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Estimation of the Weibull tail-coefficient with linear combination of upper order statistics

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Abstract

We present a new family of estimators of the Weibull tail-coefficient. The Weibull tail-coefficient is defined as the regular variation coefficient of the inverse failure rate function. Our estimators are based on a linear combination of log-spacings of the upper order statistics. Their asymptotic normality is established and illustrated for two particular cases of estimators in this family. Their finite sample performances are presented on a simulation study.

Keywords: Weibull tail-coefficient, extreme-values, order statistics, regular variations.

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1 Introduction

Weibull tail-distributions encompass a variety of light tailed distributions, i.e. distributions in the Gumbel maximum domain of attraction, see [13] for further details. Weibull tail-distributions include for instance Weibulls, Gaussians, gammas and logistic. The purpose of this paper is to study the estimation of a tail parameter associated with these distributions. More precisely, a cumulative distribution function $F$ has a Weibull tail if its logarithmic tail satisfies the following property: There exists $\theta > 0$ such that for all $\lambda > 0,$

$$\lim_{t \to \infty} \frac{\log(1 - F(\lambda t))}{\log(1 - F(t))} = \lambda^{1/\theta}.$$  \hspace{1cm} (1)

The parameter of interest $\theta$ is called the Weibull tail-coefficient. Such distributions are of great use to model large claims in non-life insurance [5]. In the particular case where $\log(1 - F(\lambda t))/\log(1 - F(t)) = \lambda^{1/\theta}$ for all $t > 0$ and $\lambda > 0$, estimating $\theta$ reduces to estimating the shape parameter of a Weibull distribution. In this context, simple and efficient methods exist, see for instance [2], Chapter 4 for a review on this topic. Otherwise, dedicated estimation methods have been proposed since the relevant information on the Weibull tail-coefficient is only contained in the extreme upper part of the sample. A first direction was investigated in [7] where an estimator based on the record values is proposed. Another family of approaches \cite{4, 6, 9, 11, 15, 16, 18} consists of using the $k_n$ upper order statistics where $(k_n)$ is an intermediate sequence of integers i.e. such that

$$\lim_{n \to \infty} k_n = \infty \text{ and } \lim_{n \to \infty} k_n/n = 0.$$  \hspace{1cm} (2)

Note that, since $\theta$ is defined only by an asymptotic behavior of the tail, the estimator should use the only extreme-values of the sample and thus the second part of (2) is required. The estimators considered here belong to this approach. Let $(X_i)_{1 \leq i \leq n}$ be a sequence of independent and identically distributed random variables with cumulative distribution function $F$. Denoting by $X_{1,n} \leq \ldots \leq X_{n,n}$ the corresponding order statistics, our family of estimators is

$$\hat{\theta}_n(\alpha) = \sum_{i=1}^{k_n-1} \alpha_{i,n}(\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})) \left/ \sum_{i=1}^{k_n-1} \alpha_{i,n}(\log \log (n/i) - \log \log (n/k_n)) \right.$$

$$\hspace{1cm} \left. + \varepsilon_{i,n} \right)$$  \hspace{1cm} (3)

with weights $\alpha_{i,n} = W(i/k_n) + \varepsilon_{i,n}$ defined from $W$ a smooth score function and $(\varepsilon_{i,n})_{1 \leq i \leq k_n-1}$ a non-random sequence. We refer to [10, 25] for similar works in the context of the estimation of the extreme-value index.

In Section 2 we state the asymptotic normality of these estimators. In Section 3, we provide two examples of weights. The first one leads to the estimator of $\theta$ proposed by Beirlant et al. [6]. The second one gives rise to a new estimator for Weibull tail-distributions. The behavior of these two estimators is investigated on finite sample situations. Finally, proofs are given in Section 4.
2 Asymptotic normality

Consider the failure rate $H = -\log(1 - F)$. Writing $H^-$ its generalized inverse $H^{-}(t) = \inf\{x, \, H(x) \geq t\}$, assumption (1) is equivalent to:

(A.1) $H^{-}(t) = t^\theta \ell(t),$

where $\ell$ is a slowly varying function i.e. such that $\ell(\lambda t)/\ell(t) \to 1$ as $t \to \infty$ for all $\lambda > 0$.

The inverse failure rate function $H^-$ is said to be regularly varying at infinity with index $\theta$ and this property is denoted by $H^- \in R_{\theta}$. We refer to [8] for more information on regular variation theory. As a comparison, Pareto type distributions satisfy $\left(1/(1 - F)\right)^- \in R_{\gamma}$, and $\gamma > 0$ is the so-called extreme-value index. As often in extreme-value theory, (A.1) is not sufficient to prove a central limit theorem for $\hat{\alpha}$. It needs to be strengthened with a second order condition on $\ell$, namely that there exist $\rho \leq 0$ and a function $b$ with limit 0 at infinity such that

(A.2) $\log \left(\ell(\lambda t)/\ell(t)\right) \sim b(t) \int_1^\lambda u^{\rho-1} du,$

uniformly locally on $\lambda > 1$ and as $t \to \infty$. The second order parameter $\rho \leq 0$ tunes the rate of convergence of $\ell(\lambda t)/\ell(t)$ to 1. The closer $\rho$ is to 0, the slower is the convergence. Condition (A.2) is the cornerstone in all proofs of asymptotic normality for extreme-value estimators.

It is used in [20, 19, 3] to prove the asymptotic normality of estimators of the extreme-value index $\gamma$. Table 1 shows that many distributions satisfy (A.1) and (A.2). Among them, Extended Weibull distributions, introduced in [21], encompass gamma, Gaussian and Benktander II distributions. We refer to [12], Table 3.4.4, for the derivation of $b(x)$ and $\rho$ in each case. Other examples are the Weibull, logistic and extreme-value (with shape parameter $\gamma = 0$) distributions.

Throughout the paper, we write $\text{Id}$ for the identity function. In particular, if $f$ is a function and $p$ a real number, the inequality $f \leq \text{Id}^p$ means $f(t) \leq t^p$ for any $p$ where it is defined. For general L-estimators, conditions on the weights are required to obtain a central limit theorem (see for instance [23]). Our assumptions are the following:

(A.3) $W$ is defined and continuously differentiable on the open unit interval,

(A.4) There exist $M > 0$, $0 \leq q < 1/2$ and $p < 1$ such that $|W| \leq M\text{Id}^{-q}$ and $|W'| \leq M\text{Id}^{-p-q}$ on the open unit interval.

Similar conditions have been introduced in the context of the estimation of the extreme-value index [10, 25]. To write the limiting variance of $\hat{\theta}_n(\alpha)$, we introduce two quantities:

$\mu(W) = \int_0^1 W(x) \log(1/x) dx,$

$\sigma^2(W) = \int_0^1 \int_0^1 W(x)W(y) \frac{\min(x,y) - xy}{xy} dxdy.$
We also define \( \| \varepsilon \|_{n, \infty} = \max_{i=1, \ldots, k_n-1} |\varepsilon_{i,n}| \). We are now in position to state our main result. Its proof is postponed to Section 4.

**Theorem 1** Suppose (A.1)–(A.4) hold. If \((k_n)\) is any intermediate sequence such that
\[
k_n^{1/2} b(\log(n)) \to \lambda \text{ and } k_n^{1/2} \max\{1/\log(n), \|\varepsilon\|_{n, \infty}\} \to 0, \tag{4}
\]
then
\[
k_n^{1/2}(\hat{\theta}_n(\alpha) - \theta) \xrightarrow{d} \mathcal{N}(\lambda, \theta^2\sigma^2(W)/\mu^2(W)).
\]
Clearly, the bias of the estimator is driven by the function \( b \). This bias term asymptotically vanishes if \( \lambda = 0 \). Some applications of this result are given in the next section, Corollary 1 and Corollary 2. The importance of the bias term is also illustrated on finite sample situations. Finally, note that condition (4) implies \( k_n/n \to 0 \).

## 3 Comparison of two estimators

First, we show in Paragraph 3.1, that our family of estimators (3) encompasses the Hill type estimator \( \hat{\theta}_n^H \) proposed in [6]. Moreover, it will appear in Corollary 1 that the asymptotic normality of \( \hat{\theta}_n^H \) stated in [18], Theorem 2 is a consequence of our main result Theorem 1.

Second, in Paragraph 3.2, we use our framework to exhibit a new estimator of the Weibull tail-coefficient and to establish its asymptotic normality in Corollary 2. In the third paragraph, we show that the new estimator performs as well as the Hill one.

### 3.1 Hill type estimator

Beirlant et al. [18] propose the following estimator of the Weibull tail-coefficient:
\[
\hat{\theta}_n^H = \frac{\sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n}))}{\sum_{i=1}^{k_n-1} (\log \log (n/i) - \log \log (n/k_n))}. \tag{5}
\]
Clearly, \( \hat{\theta}_n^H \) is a particular case of \( \hat{\theta}_n(\alpha) \) with \( W(x) = 1 \) for all \( x \in [0,1] \) and \( \varepsilon_{i,n} = 0 \) for all \( i = 1, \ldots, k_n \). The asymptotic normality of \( \hat{\theta}_n^H \), established in Theorem 2 of [18], can be obtained as a consequence of Theorem 1:

**Corollary 1** Suppose (A.1) and (A.2) hold. If \((k_n)\) is an intermediate sequence such that
\[
k_n^{1/2} \max\{b(\log(n)), 1/\log(n)\} \to 0, \text{ then } k_n^{1/2}(\hat{\theta}_n^H - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2).
\]

### 3.2 Zipf estimator

We propose a new estimator of the Weibull tail-coefficient based on a quantile plot adapted to our situation. It consists of drawing the pairs \((\log \log (n/i), \log (X_{n-i+1,n}))\) for \( i = 1, \ldots, n-1 \). The resulting graph should be approximatively linear (with slope \( \theta \)), at least for the large
values of \( i \). Thus, we introduce \( \hat{\theta}^Z_n \) the least square estimator of \( \theta \) based on the \( k_n \) largest observations:

\[
\hat{\theta}^Z_n = \frac{k_n^{-1} \sum_{i=1}^{k_n-1} (\log \log (n/i) - \zeta_n) \log(X_{n-i+1,n})}{\sum_{i=1}^{k_n-1} (\log \log (n/i) - \zeta_n) \log \log (n/i)}, \tag{6}
\]

where

\[
\zeta_n = \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \log (n/i).
\]

This estimator is similar to the Zipf estimator for the extreme-value index proposed by Kratz and Resnick [22] and Schultze and Steinebach [24]. We prove in Section 4 that \( \hat{\theta}^Z_n \) belongs to family (3) and thus apply Theorem 1 to obtain its asymptotic normality:

**Corollary 2** Suppose (A.1) and (A.2) hold. If \( (k_n) \) is an intermediate sequence such that \( k_n^{1/2} \max\{b(\log(n)), \log^2(k_n)/\log(n)\} \to 0 \), then \( k_n^{1/2}(\hat{\theta}^Z_n - \theta) \to_d N(0, 2\theta^2) \).

### 3.3 Numerical experiments

The finite sample performance of the estimators \( \hat{\theta}^Z_n \) and \( \hat{\theta}^H_n \) are investigated on 6 different distributions: \( \Gamma(0.5, 1) \), \( \Gamma(1.5, 1) \), \( \mathcal{N}(1.2, 1) \), \( \mathcal{L} \), \( \mathcal{W}(2.5, 2.5) \) and \( \mathcal{W}(0.4, 0.4) \), see Table 1 for their parameterizations.

We limit ourselves to these two estimators, since it is shown in [18] that \( \hat{\theta}^H_n \) gives better results than the other approaches [9, 4]. In each case, \( N = 200 \) samples \((X_{n,i})_{i=1,\ldots,N}\) of size \( n = 500 \) were simulated. On each sample \((X_{n,i})\), the estimates \( \hat{\theta}^Z_{n,i}(k_n) \) and \( \hat{\theta}^H_{n,i}(k_n) \) are computed for \( k_n = 2, \ldots, 250 \). Finally, the Hill-type plots are built by drawing the points

\[
\left(k_n, \frac{1}{N} \sum_{i=1}^{N} \hat{\theta}^Z_{n,i}(k_n)\right) \quad \text{and} \quad \left(k_n, \frac{1}{N} \sum_{i=1}^{N} \hat{\theta}^H_{n,i}(k_n)\right).
\]

We also present the associated MSE (mean square error) plots obtained by plotting the points

\[
\left(k_n, \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\theta}^Z_{n,i}(k_n) - \theta\right)^2\right) \quad \text{and} \quad \left(k_n, \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\theta}^H_{n,i}(k_n) - \theta\right)^2\right).
\]

The results are presented on figures 1–6. It appears that for both estimates the sign of the bias is driven by the function \( b \) in (A.2). In all plots, \( \hat{\theta}^Z_n \) appears to vary more smoothly in terms of \( k_n \) than \( \hat{\theta}^H_n \), a feature which we find appealing. The results obtained with the two estimators are very similar on Weibull distributions (figure 5 and figure 6), especially in terms of mean square error. In other cases, i.e. gamma, Gaussian and logistic distributions (figures 1–4), \( \hat{\theta}^Z_n \) gives better results in terms of bias and mean square error.

### 4 Proofs

Throughout this section, we assume that \( (k_n) \) is an intermediate sequence and, for the sake of simplicity, we note \( k \) for \( k_n \). Let us also introduce \( K_\rho(\lambda) = \int_1^\lambda u^{\rho-1} du \) for \( \lambda \geq 1 \) and
\[ J(x) = W(1 - x) \text{ for } x \in (0, 1). \] The following notations will prove useful: \( E_{n-k+1,n} \) is the \((n - k + 1)\)th order statistics associated to \(n\) independent standard exponential variables and \((F_{i,k-1})_{1 \leq i \leq k-1}\) are order statistics, independent from \(E_{n-k+1,n}\), generated by \(k - 1\) independent standard exponential variables. The next lemma presents an expansion of \(\hat{\theta}_n(\alpha)\).

**Lemma 1** Under (A.1) and (A.2), \(\hat{\theta}_n(\alpha)\) has the same distribution as

\[
\frac{\theta T_n^{(2,0)} + (1 + o_P(1)) b(E_{n-k+1,n}) T_n^{(2,\rho)}}{T_n^{(1)}},
\]

where we have defined

\[
T_n^{(1)} = \frac{1}{k - 1} \sum_{i=1}^{k-1} \alpha_{i,n} \log \left( \frac{E_{n-i+1,n}}{E_{n-k+1,n}} \right) \text{ and}
\]

\[
T_n^{(2,\rho)} = \frac{1}{k - 1} \sum_{i=1}^{k-1} \alpha_{k-i,n} K_\rho \left( 1 + \frac{F_{i,k-1}}{E_{n-k+1,n}} \right), \quad \rho \leq 0.
\]

**Proof:** Using the quantile transform, the order statistics \((X_{i,n})_{1 \leq i \leq n}\) have the same distribution as \((H^{-1}(E_{i,n}))_{1 \leq i \leq n}\). Thus, (A.2) yields that the numerator of \(\hat{\theta}_n(\alpha)\) in (3) has the same distribution as

\[
\frac{1}{k - 1} \sum_{i=1}^{k-1} \alpha_{i,n} \log \left( \frac{E_{n-i+1,n}}{E_{n-k+1,n}} \right) + (1 + o_P(1)) b(E_{n-k+1,n}) \frac{1}{k - 1} \sum_{i=1}^{k-1} \alpha_{i,n} K_\rho \left( \frac{E_{n-i+1,n}}{E_{n-k+1,n}} \right).
\]

The Rényi representation asserts that \((E_{n-i+1,n}/E_{n-k+1,n})_{1 \leq i \leq k-1}\) has the same distribution as \((1 + F_{k-i,k-1}/E_{n-k+1,n})_{1 \leq i \leq k-1}\), see [1], p. 72. Therefore, the numerator of \(\hat{\theta}_n(\alpha)\) has the same distribution as

\[
\frac{1}{k - 1} \sum_{i=1}^{k-1} \alpha_{i,n} \log \left( 1 + \frac{F_{k-i,k-1}}{E_{n-k+1,n}} \right)
\]

\[
+ \ (1 + o_P(1)) b(E_{n-k+1,n}) \frac{1}{k - 1} \sum_{i=1}^{k-1} \alpha_{i,n} \log \left( 1 + \frac{F_{k-i,k-1}}{E_{n-k+1,n}} \right).
\]

Changing \(i\) to \(k - i\) in the above formula and remarking that \(K_0\) is the logarithm function conclude the proof.

The following lemma provides an expansion of

\[
\tau_n = \frac{1}{k - 1} \sum_{i=1}^{k-1} \log \left( \frac{n}{i} \right) - \log \left( \frac{n}{k} \right),
\]

which frequently appears in the proofs.

**Lemma 2** The following expansion holds:

\[
\tau_n = \frac{1}{\log(n/k)} \left\{ 1 + O \left( \frac{\log(k)}{k} \right) + O \left( \frac{1}{\log(n/k)} \right) \right\}.
\]
Proof: We write $\tau_n$ as the sum

$$
\frac{1}{\log(n/k)} \frac{1}{k-1} \sum_{i=1}^{k-1} \log(k/i) + \frac{1}{k-1} \sum_{i=1}^{k-1} \left\{ \log \left( 1 + \frac{\log(k/i)}{\log(n/k)} \right) - \frac{\log(k/i)}{\log(n/k)} \right\}.
$$

Since

$$
\frac{1}{k-1} \sum_{i=1}^{k-1} \log(i/k) = \frac{1}{k-1} \log \left( \frac{k!}{k^k} \right),
$$

Stirling’s formula shows that the first term is

$$
\frac{1}{\log(n/k)} \left( 1 + O\left( \frac{\log(k)}{2k} \right) \right).
$$

The inequality $-x^2/2 \leq \log(1 + x) - x \leq 0$, valid for nonnegative $x$ shows that the second term is of order at most

$$
\frac{1}{\log^2(n/k)} \frac{1}{k-1} \sum_{i=1}^{k-1} \log^2(i/k) = O \left( \frac{1}{\log^2(n/k)} \right),
$$

since the above Riemann sum converges to 2 as $k \to \infty$. The result follows. \hfill \blacksquare

The next lemmas are dedicated to the study of the different terms appearing in Lemma 1. First, we focus on the non-random term $T_n^{(1)}$.

Lemma 3 Under (A.1)–(A.4), the following expansion hold:

$$
T_n^{(1)} = \frac{\mu(W)}{\log(n/k)} \left\{ 1 + O \left( \log(k)k^q-1 \right) + O \left( \frac{1}{\log(n/k)} \right) + O(\|\varepsilon\|_{n,\infty}) \right\}.
$$

Proof: Clearly, $T_n^{(1)}$ can be rewritten as the sum

$$
\frac{1}{k-1} \sum_{i=1}^{k-1} \varepsilon_{i,n} \log \left( 1 + \frac{\log(k/i)}{\log(n/k)} \right) + \frac{1}{k-1} \sum_{i=1}^{k-1} W(i/k) \log \left( 1 + \frac{\log(k/i)}{\log(n/k)} \right).
$$

The absolute value of the first term is less than $\|\varepsilon\|_{n,\infty} \tau_n$, which is $O(\|\varepsilon\|_{n,\infty}/\log(n/k))$, by Lemma 2. The second term can be expanded as

$$
\frac{1}{\log(n/k)} \frac{1}{k-1} \sum_{i=1}^{k-1} W(i/k) \log(k/i) + \frac{1}{k-1} \sum_{i=1}^{k-1} W(i/k) \left\{ \log \left( 1 + \frac{\log(k/i)}{\log(n/k)} \right) - \frac{\log(k/i)}{\log(n/k)} \right\}
$$

$$
=: T_n^{(1,1)} + T_n^{(1,2)}.
$$

For $x \in (0,1)$, define $H(x) = W(x) \log(1/x)$. The Riemann sum $T_n^{(1,1)}$ can be compared to $\mu(W)$ by:

$$
|T_n^{(1,1)} - \mu(W)| \leq \frac{1}{2k^2} \sum_{i=1}^{k-1} \sup_{i/k \leq x \leq (i+1)/k} |H'(x)| + \int_0^{1/k} |H(x)| dx + O(1/k). \quad (7)
$$

Assumption (A.4) implies that there exists a positive $M'$ such that $|H'| \leq M'Id^{-q-1}$ on the open unit interval, and thus the first term of (7) is bounded above by

$$
\frac{M'}{2k} \left( \int_{1/k}^1 t^{-q-1} dt + k^q \right) = \begin{cases} 
O \left( k^{q-1} \right) & \text{if } q \neq 0, \\
O \left( k^{-1} \log(k) \right) & \text{otherwise}.
\end{cases}
$$
Assumption (A.4) also yields $|H| \leq M d^{-q} \log(1/x)$ on the open unit interval and thus the second term in (7) is $O(k^{-1} \log(k))$. It follows that

$$T_n^{(1,1)} = \mu(W) + O(k^{-1} \log(k)).$$

(8)

Besides, the well-known inequality $|\log(1+x) - x| \leq x^2/2$, valid for all nonnegative $x$ together with (A.4) show that $|T_n^{(1,2)}|$ is bounded by

$$|T_n^{(1,2)}| \leq \frac{M}{2 \log^2(n/k)} \frac{1}{k-1} \sum_{i=1}^{k-1} (i/k)^{-q} \log^2(k/i) = O\left(\frac{1}{\log^2(n/k)}\right),$$

(9)

since the above Riemann sum converges to a finite integral. Collecting (8) and (9) gives the result.

Second, we focus on the random term $T_n^{(2,\rho)}$.

**Lemma 4** Let $\xi$ be standard Gaussian random variable. Under (A.1)–(A.4), the following expansion hold for all non-positive $\rho$:

$$T_n^{(2,\rho)} \overset{d}{=} \frac{\mu(W)}{E_{n-k+1,n}} \left\{ 1 + \frac{\sigma(W)}{\mu(W)} k^{-1/2} \xi(1 + o_P(1)) + O_P\left(\frac{1}{\log(n/k)}\right) + O_P\left(|\varepsilon|_{n,\infty}\right) \right\}.$$  

**Proof**: Note that $T_n^{(2,\rho)}$ can be written as the sum

$$\frac{1}{k-1} \sum_{i=1}^{k-1} \varepsilon_{k-i,n} K_\rho \left( 1 + \frac{F_{i,k-1}}{E_{n-k+1,n}} \right) + \frac{1}{k-1} \sum_{i=1}^{k-1} J(i/k) K_\rho \left( 1 + \frac{F_{i,k-1}}{E_{n-k+1,n}} \right).$$

(10)

Since $0 \leq K_\rho(1+x) \leq x$ for all nonnegative $x$, the absolute value of the first term is bounded by

$$|\varepsilon|_{n,\infty} \frac{1}{k-1} \sum_{i=1}^{k-1} K_\rho \left( 1 + \frac{F_{i,k-1}}{E_{n-k+1,n}} \right) \leq |\varepsilon|_{n,\infty} \frac{1}{k-1} \sum_{i=1}^{k-1} \frac{F_{i,k-1}}{E_{n-k+1,n}},$$

which has the same distribution as

$$\frac{1}{E_{n-k+1,n}} \frac{1}{k-1} \sum_{i=1}^{k-1} F_i = \frac{1}{E_{n-k+1,n}} O_P\left(|\varepsilon|_{n,\infty}\right),$$

from the law of large numbers. The second term of (10) can be expanded as

$$\frac{1}{E_{n-k+1,n}} \frac{1}{k-1} \sum_{i=1}^{k-1} J(i/k) F_{i,k-1} + \frac{1}{k-1} \sum_{i=1}^{k-1} J(i/k) \left\{ K_\rho \left( 1 + \frac{F_{i,k-1}}{E_{n-k+1,n}} \right) - \frac{F_{i,k-1}}{E_{n-k+1,n}} \right\}$$

$$=: \frac{T_n^{(2,\rho,1)}}{E_{n-k+1,n}} + T_n^{(2,\rho,2)}.$$  

Now, (A.3) and (A.4) imply that the L-statistics $T_n^{(2,\rho,1)}$ satisfies the conditions of [23] and thus is asymptotically Gaussian. More precisely, we have

$$T_n^{(2,\rho,1)} \overset{d}{=} \mu(W) + \sigma(W) k^{-1/2} \xi(1 + o_P(1)).$$

(11)
The upper bound on $T_n^{(2,\rho,2)}$ is obtained by remarking that $|K_\rho(1+x) - x| \leq (1-\rho)x^2/2$ for all nonnegative $x$. It follows that $T_n^{(2,\rho,2)}$ is bounded above by

$$\frac{1}{k-1} \sum_{i=1}^{k-1} |J(i/k)| K_\rho \left( 1 + \frac{F_{i,k-1}}{E_{n-k+1,n}} \right) - \frac{F_{i,k-1}}{E_{n-k+1,n}} \leq \frac{1-\rho}{2E_{n-k+1,n}} \frac{1}{k-1} \sum_{i=1}^{k-1} |J(i/k)| F_{i,k-1}^2.$$

Now, $E_{n-k+1,n}$ is equivalent to $\log(n/k)$ in probability and

$$\frac{1}{k-1} \sum_{i=1}^{k-1} |J(i/k)| F_{i,k-1}^2 = O_P(1),$$

from the results of [23] on L-statistics. Thus

$$T_n^{(2,\rho,2)} = \frac{1}{E_{n-k+1,n}} O_P \left( \frac{1}{\log(n/k)} \right), \quad (12)$$

and then collecting (11) and (12), the second term of (10) is

$$\frac{1}{E_{n-k+1,n}} \left( \mu(W) + \sigma(W)k^{-1/2} \xi(1 + o_P(1)) + O_P \left( \frac{1}{\log(n/k)} \right) \right),$$

and the result follows.

We are now in position to prove Theorem 1 and Corollary 2.

**Proof of Theorem 1.** From Lemma 1, $k^{1/2}(\hat{\theta}_n(\alpha) - \theta)$ has the same distribution as

$$\theta k^{1/2} \left( \frac{T_n^{(2,0)}}{T_n^{(1)}} - 1 \right) + k^{1/2} b(E_{n-k+1,n}) \frac{T_n^{(2,\rho)}}{T_n^{(1)}} (1 + o_P(1)).$$

Now, $E_{n-k+1,n}$ is equivalent to $\log(n/k)$ in probability which is also equivalent to $\log(n)$, see Lemma 5.1 in [15]. Since $|b|$ is regularly varying (see [17]), $b(E_{n-k+1,n})$ is equivalent to $b(\log(n))$ in probability. As a consequence, $k^{1/2}(\hat{\theta}_n(\alpha) - \theta)$ has the same distribution as

$$\theta k^{1/2} \left( \frac{T_n^{(2,0)}}{T_n^{(1)}} - 1 \right) + k^{1/2} b(\log(n)) \frac{T_n^{(2,\rho)}}{T_n^{(1)}} (1 + o_P(1)). \quad (13)$$

Let us consider the first term of this sum. Lemma 3, Lemma 4 and condition (4) entail that, for all non-positive $\rho$, the ratio $T_n^{(2,\rho)}/T_n^{(1)}$ has the same distribution as

$$\frac{\log(n/k)}{E_{n-k+1,n}} \left\{ 1 + \frac{\sigma(W)}{\mu(W)} k^{-1/2} \xi(1 + o_P(1)) \right\}.$$

Now, Lemma 1 in [18] asserts a central limit theorem for order statistics of an exponential sample, and thus

$$\frac{\log(n/k)}{E_{n-k+1,n}} \xrightarrow{d} 1 + O_P \left( \frac{k^{-1/2}}{\log(n)} \right).$$

Consequently, the first term of (13) converges in distribution to $\mathcal{N}(0, \theta^2 \sigma^2(W)/\mu^2(W))$. We also have that the second term of (13) converges to $\lambda$ in probability and the result is proved. 

\[\square\]
Proof of Corollary 2. First remark that (6) can be rewritten as
\[
\tilde{\theta}_n^Z = \frac{\sum_{i=1}^{k-1} \alpha_{i,n}^Z (\log(X_{n-i+1,n}) - \log(X_{n-k+1,n}))}{\sum_{i=1}^{k-1} \alpha_{i,n}^Z (\log \log (n/i) - \log \log (n/k))},
\]
where
\[
\alpha_{i,n}^Z = \log(n/k) \left( \log \log (n/i) - \zeta_n \right)
= \log(n/k) \left( \log \left( 1 + \frac{\log(k/i)}{\log(n/k)} \right) - \tau_n \right)
= \log(k/i) + O \left( \frac{\log^2(k)}{\log(n)} \right) - \log(n/k) \tau_n,
= \log(k/i) - 1 + O \left( \frac{\log^2(k)}{\log(n)} \right) + O \left( \frac{\log(k)}{k} \right),
\]
uniformly on \(i = 1, \ldots, k\) with Lemma 2. Therefore, we have \(\alpha_{i,n}^Z = W(i/k) + \varepsilon_{i,n}\) with \(W(x) = -(\log(x) + 1)\) and \(\varepsilon_{i,n} = O(\log^2(k)/\log(n)) + O(\log(k)/k)\), uniformly on \(i = 1, \ldots, k\). Then, it is easy to check that \(W\) satisfies conditions \(\text{(A.3)}\) and \(\text{(A.4)}\) and that \(\mu(W) = 1\) and \(\sigma^2(W) = 2\). □

Acknowledgment

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<table>
<thead>
<tr>
<th>Distribution</th>
<th>$1 - F(x)$</th>
<th>$\theta$</th>
<th>$b(x)$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weibull $W(\alpha, \lambda)$</td>
<td>$\exp(-(x/\lambda)^\alpha)$</td>
<td>$1/\alpha$</td>
<td>0</td>
<td>$-\infty$</td>
</tr>
<tr>
<td>Extended Weibull $\mathcal{E}W(\tau, \beta, \gamma)$</td>
<td>$r(x) \exp(-\beta x^\tau)$</td>
<td>$1/\tau$</td>
<td>$-\frac{\gamma \log x}{\bar{\tau}x}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>Gaussian $\mathcal{N}(\mu, \sigma^2)$</td>
<td>$\frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_a^\infty \exp \left( -\frac{(t-\mu)^2}{2\sigma^2} \right) dt$</td>
<td>$1/2$</td>
<td>$\frac{1}{4} \frac{\log x}{x}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>Gamma $\Gamma(\beta, \alpha)$</td>
<td>$\frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} \exp(-\beta t) dt$</td>
<td>1</td>
<td>$(1 - \alpha) \frac{\log x}{x}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>Benktander II $B(\alpha, \tau)$</td>
<td>$x^{\tau-1} \exp \left( -\frac{\alpha}{\tau} x^{\tau} \right)$</td>
<td>$1/\tau$</td>
<td>$\frac{(1-\tau) \log x}{\tau x}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>Logistic $\mathcal{L}$</td>
<td>$\frac{2}{1+\exp x}$</td>
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<td>$-\frac{\log 2}{x}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>Extreme Value $\mathcal{E}VD(\mu)$</td>
<td>$1 - \exp(-\exp(\mu - x))$</td>
<td>1</td>
<td>$-\frac{\mu}{x}$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

Table 1: Some Weibull tail-distributions
References


(a) Mean as a function of $k_n$  
(b) Mean square error as a function of $k_n$.

Figure 1: Comparison of estimates $\hat{\theta}^Z_n$ (solid line) and $\hat{\theta}^H_n$ (dashed line) for the $\Gamma(0.5, 1)$ distribution. In (a), the straight line is the true value of $\theta$.

(a) Mean as a function of $k_n$  
(b) Mean square error as a function of $k_n$.

Figure 2: Comparison of estimates $\hat{\theta}^Z_n$ (solid line) and $\hat{\theta}^H_n$ (dashed line) for the $\Gamma(1.5, 1)$ distribution. In (a), the straight line is the true value of $\theta$. 
Figure 3: Comparison of estimates $\hat{\theta}_Z^n$ (solid line) and $\hat{\theta}_H^n$ (dashed line) for the $\mathcal{N}(1.2, 1)$ distribution. In (a), the straight line is the true value of $\theta$.

Figure 4: Comparison of estimates $\hat{\theta}_Z^n$ (solid line) and $\hat{\theta}_H^n$ (dashed line) for the $\mathcal{L}$ distribution. In (a), the straight line is the true value of $\theta$. 
(a) Mean as a function of $k_n$  
(b) Mean square error as a function of $k_n$.

Figure 5: Comparison of estimates $\hat{\theta}^Z_n$ (solid line) and $\hat{\theta}^H_n$ (dashed line) for the $\mathcal{W}(2.5, 2.5)$ distribution. In (a), the straight line is the true value of $\theta$.

(a) Mean as a function of $k_n$  
(b) Mean square error as a function of $k_n$.

Figure 6: Comparison of estimates $\hat{\theta}^Z_n$ (solid line) and $\hat{\theta}^H_n$ (dashed line) for the $\mathcal{W}(0.4, 0.4)$ distribution. In (a), the straight line is the true value of $\theta$. 

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