On a class of strongly stabilizable systems of neutral type

Rabah Rabah, Grigory M. Sklyar

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R. Rabah [a] G. M. Sklyar [b,1]

[a] *IRCCyN UMR 6597, 1 rue de la Noë BP 92101
44321 Nantes Cedex 3, France*

[b] *Institute of Mathematics, University of Szczecin,
70451 Szczecin, Wielkopolska 15, Poland*

Abstract

We consider the strong stabilizability problem for delayed system of neutral type. For simplicity the case of one delay in state is studied. We separate a class of such systems and give a constructive solution of the problem in this case, without the derivative of the localized delayed state. Our results are based on an abstract theorem on the strong stabilizability of contractive systems in Hilbert space. An illustrating example is also given.

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1 Introduction

The problems of stability and stabilizability are of great importance in the theory of delayed systems [1–3]. In this context note that the majority of works deals with so-called exponential stability or stabilizability. In this case the conditions of stability (stabilizability) are well explored for the both systems with ordinary delay and systems of neutral type [1–3,18,4]. Note also that the mentioned type of stability is similar to the stability for finite-dimensional linear systems. However, for systems of neutral type there appears an essentially different kind of stability – the so-called strong stability.

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Email addresses: `rabah@emn.fr`, `sklar@sus.univ.szczecin.pl` (G. M. Sklyar).

1 Also with Dept. of Mechanics and Mathematics, Kharkov University, 4 Svobody sqr., Kharkov, 61077, Ukraine

Consider a system of the form

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + A_{-1}\dot{x}(t-1), \tag{1}$$

where $x \in \mathbf{R}^n, A_0, A_1, A_{-1}$ are $n \times n$-matrices. It is well known [1] that (1) is exponentially stable iff

(i) $\Re(\sigma) \leq -\alpha < 0$, where $\sigma = \{\lambda : \det(\lambda I - A_{-1}\lambda e^{-\lambda} - A_0 - A_1 e^{-\lambda}) = 0\}$, and $\Re(\sigma)$ is the real part of all values in σ.
In particular, the condition (i) holds [3] if $\|A_{-1}\| < 1$ and $\Re(\sigma) < 0$.

On the other hand, the system (1) is unstable if matrix A_{-1} possesses at least one eigenvalue μ such that $|\mu| > 1$. At the same time, it turns out [2,5] that asymptotic stability of (1) is possible under a noticably weaker condition $\Re(\sigma) < 0$, which, in particular, may appear when there are some eigenvalues μ_j, $j = 1,..,k$ of matrix A_{-1} such that $|\mu| = 1$, $j = 1,..,k$. As is shown in [5], in this case solutions of the system (1) decay essentially slower than exponentials, namely as functions $1/t^{\beta}$, $\beta > 0$.

An explanation of this effect can be found using the model of neutral type systems as abstract differential equations in Banach space and the results of the theory of strong asymptotic stability originated in [6–8] (see also the bibliography in [9]).

Note also that the results on the strong stability find it natural application in the control theory for analysis of the strong stabilizability of contractive semigroups, for example [10–14]

The main goal of the present paper is to show an extention of the stabilizability theory to the case of control systems of neutral type. To justify this point we consider a special class of neutral type systems (including, however, all the one-dimensional systems) and give a constructive solution of the strong stabilizability problem for this class based on an abstract theorem from [11].

The paper is organized as follows. In Section 1 we formulate the problem on the strong stabilizability and interprete it in the language of abstract control systems. In Section 2 an operator analysis of obtained system is given. In the Section 3 the main results are given. We design explicitely a strong stabilizing control and give an illustrating example.

Finally notice that the analysis of the strong stabilizability problem under some more general assumption is to be given in one of our forthcoming works.

2 The model and the statement of stabilizability problem

For simplicity we consider a control neutral type system with one delay in state

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + A_{-1}\dot{x}(t-1) + Bu(t), \tag{2}$$

where $x \in \mathbf{R}^n, u \in R^r, A_j, j = 0, 1, -1$ are $n \times n$-matrices, B is a $n \times r$-matrix.

The stabilizability problem consists in the determination of linear feedback control law $u = p(x(\cdot))$ such that the closed-loop system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + A_{-1}\dot{x}(t-1) + Bp(x(\cdot))$$

becomes a asymptotic stable one. In order to formulate the problem more precisely let us go over to an abstract functional model of the system (2). Following Yamamoto and Ueshima [15] (see also [4]) we put $z_t(\cdot) : \theta \mapsto x(t+\theta), \theta \in [-1, 0]$ and $y(t) = x(t) - A_{-1}x(t-1)$. Let $Z = \mathbf{C}^n \times L_2[(-1, 0), \mathbf{C}^n]$. Introduce an operator $\mathcal{A} : \mathcal{D}(A) \to Z$ defined by

$$\mathcal{A}\begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} = \begin{pmatrix} A_0 q + (A_1 + A_0 A_{-1})\varphi(-1) \\ \frac{\partial}{\partial\theta}\varphi(\cdot) \end{pmatrix},$$

where

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} : q = \varphi(0) - A_{-1}\varphi(-1), \quad \varphi(\cdot) \in W_2^{(1)}[(-1, 0), \mathbf{C}^n] \right\}.$$

With these notations the system (2) can be rewritten as

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \mathcal{A}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} + \mathcal{B}u(t), \tag{3}$$

where $\mathcal{B} = \binom{B}{0}$ is a linear operator $\mathcal{B} : \mathbf{C}^n \to Z$.

It is known [15,16] that $\mathcal{A}$ generates a C_0-semigroup in Z and that its spectrum $\sigma(\mathcal{A})$ is the set

$$\sigma(\mathcal{A}) = \sigma = \{\lambda : \det(\lambda I - A_{-1}\lambda e^{-\lambda} - A_0 - A_1 e^{-\lambda}) = 0\}.$$

and consists of eigenvalues only. Denote further by $\sum$ the set of all nonzero eigenvalues of matrix A_{-1}. Then [2] for any $\mu \in \sum$ the set σ includes a family of eigenvalues

$$\sum\nolimits^{\mu} = \{\lambda_k^{\mu} = \log|\mu| + i(Arg\,\mu + 2\pi k) + \bar{o}(1), \quad k \in \mathbf{Z}\}, \tag{4}$$

where $\bar{o}$ is meant as $k \to \pm\infty$.

The substitution of a feedback control $u = p(x(\cdot))$ into (2) leads to transformation of system (3) to the form

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \tilde{\mathcal{A}}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix}, \tag{5}$$

where $\tilde{\mathcal{A}}$ is a perturbation of the infinitesimal operator $\mathcal{A}$ by an operator of the form $\mathcal{B}P$, where $\mathcal{P} : Z \to \mathbf{C}^n$. There are three different kinds of such a perturbation.

1. *Perturbation with a bounded operator* corresponds to the case when we admit feedback controls

$$u = \mathcal{P}(x(\cdot)) = P(x(t) - A_{-1}x(t-1)) + \int_{-1}^{0} \hat{P}(\theta)x(t+\theta)d\theta,$$

where P is a real $(r \times n)$-matrix, $\hat{P}(\theta), \theta \in [-1, 0]$ is a real square-integrable $(r \times n)$-matrix-function.

In this case $\mathcal{B}P$ is a bounded operator and so [17] the perturbation operator $\tilde{\mathcal{A}}$ is infinitesimal and $\mathcal{D}(\tilde{\mathcal{A}}) = \mathcal{D}(\mathcal{A})$. Note, however, that possibilities of stabilization in this class of controls are rather restricted.

2. *Perturbation with an operator bounded with respect to* $\mathcal{A}$ corresponds to the choice

$$u = \mathcal{P}(x(\cdot)) = \int_{-1}^{0} \tilde{P}(\theta)\dot{x}(t+\theta)d\theta + \int_{-1}^{0} \hat{P}(\theta)x(t+\theta)d\theta,$$

where $\tilde{P}(\theta), \hat{P}(\theta), \theta \in [-1, 0]$ are real square-integrable $(r\times n)$-matrix-function. In this case one can easily check that operator $\mathcal{B}P$ is a bounded with respect to $\mathcal{A}$ [17] i.e., for some $a, b > 0$

$$\left\| \mathcal{B}P\begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \right\| \le a \left\| \mathcal{A}\begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \right\| + b \left\| \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \right\|.$$

This implies $\mathcal{D}(\tilde{\mathcal{A}}) = \mathcal{D}(\mathcal{A})$. At the same time, the infinitesimality of $\tilde{\mathcal{A}}$ must be proved separately (see [17]). One can observe, however, an important for our further purpose particular case when this infinitesimality is obvious. Let

$$u = \mathcal{P}(x(\cdot)) = P_0x(t) + P_1x(t-1) + \int_{-1}^{0} \hat{P}(\theta)x(t+\theta)d\theta,$$

P_0, P_1 are $(r \times n)$-matrices and $\hat{P}(\theta)$ is a square-integrable $(r \times n)$-matrix-function. Then the operator $\tilde{\mathcal{A}}$ can be represented as a perturbation of the infinitesimal operator $\tilde{\mathcal{A}}_1$:

$$\tilde{\mathcal{A}}_1 \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} = \begin{pmatrix} (A_0 + BP_0)q + ((A_1 + BP_1) + (A_0 + BP_0)A_{-1})\varphi(-1) \\ \frac{\partial}{\partial\theta}\varphi(\cdot) \end{pmatrix}, \quad (6)$$

by a bounded operator. So it is also infinitesimal.

Consider the possibilities of stabilization by feedback controls of class 2. It can be proved that the spectrum $\sigma(\tilde{\mathcal{A}}) = \tilde{\sigma}$ of the perturbed operator $\tilde{\mathcal{A}}$ is given by

$$\tilde{\sigma} = \Big\{ \quad \lambda \quad | \quad det \Big(\lambda I - A_{-1}\lambda e^{-\lambda} - A_0 - A_1 e^{-\lambda}$$
$$+B\lambda \int_{-1}^{0} e^{-\lambda\theta}\tilde{P}(\theta)\dot{x}(t+\theta)d\theta + B\int_{-1}^{0} e^{-\lambda\theta}\hat{P}(\theta)x(t+\theta)d\theta\Big)\Big\} = 0$$

and then it also includes the families $\sum^{\mu}$ of the form (4) for any $\mu \in \sum$. This means that the exponential stability of the closed-loop system (5) is possible only in the case when $|\mu| < 1$ for all $\mu \in \sum$. On the other hand, it is clear that the system (5) is unstable if there exists at least one $\mu \in \sum$ such that $|\mu| > 1$. It remains one more case to be examined. Let

(a1) $\sum \subset \{w : |w| \leq 1\}$ and there exists $\mu \in \sum : |\mu| = 1$.

In this case system (5) cannot be stable exponentially but probably can be strongly stable. That leads us to the following statement:

Problem of Strong Stabilizability (PSS) *Let matrix A_{-1} satisfy (a1). Find conditions on system (2) (or (3)) under which there exists a feedback control of class 2 such that the perturbed operator $\tilde{\mathcal{A}}$ in (5) is infinitesimal and all the solutions of this equation tend to* 0 *as $t \to +\infty$ in the norm of Z.*

We consider PSS in the further sections. Now we mention one more way to formulate the stabilizability problem.

3. *Perturbation with an operator unbounded with respect to $\mathcal{A}$.*

It is shown [18] that the possibilities of stabilization of system (2) are essientally wider if we admit feedback controls of the form

$$u = \mathcal{P}(x(\cdot)) = P_{-1}\dot{x}(t-1)) + \int_{-1}^{0} \tilde{P}(\theta)\dot{x}(t+\theta)d\theta + \int_{-1}^{0} \hat{P}(\theta)x(t+\theta)d\theta. \quad (7)$$

This kind of stabilization is out of our consideration. We only notice the use of a control (7) means, from the operator point of view, a perturbation of $\mathcal{A}$ by an operator $\mathcal{B}P$ which is not bounded with respect to $\mathcal{A}$. In particular, that implies $\mathcal{D}(\tilde{\mathcal{A}}) \neq \mathcal{D}(\mathcal{A})$. So even if we prove infinitesimality of $\tilde{\mathcal{A}}$, the domains of solutions of the initial and closed-loop systems are different.

3 Operator analysis of the model

We consider PSS and supplement (a1) by the following assumptions characterizing the class of systems (2) we deal with:

(a2) All the eigenvalues $\mu \in \sum$ such that $|\mu| = 1$ are simple in the sense that there are no Jordan chains corresponding to such eigenvalues.

(a3) The finite-dimensional system

$$\dot{x}(t) = A_0 x(t) + Bu(t), \quad x \in R^n, u \in R^r \tag{8}$$

is controlable, i.e. $rank\,(\,B, A_0 B, ..., A_0^{n-1} B\,) = n$. In particular, this implies that (8) is stabilizable, i.e. there exists a linear feedback control $u = P_0^0 x$ such that $\Re(\sigma)(A + BP_0^0) < 0$.

(a4) $rank\,(\,A_1 + A_0 A_{-1} \quad B\,) = rank B$.

Let us put into (2) a control $u(t) = P_0 x(t) + P_1 x(t-1) + v(t)$. That leads to replace (3) by the system

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \tilde{\mathcal{A}}_1 \begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} + \mathcal{B}v(t),$$

where $\tilde{\mathcal{A}}_1$ is given by (6). Due to (a4) for any $P_0 \in \mathbf{R}^{(r\times n)}$ there exists unique $P_1 = P_1(P_0) \in \mathbf{R}^{(r\times n)}$ such that

$$A_1 + BP_1 + (A_0 + BP_0)A_{-1} = (A_1 + A_0 A_{-1}) + BP_0 A_{-1} + BP_1 = 0.$$

For this choice of P_1 operator $\tilde{\mathcal{A}}_1$ takes the form

$$\tilde{\mathcal{A}}_1 \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} = \begin{pmatrix} A_0 + BP_0 & 0 \\ 0 & \frac{\partial}{\partial\theta} \end{pmatrix} \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix}. \tag{9}$$

Proposition 1 *Let $\tilde{\mathcal{A}}_1$ be given by (9). Then*

i) $\sigma(\tilde{\mathcal{A}}_1) = \sigma(A_0 + BP_0) \cup \log \sum$.

ii) Under the assumption $\sigma(A_0 + BP_0) \cap \log \sum = \emptyset$ the set of eigenvectors of $\tilde{\mathcal{A}}_1$ is as follows:

a) to each eigenvector $d \in \mathbf{C}^n$ of $A_0 + BP_0$ with eigenvalue λ there corresponds an eigenvector

$$\tilde{d} = \begin{pmatrix} d \\ (I - e^{-\lambda} A_{-1})^{-1} e^{\lambda\theta} \end{pmatrix}$$

of $\tilde{\mathcal{A}}_1$ with the same eigenvalue;

b) to each eigenvector $g \in \mathbf{C}^n$ of A_{-1} with eigenvalue μ there corresponds a family $\{\tilde{g}_k\}_{k\in\mathbf{Z}}$ of eigenvectors of $\tilde{\mathcal{A}}$:

$$\tilde{g}_k = \begin{pmatrix} (I - e^{-\lambda_k^\mu} A_{-1}) g \\ e^{\lambda_k^\mu \theta} g \end{pmatrix} = \begin{pmatrix} 0 \\ e^{\lambda_k^\mu \theta} g \end{pmatrix},$$

where $\lambda_k^\mu = \log|\mu| + i(Arg\,\mu + 2\pi k)$, $k \in \mathbf{Z}$ is the eigenvalue corresponding to $\tilde{g}_k$.

Proof: Let $\binom{q}{\varphi(\cdot)}$ be an arbitrary eigenvector of $\tilde{\mathcal{A}}_1$ and λ be the corresponding eigenvalue. Taking into account (9) we have $\varphi'(\theta) = \lambda\varphi(\theta)$ and $(A_0 + BP_0)q = \lambda q$.

From the first equality we obtain $\varphi(\theta) = e^{\lambda\theta}c$, $c \in \mathbf{C}^n, c \neq 0$. Since $q = \varphi(0) - A_{-1}\varphi(1)$, then the second equality yields $(A_0 + BP_0)(c - A_{-1}e^{-\lambda}c) - \lambda(c - A_{-1}e^{-\lambda}c) = (A_0 + BP_0 - \lambda I)(I - A_{-1}e^{-\lambda})c = 0$. Therefore, either $(I - A_{-1}e^{-\lambda})c$ is an eigenvector for $A_0 + BP_0$ corresponding to λ or c is an eigenvector of A_{-1} corresponding to e^λ. Analysis of this alternative completes the proof. ∎

Using (a3) one can choose $P_0 \in \mathbf{R}^{(r\times n)}$ in such a way that the spectum $\sigma(A_0 + BP_0)$ consists of n distinct negative eigenvalues which do not belong to the set $\log \sum$. Let further P_0^0 be such a matrix, $P_1^0 = P_1(P_0^0)$ and $\tilde{\mathcal{A}}_1$ be the operator (6) corresponding to the choice $P_0 = P_0^0$, $P_1 = P_1^0$. Then, by Proposition 1, the spectrum $\sigma(\tilde{\mathcal{A}}_1)$ belongs to the semiplane $\{\lambda : \Re(\lambda) \leq 0\}$. Our next goal is to prove that the system

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \tilde{\mathcal{A}}_1^0 \begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} + \mathcal{B}v(t), \tag{10}$$

is strongly stabilizable by linear bounded controls. To show that, we first prove dissipativity of the operator $\tilde{\mathcal{A}}_1$ in some equivalent norm in Z.

Let $d_j, j = 1, ..., n$ be eigenvectors of $A_0 + BP_0$, $(A_0 + BP_0)d_j = \lambda_j d_j$ $\lambda_j < 0$, $j = 1, ..., n$. Denote by D a nonsingular matrix $D = (d_1, d_2, ..., d_n)$. Next observe that, due to $(a1) - (a2)$, matrix A_{-1} can be represented in the form

$$A_{-1} = GJG^{-1}, \tag{11}$$

where G is a nonsingular matrix and J is a contraction, $\|J\| \leq 1$. As the matrix J one can take, for example, a block diagonal form of A_{-1} which boxes are

$$J_k = \begin{pmatrix} \mu_k & \nu_k & 0 & \dots & 0 \\ 0 & \mu_k & \nu_k & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mu_k \end{pmatrix},$$

where $|\nu_k| \leq 1 - |\mu_k|$, $k = 1, .., \ell$ (note that all the eigenvalues μ_k of A_{-1} such that $|\mu_k| = 1$ are simple $(a2)$). Finally introduce a linear transformation $F : \mathbf{C}^n \to L_2[(-1, 0), \mathbf{C}^n]$ defined by

$$Fq = F\left(\sum_{j=1}^{n} q_j d_j\right) = -G^{-1} \sum_{j=1}^{n} q_j \left(I - A_{-1}e^{-\lambda_j}\right)^{-1} e^{\lambda_j \theta} d_j, \tag{12}$$

$q \in \mathbf{C}^n$. Consider now a linear bounded operator $T : Z \to Z$ given by

$$T = \begin{pmatrix} D^{-1} & 0 \\ F & G^{-1} \end{pmatrix}$$

and the corresponding equivalent Hilbert norm $\|\cdot\|_T$ in Z defined by

$$\left\|\begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix}\right\|_T = \left\|T\begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix}\right\| = \sqrt{\|D^{-1}q\|^2 + \int_{-1}^{0} |(Fq)(\theta) + G^{-1}\varphi(\theta)|^2 d\theta}.$$

This new norm allows to get dissipativity of the operator $\tilde{\mathcal{A}}_1^0$.

Proposition 2 *Operator $\tilde{\mathcal{A}}_1^0$ is dissipative in the norm $\|\cdot\|_T$, i.e.*

$$\Re\left(\left\langle \tilde{\mathcal{A}}_1^0 \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix}, \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix}\right\rangle_T\right) \leq 0, \quad \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \in \mathcal{D}(A).$$

Proof: We have

$$\begin{aligned} &\left\langle \tilde{\mathcal{A}}_1^0 \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix}, \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix}\right\rangle_T = \left\langle D^{-1}(A_0 + BP_0^0)q, D^{-1}q\right\rangle + \\ &\int_{-1}^{0} \left\langle (F(A_0 + BP_0^0)q)(\theta) + G^{-1}\varphi'(\theta), (Fq)(\theta) + G^{-1}\varphi(\theta)\right\rangle d\theta. \end{aligned} \tag{13}$$

Let $q = \sum_{j=1}^{n} q_j d_j$. Then $\langle D^{-1}(A_0 + BP_0^0)q, D^{-1}q\rangle = \sum_{j=1}^{n} \lambda_j \|d_j\|^2$ and, therefore,

$$\Re\left(\left\langle D^{-1}(A_0 + BP_0^0)q, D^{-1}q\right\rangle\right) = \sum_{j=1}^{n} \Re(\lambda_j)\|d_j\|^2 \leq 0. \tag{14}$$

Let us denote the second term of (13) by R. Taking into account (12) we have

$$(F(A_0 + BP_0^0)q)(\theta) = G^{-1}\sum_{j=1}^{n} q_j \left(I - A_{-1}e^{-\lambda_j}\right)^{-1} e^{\lambda_j\theta} d_j = \frac{d}{d\theta}(Fq)(\theta).$$

Therefore

$$\begin{aligned} R = & \int_{-1}^{0} \left\langle \frac{d}{d\theta}[(Fq)(\theta) + G^{-1}\varphi(\theta)], (Fq)(\theta) + G^{-1}\varphi(\theta) \right\rangle d\theta \\ = & \|(Fq)(\theta) + G^{-1}\varphi(\theta)\|^2|_{-1}^{0} - \\ & \int_{-1}^{0} \left\langle (Fq)(\theta) + G^{-1}\varphi(\theta), \frac{d}{d\theta}[(Fq)(\theta) + G^{-1}\varphi(\theta)] \right\rangle d\theta \end{aligned}$$

and

$$\Re(R) = \frac{1}{2}\left(\|(Fq)(0) + G^{-1}\varphi(0)\|^2 - \|(Fq)(-1) + G^{-1}\varphi(-1)\|^2\right). \quad (15)$$

Note that

$$\varphi(0) - A_{-1}\varphi(-1) = q = \sum_{j=1}^{n} q_j d_j. \quad (16)$$

Let $\varphi(\theta) = \sum_{j=1}^{n} q_j \left(I - A_{-1}e^{-\lambda_j}\right)^{-1} e^{\lambda_j\theta} d_j + \psi(\theta)$. Then one can easily check that (16) implies

$$\psi(0) = A_{-1}\psi(-1). \quad (17)$$

From (12) we obtain

$$\begin{aligned} & (Fq)(\theta) + G^{-1}\varphi(\theta) = \\ & -G^{-1}\left(\sum_{j=1}^{n} q_j \left(I - A_{-1}e^{-\lambda_j}\right)^{-1} e^{\lambda_j\theta} d_j - \varphi(\theta)\right) = G^{-1}\psi(\theta). \end{aligned}$$

Hence, taking into account (17), relation (15) can be rewritten as $\Re(R) = \frac{1}{2}(\|G^{-1}\psi(0)\|^2 - \|G^{-1}\psi(-1)\|^2) = \frac{1}{2}(\|G^{-1}A_{-1}\psi(-1)\|^2 - \|G^{-1}\psi(-1)\|^2)$. Let us substitute $G^{-1}\psi(-1) = w$ and make use of (11). That yields

$$\Re(R) = \frac{1}{2}\left(\|Jw\|^2 - \|w\|^2\right) \leq 0. \quad (18)$$

Comparison of (13), (14) and (18) completes the proof. ∎

Corollary 3 *It follows from Proposition 2 that the semigroup* $\{e^{\tilde{\mathcal{A}}_1^0 t}\}_{t\geq 0}$ *is contractive in the norm* $\|\cdot\|_T$*. In fact,*

$$\frac{d}{dt}\|e^{\tilde{\mathcal{A}}_1^0 t}\|_T^2 = \Re\left(\left\langle \tilde{\mathcal{A}}_1^0 e^{\tilde{\mathcal{A}}_1^0 t}\begin{pmatrix} q \\ \varphi(\cdot)\end{pmatrix}, e^{\tilde{\mathcal{A}}_1^0 t}\begin{pmatrix} q \\ \varphi(\cdot)\end{pmatrix}\right\rangle_T\right) \leq 0.$$

This means that (10) is a contractive system in the space Z *with norm* $\|\cdot\|_T$ *(see [11]).*

4 The strong stabilizability

In order to analyze strong stabilizability of (10) we make use of the following theorem on the strong stabilizability of contractive systems [11, Theorem 5]:

Consider a system of the form

$$\frac{d}{dt}x = Ax + Bu, \quad x \in H, u \in U,$$

where H and U are Hilbert spces, the operator A generates a strongly continuous contractive semigroup $\{e^{At}\}_{t\geq 0}$ and $B \in [U, H]$. Let there exists $t_0 > 0$ such that the set $\sigma(e^{At_0}) \cap \{w \in \mathbf{C} | |w| = 1\}$ is at most countable. Then the system is strongly stabilizable (with the aid of linear bounded controls) if and only if there does not exist an eigenvector x_0 of the operator A corresponding to an eigenvalue $\lambda_0 (\Re(\lambda_0) = 0)$ such that $x_0 \in Ker\, B^*$. If this condition holds then the strong stabilizing control can be chosen as

$$u = -B^* x.$$

We showed that the semigroup $\{e^{\tilde{\mathcal{A}}_1^0 t}\}_{t\geq 0}$ is contractive in the space Z with norm $\|\cdot\|_T$. It is known [19] that

$$\sigma(e^{\tilde{\mathcal{A}}_1^0 t_0}) = \overline{\exp(t_0 \sigma(\tilde{\mathcal{A}}_1^0))}$$

($\overline{S}$ means the closure of S). From Proposition 1 we have for $t_0 = 1$:

$$\exp(\sigma(\tilde{\mathcal{A}}_1^0)) = \exp(\sigma(A^0 + BP_0^0)) \cup \sum$$

and, therefore, this set is finite. Hence the set

$$\sigma(e^{\tilde{\mathcal{A}}_1^0}) \cap \{w \in \mathbf{C} : |w| = 1\} \subset \sigma(e^{\tilde{\mathcal{A}}_1^0}) = \exp(\sigma(\tilde{\mathcal{A}}_1^0))$$

is also finite. Thus, basing on the [11, Theorem 5] we conclude that the system (10) is strongly stabilizable (notice that stabilizabilities in norms $\|\cdot\|$ and $\|\cdot\|_T$

are equivalent) iff there does not exist an eigenvector f_0 of the operator $\tilde{\mathcal{A}}_1^0$ corresponding to a pure imaginary eigenvalue such that

$$f_0 \in Ker\, \mathcal{B}_T^*, \tag{19}$$

where $\mathcal{B}_T^* : (Z, \|\cdot\|_T) \to \mathbf{C}^r$ is the adjoint operator to $\mathcal{B}$ in the norm $\|\cdot\|_T$. If (19) holds then one can choose

$$v = -\mathcal{B}_T^* \begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} \tag{20}$$

as a strong stabilizing control.

Let $u = \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} \in \mathbf{C}^r$, $\left(\begin{smallmatrix} q \\ \varphi(\cdot) \end{smallmatrix}\right) \in Z$ and $q = \sum_{j=1}^n q_j d_j$, $b_k = \sum_{j=1}^n b_{jk} d_j$, $k = 1,..,r$, where b_k is the k-th column of B, $\{d_j\}_{j=1}^n$ is the eigenbasis of $A_0 + BP_0^0$ appearing in definition of the operator T. In order to find $\mathcal{B}_T^*$ we observe

$$\left\langle \mathcal{B}u, \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \right\rangle_T = \left\langle T\begin{pmatrix} Bu \\ 0 \end{pmatrix}, T\begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} D^{-1}Bu \\ (FBu)(\cdot) \end{pmatrix}, \begin{pmatrix} D^{-1}q \\ (Fq)(\cdot) + G^{-1}\varphi(\cdot) \end{pmatrix} \right\rangle = \left\langle u, B^* D^{-1*} D^{-1} q \right\rangle_{\mathbf{C}^n}$$

$$+ \sum_{k=1}^r u_k \sum_{i,j=1}^n \frac{1 - e^{-\lambda_i - \lambda_j}}{\lambda_i + \lambda_j} \left\langle G^{-1}\left(I - A_{-1}e^{-\lambda_i}\right)^{-1} d_i, G^{-1}\left(I - A_{-1}e^{-\lambda_j}\right)^{-1} d_j \right\rangle_{\mathbf{C}^n}$$

$$- \sum_{k=1}^r u_k \sum_{j=1}^n \left\langle d_j, \left(I - A_{-1}^* e^{-\lambda_j}\right)^{-1} G^{-1*} G^{-1} \int_{-1}^{0} e^{\lambda_j \theta} \varphi(\theta) d\theta \right\rangle_{\mathbf{C}^n} b_{jk}.$$

Let Q be an $(n \times n)$-matrix which j-th row is

$$e^{\lambda_j \theta} d_j^* \left(I - A_{-1}^* e^{-\lambda_j}\right)^{-1} G^{-1*} G^{-1}, j = 1,..,n.$$

With these notations, taking into account that

$$\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = D^{-1} q, \quad \begin{pmatrix} b_{11} & \dots & b_{1r} \\ \dots & \dots & \dots \\ b_{11} & \dots & b_{1r} \end{pmatrix} = D^{-1} B,$$

we obtain

$$\left\langle \mathcal{B}u, \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \right\rangle_T = \left\langle u, B^* D^{-1*} D^{-1} q \right\rangle_{\mathbf{C}^n}$$

$$+ \left\langle u, B^* D^{-1*} Q D^{-1} q \right\rangle_{\mathbf{C}^n} - \left\langle u, \int_{-1}^{0} B^* D^{-1*} \hat{Q}(\theta) \varphi(\theta) d\theta \right\rangle_{\mathbf{C}^n}.$$

Thus, the stabilizing control (20) is of the form

$$v = -\mathcal{B}_T^* \begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = -B^* D^{-1*} \left((I+Q)D^{-1}y(t) - \int\limits_{-1}^{0} \hat{Q}(\theta) z_t(\theta) d\theta \right). \quad (21)$$

Now let us analyze the condition (19). Among all the eigenvalues of $\tilde{\mathcal{A}}_1^0$ the pure imaginary ones are (see Proposition 1)

$$\lambda_k^\mu = \log|\mu| + i(Arg\mu + 2\pi k), \quad k \in \mathbf{Z}, \quad (22)$$

for $\mu \in \sum$ such that $|\mu| = 1$. For the corresponding eigenvectors $\tilde{g}_k = \begin{pmatrix} 0 \\ e^{\lambda_k^\mu \theta} g \end{pmatrix}, \quad k \in \mathbf{Z}$ we have

$$\begin{aligned} \mathcal{B}_T^* \tilde{g}_k &= B^* D^{-1*} \int\limits_{-1}^{0} \hat{Q}(\theta) e^{\lambda_k^\mu \theta} g d\theta \\ &= B^* D^{-1*} \begin{pmatrix} d_1^* \left(I - A_{-1}^* e^{-\lambda_1}\right)^{-1} \frac{1-e^{-\lambda_1 - \lambda_k^\mu}}{\lambda_1 + \lambda_k^\mu} \\ \cdots \\ d_n^* \left(I - A_{-1}^* e^{-\lambda_n}\right)^{-1} \frac{1-e^{-\lambda_n - \lambda_k^\mu}}{\lambda_n + \lambda_k^\mu} \end{pmatrix} G^{-1*} G^{-1} g. \end{aligned} \quad (23)$$

In (23) g is an eigenvector of A_{-1} corresponding to eigenvalue μ. This implies that $G^{-1*}G^{-1}g$ is an eigenvector of A_{-1}^* corresponding to complex conjugate eigenvalue $\bar{\mu}$. Indeed, taking into account (11) we get

$$\begin{aligned} \left\langle J^* G^{-1} g, G^{-1} g \right\rangle &= \left\langle G^* A_{-1}^* G^{-1*} G^{-1} g, G^{-1} g \right\rangle \\ &= \left\langle G^{-1*} G^{-1} g, A_{-1} g \right\rangle \\ &= \bar{\mu} \left\langle G^{-1} g, G^{-1} g \right\rangle. \end{aligned} \quad (24)$$

Since the adjoint operator J^* is also a contraction then (24) yields $J^*G^{-1}g = \bar{\mu}G^{-1}g$. From here and (11) we get $A_{-1}^* G^{-1*} G^{-1} g = \bar{\mu} G^{-1*} G^{-1} g$. This fact and the observation that $e^{-\lambda_k^\mu} = \bar{\mu}$, $\overline{\lambda_k^\mu} = -\lambda_k^\mu$, $k \in \mathbf{Z}$ allow to rewrite (23) as

$$\mathcal{B}_T^* \tilde{g}_k = B^* D^{-1*} \begin{pmatrix} \frac{1}{\lambda_1 + \lambda_k^\mu} d_1^* \\ \cdots \\ \frac{1}{\lambda_n + \lambda_k^\mu} d_n^* \end{pmatrix} g = B^* R_{\lambda_k^\mu}^* (A + BP_0^0) g, \quad (25)$$

where $R_\lambda(A_0 + BP_0^0) = (A_0 + BP_0^0 - \lambda I)^{-1}$ is the resolvent of matrix $A_0 + BP_0^0$. With respect to formulas (21) and (25) the necessary and sufficient conditions of the strong stabilizability for the system (10) take the following form:

Theorem 4 *System (10) is strongly stabilizable (with the aid of the of bounded controls) iff there do not exist an eigenvector g of matrix A_{-1} corresponding to an eigenvalue $\mu \in \sum$, $|\mu| = 1$ and $k \in \mathbf{Z}$ such that*

$$B^* R^*_{\lambda_k^\mu}(A_0 + BP_0^0)g = 0,$$

where λ_k^μ is given by (22). Under this condition a stabilizing control is given by (21).

Remark. *Let P_0 be a $(r \times n)$-matrix and let $\lambda \in \mathbf{C}$ be such that $\lambda \notin \sigma(A_0 + BP_0^0) \cup \sigma(A_0 + BP_0)$. Let us precise that*

$$\begin{aligned} R^*_\lambda(A_0 + BP_0^0) - R^*_\lambda(A_0 + BP_0) &= R^*_\lambda(A_0 + BP_0)(P_0^* - P_0^{0*})B^* R^*_\lambda(A_0 + BP_0^0) \\ &= R^*_\lambda(A_0 + BP_0^0)(P_0^* - P_0^{0*})B^* R^*_\lambda(A_0 + BP_0). \end{aligned}$$

From this identity one can easily infer that for given μ, g, λ_k^μ the relation $B^* R^*_{\lambda_k^\mu}(A_0 + BP_0^0)g = 0$ holds if and only if $B^* R^*_{\lambda_k^\mu}(A_0 + BP_0)g = 0$ for an arbitrary P_0 such that $\lambda_k^\mu \notin \sigma(A + BP_0)$.

The following theorem is the main result of the paper.

Theorem 5 *Let a system (2) satisfy the assumptions $(a1) - (a4)$. Then this system is strongly stabilizable with the aid of feedback controls of class 2 if and only if for an arbitrarily chosen matrix P_0 such that*

$$\sigma(A_0 + BP_0) \cap \log(\sum) \cap (i\mathbf{R}) = \emptyset$$

there do not exist an eigenvector g of A_{-1} corresponding to an eigenvalue $\mu \in \sum$, $|\mu| = 1$ and $k \in \mathbf{Z}$ such that

$$B^* R^*_{\lambda_k^\mu}(A_0 + BP_0)g = 0, \tag{26}$$

where λ_k^μ is given by (22). Under this condition the strong stabilization can by achived by the choice of control:

$$u = P_0^0 x(t) + P_1^0 x(t-1) + v,$$

where P_0^0 and P_1^0 are defined in Section 2 and v is given by (21):

$$v = -B^* D^{-1*}\left(\Big((I+Q)D^{-1}(x(t) - A_{-1}x(t-1))\Big) - \int_{-1}^{0} \hat{Q}(\theta)x(\theta)d\theta\right).$$

Proof: Sufficiency follows directly from Theorem 4 and the Remark to this theorem.

Let us prove the necessity. Assume that there exists a control $u = \mathcal{P}\left(\begin{smallmatrix} y(t) \\ z_t(\cdot) \end{smallmatrix}\right)$ of class 2 which strongly stabilizes system (3). This means that the operator $\tilde{\mathcal{A}} = \mathcal{A} + \mathcal{B}P$ with $\mathcal{D}(\tilde{\mathcal{A}}) = \mathcal{D}(A)$ is infinitesimal and the semogroup $\{e^{\tilde{\mathcal{A}}t}\}_{t\geq 0}$ is strongly asymptotic stable. Then $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_1^0 + \mathcal{B}P_1$, where $\mathcal{B}P_1$ is an operator bounded with respect to $\mathcal{A}$. If (26) does not hold then (see Remark) there exist an eigenvector g of A_{-1} corresponding to an eigenvalue $\mu \in \sum$, $|\mu| = 1$ and $k \in \mathbf{Z}$ such that $B^* R^*_{\lambda_k^\mu}(A_0 + BP_0^0)g = 0$, This implies (see (25)) that the eigenvector $\tilde{g}_k$ of $\tilde{\mathcal{A}}_1^0$, i.e. $\tilde{\mathcal{A}}_1^0 \tilde{g}_k = \lambda_k^\mu \tilde{g}_k$ belongs to $Ker\,\mathcal{B}_T^*$.

Let us show that $\tilde{g}_k$ is an eigenvector of the operator $(\tilde{\mathcal{A}}_1^0)_T^*$, adjoint to $\tilde{\mathcal{A}}_1^0$ in the norm $\|\cdot\|_T$, and the corresponding to $\tilde{g}_k$ eigenvalue of $(\tilde{\mathcal{A}}_1^0)_T^*$ equals $\overline{\lambda_k^\mu} = -\lambda_k^\mu$. Let $f \in \mathcal{D}(\tilde{\mathcal{A}}_1^0) = \mathcal{D}(A)$ and $w \in \mathbf{C}$. Then using the dissipativity of $\tilde{\mathcal{A}}_1^0$ (see Proposition 2) we have

$$\begin{aligned} 0 &\geq \Re\left(\left\langle(\tilde{\mathcal{A}}_1^0 - \lambda_k^\mu I)(\tilde{g}_k + wf), (\tilde{g}_k + wf)\right\rangle_T\right) \\ &= \Re\left(w\left\langle(\tilde{\mathcal{A}}_1^0 - \lambda_k^\mu I)f, \tilde{g}_k\right\rangle_T\right) + |w|^2 \Re\left(\left\langle(\tilde{\mathcal{A}}_1^0 - \lambda_k^\mu I)f, f\right\rangle_T\right). \end{aligned}$$

Let us put $w = \alpha\overline{\left\langle(\tilde{\mathcal{A}}_1^0 - \lambda_k^\mu I)f, \tilde{g}_k\right\rangle_T}$, $\alpha \in \mathbf{R}$. This leads to the inequality

$$\alpha\left|\left\langle(\tilde{\mathcal{A}}_1^0 - \lambda_k^\mu I)f, \tilde{g}_k\right\rangle_T\right|^2 (1 + \alpha\Re\left\langle(A - \lambda_k^\mu I)\,f, f\right\rangle) \leq 0$$

which holds for all $\alpha \in \mathbf{R}$. It follows from here that

$$\left\langle(\tilde{\mathcal{A}}_1^0 - \lambda_k^\mu I)f, \tilde{g}_k\right\rangle_T = 0, \text{ for all } f \in \mathcal{D}(\tilde{\mathcal{A}}_1^0).$$

The later relation means that $\tilde{g}_k \in \mathcal{D}((\tilde{\mathcal{A}}_1^0)_T^*)$ and

$$(\tilde{\mathcal{A}}_1^0)_T^* \tilde{g}_k = \overline{\lambda_k^\mu}\tilde{g}_k = -\lambda_k^\mu \tilde{g}_k.$$

Since in addition

$$\tilde{g}_k \in Ker\,\mathcal{B}_T^* \subset Ker\,(\mathcal{B}P_1)_T^*$$

then $\tilde{g}_k \in \mathcal{D}(\tilde{\mathcal{A}}_T^*)$ and

$$\tilde{\mathcal{A}}_T^* \tilde{g}_k = (\tilde{\mathcal{A}}_1^0)_T^* \tilde{g}_k + (\tilde{\mathcal{B}P_1})_T^* \tilde{g}_k = -\lambda_k^\mu \tilde{g}_k.$$

Hence

$$\left(e^{\tilde{\mathcal{A}}t}\right)_T^* \tilde{g}_k = e^{-\lambda_k^\mu t}\tilde{g}_k$$

and, as a consequence,

$$\left\langle e^{\tilde{\mathcal{A}}t}\tilde{g}_k, \tilde{g}_k\right\rangle_T = \left\langle \tilde{g}_k, \left(e^{\tilde{\mathcal{A}}t}\right)_T^* \tilde{g}_k\right\rangle_T = e^{\lambda_k^\mu t}\|\tilde{g}_k\|_T^2, \quad t \geq 0.$$

Thus

$$\|e^{\tilde{\mathcal{A}}t}\tilde{g}_k\|_T \geq \|\tilde{g}_k\|_T \not\to 0 \quad \text{as } t \to +\infty.$$

This contradiction completes the proof. ∎

Remark. *Assume that* $rank\, B = n$*. In this case one can easily observe that assumptions (a3)-(a4) are satisfied automatically. Besides, the condition (26) from Theorem 2 is also always satisfied. So any system (2) with* $rank\, B = n$ *and (a1)-(a2) is strongly stabilizable.*

Example

Consider the following one-dimensional system

$$\dot{x}(t) = -x(t) + x(t-1) + \dot{x}(t-1) + u(t). \tag{27}$$

It is shown in [20] that this system is not exponentially stabilizable by a feedback of clas 2, because only a finite part of spectrum of the closed-loop system can be moved to a semiplane $\{\lambda : \Re(\lambda) \leq \alpha < 0\}$

Now observe that (27) is strongly stabilizable due to the Theorem 5. In fact, for this system we have: $n = 1, A_0 = -1, A_1 = A_{-1} = 1, B = 1, \sum = \{-1\}$ which is simple eigenvalue. Since $rank\, B = 1 = n$ and (a1)-(a2) are satisfied the (27) is strongly stabilizable. Let us find a stabilizing control. Since $A_1 + A_0 A_{-1} = 0$ and $\sigma(A_0) = \{-1\}$ is real negative we can put $P_0^0 = P_1^0 = 0$. Further simple calculations give $G = 1$, $D = 1, (Fq)(\theta) = -(1-e)^{-1} e^{-\theta} q$ and, therefore,

$$Q = \frac{1}{2}\left(\frac{e+1}{e-1}\right), \quad \hat{Q}(\theta) = \frac{e^{-\theta}}{e-1}.$$

Thus, a stabilizing control from Theorem 5 for our system takes the form

$$u = -\left(1 + \frac{1}{2}\frac{e+1}{e-1}\right)(x(t) - x(t-1)) - \frac{1}{e-1}\int_{-1}^{0} e^{-\theta} x(t+\theta) d\theta.$$

5 Conclusion

For linear systems of neutral type we gave a characterization of a class of strong stabilizable systems by relatively bounded feedback law. No derivative of the state is needed in the feedback. The contrepart is that the stabilizability is not exponential. As a perspective, one can expect that this technique may be used for more general systems with delay of neutral type. using the same infinite dimensional abstract framework.

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