -\frac{N-r}{N+1} p^\bot_r h_N^* \]

Proof. From (120), one has

(129) \[ \sum_{n > r} p_{n-r}(p^\bot_r h_N^*) = \sum_{n > r} \sum_{\rho \vdash n-r} (-1)^l(\rho)(n+1)^{l(\rho)-1} \frac{P_\rho}{z_\rho} \]

for \( \mu = \rho \cup (n-r) \) and \( k = n-r \), we obtain:

(130) \[ \sum_{n > r} p_{n-r}(p^\bot_r h_N^*) = \sum_{\mu \vdash n-r} \sum_{k \in \mu} (-1)^l(\mu)(N+1)^{l(\mu)-1} \frac{P_\mu}{z_{\mu \setminus (k)}} \]

(131) \[ \sum_{n > r} p_{n-r}(p^\bot_r h_N^*) = \sum_{\mu \vdash n-r} (-1)^l(\mu)(N+1)^{l(\mu)-1} \left( \sum_{k \in \mu} \frac{1}{z_{\mu \setminus (k)}} \right) P_\mu \]
From (124), one has

\[ \sum_{k \in \mu} \frac{1}{z_{\mu}(k)} = \sum_{k \in \mu} \frac{km_{\mu}(k)}{z_{\mu}} \]

Hence,

\[ \sum_{k \in \mu} \frac{1}{z_{\mu}(k)} = \frac{1}{z_{\mu}} \sum_{k \in \mu} km_{\mu}(k) = \frac{\vert \mu \vert}{z_{\mu}} \]

So we can rewrite (131) as

\[ \sum_{n \in \nu} p_{n-r}^{L}(p_{n}^{1}h_{N}^{*}) = \sum_{\mu \in N-r} (-1)^{(\mu)}(N + 1)^{(\mu)} - N - r \frac{z_{\mu}}{p_{\mu}} \]

\[ \sum_{n \in \nu} p_{n-r}^{L}(p_{n}^{1}h_{N}^{*}) = \frac{N - r}{N + 1} \sum_{\mu \in N-r} (-1)^{(\mu)}(N + 1)^{(\mu)} \frac{z_{\mu}}{p_{\mu}} \]

From (120), we get the required formula.

\[ \text{Proof – (of Theorem 5.2)} \]

Since \( b_{r} = -q_{r} \) and \( g_{r} = -p_{r} \), one has

\[ \langle b_{\lambda} *_{\gamma} b_{r}, h_{N}^{*} \rangle = -\langle b_{\lambda}, q_{r}^{+}h_{N}^{*} \rangle \]

Then, from (109) and lemma 5.3,

\[ \langle b_{\lambda} *_{\gamma} b_{r}, h_{N}^{*} \rangle = -\langle b_{\lambda}, q_{r}^{+}h_{N}^{*} \rangle \sum_{n \in \nu} p_{n-r}^{L}(p_{n}^{1}h_{N}^{*}) \]

\[ \langle b_{\lambda} *_{\gamma} b_{r}, h_{N}^{*} \rangle = -\langle b_{\lambda}, q_{r}^{+}h_{N}^{*} \rangle - \gamma \langle b_{\lambda}, \sum_{n \in \nu} p_{n-r}^{L}(p_{n}^{1}h_{N}^{*}) \rangle \]

\[ \langle b_{\lambda} *_{\gamma} b_{r}, h_{N}^{*} \rangle = -\langle b_{\lambda}, q_{r}^{+}h_{N}^{*} \rangle + \gamma \frac{N - r}{N + 1} \langle b_{\lambda}, p_{r}^{+}h_{N}^{*} \rangle \]

\[ \langle g_{\lambda} b_{r}, h_{N}^{*} \rangle = -\langle b_{\lambda}, q_{r}^{+}h_{N}^{*} \rangle + \gamma \frac{N - r}{N + 1} \langle g_{\lambda}, p_{r}^{+}h_{N}^{*} \rangle \]

\[ \langle g_{\lambda} b_{r}, h_{N}^{*} \rangle = -\langle b_{\lambda}, q_{r}^{+}h_{N}^{*} \rangle + \gamma \frac{N - r}{N + 1} \langle g_{\lambda}, p_{r}^{+}h_{N}^{*} \rangle - \gamma \frac{N - r}{N + 1} \langle g_{\lambda} p_{r}, h_{N}^{*} \rangle \]

\[ \langle g_{\lambda} b_{r}, h_{N}^{*} \rangle = \left( 1 - \gamma \frac{N - r}{N + 1} \right) \langle g_{\lambda} b_{r}, h_{N}^{*} \rangle \]

From that, we deduce (119).

6. A deformation of the Witt algebra

6.1. A simpler multiplicative formula in \( \mathcal{H}_{\gamma}' \). Now let us expand \( q_{k} *_{\gamma} q_{n} \) on the \( q_{\mu} \). One has

\[ q_{k} *_{\gamma} q_{n} = \sum_{\mu} \frac{1}{z_{\mu}} \langle q_{k} *_{\gamma} q_{n}, p_{\mu} \rangle q_{\mu} \]

so that

\[ q_{k} *_{\gamma} q_{n} = \sum_{\mu} \frac{1}{z_{\mu}} \langle q_{k}, q_{n}^{+}(p_{\mu}) \rangle q_{\mu} \]
From (109), deduce that

\[
q_n^\perp p_\mu = p_n^\perp p_\mu + \gamma \sum_{r > n} p_{r-n}^\perp p_\mu
\]

so that

\[
q_n^\perp p_\mu = p_n^\perp p_\mu + \gamma \sum_{r > n} r p_{r-n} \frac{\partial p_\mu}{\partial p_r}
\]

\[
= p_n^\perp p_\mu + \gamma \sum_{r > n} r m_r(\mu)p_{r\setminus(r-n)}
\]

Hence, \(q_\mu\) gives to \(q_\kappa \ast \gamma q_n\) a contribution to the factor of \(\gamma\) only if \(\mu = (k + n)\). In this case we have

\[
\langle q_k, q_n^\perp p_{k+n} \rangle = \gamma (k + n)k
\]

Hence,

\[
q_k \ast \gamma q_n = q_{(k,n)} + k\gamma q_{k+n}
\]

this formula also completely determines the multiplicative structure of \(\mathcal{H}'_\gamma\). In the case \(\gamma = 1\), we can see that it is coherent with the result of [3].

6.2. Lie algebra structure corresponding to \(\mathcal{H}'_\gamma\). Now, suppose that \(\gamma \neq 0\). Since \(\mathcal{H}'_\gamma\) is a connected cocommutative Hopf algebra, it is the universal enveloping algebra of a Lie algebra \(\mathcal{L}_\gamma\). From (143), its bracket is determined by

\[
[q_k, q_n]_\gamma = \gamma (k - n)q_{k+n}.
\]

Denote by \(d_{n,\gamma}\) the differential operator

\[
d_{n,\gamma} = t^{1-n\gamma} \frac{d}{dt}
\]

The \(d_{n,\gamma}\) satisfy the same relation (144) as the \(q_n\) of \(\mathcal{L}_\gamma\), so that \(\mathcal{H}'_\gamma\) can be interpreted as a Lie algebra of differential operators: it is the Lie algebra generated by the \(d_{n,\gamma}\). In the case \(\gamma = 1\) one has \(d_{n,1} = t^{1-n} \frac{d}{dt}\). These operators generate \(\mathcal{L}_1\), that is called the Witt algebra. It is known that the universal enveloping algebra of the Witt algebra is the dual of the Fàa di Bruno algebra. Note also that from the commutativity of \(\mathcal{H}'_0\), (144) is also valid in the degenerate case \(\gamma = 0\).

References


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