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Linear Theory of Shells Taking into Account Surface Stresses

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Recently, the theory of elasticity with surface stresses \cite{1} was developed for nanomechanical problems. Within this theory, the surface stresses $\tau$ act on the body boundary or its portion alongside with the stress tensor $\sigma$ defined in its volume $V$ and on its surface $\Omega$. The tensor $\tau$ generalizes the scalar surface tension known in hydromechanics in the case of solids. The introduction of surface stresses makes it possible to describe, in particular, the size effect characteristic for nanomaterials.

Mathematical investigations of surface stresses in solids and liquids go back to the works of Laplace, Young, Gibbs, etc. (see, for example, review \cite{2}). The foundations of mechanics of solids with consideration of the surface stresses were developed in \cite{3,4,5}, etc. From the mechanical viewpoint, the model of surface stresses proposed in \cite{3} is equivalent to that for a deformable body with an elastic membrane glued on its surface, and the tensor $\tau$ can be considered as the stress resultants tensor operating in this membrane. Model \cite{3} is generalized in \cite{5} to the case of taking into account the bending stiffness of the body surface and actually reduced to the presence of a shell on the body surface described by the equations of a theory of the Kirchhoff–Love type.

As applied to mechanical problems of objects such as nanodimensional beams, plates, and shells, the continuum approaches based on the Kirchhoff–Love, Reissner–Mindlin, etc., models of plates and shells are widely used in the literature. In this case, a substantial element of the theory is the modification of constitutive equations with taking into account the behavior features of nanomaterials. In particular, the problems of mechanics of nanodimensional plates with taking into account the surface stresses were discussed in \cite{6,7,8,9,10,11,12}.

In this work, generalizing the results \cite{12} to the case of the linear theory of elastic shells taking into account the transverse shear, we obtained the equations of equilibrium and constitutive equations for the stress resultants and couples tensors while taking into account the surface stresses acting on the shell surfaces. The effective shell stiffness, in particular, the bending stiffness $D_{\text{eff}}$, depends here also on the surface elastic moduli, which is substantial for nanodimensional thicknesses.

I. For the formulation of boundary-value problems of the theory of elasticity with surface stresses, we use the variational method. Let the elastic body occupy the region $V \in \mathbb{R}^3$ bounded by a smooth surface $\Omega$. Without restriction of generality, we assume that the displacement–vector field $u$ is zero at the boundary portion $\Omega_1 \subset \Omega$:

$$u_{|\Omega_1} = 0,$$

and, at the remaining boundary portion $\Omega_2 = \Omega \setminus \Omega_1$, the external loads $\varphi$ are set, and the surface stresses $\tau$ act.

Further, for simplicity, we restrict ourselves to the consideration of an isotropic body. The strain energy density of an elastic body is given by the formula

$$W = W(\varepsilon) = \frac{1}{2} \lambda \text{tr}^2 \varepsilon + \mu \varepsilon : \varepsilon,$$

$$\varepsilon = \varepsilon(u) = \frac{1}{2} (V u + V u^T),$$

where $\varepsilon$ is the strain tensor, $\lambda$ and $\mu$ are the Lamé constants, $V$ is the spatial gradient operator, the dot designates the scalar product, $\varepsilon : \varepsilon = \text{tr}(\varepsilon \cdot \varepsilon)$, $\text{tr}$ is the trace-calculation operator.

For an isotropic body, the surface energy $U$ set on $\Omega_2$ can be written as \cite{3}

$$U = U(\varepsilon) = \tau_0 \text{tr} \varepsilon + \frac{1}{2} \lambda \text{tr}^2 \varepsilon + \mu \varepsilon : \varepsilon,$$

$$\varepsilon = \varepsilon(u) = \frac{1}{2} ((V_\Lambda u_\Lambda) \cdot A + A \cdot (V_\Lambda u_\Lambda)^T).$$

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Further, we require fulfillment of the conditions of positive definiteness of the functions $W(\varepsilon)$ and $U(\varepsilon)$:

$$W(\varepsilon) > 0 \text{ for all } \varepsilon \neq 0, \quad U(\varepsilon) > 0 \text{ for all } \varepsilon \neq 0.$$  \hspace{1cm} (4)

The fulfillment of inequalities (4) results in the following restrictions on the elastic moduli:

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \quad \mu_S > 0, \quad \mu + \lambda_S > 0.$$  \hspace{1cm} (5)

It is necessary to note that assumptions (4) should be fulfilled independently from each other. If, in particular, we assume that $U < 0$ for certain deformations, or inequalities (5) are violated, it is possible to show that $J(u)$ proves to be unbounded from below.

2. For the transition to the shell-theory equations, we consider a three-dimensional body, one of the characteristic sizes of which is much less than others—the so-called shell-like body (Fig. 1). The shell-like body volume $V$ is bounded by two faces $\Omega_+$ and the lateral surface $\Omega$. We introduce also a middle (base) surface $\omega$, which is equidistant from $\Omega$. The lateral surface $\Omega$ represents the ruled surface formed by the motion of the normal $n$ to $\omega$ along its contour $\gamma = \partial \omega$. It is convenient to present the radius vector $r$ of the shell-body points as $[13, 14]$

$$r = r(q^1, q^2) + zn,$$

where $\rho$ is the radius vector of points of the base surface $\omega$, $n$ is the vector of the normal to $\omega$, $z$ is the coordinate counted from the normal to $\omega$, $z \in [-h/2, h/2]$, $h$ is the shell thickness, and $q^1$ and $q^2$ are the Gaussian coordinates at $\omega$. The radius vectors of $\Omega$ are $r_z = \rho \pm nh/2$, respectively.

We display certain auxiliary formulas related to the description of tensor fields near the surface $\omega$ [13, 14]. The basic and dual bases and the surface nabla operator at $\omega$ are given by the formulas

$$\rho_\alpha = \frac{\partial \rho}{\partial q^\alpha}, \quad \rho_\alpha \cdot \rho^\beta = \delta_\alpha^\beta, \quad \alpha, \beta = 1, 2,$$

$$\nabla_S = \rho_\alpha \frac{\partial}{\partial q^\alpha},$$

where $\delta_\alpha^\beta$ is the Kronecker delta. We use the quantities $q^1, q^2$, and $z$ as the curvilinear coordinates in the vicinity of $\omega$. Then the following formulas hold:

$$r_\alpha = \frac{\partial r}{\partial q^\alpha} = \rho_\alpha + z \frac{\partial n}{\partial q^\alpha} = (A - zB) \cdot \rho_\alpha,$$

$$r_3 = r^3 = n,$$

$$r^\alpha = (A - zB)^{-1} \cdot \rho^\alpha, \quad r_\alpha \cdot r^\beta = \delta_\alpha^\beta,$$

$$B = -\nabla_S n,$$

$$\nabla = r^\alpha \frac{\partial}{\partial q^\alpha} + n \frac{\partial}{\partial z} = (A - zB)^{-1} \cdot \nabla_S + n \frac{\partial}{\partial z},$$

$$\nabla_S = (A + \frac{h}{2}B)^{-1} \cdot \rho_\alpha \frac{\partial}{\partial q^\alpha} = (A + \frac{h}{2}B)^{-1} \cdot \nabla_S,$$
forming them while taking into account the assumption that $h\|B\| \ll 1$, we obtain
\[
\nabla_S \cdot T^* + q = 0, \quad \nabla_S \cdot M^* + T^*_S + m = 0, \tag{9}
\]
where the effective stress resultants and couples tensors $T^*$ and $M^*$ are introduced
\[
T^* = T + T_S, \quad M^* = M + M_S, \tag{10}
\]
\[
T_S = \tau_+ + \tau_-, \quad M_S = -\frac{h}{2}(\tau_+ - \tau_-) \times n.
\]
For description the shell deformations, we assume the displacement-field approximation linear in thickness used in the theory of plates and shells taking into account the transverse shear (see, for example, [15]):
\[
u(q^1, q^2, z) = w(q^1, q^2) - z\theta(q^1, q^2), \quad n \cdot \theta = 0.
\]
Here it is assumed that the rotation vector $\theta$ is kinematically independent of the displacement vector of the shell mid-surface $w$. Equation (11) leads to the formula
\[
u_S^\pm = w + \frac{h}{2}\theta, \quad \epsilon_\pm = \epsilon + \frac{h}{2}\kappa, \tag{12}
\]
where
\[
e = \frac{1}{2}(\nabla_S w \cdot A + A \cdot (\nabla_S w))^T,
\]
\[
\kappa = \frac{1}{2}(\nabla_S \theta \cdot A + A \cdot (\nabla_S \theta))^T
\]
are the two-dimensional tensors of extension–shear and bending–torsion deformations. Using Eq. (12), we obtain for the surface stresses $\tau_\pm$ the expressions
\[
\tau_\pm = \tau_0^\pm A + \lambda_\pm^S At r e \epsilon + 2\mu_\pm^S \epsilon \pm + \frac{h}{2}(\lambda_\pm^S At r k + 2\mu_\pm^S \kappa).
\]
In the case of the shell with identical surface properties, i.e., when $\tau_0^+ = \tau_0^-, \mu_+ = \mu_-, \lambda_+ = \lambda_-, \lambda_+ = \lambda_-$, we obtain the stress resultants and couples tensors generated by the surface-stress action:
\[
T_S = 2\tau_0 A + C_1^S \epsilon + C_2^S At r e,
\]
\[
M_S = -[D_1^S \kappa + D_2^S At r k] \times n, \tag{13}
\]
\[
C_1^S = 4\mu_+^S, \quad C_2^S = 2\lambda_+^S, \quad D_1^S = h^2 \mu_+^2, \quad D_2^S = \frac{h^2 \lambda_+^S}{2}.
\]
With taking into account Eqs. (10), it follows from Eq. (13) that the surface stresses render no effect on the transverse shear forces because $T_S \cdot n = 0$ and have little or no effect on the stiffness and the transverse shear of the shell. From Eq. (13), it can be seen also that the residual surface stresses $\tau_0$ do not affect the shell stiffness, although, naturally they affect its stress state.
For \( \mathbf{T} \) and \( \mathbf{M} \), we accept the constitutive equations in one of the simplest forms given, for example, in [15]:

\[
\mathbf{T} \cdot \mathbf{A} - \frac{1}{2}(\mathbf{M} \cdot \mathbf{B})\mathbf{A} \times \mathbf{n} = \frac{\partial W_S}{\partial \epsilon}, \\
\mathbf{T} \cdot \mathbf{n} = \frac{\partial W_S}{\partial \gamma}, \quad \mathbf{M} = \frac{\partial W_S}{\partial \kappa},
\]

(14)

Here, \( W_S \) is the surface strain energy density, \( \mathbf{C} \) and \( \mathbf{D} \) are the fourth-order tensors determining the tangential and bending stiffness of the shell, \( \gamma \) is the transverse-shear vector: \( \gamma = \nabla_S (\mathbf{w} \cdot \mathbf{n}) - \mathbf{9} \), while \( \Gamma \) is the transverse-shear stiffness. For an isotropic shell [15]

\[
\mathbf{C} = C_{11} \mathbf{a}_1 \mathbf{a}_1 + C_{22} (\mathbf{a}_2 \mathbf{a}_2 + \mathbf{a}_4 \mathbf{a}_4), \\
\mathbf{D} = D_{22} (\mathbf{a}_2 \mathbf{a}_2 + \mathbf{a}_4 \mathbf{a}_4) + C_{33} \mathbf{a}_3 \mathbf{a}_3,
\]

where

\[
\mathbf{a}_1 = \mathbf{A} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2, \\
\mathbf{a}_2 = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2, \\
\mathbf{a}_3 = -\mathbf{A} \times \mathbf{n} = \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1, \\
\mathbf{a}_4 = \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1,
\]

\( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) are the unit vectors lying in the plane tangent to \( \omega (\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{n} = \mathbf{e}_2 \cdot \mathbf{n} = 0) \). The components \( C_{11}, C_{22}, D_{22}, D_{33} \), and \( \Gamma \) are given by the formulas [15]

\[
C_{11} = \frac{Eh}{2(1 - \nu)}, \quad C_{22} = \frac{Eh}{2(1 + \nu)}, \\
D_{22} = \frac{Eh^3}{24(1 + \nu)}, \quad D_{33} = \frac{Eh^3}{24(1 - \nu)}, \quad \Gamma = k \mu h,
\]

\[
E = 2\mu(1 + \nu), \quad \nu = \frac{\lambda}{2(\lambda + \mu)}, \\
C = C_{11} + C_{22} = \frac{Eh}{1 - \nu^2},
\]

where \( C \) and \( D \) are the tangential and bending stiffness parameters, \( E \) and \( \nu \) are the Young’s modulus and Poisson ratio of the shell material, respectively; and \( k \) is the analogue of the transverse-shear factor [15]. The effective tangential and bending stiffness parameters are equal to

\[
C_{\text{eff}} = C_1 + C_2 = C + 4\mu S + 2\lambda S, \\
D_{\text{eff}} = D_1 + D_2 = D + h^2 \mu S + \frac{h^2 \lambda S}{2}.
\]

Constitutive equations (14) and (15) make it possible to write the equilibrium equations for the shell and plate with taking into account surface stresses (9) in terms of displacements \( \mathbf{w} \) and rotations \( \mathbf{9} \). In particular, the equation for the deflection \( \mathbf{w} = \mathbf{w} \cdot \mathbf{i}_1 \) in the case of the plate \( (\mathbf{n} = \mathbf{i}_1) \) can be reduced to the form

\[
D_{\text{eff}} \Delta \mathbf{w} = \nabla_S \cdot \mathbf{m} - \frac{D_{\text{eff}}}{\Gamma} q_n + q_n,
\]

\[
q_n = \mathbf{q} \cdot \mathbf{i}_1, \quad \Delta = \nabla_S \cdot \nabla_S.
\]

For the quantitative estimate of the results of the surface-stress effect, we use the data for aluminum [1]: \( \mu = 34.7 \ \text{GPa}, \ \nu = 0.3, \ \lambda^S = -3.48912 \ \text{N/m}, \) and \( \mu^S = 6.2178 \ \text{N/m}. \) We consider the dependences of \( D_{\text{eff}}, C_1, C_2, D_1, \) and \( D_2 \) on the thickness \( h \). The plot for the bending stiffness \( D_{\text{eff}} \) is shown in Fig. 2, and the plots for the dimensionless values \( \bar{C}_1 = \frac{C_1}{(1 - \nu)}, \ \bar{C}_2 = \frac{C_2}{C\nu}, \ \bar{D}_1 = \frac{D_1}{D(1 - \nu)}, \) and \( \bar{D}_2 = \frac{D_2}{D\nu} \) are shown in Fig. 3.

As it follows from Figs. 2 and 3, the surface-stress effect is almost negligible for \( h > 50 \ \text{nm} \). They render the greatest effect for \( h < 20 \ \text{nm} \). In addition, it can be seen that the surface stresses differently affect the stiffness parameters—some of them increase \( (C_1, D_1) \),
while others \((C_2, D_2)\) decrease, the thicknesses being zero for certain values. Condition (5) guarantees a positive sign for \(C_{\text{eff}}\) and \(D_{\text{eff}}\) for arbitrary values of \(h\).

Thus, we obtained the two-dimensional equilibrium equations for plates and shells with taking into account the transverse shear and the presence of surface stresses. We presented the relations for the stress resultants and couples tensors and found the expressions for effective stiffness parameters of shells. In particular, it was shown that the plate stiffness substantially changes with taking into account the surface stresses, which agrees with the results of the theoretical analysis and the experimental data known in the literature (see, for example, [1]). In particular, it is shown that the shell bending stiffness substantially grows for the nanometer thicknesses.

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REFERENCES


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