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Abstract

The asymtoptic stability properties of neutral type systems are studied mainly in the critical case when the exponential stability is not possible. We consider an operator model of the system in Hilbert space and use recent results on the existence of a Riesz basis of invariant finite-dimensional subspaces in order to verify its dissipativity. The main results concern the conditions of asymptotic non exponential stability. We show that the property of asymptotic stability is not determinated only by the spectrum of the system but essentially depends on the geometric spectral characteristic of its main neutral term. Moreover, we present an example of two systems of neutral type which have both the same spectrum in the open left-half plane and the main neutral term but one of them is asymptotically stable while the other is unstable.

Keywords. Neutral type systems, exponential stability, strong stability, infinite dimensional systems.

Mathematical subject classification. 34K06, 34K20, 34K40, 93C23.

1 Introduction

A number of applied problems from physics, mechanics, biology and other fields can be described by partial differential or delay differential equations. This leads to the construction and study of infinite-dimensional dynamical systems. In our work we are interested in stability theory of systems with a delayed argument. It is almost impossible to overview the huge literature on the subject, so we only cite some monographs [3,9,13,16,17] which are among the main references. The most used methods and approaches are developed for retarded differential equations while the case of neutral type systems remains more difficult and less studied so far. Our attention is attracted by the fact that in the case of neutral type systems one meets two essentially different type of stability: exponential and strong asymptotic non-exponential stability. The last type of stability is impossible for retarded systems, but may occur for neutral type systems. In particular as it is shown in [5] for a high order differential equation of neutral type the smooth solutions decay essentially slower that exponential, namely as function $1/t^{\beta}, \beta > 0$. One of the explanations of this fact is that a neutral type equation may have an infinite sequence of roots of the characteristic equation with negative real parts approaching to zero. It is obvious that in such a case the equation is not exponentially stable and one needs more subtle methods in order to characterize this type of asymptotic stability.

Our approach is based on the general theory of C_0-semigroups of linear bounded operators (see e.g. [31]).

Let us give the precise description of the system and the operator model under consideration. We study the following neutral type system

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + \int_{-1}^{0} A_2(\theta)\dot{z}(t+\theta)\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z(t+\theta)\mathrm{d}\theta \tag{1}$$

where A_{-1} is constant $n \times n$-matrix,$\det A_{-1} \neq 0$, A_2, A_3 are $n \times n$-matrices whose elements belong to $L_2(-1,0)$. This equation occurs, for example, when a system of neutral type is stabilized. Even if the initial system contains pointwise delays only, then the set of natural feedback laws contains distributed delays (see e.g., [20,21]), so the corresponding closed-loop system takes the form (1).

We do not consider here the case of mixed retarded-neutral type systems, i.e. when $A_{-1} \neq 0, \det A_{-1} = 0$, and limit ourselves to one principal neutral term.

One of the main questions for the construction of an operator model and a corresponding dynamical system is the choice of the phase space. In [13], the framework is based on the description of the neutral type system in the space of continuous functions $C([-1,0];\mathbb{C}^n)$. The essential result in this framework is that the exponential stability is characterized by the condition that the spectrum is in the open left-half plane and bounded away from the imaginary axis (see also [14, Theorem 6.1]). The case when the spectrum is not bounded away from the imaginary axis is much more complicated. It has been shown in [12] that a linear neutral differential equation can have unbounded solutions even though the associated characteristic equation has only purely imaginary roots (see also [8,11,38]). In [5] a lower and upper estimations for the behavior of smooth solutions are given. This suggests that non exponential stability may occur if the characteristic roots are not bounded away from zero. Such a behavior is impossible for ordinary or retarded linear differential equations.

The main purpose of this paper is to characterize the asymptotic stability in the critical case when the exponential one is not possible.

We consider the operator model of neutral type systems introduced by Burns and al. in product spaces. This approach was also used in [33] for the construction of a spectral model. In [37] the authors consider the particular case of discrete delay, which served as a model in [22,23] to characterize the stabilizability of a class of systems of neutral type. The distributed delay case of the system (1) was considered by authors of the present paper in [24].

The state space is $M_2(-1,0;\mathbb{C}^n) = \mathbb{C}^n \times L_2(-1,0;\mathbb{C}^n)$, briefly M_2, and permits (1) to be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \mathcal{A}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \begin{pmatrix} \int_{-1}^{0} A_2(\theta)\dot{z}_t(\theta)\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z_t(\theta)\mathrm{d}\theta \\ \mathrm{d}z_t(\theta)/\mathrm{d}\theta \end{pmatrix}, \quad (2)$$

where the domain of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) = \{(y, z(\cdot)) : z \in H^1(-1,0;\mathbb{C}^n), y = z(0) - A_{-1}z(-1)\} \subset M_2$$

and the operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $\mathrm{e}^{\mathcal{A}t}$. The relation between the solutions of the delay system (1) and the system (2) is $z_t(\theta) = z(t+\theta)$.

In the particular case when $A_2(\theta) = A_3(\theta) = 0$, we use the notation $\bar{\mathcal{A}}$ for $\mathcal{A}$.

We will show that the properties of $\bar{\mathcal{A}}$ can be expressed in terms of the properties of matrix A_{-1} only. We will show also that some important properties of $\mathcal{A}$ are close to those of $\bar{\mathcal{A}}$.

For the dynamical system $(\mathrm{e}^{\mathcal{A}t}, M_2)$, we consider the problem of strong asymptotic stability: *the system is said to be strongly asymptotically stable if for all* $x = (y, z(\cdot)) \in M_2$, $\lim_{t\to+\infty} \mathrm{e}^{\mathcal{A}t}x = 0$.

In contrast to [5], the strong asymptotic stability of the dynamical system $(\mathrm{e}^{\mathcal{A}t}, M_2)$ means the convergence to 0 of all the solutions even if these solutions are not smooth. This is important for the purpose of further investigations, namely, the stabilizability problem.

The fundation for our investigations is the powerful general Theorem 20 on the strong asymptotic stability of an abstract C_0-semigroup in Banach spaces. However the verification of the conditions of stability in Theorem 20 is rather difficult in a Banach space setting. At the same time, the structure of Hilbert space gives more advanced techniques [1,10]. We use also results on stability and stabilizability of abstract linear systems in Hilbert spaces [27,28]. This motivates the choice of M_2 as a phase space.

One of the main ideas in this work is to involve the notion of Riesz bases in the analysis of the stability of neutral type systems [24,26]. Usually, one looks for a Riesz basis of eigenvectors or more generally, a Riesz basis of generalized eigenvectors (eigen- and rootvectors). In this connection we refer to [33] where for a more general system of neutral type a condition for generalized eigenspaces to form a Riesz basis is given. It is noted in [33] that if the spectrum of the system is contained in a vertical strip and $\inf\{|\lambda-\lambda'|,\ \lambda,\lambda' \in \sigma(\mathcal{A}), \lambda \neq \lambda'\} > 0$, then the generalized eigenspaces form a Riesz basis. We recall [10] that a basis $\{V_k\}$ of subspaces is called a basis equivalent to orthogonal (a Riesz basis) if there are an orthogonal basis of subspaces $\{W_k\}$ and a linear bounded invertible (with bounded inverse) operator R, such that $RV_k = W_k$ (see also discussion in [26]).

A simple example [24] proves that the neutral type system (1) (or (2)) does not always possess such a basis since the eigenvalues may be not separated.

One of our main ideas is that in spite of the fact that system (1) does not possess a Riesz basis of generalized eigenspaces, it possesses (see Section 2) a Riesz basis of finite-dimensional subspaces which are invariant under the evolution semigroup. Moreover, this basis is quadratically close to the basis of generalized eigenspaces of the operator $\bar{\mathcal{A}}$, and the last basis is found in a constructive way (Theorem 7). It is important to note that in contrast to [33, Theorem 4.8] for our case we do not assume but prove that generalized eigenspaces of $\bar{\mathcal{A}}$ and the finite-dimensional invariant subspaces of $\mathcal{A}$ are complete in the whole space (see Theorems 7 and 16). Apart from its own value this result gives the way for studying the stability properties.

The existence of a Riesz basis of *invariant* subspaces means that we can split our infinite-dimensional dynamical system on a family of finite-dimensional ones to study them separately. We found these invariant subspaces as the images of spectral projectors given by Schwartz integrals over certain circles with centers in the points $\lambda^{(k)} = \ln|\mu| + \mathrm{i}(\arg\mu + 2\pi k), \mu \in \sigma(A_{-1}); k \in \mathbb{Z}$.

In [34–36], the spectrum of the operator $\mathcal{A}$ is needed in the construction of the Riesz basis of subspaces of solutions of the system like (1) in Sobolev spaces. As one can mention, in our approach, points $\lambda^{(k)}$ are defined by the matrix A_{-1} only. This fact gives us an idea to infer as much information as possible on the stability of (1) in terms of the properties of the constant matrix A_{-1}.

In this context, it is natural to consider three cases: $\mu_{\max} > 1, \mu_{\max} < 1$ and $\mu_{\max} = 1$, where $\mu_{\max} = \max\{|\mu|,\ \mu \in \sigma(A_{-1})\}$. As a consequence of the Theorem on the location of the spectrum of operator $\mathcal{A}$ (see Theorem 2 below and for the simplest case of one discrete delay see [30] and [19, Proposition 1.4, p.35]), we can conclude that in the case $\mu_{\max} > 1$ there are eigenvalues of $\mathcal{A}$ with positive real part, so the system is unstable. On the other hand, the condition $\mu_{\max} < 1$ means that if the system is stable, then the stability is of exponential type. So we apply our main efforts to the difficult case $\mu_{\max} = 1$. Our main result on the stability can be formulated as follows.

Assume that $\sigma(\mathcal{A}) \subset \{\lambda : \operatorname{Re}\lambda < 0\}$ and $\max\{|\mu| : \mu \in \sigma(A_{-1})\} = 1$, where by $\operatorname{Re}\lambda$ we denote the real part of λ. Let us put $\sigma_1 \equiv \sigma(A_{-1}) \cap \{\lambda : |\lambda| = 1\}$. Then the following three mutually exclusive possibilities hold true:

i) σ_1 consists of simple eigenvalues only i.e., to each eigenvalue corresponds an one-dimensional eigenspace and there are no rootvectors. The system (2) is asymptotically stable.
ii) The matrix A_{-1} has a Jordan block, corresponding to an $\mu \in \sigma_1$. In this case the system (2) is unstable.
iii) There are no Jordan blocks, corresponding to eigenvalues in σ_1, but there exists an $\mu \in \sigma_1$ whose eigenspace is at least two-dimensional. In this case the system (2) can be either stable or unstable. Moreover, there exist two systems with the same spectrum, such that one of them is stable while the other one is unstable.

The paper is organized as follows. Section 2 is devoted to the analysis of the fundamental properties of the system: semigroup property, analysis of the spectrum, computation of the resolvent of the operator $\mathcal{A}$ and basis property of invariant subspaces. Results of this section have been partially announced in [24]. Section 3 contains the stability analysis: necessary and sufficient conditions of stability and the open problem: when the matrix A_{-1} has multiple eigenvalues of module 1 without Jordan chain, then the system can be either asymptotically stable or unstable.

2 Preliminaries.

2.1 C_0-semigroup property

In this paragraph we recall that the operator $\mathcal{A}$ generates a C_0-semigroup in the general case and a group when $\det A_{-1} \neq 0$. For more details, see [6] for a more general case and [25] for our case.

Suppose that the initial conditions for the system (1) are $z(0) = y$ and $\varphi(t) = z_0(t), t \in [-1, 0[$ and let us put $z_t(\theta) = z(t+\theta), \theta \in [-1, 0[$. The semigroup generated by $\mathcal{A}$ is given by

$$\mathrm{e}^{\mathcal{A}t}\begin{pmatrix} y \\ z_0(\cdot) \end{pmatrix} = \begin{pmatrix} z_t(0) - A_{-1}z_t(-1) \\ z_t(\cdot) \end{pmatrix} = \begin{pmatrix} z(t) - A_{-1}z(t-1) \\ z(t+\cdot) \end{pmatrix}.$$

If $\begin{pmatrix} y \\ z_0(\cdot) \end{pmatrix} \in \mathcal{D}(\mathcal{A})$, this gives a strong (classical) solution of the system (2) and an absolutely continuous solution of the system (1). For $\begin{pmatrix} y \\ z_0(\cdot) \end{pmatrix} \in M_2$, it is a generalized or weak solution ([6], [32]).

2.2 Analysis of the spectrum

We start with the expression of the explicit form of the resolvent $R(\mathcal{A}, \lambda)$. We need the following

$$\Delta_{\mathcal{A}}(\lambda) = \Delta(\lambda) = -\lambda I + \lambda \mathrm{e}^{-\lambda} A_{-1} + \lambda \int_{-1}^{0} \mathrm{e}^{\lambda s} A_2(s)\mathrm{d}s + \int_{-1}^{0} \mathrm{e}^{\lambda s} A_3(s)\mathrm{d}s. \quad (3)$$

It is easy to see that $\det \Delta(\lambda) \not\equiv\ 0$, namely $\det \Delta(\lambda) \neq 0$ if $\operatorname{Re} \lambda$ is sufficiently large. A more precise description of the set $\{\lambda : \det \Delta(\lambda) = 0\}$ is given in Theorem 2.

Proposition 1 *The resolvent of $\mathcal{A}$ is given by*

$$R(\mathcal{A}, \lambda)\begin{pmatrix} z \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} A_{-1}\mathrm{e}^{-\lambda}\int_{-1}^{0} \mathrm{e}^{-\lambda s}\psi(s)\mathrm{d}s + (I - A_{-1}\mathrm{e}^{-\lambda})\Delta^{-1}(\lambda)D(z,\psi) \\ \int_0^\theta \mathrm{e}^{\lambda(\theta - s)}\psi(s)\mathrm{d}s + \mathrm{e}^{\lambda\theta}\Delta^{-1}(\lambda)D(z,\psi) \end{pmatrix}, \quad (4)$$

where $D(z, \psi) = z + \lambda\mathrm{e}^{-\lambda}A_{-1}\int_{-1}^{0}\mathrm{e}^{-\lambda s}\psi(s)\mathrm{d}s - \int_{-1}^{0}A_2(s)\psi(s)\mathrm{d}s - \int_{-1}^{0}\{\lambda A_2(\theta) + A_3(\theta)\}\mathrm{e}^{\lambda\theta}\{\int_0^\theta \mathrm{e}^{-\lambda s}\psi(s)\mathrm{d}s\}\mathrm{d}\theta$, *and for* λ *such that* $\det \Delta(\lambda) \neq 0$.

PROOF: To compute the resolvent, we have to consider the equation

$$\begin{aligned}(\mathcal{A} - \lambda I)\begin{pmatrix} y \\ \varphi(\cdot) \end{pmatrix} &= \begin{pmatrix} \int_{-1}^{0} A_2(\theta)\dot{\varphi}(\theta)\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)\varphi(\theta)\mathrm{d}\theta - \lambda\varphi(0) + \lambda A_{-1}\varphi(-1) \\ \dot{\varphi}(\theta) - \lambda\varphi(\theta) \end{pmatrix} \\ &= \begin{pmatrix} z \\ \psi(\cdot) \end{pmatrix}.\end{aligned} \quad (5)$$

From the second line we get $\varphi(\theta) = \mathrm{e}^{\lambda\theta}\varphi(0) + \int_0^\theta \mathrm{e}^{\lambda(\theta-s)}\psi(s)\mathrm{d}s$. It gives $\dot{\varphi}(\theta) = \lambda\mathrm{e}^{\lambda\theta}\varphi(0) + \lambda\int_0^\theta \mathrm{e}^{\lambda(\theta-s)}\psi(s)\mathrm{d}s + \psi(\theta)$. Let us replace this in the first line of (5) and use $\varphi(-1) = \mathrm{e}^{-\lambda}\varphi(0) - \int_{-1}^{0}\mathrm{e}^{-\lambda s}\psi(s)\mathrm{d}s \cdot \mathrm{e}^{-\lambda}$. Collecting all the terms with

$\varphi(0)$ we get $\Delta(\lambda)\varphi(0) = D(z,\psi)$, where $D(z,\psi)$ is defined in the statement of the Proposition. Hence $\varphi(0) = \Delta^{-1}(\lambda)D(z,\psi)$ for $\det\Delta(\lambda) \neq 0$, which gives the second line of (4). The first line of (4) follows from the definition of the domain $\mathcal{D}(\mathcal{A})$, i.e. $y = \varphi(0) - A_{-1}\varphi(-1)$, which ends the proof of Proposition 1. $\square$

In the sequel we will consider the matrix A_{-1} in a Jordan basis and change the norm in $\mathbb{C}^n$ such that the corresponding eigen- and rootvectors of A_{-1} form an orthogonal basis.

Let us denote by $\mu_1, ..., \mu_\ell$, $\mu_i \neq \mu_j$ if $i \neq j$, the eigenvalues of A_{-1} and the dimensions of their rootspaces by $p_1, ..., p_\ell$, $\sum_{k=1}^{\ell} p_k = n$. Consider the points $\lambda_m^{(k)} = \ln|\mu_m| + \mathrm{i}(\arg\mu_m + 2\pi k), m = 1, .., \ell; k \in \mathbb{Z}$ and the circles $L_m^{(k)}$ of fixed radius $r \leq r_0 = \frac{1}{3}\min\{|\lambda_m^{(k)} - \lambda_i^{(j)}|, (m,k) \neq (i,j)\}$ centered at $\lambda_m^{(k)}$.

Now we need a detailed description of the location of the spectrum of $\mathcal{A}$ (in the simplest case of one discrete delay see [30] and [19, Proposition 1.4, p.35]). We prove that for any fixed $m = 1, .., \ell$, the total multiplicity of eigenvalues of $\mathcal{A}$ inside the circles $L_m^{(k)}$ is independent of k (for $|k|$ large enough) and is equal to the multiplicity of μ_m as an eigenvalue of matrix A_{-1}. This fact plays an important role in our investigations.

Theorem 2 *The spectrum of $\mathcal{A}$ consists of the eigenvalues only which are the roots of the equation $\det\Delta(\lambda) = 0$, where $\Delta(\lambda)$ is given by (3). The corresponding eigenvectors of $\mathcal{A}$ are $\varphi = \left(\begin{smallmatrix} C - \mathrm{e}^{-\lambda}A_{-1}C \\ \mathrm{e}^{\lambda\theta}C \end{smallmatrix}\right)$, with $C \in \mathrm{Ker}\Delta(\lambda)$.*
There exists N_1 such that for any k, such that $|k| \geq N_1$, the total multiplicity of the roots of the equation $\det\Delta(\lambda) = 0$, contained in the circle $L_m^{(k)}$, equals p_m.

Remark 3 *We notice that this characterization of the spectrum holds only in the case $\det A_{-1} \neq 0$ (see also [19, Proposition 1.4, p.35])*

Remark 4 *In the simplest case of discrete neutral system*

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + A_1 z(t) + A_2 z(t-1),$$

we can combine the results of Theorem 2 with [19, Proposition 1.4, p.35] to get an analogous statement to Theorem 2 with a sequence of circles $\widetilde{L}_m^{(k)}$ of vanishing radius, i.e. $\sup_{m=1,...,\ell}\widetilde{r}_m^{(k)} \to 0$, when $|k| \to \infty$.

PROOF: It is easy to see from the explicit form of $R(\mathcal{A},\lambda)$ and the compactness of embedding of $H^1(-1,0;\mathbb{C}^n)$ into $L_2(-1,0;\mathbb{C}^n)$, that $R(\mathcal{A},\lambda)$ is compact. It gives that $\mathcal{A}$ has point spectrum only. Each eigenvalue is a root of $\det\Delta(\lambda)$ of finite multiplicity. Calculations give the form of eigenvectors of $\mathcal{A}$, taking into account the explicit definition of $\mathcal{D}(\mathcal{A})$.

To describe the location of the spectrum of $\mathcal{A}$ we use Rouché theorem.

More precisely, for sufficiently large k and any m we show that $|f_1(\lambda)| > |f_2(\lambda)|$ for any $\lambda \in L_m^{(k)}$ and

$$\begin{aligned} f_1(\lambda) &= \det(A_{-1} - \mathrm{e}^{\lambda} I), \\ f_2(\lambda) &= \det(A_{-1} - \mathrm{e}^{\lambda} I) \\ &\quad - \det\left((A_{-1} - \mathrm{e}^{\lambda} I) + \mathrm{e}^{\lambda} \int_{-1}^{0} \mathrm{e}^{\lambda s} A_2(s) \mathrm{d}s + \mathrm{e}^{\lambda} \lambda^{-1} \int_{-1}^{0} \mathrm{e}^{\lambda s} A_3(s) \mathrm{d}s \right). \end{aligned}$$

Thus, $f_1 - f_2$ will have the same number of roots inside $L_m^{(k)}$ as function f_1 has. On the other hand the roots of $f_1(\lambda) - f_2(\lambda)$ are the same as the roots of $\det \Delta(\lambda)$ for $\lambda \in L_m^{(k)}$ and for sufficiently large k.

Let us rewrite f_2 as follows:

$$f_2(\lambda) = \det(A_{-1} - \mathrm{e}^{\lambda} I) \left[1 - \det(I + (A_{-1} - \mathrm{e}^{\lambda} I)^{-1} L(\lambda)) \right],$$

where

$$L(\lambda) = \mathrm{e}^{\lambda} \int_{-1}^{0} \mathrm{e}^{\lambda s} A_2(s) \mathrm{d}s + \mathrm{e}^{\lambda} \lambda^{-1} \int_{-1}^{0} \mathrm{e}^{\lambda s} A_3(s) \mathrm{d}s. \tag{6}$$

To show that $|f_1(\lambda)| > |f_2(\lambda)|$ it is sufficient to get

$$|1 - \det(I + (A_{-1} - \mathrm{e}^{\lambda} I)^{-1} L(\lambda))| < 1. \tag{7}$$

We will show that

$$\|(A_{-1} - \mathrm{e}^{\lambda} I)^{-1} L(\lambda))\| \le \nu \tag{8}$$

for sufficiently small ν (and all large k) which gives (7).

Remark 5 *For a matrix D one has $\det D = \prod \xi_k$, where $\{\xi_k\} = \sigma(D)$- spectrum of D. Let us denote by $\{\nu_1, ..., \nu_n\}$ the eigenvalues of $(A_{-1} - \mathrm{e}^{\lambda} I)^{-1} L(\lambda))$. Hence (8) implies $|\nu_i| \le \nu$ for all i=1,...,n and so $\det(I + (A_{-1} - \mathrm{e}^{\lambda} I)^{-1} L(\lambda)) = \prod(1 + \nu_i)$. We estimate (see (7) and (8)): $|1 - \det(I + (A_{-1} - \mathrm{e}^{\lambda} I)^{-1} L(\lambda))| = |1 - \prod(1 + \nu_i)| \le |1 - (1 + \nu)^n| \le C \cdot \nu$ for sufficiently small $\nu > 0$. So (8) gives (7).*

Since $\|(A_{-1} - \mathrm{e}^{\lambda} I)^{-1} L(\lambda))\| \le \|(A_{-1} - \mathrm{e}^{\lambda} I)^{-1}\| \|L(\lambda))\|$ we need two estimates:

$$\|(A_{-1} - \mathrm{e}^{\lambda} I)^{-1}\| \le C_1, \tag{9}$$

for some $C_1 > 0$ and

$$\|L(\lambda))\| \leq \alpha_{m,k}, \qquad \lambda \in L_m^{(k)}, \qquad \lim_{|k|\to\infty} \alpha_{m,k} = 0. \tag{10}$$

We have $\lambda = \lambda_m^{(k)} + re^{i\varphi}, \varphi \in [0, 2\pi]$ and hence $|\mu_m - \mathrm{e}^\lambda| = |\mu_m - \mu_m \mathrm{e}^{re^{i\varphi}}| = |\mu_m||1 - \mathrm{e}^{re^{i\varphi}}| \geq C_0 > 0$ for all m, k and $\lambda \in L_m^k$. We use here the assumption that $\det A_{-1} \neq 0$ which implies $\min |\mu_m| > 0$. Using the fact that A_{-1} has a Jordan form and a well known fact that for a Jordan block

$$B = \begin{pmatrix} \mu & 1 & 0 & \dots & 0 \\ 0 & \mu & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix}$$

one has

$$(B - \mathrm{e}^\lambda I)^{-1} = \begin{pmatrix} (\mu - \mathrm{e}^\lambda)^{-1} & -(\mu - \mathrm{e}^\lambda)^{-2} & \dots & (-1)^{n-1}(\mu - \mathrm{e}^\lambda)^{-n} \\ 0 & (\mu - \mathrm{e}^\lambda)^{-1} & \dots & (-1)^{n-2}(\mu - \mathrm{e}^\lambda)^{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (\mu - \mathrm{e}^\lambda)^{-1} \end{pmatrix},$$

and we deduce (9).

To obtain (10) we write $\lambda \in L_m^{(k)}$ as $\lambda = \widetilde{\lambda} + \mathrm{i} \cdot 2\pi k$, with $\widetilde{\lambda} \in L_m^{(0)}$.

To estimate $\|L(\lambda))\|$ it is enough to consider

$$\int_{-1}^{0} \mathrm{e}^{\lambda s} A_i(s) \mathrm{d}s = \int_{-1}^{0} \mathrm{e}^{\mathrm{i}\cdot 2\pi k} (A_i(s) \mathrm{e}^{\widetilde{\lambda} s}) \mathrm{d}s. \tag{11}$$

Here functions $A_i(s)\mathrm{e}^{\widetilde{\lambda} s}$, $i = 2, 3$, belong to $L_2(-1, 0)$ and do not depend on k. The functions $\{\mathrm{e}^{\mathrm{i}\cdot 2\pi k}\}_{k\in\mathbb{Z}}$ form the trigonometric basis of $L_2(-1, 0)$. So the integral (11) is the Fourier coefficient of $A_i(s)\mathrm{e}^{\widetilde{\lambda} s}$. It implies the estimate (10).

By (9), (10) we obtain (8). We apply Rouché theorem and this completes the proof of Theorem 2. $\square$

2.3 The Riesz basis property

The results of this section form the technical foundation for all our considerations about stability properties.

Recall that in the particular case where $A_2(\theta) = A_3(\theta) = 0$, we use the notation $\bar{\mathcal{A}}$ for $\mathcal{A}$. We will show that the properties of $\bar{\mathcal{A}}$ can be expressed in terms of the properties of matrix A_{-1} only. The basis properties of the operator $\mathcal{A}$ will be deduced from the ones of $\bar{\mathcal{A}}$.

2.3.1 Basis property of eigen- and rootvectors of the operator $\bar{\mathcal{A}}$.

Let ν_m be the number of Jordan blocks, corresponding to $\mu_m \in \sigma(A_{-1})$. Denote by $p_{m,j}, j = 1, .., \nu_m$, $\sum_{j=1}^{\nu_m} p_{m,j} = p_m$, the orders of these blocks and by $\{C_{m,j}^0, ..., C_{m,j}^{p_{m,j}-1}\}$ the orthonormal system of corresponding eigen- and rootvectors, i.e.

$$
\begin{aligned}
&(A_{-1} - \mu_m I)C_{m,j}^0 = 0,\\
&(A_{-1} - \mu_m I)C_{m,j}^1 = C_{m,j}^0,\\
&(A_{-1} - \mu_m I)C_{m,j}^s = C_{m,j}^{s-1},\ s = 1, ..., p_{m,j} - 1.
\end{aligned} \tag{12}
$$

If there exists $k \in \{1, \ldots, \ell\}$ such that $\mu_k = 1 \in \sigma(A_{-1})$ we denote by K the rootspace of A_{-1} corresponding to the eigenvalue 1 and put $K = \{0\}$ otherwise. Finally, let $K_1 = K^\perp = Lin\{C_{m,j}^d, m \in \{0, \ldots, \ell\} : \mu_m \neq 1; j = 1, \ldots, \nu_m; d = 0, \ldots, p_{m,j} - 1\}$. In order to describe eigen- and rootvectors of the operator $\bar{\mathcal{A}}$ (see Theorem 7) we need the following lemma.

Lemma 6 *Let $\nu(1) = 0$ if $1 \notin \sigma(A_{-1})$ and $\nu(1) = \nu_k$ if for some $k \in \{0, \ldots, \ell\}$ we have $\mu_k = 1$, then*

i) for any $y \in \mathbb{C}^n$ there exists the unique polynomial vector $P_y(\theta)$ of the form

$$P_y(\theta) = \sum_{j=1}^{\nu(1)} \sum_{d=0}^{p_{k,j}-1} C_{k,j}^d \sum_{i=0}^{p_{k,j}-d-1} \alpha_y^{i,j} \frac{\theta^{p_{k,j}-d-i}}{(p_{k,j} - d - i)!} + \gamma_y,$$

where $\gamma_y \in K_1$, such that

$$y = P_y(0) - A_{-1} P_y(-1). \tag{13}$$

ii) the mapping $y \overset{D}{\mapsto} P_y(\cdot)$ is a linear bounded operator $D : \mathbb{C}^n \to H^1(-1, 0; \mathbb{C}^n)$.

In the particular case when $1 \notin \sigma(A_{-1})$ the mapping D is given by $P_y(\theta) = (I - A_{-1})^{-1} y$.

PROOF: To prove i) we observe that

$$
\begin{aligned}
&P_y(0) - A_{-1}P_y(-1) = \\
&\quad = \sum_{j=1}^{\nu(1)} \sum_{d=0}^{p_{k,j}-1} (C_{k,j}^d + C_{k,j}^{d-1}) \\
&\quad \times \left(\alpha_y^{0,j} \tfrac{(-1)^{p_{k,j}-d+1}}{(p_{k,j}-d)!} + \alpha_y^{1,j} \tfrac{(-1)^{p_{k,j}-d}}{(p_{k,j}-d-1)!} + \cdots + \alpha_y^{p_{k,j}-d-1,j} \right) \\
&\quad +(I - A_{-1})\gamma_y,
\end{aligned}
\tag{14}
$$

where $C_{k,j}^{-1} = 0$. On the other hand, the vector y is decomposed as

$$
y = \sum_{j=1}^{\nu(1)} \sum_{d=0}^{p_{k,j}-1} C_{k,j}^d \xi_y^{p_{k,j}-d-1,j} + \delta_y, \qquad \delta_y \in K_1. \tag{15}
$$

Note that K_1 is an invariant subspace for A_{-1} and the restriction $(I - A_{-1})_{|K_1}$ is an invertible operator. Equating (14) and (15) yields $\delta_y = (I - A_{-1})\gamma_y$ and

$$
\begin{aligned}
\xi_y^{0,j} &= \alpha_y^{0,j}, \\
\xi_y^{1,j} &= \alpha_y^{0,j} + \alpha_y^{1,j} - \tfrac{1}{2}\alpha_y^{0,j}, \\
\vdots\ &\ \ \vdots \\
\xi^{p_{k,j}-1,j} &= \alpha_y^{p_{k,j}-2,j} - \tfrac{1}{2}\alpha_y^{p_{k,j}-3,j} + \cdots + \tfrac{(-1)^{p_{k,j}-2}}{(p_{k,j}-1)!}\alpha_y^{0,j} \\
&\quad +\alpha_y^{p_{k,j}-1,j} - \tfrac{1}{2}\alpha_y^{p_{k,j}-2,j} + \cdots + \tfrac{(-1)^{p_{k,j}-1}}{p_{k,j}!}\alpha_y^{0,j},
\end{aligned}
$$

$$j = 1, \ldots, \nu(1),$$

and, therefore, we have $\gamma_y = (I - A_{-1})_{|K_1}^{-1}\delta_y$ and

$$
\begin{pmatrix} \alpha_y^{0,j} \\ \alpha_y^{1,j} \\ \vdots \\ \alpha_y^{p_{k,j}-1,j} \end{pmatrix}
=
\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ * & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \ldots & 1 \end{pmatrix}^{-1}
\begin{pmatrix} \xi_y^{0,j} \\ \xi_y^{1,j} \\ \vdots \\ \xi_y^{p_{k,j}-1,j} \end{pmatrix}, \quad j = 1, \ldots, \nu(1).
$$

The linearity of the mapping D is obvious and we have already shown that it is invertible. By construction we obtain relation (13).

If $1 \notin \sigma(A_{-1})$ then putting $P_y(\theta) = c$, with a constant vector c, we get

$$
P_y(0) - A_{-1}P_y(-1) = c - A_{-1}c
$$

and then if $c=(I-A_{-1})^{-1}y$ we get (13). $\square$

This allows to state the following result.

Theorem 7 *The spectrum of $\bar{\mathcal{A}}$ only consists of the eigenvalues which are the roots of the equation* $\det\Delta_{\bar{\mathcal{A}}}(\lambda)=\det(\lambda-\lambda\mathrm{e}^{-\lambda}A_{-1})=0$, *i.e.*

$$\sigma(\bar{\mathcal{A}})=\{\lambda_m^{(k)}=\ln|\mu_m|+\mathrm{i}(\arg\mu_m+2k\pi)\}\cup\{0\},$$

where $\mu_m\in\sigma(A_{-1}), m=1,\dots,\ell$. The corresponding generalized eigenvectors are of two forms. To each $\{\lambda_m^{(k)}\}$ and each Jordan chain $\{C_{m,j}^d\}$ of the orthonormal eigen- and rootvectors of the matrix A_{-1} (see (12)) corresponds the Jordan chain of $\bar{\mathcal{A}}$: $v_{m,j}^{(k),0}, v_{m,j}^{(k),1},\dots,v_{m,j}^{(k),p_{m,j}-1}$, i.e. the vectors v^s (the indices k,m,j are omitted) verify the relation $(\bar{\mathcal{A}}-\lambda I)v^s=v^{s-1}$. They are given by

$$v_{m,j}^{(k),s}=\begin{pmatrix}0\\ \mathrm{e}^{\lambda_m^{(k)}\theta}P^s(\theta)\end{pmatrix},\quad P^s(\theta)=P_{m,j}^s(\theta)=\sum_{d=0}^{s}C_{m,j}^d\left(\sum_{i=0}^{s-d}\beta_d^{i,s}\frac{\theta^i}{i!}\right),\tag{16}$$

where $s=0,..,p_{m,j}-1$, $m=1,\dots,\ell$, $k\in\mathbb{Z}$, $j=1,\dots,\nu_m$.

Besides, to $\lambda=0\in\sigma(\bar{\mathcal{A}})$ correspond n generalized eigenvectors of another form

$$\begin{pmatrix}e_i\\ P_{e_i}(\theta)\end{pmatrix},\tag{17}$$

where $\{e_i\}_{i=1}^n$ is an arbitrary orthogonal basis in $\mathbb{C}^n$ and the polynomial $P_y(\theta)$ is described in Lemma 6.

The collection (16) and (17) constitutes a Riesz basis in M_2 which becomes an orthogonal basis if we choose the equivalent norm

$$\|(y,z(\cdot)\|_1^2=\|y\|^2+\int_{-1}^{0}\|T(P_y(\theta)-z(\theta))\|^2\mathrm{d}\theta,\tag{18}$$

where T is a bounded operator in $L_2(-1,0;\mathbb{C}^n)$ with a bounded inverse.

Let us first discuss the formulation of this result.

Remark 8 *We emphasize, that the polynomials $P_{m,j}^s(\theta)$ in (16) do not depend on index k. So vectors $v_{m,j}^{(k),s}$ in (16) only differ (for different k) in the exponential $\mathrm{e}^{\lambda_m^{(k)}\theta}$. This property is essential for our investigations.*

Remark 9 *Note that if the Jordan basis $\{C^d_{m,j}\}$ is not orthogonal in the initial norm, then one can change the scalar product in $\mathbb{C}^n$ to achieve its orthogonality. So the assumption on $\{C^d_{m,j}\}$ to be orthogonal is not essential for the property of generalized eigenvectors of $\bar{\mathcal{A}}$ to form a Riesz basis.*

Remark 10 *We notice that the rootspace of $\bar{\mathcal{A}}$ corresponding to 0 has essentially different structures in cases $1 \notin \sigma(A_{-1})$ and $1 \in \sigma(A_{-1})$. The difference is in the family (17). Let us consider this in more details.*

Vector $\binom{e}{\varphi(\theta)}$ is an eigenvector of $\bar{\mathcal{A}}$, corresponding to 0 iff

$$\bar{\mathcal{A}} \begin{pmatrix} e \\ \varphi(\theta) \end{pmatrix} = 0, \qquad \begin{pmatrix} e \\ \varphi(\theta) \end{pmatrix} \in \mathcal{D}(\mathcal{A}).$$

The first property gives $\varphi(\theta) \equiv C = const$, and the second one implies $e = \varphi(0) - A_{-1}\varphi(-1) = (I - A_{-1})C$. So the number of linearly independent vectors $\{e_i\}$ ($e_i \neq 0$) which can be the first (nonzero) coordinate of an eigenvector $\binom{e}{\varphi(\theta)}$ is equal *to* $\dim Im(I - A_{-1})$.

In the case $1 \notin \sigma(A_{-1})$, we have $\dim Im(I - A_{-1}) = n$, *so there are n eigenvectors (of $\bar{\mathcal{A}}$ corresponding to 0) with the first nonzero coordinate. This means that in addition to the set of eigen- and rootvectors from (16) one has n eigenvectors from (17), i.e. with the first (nonzero) coordinate.*

In the case $1 \in \sigma(A_{-1})$, we have $\dim Im(I - A_{-1}) < n$, *so the number of eigenvectors is less than n. Moreover, there is* no *rootvector (of $\bar{\mathcal{A}}$) which satisfies*

$$(\bar{\mathcal{A}} - 0 \cdot I) \begin{pmatrix} y \\ \psi(\theta) \end{pmatrix} = \begin{pmatrix} e \\ \varphi(\theta) \end{pmatrix}, \qquad e \neq 0$$

since $\bar{\mathcal{A}} \binom{y}{\psi(\theta)} = \binom{0}{\mathrm{d}/\mathrm{d}\theta\psi(\theta)}$, because the first coordinate must be zero.

More precisely, the subspace $C^n \ominus Im(I - A_{-1})$ is the linear span of all rootvectors of the highest order of A_{-1}, corresponding to $\mu = 1 \in \sigma(A_{-1})$, i.e. $\{C^{p_{m,j}}_{m,j}\}$. In this case the vectors from (17) with the first coordinate from $C^n \ominus Im(I - A_{-1})$ are rootvectors of $\bar{\mathcal{A}}$ which "continue" the corresponding sequence from (16). They are the rootvectors of highest orders in these sequences.

We can illustrate this fact, for example, on particular cases. For simplicity, we choose $\{e_i\}_{i=1}^n = \{C^d_{m,j}\}$ (see vectors (17)).

1. Assume that $\mu_m = 1 \in \sigma(A_{-1})$ is a simple root ($\nu_m = 1$ and $p_{m,j} = 1$). Hence from the family (16) we have one eigenvector

$$v^0_{m,j} = \begin{pmatrix} 0 \\ \mathrm{e}^{\lambda^{(0)}_m \theta} P^0(\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ C^0_{m,j} \end{pmatrix},$$

and the corresponding rootvector from the family (17)

$$v_{m,j}^1 = \begin{pmatrix} C_{m,j}^0 \\ C_{m,j}^0 \theta \end{pmatrix}.$$

It is easy to check that $(\bar{\mathcal{A}} - 0 \cdot I)v_{m,j}^1 = v_{m,j}^0$ *and* $v_{m,j}^1 \in \mathcal{D}(\bar{\mathcal{A}})$*. Here we are interested in the rootvector with the first (nonzero) coordinate* $C_{m,j}^0$ *since only this vector* does not *belong to* $Im\,(I - A_{-1})$.

2. Assume $\nu_m = 1$ *and* $p_{m,j} = 2$*, i.e. from the family (16) we get two vectors*

$$v_{m,j}^0 = \begin{pmatrix} 0 \\ C_{m,j}^0 \end{pmatrix}, \qquad v_{m,j}^1 = \begin{pmatrix} 0 \\ C_{m,j}^0 \theta + C_{m,j}^1 \end{pmatrix}.$$

In this case we are interested in the rootvector with the first (nonzero) coordinate $C_{m,j}^1$ *since only this vector* does not *belong to* $Im\,(I - A_{-1})$*. The corresponding rootvector (from the family (17)) is*

$$v_{m,j}^2 = \begin{pmatrix} C_{m,j}^1 \\ C_{m,j}^0 \frac{\theta^2}{2} + C_{m,j}^1 (\theta + \frac{1}{2}) \end{pmatrix}.$$

Proof of the Theorem: We prove by induction.

Step 1. First, check by direct calculation that the vector $v^0 = \begin{pmatrix} 0 \\ \mathrm{e}^{\lambda_m^{(k)}\theta} C_{m,j}^0 \end{pmatrix}$ belongs to $\mathcal{D}(\bar{\mathcal{A}})$ and satisfies $(\bar{\mathcal{A}} - \lambda I)v^0 = 0$.

Step 2. $s \to s+1$.

Consider $P^s(\theta) = \sum_{d=0}^{s} C_{m,j}^d \left(\sum_{i=0}^{s-d} \beta_d^i \frac{\theta^i}{i!} \right)$ and $P^{s+1}(\theta) = \sum_{d=0}^{s+1} C_{m,j}^d \left(\sum_{i=0}^{s+1-d} \gamma_d^i \frac{\theta^i}{i!} \right)$. We omit some indices which do not change in this part of the proof (we write $\beta_d^i = \beta_d^{i,s}$ and $\gamma_d^i = \beta_d^{i,s+1}$). The coefficients β_d^i are known and we are looking for γ_d^i. It is easy to check, that the property $(\bar{\mathcal{A}} - \lambda_m^{(k)} I)v^{s+1} = v^s$ implies $\mathrm{d}/\mathrm{d}\theta P^{s+1}(\theta) = P^s(\theta)$, i.e.

$$\sum_{d=0}^{s} C_{m,j}^d \left(\sum_{i=1}^{s+1-d} \gamma_d^i \frac{\theta^{i-1}}{(i-1)!} \right) = \sum_{d=0}^{s} C_{m,j}^d \left(\sum_{i=0}^{s-d} \beta_d^i \frac{\theta^i}{i!} \right).$$

Since $\{C_{m,j}^d\}$ are linearly independent, it follows that

$$\sum_{i=1}^{s+1-d} \gamma_d^i \frac{\theta^{i-1}}{(i-1)!} = \sum_{i=0}^{s-d} \beta_d^i \frac{\theta^i}{i!}.$$

It implies

$$\gamma_d^{i+1} = \beta_d^i, \quad i = 0, ..., s-d, \quad d = 0, ..., s. \tag{19}$$

So, we only need to find γ_d^0, $\quad d = 0, ..., s+1$ i.e, the coefficients of θ^0 in $P^{s+1}(\theta)$. Consider $P^{s+1}(0) = \sum_{d=0}^{s+1} C_{m,j}^d \gamma_d^0$ and let us use the fact that $v^{s+1} \in \mathcal{D}(\bar{\mathcal{A}})$. It gives

$$0 = P^{s+1}(0) - A_{-1}\mathrm{e}^{-\lambda_m^{(k)}} P^{s+1}(-1)$$

or equivalently, multiplying by $\mu_m = \mathrm{e}^{\lambda_m^{(k)}}$, we get

$$\mu_m P^{s+1}(0) - A_{-1} P^{s+1}(-1) = 0.$$

We obtain, using (19),

$$\mu \sum_{d=0}^{s+1} C_{m,j}^d \gamma_d^0 - A_{-1} \sum_{d=0}^{s+1} C_{m,j}^d \left(\sum_{i=1}^{s+1-d} \beta_d^{i-1} \frac{(-1)^i}{i!} + \gamma_d^0 \right) = 0.$$

Collecting the terms with γ_d^0 and using $(A_{-1} - \mu_m) C_{m,j}^d = C_{m,j}^{d-1}$, $(A_{-1} - \mu_m) C_{m,j}^0 = 0$, one gets

$$\sum_{d=1}^{s+1} C_{m,j}^{d-1} \gamma_d^0 + \sum_{d=0}^{s+1} \left(C_{m,j}^{d-1} + \mu C_{m,j}^d \right) \sum_{i=1}^{s+1-d} \beta_d^{i-1} \frac{(-1)^i}{i!} = 0.$$

Here we denote $C_{m,j}^{-1} = 0$. Collecting the coefficients of $C_{m,j}^d$, we arrive for each $d = 0, ..., s$ to equation

$$\gamma_{d+1}^0 + \sum_{i=1}^{s-d} \beta_{d+1}^{i-1} \frac{(-1)^i}{i!} + \mu \sum_{i=1}^{s+1-d} \beta_d^{i-1} \frac{(-1)^i}{i!} = 0 \tag{20}$$

which is the formula for γ_{d+1}^0. The coefficient γ_0^0 can be chosen arbitrarily, say $\gamma_0^0 = 0$ for simplicity. This and (19) give all coefficients γ_d^i so the existence of polynomial $P^{s+1}(\theta)$ with the desired properties is proved. These polynomials give the sequence of eigen- and rootvectors (see (16)).

Remark 11 *It follows from (20) that* $\gamma_{s+1}^0 = \mu_m \beta_s^0$ *or, more precisely,* $\beta_{s+1}^{0,s+1} = \mu_m \beta_s^{0,s} = \mu_m^{s+1} \beta_0^{0,0}$ *for all* $s = 0, ..., p_{m,j} - 1$.

Remark 12 *If* $\beta_0^{0,0} = \alpha \in \mathbb{C}$ *(see the definition of* P^s *in (16)), then* $\beta_0^{s,s} = \alpha$ *for all* $s = 0, ..., p_{m,j} - 1$. *We remind that* $\beta_0^{s,s}$ *is the coefficient of* $\frac{\theta^s}{s!}$, *where* s *is the highest order of* θ *in* P^s.

Now we prove that the sequence of eigen- and rootvectors (16), (17) forms a Riesz basis in M_2. First we show that the functions

$$\begin{gathered} \{ \mathrm{e}^{\lambda_m^{(k)}\theta} C_{m,j}^0, \ \mathrm{e}^{\lambda_m^{(k)}\theta} C_{m,j}^1, \ldots, \ \mathrm{e}^{\lambda_m^{(k)}\theta} C_{m,j}^{p_{m,j}-1} \}, \\ m = 1, ..., \ell; j = 1, ..., \nu_m; k \in \mathbb{Z}. \end{gathered} \tag{21}$$

form an orthogonal basis of $L_2(-1,0;\mathbb{C}^n)$. This follows from the fact that $\{C^i_{m,j}\}$ forms an orthogonal basis of $\mathbb{C}^n$ and $\{\,\mathrm{e}^{\lambda_m^{(k)}\theta}\}_{k\in\mathbb{Z}}$ - an orthogonal basis of $L_2(-1,0;\mathbb{C})$, for each m. We use here that $\lambda_m^{(k)}-\lambda_m^{(j)}=\mathrm{i}\cdot 2\pi(k-j)$.

Now in each subspace $L_2(-1,0;\mathcal{L}_{m,j})$, $\mathcal{L}_{m,j}=Lin\,\{C^d_{m,j}\}_{d=0}^{p_{m,j}-1}$ we consider the operator $\widetilde{T}_{m,j}=\{t_{s,d}(\theta)\}$; $s,d=0,..,p_{m,j}-1$, where elements $t_{s,d}(\theta)$ are polynomials (see the definition of P^s in (16)) such that $t_{s,d}(\theta)=0$ for $s<d$ and $t_{s,d}(\theta)=\sum_{i=0}^{s-d}\beta_d^{i,s}\frac{\theta^i}{i!}$ for $s\geq d$.

$$\widetilde{T}_{m,j}=\begin{pmatrix} \beta_0^{0,0} & 0 & \dots & \dots & 0\\ t_{2,1}(\theta) & \beta_1^{0,1} & 0 & \dots & 0\\ * & * & \beta_2^{0,2} & 0 & 0\\ \dots & \dots & \dots & \dots & 0\\ t_{p_{m,j}-1,1}(\theta) & * & \dots & * & \beta^{0,p_{m,j}-1}_{p_{m,j}-1}\end{pmatrix}.$$

This operator maps the group of functions

$$\{\,\mathrm{e}^{\lambda_m^{(k)}\theta}C^0_{m,j},\ \mathrm{e}^{\lambda_m^{(k)}\theta}C^1_{m,j},\dots,\ \mathrm{e}^{\lambda_m^{(k)}\theta}C^{p_{m,j}-1}_{m,j}\}$$

into the group (see (16))

$$\{\,\mathrm{e}^{\lambda_m^{(k)}\theta}P^0(\theta),\ \mathrm{e}^{\lambda_m^{(k)}\theta}P^1(\theta),\dots,\ \mathrm{e}^{\lambda_m^{(k)}\theta}P^{p_{m,j}-1}(\theta)\}.$$

The operator $\widetilde{T}_{m,j}$ is bounded and has bounded inverse since it is a polynomial matrix whose determinant is a constant different from zero. Such matrix is called an *elementary polynomial* (see e.g. [4, Theorem 53.1, p.142]). In our case $\det\widetilde{T}_{m,j}=\prod_{s=0}^{p_{m,j}-1}\beta_s^{0,s}=\mu^{1+2+..+(p_{m,j}-1)}\cdot\beta_0^{0,0}\neq 0$ (see Remark 11).

Thus the diagonal operator $\widetilde{T}=\mathrm{diag}\,[\widetilde{T}_{m,j}]$ is bounded, has bounded inverse and maps the orthogonal basis (21) into the Riesz basis (in $L_2(-1,0;\mathbb{C}^n)$):

$$\{\,\mathrm{e}^{\lambda_m^{(k)}\theta}P^0_{m,j}(\theta),\ \mathrm{e}^{\lambda_m^{(k)}\theta}P^1_{m,j}(\theta),\dots,\ \mathrm{e}^{\lambda_m^{(k)}\theta}P^{p_{m,j}-1}_{m,j}(\theta)\},$$
$$m=1,..,\ell;\,j=1,..,\nu_m;\,k\in\mathbb{Z}.$$

In order to define the new norm (see (18)) we put $T=\widetilde{T}^{-1}$.

Since the collection (17) consists of n linearly independent vectors which do not belong to the subspace $\{0\}\times L_2(-1,0;\mathbb{C}^n)\subset M_2$ we then conclude that (16), (17) is a Riesz basis in M_2.

Let us show that vectors (17) are eigen- or rootvectors corresponding to the eigenvalue 0 for operator $\bar{\mathcal{A}}$. First of all, due to Lemma 6 these vectors belong to $\mathcal{D}(\bar{\mathcal{A}})$. If $K=\{0\}$ it is obvious that $\bar{\mathcal{A}}\begin{pmatrix} e_i\\ P_{e_i}(\theta)\end{pmatrix}=0$ (see Lemma 6) and, therefore, (17) is a collection of eigenvectors corresponding to 0. If $K \neq \{0\}$ we notice that $K = Lin\{C_{m,j}^{p_{m,j}-1}\}_{j=1}^{\nu_m}$, where $C_{m,j}^{p_{m,j}-1}$ are rootvectors (of A_{-1}) of the highest orders, corresponding to $\mu_m = 1$. Let us first consider $y = C_{m,j}^{p_{m,j}-1}$. In the same way as in the construction of polynomials $P_{m,j}^{s}$ (see (16)) we look for a rootvector such that

$$\bar{\mathcal{A}} \begin{pmatrix} C_{m,j}^{p_{m,j}-1} \\ P(\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ P_{m,j}^{p_{m,j}-1}(\theta) \end{pmatrix}.$$

It is easy to see that the property $\begin{pmatrix} C_{m,j}^{p_{m,j}-1} \\ P(\theta) \end{pmatrix} \in \mathcal{D}(\mathcal{A})$ and Lemma 6 imply $P(\theta) = P_y(\theta)$ for $y = C_{m,j}^{p_{m,j}-1}$ (by the uniqueness of polynomial $P_y(\theta)$). In general, $y = y^1 + y^2$, $y^1 \in K$, $y^2 \in K^{\perp}$ and linearity of the mapping $y \to P_y(\cdot)$ (see Lemma 6 item ii)) gives the rootvectors of the form (17).

It remains to prove that the basis (16), (17) is orthogonal in the norm $\|\cdot\|_1$. First we observe that

$$\|(0, z(\cdot))\|_1 = \|Tz(\cdot)\|_{L_2(-1,0;\mathbb{C}^n)}.$$

Hence all the vectors of the collection (16) are orthogonal to each others. Then we note

$$\left\langle \begin{pmatrix} y \\ P_y(\cdot) \end{pmatrix}, \begin{pmatrix} 0 \\ z(\cdot) \end{pmatrix} \right\rangle_1 = (y, 0) + \int_{-1}^{0} (T(P_y(\theta) - P_y(\theta)), Tz(\theta))\, \mathrm{d}\theta = 0.$$

Therefore, any vector from (16) is orthogonal to each from (17). And finally, we get

$$\left\langle \begin{pmatrix} e_i \\ P_{e_i}(\cdot) \end{pmatrix}, \begin{pmatrix} e_j \\ P_{e_j}(\cdot) \end{pmatrix} \right\rangle_1 = (e_i, e_j) = 0$$

as $i \neq j$. This completes the proof of the theorem. $\square$

2.3.2 Basis property of finite-dimensional invariant subspaces

Let us recall [10] that a basis $\{V_k\}$ of subspaces is called a basis equivalent to orthogonal (a Riesz basis) if there are an orthogonal basis of subspaces $\{W_k\}$ and a linear bounded invertible (with bounded inverse) operator R, such that $RV_k = W_k$. See also the discussion in [26].

In order to give a complete proof of the main result of this section, we need the following important Lemma and Theorem.

Lemma 13 *For any $m = 1,..,\ell$ and $|k| > N_1$ (N_1 is defined in Theorem 2) the following estimate holds*

$$\sup_{\lambda\in L_m^{(k)}} \|R(\mathcal{A},\lambda) - R(\bar{\mathcal{A}},\lambda)\| \le \gamma_k, \quad \textit{with} \quad \sum_{|k|>N_1} \gamma_k^2 < \infty. \tag{22}$$

Proof: Using the explicit form of resolvent (4), we get

$$[R(\mathcal{A},\lambda) - R(\bar{\mathcal{A}},\lambda)] \begin{pmatrix} z \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} (I-A_{-1}\mathrm{e}^{-\lambda})\{\Delta_{\mathcal{A}}^{-1}(\lambda)D_{\mathcal{A}} - \Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)D_{\bar{\mathcal{A}}}\} \\ \mathrm{e}^{\lambda\theta}\{\Delta_{\mathcal{A}}^{-1}(\lambda)D_{\mathcal{A}} - \Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)D_{\bar{\mathcal{A}}}\} \end{pmatrix}, \tag{23}$$

where

$D_{\mathcal{A}} = D_{\mathcal{A}}(z,\psi) = z + \lambda\mathrm{e}^{-\lambda}A_{-1}\int_{-1}^{0}\mathrm{e}^{-\lambda s}\psi(s)\mathrm{d}s - \int_{-1}^{0}A_2(s)\psi(s)\mathrm{d}s - \int_{-1}^{0}\{\lambda A_2(\theta) + A_3(\theta)\}\mathrm{e}^{\lambda\theta}\{\int_0^{\theta}\mathrm{e}^{-\lambda s}\psi(s)\mathrm{d}s\}\mathrm{d}\theta$ and $D_{\bar{\mathcal{A}}} = D_{\bar{\mathcal{A}}}(z,\psi) = z + \lambda\mathrm{e}^{-\lambda}A_{-1}\int_{-1}^{0}\mathrm{e}^{-\lambda s}\psi(s)\mathrm{d}\theta$

Now we write

$$\{\Delta_{\mathcal{A}}^{-1}(\lambda)D_{\mathcal{A}} - \Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)D_{\bar{\mathcal{A}}}\} = [\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda) - \Delta_{\mathcal{A}}^{-1}(\lambda)]D_{\mathcal{A}} + \Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)[D_{\mathcal{A}} - D_{\bar{\mathcal{A}}}]. \tag{24}$$

Let us show that there exists $C_2 > 0$ such that for any $\lambda \in L_m^{(k)}, m = 1,..,\ell$ and large enough $|k|$, one has

$$\|\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)\| \le C_2|\lambda|^{-1}. \tag{25}$$

We have $\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda) = (-\lambda I + \lambda\mathrm{e}^{-\lambda}A_{-1})^{-1} = \lambda^{-1}(-I + \mathrm{e}^{-\lambda}A_{-1})^{-1}$ and $(-I + \mathrm{e}^{-\lambda}A_{-1})^{-1} = [(A_{-1} - \mathrm{e}^{\lambda}I)\mathrm{e}^{-\lambda}]^{-1} = (A_{-1} - \mathrm{e}^{\lambda}I)^{-1}\mathrm{e}^{\lambda}$. Hence

$$\|(-I + \mathrm{e}^{-\lambda}A_{-1})^{-1}\| = \|(A_{-1} - \mathrm{e}^{\lambda}I)^{-1}\||\mu_m||\mathrm{e}^{r\mathrm{e}^{i\varphi}}| \le C_1 \max_{m=1,...,\ell}|\mu_m|\mathrm{e}^{r} \equiv C_2,$$

where C_1 is defined in (9) and $\lambda = \lambda_m^{(k)} + r\mathrm{e}^{i\varphi}, \varphi \in [0,2\pi], r$ is the fixed radius of all circles $L_m^{(k)}$ (see the proof of (9) for details). It gives (25).

In a quite similar way we get the estimate for $\Delta_{\mathcal{A}}^{-1}(\lambda)$ similar to (25):

$$\|\Delta_{\mathcal{A}}^{-1}(\lambda)\| \le C_3|\lambda|^{-1}. \tag{26}$$

More precisely, from (3), we get

$$\begin{aligned}\Delta_{\mathcal{A}}(\lambda) &= \lambda\mathrm{e}^{-\lambda}\left(-I\mathrm{e}^{\lambda} + A_{-1} + \mathrm{e}^{\lambda}\int_{-1}^{0}\mathrm{e}^{\lambda s}A_2(s)\mathrm{d}s + \mathrm{e}^{\lambda}\lambda^{-1}\int_{-1}^{0}\mathrm{e}^{\lambda s}A_3(s)\mathrm{d}s\right) \\ &= \lambda\mathrm{e}^{-\lambda}(A_{-1} - I\mathrm{e}^{\lambda} + L(\lambda)) = \lambda\mathrm{e}^{-\lambda}(A_{-1} - I\mathrm{e}^{\lambda})\left\{I + (A_{-1} - I\mathrm{e}^{\lambda})^{-1}L(\lambda)\right\},\end{aligned}$$

where $L(\lambda)$ is defined in (6).

Hence

$$\Delta_{\mathcal{A}}^{-1}(\lambda) = \lambda^{-1}\mathrm{e}^{\lambda}\left\{I + (A_{-1} - I\mathrm{e}^{\lambda})^{-1}L(\lambda)\right\}^{-1}(A_{-1} - I\mathrm{e}^{\lambda})^{-1}.$$

Now estimates (9) and (8) with ν small enough give (26).

Remark 14 *It is easy to check for arbitrary matrices A and K, such that A and $A+K$ are nonsingular, the identity*

$$A^{-1} - (A+K)^{-1} = (A+K)^{-1}KA^{-1}.$$

We get this by multiplying the identity $(A+K) - A = K$ by $(A+K)^{-1}$ from the left and by A^{-1} from the right.

The last Remark gives

$$\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda) - \Delta_{\mathcal{A}}^{-1}(\lambda) = \Delta_{\mathcal{A}}^{-1}(\lambda)\left[\lambda\int_{-1}^{0}\mathrm{e}^{\lambda s}A_2(s)\mathrm{d}s + \int_{-1}^{0}\mathrm{e}^{\lambda s}A_3(s)\mathrm{d}s\right]\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda).$$

Using the last estimate together with (25) and (26) we deduce

for any $\lambda \in L_m^{(k)}$

$$\|\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda) - \Delta_{\mathcal{A}}^{-1}(\lambda)\| \leq C_4|\lambda|^{-1}. \tag{27}$$

We also obtain

$$\begin{aligned}\left\|[\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda) - \Delta_{\mathcal{A}}^{-1}(\lambda)]D_{\mathcal{A}}(z,\psi)\right\| \leq \\ \leq \tfrac{C}{|\lambda|}\|(z,\psi)\| + \tfrac{C}{|\lambda|}\left\|\lambda\mathrm{e}^{-\lambda}A_{-1}\textstyle\int_{-1}^{0}\mathrm{e}^{-\lambda s}\psi(s)\mathrm{d}s\right\| + \\ + \tfrac{C}{|\lambda|}\left\|\lambda\textstyle\int_{-1}^{0}\{A_2(\theta) + \lambda^{-1}A_3(\theta)\}\mathrm{e}^{\lambda\theta}\{\textstyle\int_{0}^{\theta}\mathrm{e}^{-\lambda s}\psi(s)\mathrm{d}s\}\mathrm{d}\theta\right\|.\end{aligned} \tag{28}$$

Let us first estimate the second term in (28).

We can write $\lambda \in L_m^{(k)}$ as $\lambda = \mathrm{i}\cdot 2\pi k + \hat{\lambda}$, where $\hat{\lambda} \in L_m^{(0)}$. Since $\mathrm{e}^{-\hat{\lambda}s}\psi(s) \in L_2$, from the second term in (28), that $\int_{-1}^{0}\mathrm{e}^{-\lambda s}\psi(s)\mathrm{d}s = \int_{-1}^{0}\mathrm{e}^{-\mathrm{i}\cdot 2\pi ks}\mathrm{e}^{-\hat{\lambda}s}\psi(s)\mathrm{d}s$ is the Fourier coefficient of function $\mathrm{e}^{-\hat{\lambda}s}\psi(s)$. Notice that for any $\hat{\lambda} \in L_m^{(0)}$ one has $\|\mathrm{e}^{-\hat{\lambda}\cdot}\psi(\cdot)\|_{L^2(-1,0;\mathbb{C}^n)} \leq C\|\psi\|_{L^2(-1,0;\mathbb{C}^n)}$, i.e. the family of functions $\{\mathrm{e}^{-\hat{\lambda}\cdot}\psi(\cdot)\}_{\hat{\lambda}\in L_m^{(0)}}$ is uniformly bounded in the space $L^2(-1,0;\mathbb{C}^n)$.

Now to estimate the third term in (28), we first consider

$$\int_{-1}^{0} A_2(\theta)\mathrm{e}^{\lambda\theta}\left\{\int_{0}^{\theta}\mathrm{e}^{-\lambda s}\psi(s)\mathrm{d}s\right\}\mathrm{d}\theta=\int_{-1}^{0}\mathrm{d}\left(\int_{0}^{\theta}A_2(t)\mathrm{e}^{\lambda t}\mathrm{d}t\right)\left\{\int_{0}^{\theta}\mathrm{e}^{-\lambda s}\psi(s)\mathrm{d}s\right\}$$

Integrating by parts we get

$$-\int_{-1}^{0}A_2(t)\mathrm{e}^{\lambda t}\mathrm{d}t\int_{-1}^{0}\mathrm{e}^{-\lambda s}\psi(s)\mathrm{d}s-\int_{-1}^{0}\left\{\int_{0}^{\theta}A_2(t)\mathrm{e}^{\lambda t}\mathrm{d}t\right\}\mathrm{e}^{-\lambda\theta}\psi(\theta)\mathrm{d}\theta. \tag{29}$$

Consider the function $\chi(\theta)\equiv\int_0^\theta A_2(t)\mathrm{e}^{\lambda t}\mathrm{d}t\cdot\psi(\theta)$ and notice that $|\chi(\theta)|=|\psi(\theta)|\cdot|\int_0^\theta A_2(t)\mathrm{e}^{\lambda t}\mathrm{d}t|\le|\psi(\theta)|\cdot\int_\theta^0|A_2(t)|\mathrm{e}^{t\,Re\lambda}\mathrm{d}t\le|\psi(\theta)|\cdot\int_\theta^0|A_2(t)|\mathrm{d}t\cdot C_5$, where $C_5\equiv\exp\{\sup|Re\lambda|,\lambda\in\lambda\in L_m^{(k)},k\in Z,m=1,...,\ell\}<\infty$, since all the circles $L_m^{(k)}$ are located in a vertical strip of the complex plane. Hence we conclude $\|\chi\|_{L^2(-1,0;\mathbb{C}^n)}\le\|\psi\|_{L^2(-1,0;\mathbb{C}^n)}\|A_2\|_{L^2(-1,0;\mathbb{C}^n)}C_5$ and we arrive to the same case as in the consideration of the second term in (28), i.e. we have the Fourier coefficient of χ instead of ψ. The same calculations give an analogous estimate for the first term in (29) and the term A_3 instead of A_2 (see the third term in (28)). Moreover, the factor λ^{-1} simplifies estimation.

This gives for any $\lambda\in L_m^{(k)}$ that $\|[\Delta_{\bar{\mathcal{A}}}^{-1}(\lambda)-\Delta_{\mathcal{A}}^{-1}(\lambda)]D_{\mathcal{A}}(z,\psi)\|\cdot\|(z,\psi)\|^{-1}\le\alpha_{k,m}(\hat{\lambda}),\hat{\lambda}\in L_m^{(0)}$, such that $\sum_{\substack{k\in\mathbb{Z}\\ m=1,..,\ell}}\alpha_{k,m}^2=S(\hat{\lambda})\le S<\infty$ where S is independent of $\hat{\lambda}\in L_m^{(0)}$. Essentially the same considerations lead to an analogous estimate for the second term in (24).

Collecting this with (24)-(28) and taking into account that for all $\lambda\in L_m^{(k)}$, one has $|\lambda|^{-1}\le(2\pi|k|-\hat{C})^{-1}$ with $\hat{C}$ independent of k and m, we get (22). Lemma 13 is proved. $\square$

Theorem 15 *There exists N_0 large enough, such that for any $N\ge N_0$,*

i) the sequence of subspaces $\{V_m^{(k)}\}_{\substack{|k|\ge N\\ m=1,..,\ell}}$ form a Riesz basis of the closure of their linear span, say $\mathcal{L}_N$. Here $V_m^{(k)}=P_m^{(k)}M_2$ and $P_m^{(k)}=\frac{1}{2\pi\mathrm{i}}\int_{L_m^{(k)}}R(\mathcal{A},\lambda)\mathrm{d}\lambda$ are spectral projectors; $L_m^{(k)}$ are circles defined in Paragraph 2.2;

ii) $\mathrm{codim}\mathcal{L}_N=(2N+1)n+n=2(N+1)n$.

Proof: An essential tool of the proof is the estimation of the norm of the difference

$$P_m^{(k)}-\bar{P}_m^{(k)}=\frac{1}{2\pi\mathrm{i}}\int_{L_m^{(k)}}[R(\mathcal{A},\lambda)-R(\bar{\mathcal{A}},\lambda)]\mathrm{d}\lambda, \tag{30}$$

where $\bar{P}^{(k)}$ is the same eigenprojector corresponding to the operator $\bar{\mathcal{A}}$. This is given by the application of the Lemma 13.

From (22) we infer $\sum_{m=1}^{\ell}\sum_{|k|>N_2}\|P_m^{(k)}-\bar{P}_m^{(k)}\|^2<\infty$. Here $N_2\geq N_1$ (see Theorem 2). Using this we easily obtain N_0 such that for any $N\geq N_0$ one has $\sum_{m=1}^{\ell}\sum_{|k|>N_0}\|P_m^{(k)}-\bar{P}_m^{(k)}\|^2<1$. The last estimate means that the sequences of subspaces $\{V_m^{(k)}\}_{\substack{|k|\geq N\\ m=1,..,\ell}}$ and $\{\bar{V}_m^{(k)}\}_{\substack{|k|\geq N\\ m=1,..,\ell}}$ are quadratically close. Theorem 5.2 [10] (see also Theorem 2.20 and Corollary 2.22 in [15]) completes the proof of item i).

Let us prove item ii). We use that the sequence $\{\bar{V}_m^{(k)}\}_{\substack{k\in\mathbb{Z}\\ m=1,..,\ell}}$ forms an orthogonal basis in M_2 (see Theorem 7) and $\{\bar{V}_m^{(k)}\}_{\substack{|k|\leq N\\ m=1,..,\ell}}\cup\{V_m^{(k)}\}_{\substack{|k|>N\\ m=1,..,\ell}}$ is quadratically close to the sequence $\{\bar{V}_m^{(k)}\}_{\substack{k\in\mathbb{Z}\\ m=1,..,\ell}}$ (see the proof of item i). Hence $\{\bar{V}_m^{(k)}\}_{\substack{|k|\leq N\\ m=1,..,\ell}}\cup\{V_m^{(k)}\}_{\substack{|k|>N\\ m=1,..,\ell}}$ forms a Riesz basis in M_2 and $\dim\{\bar{V}_m^{(k)}\}_{\substack{|k|\leq N\\ m=1,..,\ell}}=2(N+1)n$.

The proof of Theorem 15 is complete. □

The following result is very important in our framework.

Theorem 16 *There exists a sequence of $\mathcal{A}$-invariant finite-dimensional subspaces which constitute a Riesz basis in M_2. More precisely, these subspaces are $\{V_m^{(k)}\}_{\substack{|k|\geq N\\ m=1,..,\ell}}$ defined in Theorem 15 and a $2(N+1)n$-dimensional subspace W_N spaned by all eigen- and rootvectors, corresponding to all eigenvalues of $\mathcal{A}$, which are outside all circles $L_m^{(k)}, |k|\geq N, m=1,..,\ell$.*

Proof: Let $X_1=\mathcal{L}_N$, where $\mathcal{L}_N$ is defined in Theorem 15. The subspace X_1 is of finite co-dimension and $\mathcal{A}$-invariant, i.e. for any $x\in\mathcal{D}(A)\cap X_1$, one has $\mathcal{A}x\in X_1$.

Step 1. Let us show that M_2 can be split into the direct sum $M_2=X_1\oplus X_2$, and the operator $\mathcal{A}$ can be presented in the triangular form $\mathcal{A}=\begin{pmatrix}\mathcal{A}_{11}&\mathcal{A}_{12}\\0&\mathcal{A}_{22}\end{pmatrix}$, where $\mathcal{A}_{11}=P_1\mathcal{A}P_1:X_1\to X_1$, $\mathcal{A}_{22}=P_2\mathcal{A}P_2:X_2\to X_2$, $\mathcal{A}_{12}=P_1\mathcal{A}P_2:X_2\to X_1$. Here P_i are projectors on X_i along $X_j, j\neq i$.

First, we have to show that there exists X_2, such that $\mathcal{D}(A)\cap X_2\neq\{0\}$. Since $\operatorname{codim}X_1<\infty$, and $\mathcal{A}$ is densely defined, we get the existence of $y_1\notin X_1, y_1\in\mathcal{D}(A)$. Now denote by $X_1^1=Lin\{y_1,X_1\}$. The same arguments give the existence of $y_2\notin X_1^1, y_2\in\mathcal{D}(A)$ and so on. Since $\operatorname{codim}X_1<\infty$, we have a finite number of y_i, whose linear span gives X_2. Hence the splitting $M_2=X_1+X_2$ and the invariance of X_1 gives the triangular form of $\mathcal{A}$, mentioned above, if we identify X_1+X_2 with $X_1\times X_2$.

Step 2. Let us show that $\sigma(\mathcal{A}_{11})\cap\sigma(\mathcal{A}_{22})=\emptyset$. Let us assume that there exists

$\hat{\mu} \in \sigma(\mathcal{A}_{11}) \cap \sigma(\mathcal{A}_{22}) \neq \emptyset$. By Theorem 15, item i), X_1 possesses a Riesz basis of invariant subspaces.

Consider the generalized eigenspace for $\mathcal{A}_{11}$ corresponding to $\hat{\mu}$ and present it as $V_{\hat{\mu}} = \mathcal{I}m(\mathcal{A}_{11} - \hat{\mu}I)|_{V_{\hat{\mu}}} + K_{\hat{\mu}}$. Here $K_{\hat{\mu}}$ is spaned by rootvectors (of $\mathcal{A}_{11}$) of the highest orders from the rootchains giving the Jordan basis of $\mathcal{A}_{11}|_{V_{\hat{\mu}}}$. Notice that the dimension of $K_{\hat{\mu}}$ equals the number of Jordan blocks of $\mathcal{A}_{11}|_{V_{\hat{\mu}}}$. Since $(\mathcal{A}_{11} - \hat{\mu}I)$ is invertible on $X_1 \ominus V_{\hat{\mu}}$, X_1 itself is presented as $X_1 = \mathcal{I}m(\mathcal{A}_{11} - \hat{\mu}I) + K_{\hat{\mu}}$.

Now consider the eigenvector $h \in X_2$ for $\mathcal{A}_{22}$ corresponding to $\hat{\mu}$, i.e. $\mathcal{A}_{22}h = \hat{\mu}h$. We are going to show that there exist $y \in X_1$ and $v \in K_{\hat{\mu}}$, such that

$$\begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & \mathcal{A}_{22} \end{pmatrix} \begin{pmatrix} y \\ h \end{pmatrix} = \hat{\mu} \begin{pmatrix} y \\ h \end{pmatrix} + \begin{pmatrix} v \\ 0 \end{pmatrix}. \tag{31}$$

Since $\mathcal{A}_{12}h \in X_1$, there exist $w \in \mathcal{I}m(\mathcal{A}_{11} - \hat{\mu}I)$ and $v \in K_{\hat{\mu}}$, such that $\mathcal{A}_{12}h = w + v$. If we substitute this decomposition into the first row of (31): $(\mathcal{A}_{11} - \hat{\mu}I)y = -\mathcal{A}_{12}h - v = -w - v + v = -w$, we can find y. As a result, we get the existence of a rootvector $\left(\begin{smallmatrix} y \\ h \end{smallmatrix}\right)$ for $\mathcal{A}$, corresponding to $\hat{\mu}$, which is of higher order than $\left(\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right)$. The contradiction gives the result of this step.

Step 3. Let us show that M_2 can be split into the direct sum $M_2 = X_1 + \widehat{X}_2$, with *an invariant* $\widehat{X}_2$. Consider finite-dimensional operator $\mathcal{A}_{22}$. There exists $\sigma(\mathcal{A}_{22}) = \{\hat{\mu}_1, .., \hat{\mu}_s\}$ with the total multiplicity equals to $\dim X_2 = \operatorname{codim} X_1$. Consider h- an eigenvector of $\mathcal{A}_{22}$ corresponding to $\hat{\mu}$ and find $y \in X_1$ such that $\mathcal{A}\left(\begin{smallmatrix} y \\ h \end{smallmatrix}\right) = \hat{\mu}\left(\begin{smallmatrix} y \\ h \end{smallmatrix}\right)$. Such y is given by $y = -(\mathcal{A}_{11} - \hat{\mu}I)^{-1}\mathcal{A}_{12}h$ (see the triangular form of $\mathcal{A}$). This is due to the property $\sigma(\mathcal{A}_{11}) \cap \sigma(\mathcal{A}_{22}) = \emptyset$, which implies $\hat{\mu} \notin \sigma(\mathcal{A}_{11})$. Exactly in the same way one can find all rootvectors of $\mathcal{A}$ for all $\hat{\mu}$. Hence the number of eigen- and rootvectors of $\mathcal{A}$ corresponding to $\sigma(\mathcal{A}_{22})$ is equal to $\dim X_2 = 2(N+1)n$. By construction, the linear span of these vectors gives an $\mathcal{A}$-invariant subspace $\widehat{X}_2$. Now Theorem 15 completes the proof of Theorem 16. $\square$

3 Stability analysis

In this section we study the stability of the system (2). We consider two notions of stability: the strong asymptotic stability and the exponential stability (see for example [7,31] and references therein for abstract systems and different concrete examples).

Definition 17 *The system (2) (or (1)) is said to be exponentially stable if* *for some positive constants M and ω one has*

$$\|e^{\mathcal{A}t}\| \leq M e^{-\omega t}.$$

It is said to be strongly asymptotically stable if

$$\forall x \in M_2, \quad \lim_{t \to +\infty} e^{\mathcal{A}t} x = 0.$$

3.1 Exponential stability.

In the following theorem, using the results on the existence of Riesz bases of subspaces (see Theorem 16) and on the location of $\sigma(\mathcal{A})$ (see Theorem 2), we (partially) reformulate in terms of the matrix A_{-1} the condition on the spectrum $\sigma(\mathcal{A})$ to be bounded away from the imaginary axis (cf. [14, Theorem 6.1]).

Theorem 18 *System (2) is exponentially stable if and only if the following conditions are verified*

i) $\sigma(\mathcal{A}) \subset \{\lambda : \operatorname{Re}\lambda < 0\}$

ii) $\sigma(A_{-1}) \subset \{\lambda : |\lambda| < 1\}$.

Proof: Suppose that system (2) is exponentially stable, i.e. there exist positive M and α such that $\|e^{\mathcal{A}t}\| \leq M e^{-\alpha t}$. Hence $\sigma(\mathcal{A}) \subset \{\lambda : \operatorname{Re}\lambda \leq -\alpha\}$ together with Theorem 2 easily give the properties $i), ii)$.

To show that the conditions of the theorem are sufficient, we use the existence of a Riesz basis of invariant subspaces $\{W_N, V_m^{(k)}, k \in \mathbb{Z}, m = 1, \ldots, \ell\}$ (see Theorem 16). Consider the norm $\|.\|_1$ where the subspaces $\{W_N, V_m^{(k)}, k \in \mathbb{Z}, m = 1, \ldots, \ell\}$ are orthogonal. The semigroup $e^{\mathcal{A}t}|_{W_N}$ is clearly exponentially stable.
Let us consider now $e^{\mathcal{A}t}|_{V_m^{(k)}}$. By construction we have

$$e^{\mathcal{A}t}|_{V_m^{(k)}} = \frac{1}{2\pi i} \int_{L_m^{(k)}} e^{\lambda t} R(\mathcal{A}, \lambda) d\lambda.$$

From the expression of the resolvent (Proposition 1) and using the same estimate as in the proof of Lemma 13 we get

$$\|R(\mathcal{A}, \lambda)\|_1 \leq C, \qquad \lambda \in L_m^{(k)}, \quad k \in \mathbb{Z}, \quad m = 1, \ldots, \ell.$$

The conditions of the Theorem give $\operatorname{Re}\lambda_m^{(k)} \leq -\varepsilon_0 < 0$ for all $k \in \mathbb{Z}$ and then

$$\|e^{\mathcal{A}t}|_{V_m^{(k)}}\|_1 \leq M_1 e^{-\varepsilon t} \int\limits_{L_m^{(k)}} \|R(\mathcal{A},\lambda)\|_1 d\lambda, \qquad k \in \mathbb{Z}, \tag{32}$$

where $0 < \varepsilon < \varepsilon_0$ and then $e^{\mathcal{A}t}|_{V_m^{(k)}}$ are uniformly exponentially stable for such m:

$$\|e^{\mathcal{A}t}|_{V_m^{(k)}}\|_1 \leq M e^{-\varepsilon t}, \qquad t \geq 0, \qquad k \in \mathbb{Z}. \tag{33}$$

Since the constant M in the last estimate is independent of m, k and $\{W_N, V_m^{(k)}\}$ form a Riesz basis we get the exponential stability. The proof is complete. $\square$

3.2 Strong asymptotic stability

A well known necessary condition is given in the following Proposition.

Proposition 19 *If system (2) is asymptotically stable, then the spectrum satisfies $\sigma(\mathcal{A}) \subset \{\lambda : \operatorname{Re}\lambda < 0\}$.*

The proof is obvious because the spectrum of $\mathcal{A}$ consists on eigenvalues only (Theorem 2).

We will essentially use the following

Theorem 20 *Let $e^{\mathcal{A}t}$, $t \geq 0$ be a C_0-semigroup in the Banach space X and $\mathcal{A}$ be the infinitesimal generator of the semigroup. Assume that $\sigma(\mathcal{A}) \cap (i\mathbb{R})$ is at most countable and the operator $\mathcal{A}^*$ has no pure imaginary eigenvalues. Then $e^{\mathcal{A}t}$ is strongly asymptotically stable if and only if one of the following conditions hold:*

- *i) There exists a norm $\|\cdot\|_1$, equivalent to the initial one $\|\cdot\|$, such that the semigroup $e^{\mathcal{A}t}$ is contractive according to this norm: $\|e^{\mathcal{A}t}x\|_1 \leq \|x\|_1$, $\forall x \in X$, $t \geq 0$;*
- *ii) The semigroup $e^{\mathcal{A}t}$ is uniformly bounded: $\exists C > 0$ such that $\|e^{\mathcal{A}t}\| \leq C$, $t \geq 0$.*

Theorem 20 was obtained first in [29] for the case of bounded operator $\mathcal{A}$, then generalized in [2,18] for the general case. The development of this theory concerns a large class of differential equations in Banach space (see [31] and references therein).

Theorem 21 *System (2) is strongly asymptotically stable if and only if* $\sigma(\mathcal{A}) \subset \{\lambda : \mathrm{Re}\,\lambda < 0\}$ *and* $\mathcal{A}$ *is dissipative in an equivalent norm.*

PROOF: We apply Theorem 20 to our system and use that, in our case, the operator $\mathcal{A}$ has eigenvalues only. $\square$

The condition $\sigma(\mathcal{A}) \subset \{\lambda : \mathrm{Re}\,\lambda < 0\}$ is necessary for asymptotic stability. Then, in order to have more precise conditions of strong stability, the main problem is to verify dissipativity of the operator $\mathcal{A}$. In our framework, the analysis of this problem is given in terms of the spectral properties of the matrix A_{-1}.

3.2.1 Conditions for stability and instability

In this section, assuming the spectrum $\sigma(\mathcal{A})$ is in the left-half plane, we present sufficient conditions (in terms of matrix A_{-1} only) for the system (2) to be stable (Theorem 23) or unstable (Theorem 24). The case when these sufficient conditions are not satisfied is much more complicated and is studied in section 3.2.2.

Theorem 2 gives that the property $\sigma(\mathcal{A}) \subset \{\lambda : \mathrm{Re}\,\lambda < 0\}$ implies $\sigma(A_{-1}) \subset \{\lambda : |\lambda| \leq 1\}$.

Let us split $\sigma(A_{-1}) = \sigma_0 \cup \sigma_1$, where $\sigma_0 = \sigma(A_{-1}) \cap \{\lambda : |\lambda| < 1\}$, and $\sigma_1 = \sigma(A_{-1}) \cap \{\lambda : |\lambda| = 1\}$.

Our main result on stability can be formulated as follows.

Theorem 22 *Assume* $\sigma(\mathcal{A}) \subset \{\lambda : \mathrm{Re}\,\lambda < 0\}$ *and* $\max\{|\mu_m| : \mu_m \in \sigma(A_{-1})\} = 1$. *Then the following three mutually exclusive possibilities exist:*

p1) *the part of the spectrum* $\sigma_1 \equiv \sigma(A_{-1}) \cap \{\lambda : |\lambda| = 1\}$ *consists of simple eigenvalues only, i.e. to each eigenvalue corresponds a one-dimensional eigenspace and there are no rootvectors. In this case system (2) is asymptotically stable.*

p2) *the matrix* A_{-1} *has a Jordan block, corresponding to* $\mu \in \sigma_1$. *In this case system (2) is unstable.*

p3) *there are no Jordan blocks, corresponding to eigenvalues in* σ_1, *but there exists* $\mu \in \sigma_1$ *whose eigenspace is at least two-dimensional. In this case system (2) can be stable as well as unstable. Moreover, there exist two systems with the same spectrum, such that one of them is stable while the other one is unstable.*

We split the proof on several assertions.

Theorem 23 *Let $\sigma(\mathcal{A}) \subset \{\lambda : \operatorname{Re}\lambda < 0\}$. Assume that the part of the spectrum σ_1 consists of simple eigenvalues only, i.e. to each eigenvalue corresponds a one-dimensional eigenspace and there are no rootvectors. Then system (2) is asymptotically stable.*

PROOF: We use the existence of a Riesz basis of invariant subspaces $\{W_N, V_m^{(k)}, k \in \mathbb{Z}, m = 1, \ldots, \ell\}$ (see Theorem 16). Consider the norm $\|.\|_1$ where the subspaces $\{W_N, V_m^{(k)}, k \in \mathbb{Z}, m = 1, \ldots, \ell\}$ are orthogonal. The semigroup $\mathrm{e}^{\mathcal{A}t}|_{W_N}$ is clearly exponentially stable and then uniformly bounded.
Let us consider now $\mathrm{e}^{\mathcal{A}t}|_{V_m^{(k)}}$. Let us distinguish two families of eigenvalues corresponding to the spectrum σ_1 and σ_0 of the matrix A_{-1}.

Suppose first that m is such that $\mu_m \in \sigma_0$, then $\operatorname{Re}\lambda_m^{(k)} \leq -\varepsilon_0 < 0$ for all $k \in \mathbb{Z}$. By the same arguments as in the proof of Theorem 18 (see relations (32 and (33)) we obtain that $\mathrm{e}^{\mathcal{A}t}|_{V_m^{(k)}}$ are uniformly exponentially stable and then uniformly bounded for such m.

Now consider the critical case when m is such that $\mu_m \in \sigma_1$. By hypothesis this part of spectrum is simple. Let x be in closed span of the corresponding subspaces $V_m^{(k)}$, then

$$x = \sum_{\substack{k\in\mathbb{Z}\\ \mu_m\in\sigma_1}} x_m^{(k)}, \qquad \mathrm{e}^{\mathcal{A}t}x = \sum_{\substack{k\in\mathbb{Z}\\ \mu_m\in\sigma_1}} \mathrm{e}^{\lambda_m^{(k)}t} x_m^{(k)},$$

This gives

$$\|\mathrm{e}^{\mathcal{A}t}x\|_1^2 = \sum_{\substack{k\in\mathbb{Z}\\ \mu_m\in\sigma_1}} \mathrm{e}^{2\operatorname{Re}\lambda_m^{(k)}t} \|x_m^{(k)}\|_1^2 \leq \|x\|_1^2,$$

and this means that $\mathrm{e}^{\mathcal{A}t}$ is uniformly bounded in the subspace generated by the corresponding subspaces $V_m^{(k)}$.

Hence we obtain

$$\|\mathrm{e}^{\mathcal{A}t}|_{V_m^{(k)}}\|_1 \leq M, \qquad k \in \mathbb{Z}, m = 1, \ldots, \ell,$$

and the constant M does not depend on k or m. This gives that the semigroup in uniformly bounded on the closed span of the subspaces $V_m^{(k)}$ and then, with the boundedness on W_N, we obtain the uniform boundedness in M_2. Then by Theorem 20 the system is strongly asymptotically stable. $\square$

Theorem 24 *If the matrix A_{-1} has a Jordan block, corresponding to $\mu \in \sigma_1$, then the system (2) is not asymptotically stable.*

PROOF: Consider $\mu = \mu_m \in \sigma_1$. By Theorem 7, points $\lambda_m^k \in \sigma(\bar{\mathcal{A}})$ belong to $i\mathbb{R}$, i.e. they are purely imaginary numbers. We denote by $\bar{v}^k = \bar{v}_m^k$ and $\bar{w}^k = \bar{w}_m^k$ the normed eigen- and rootvectors corresponding to λ_m^k (see (16) in Theorem 7).

Hence, using $\|\bar{v}^k\| = \|\bar{w}^k\| = 1$ and $|e^{\lambda_m^k T}| = 1$, one gets

$$\|e^{\bar{\mathcal{A}}T}\bar{w}^k\| = \|e^{\lambda_m^k T}(T\bar{v}^k + \bar{w}^k)\| = |e^{\lambda_m^k T}| \cdot \|T\bar{v}^k + \bar{w}^k\| \geq T - 1. \tag{34}$$

Now consider the spectral projectors $P^{(k)} = P_m^{(k)}$ and $\bar{P}^{(k)} = \bar{P}_m^{(k)}$ (see Theorem 15 and (30)) and estimate

$$\begin{aligned} \|e^{\mathcal{A}T}P^{(k)} - e^{\bar{\mathcal{A}}T}\bar{P}^{(k)}\| &= \left\| \tfrac{1}{2\pi i} \int_{L_m^{(k)}} e^{\lambda T}[R(\mathcal{A},\lambda) - R(\bar{\mathcal{A}},\lambda)]\mathrm{d}\lambda \right\| \\ &\leq \tfrac{1}{2\pi} e^{\varepsilon_0 T} \int_{L_m^{(k)}} \left\| R(\mathcal{A},\lambda) - R(\bar{\mathcal{A}},\lambda)\right\| \mathrm{d}\lambda \\ &\leq e^{\varepsilon_0 T}\gamma_k, \end{aligned} \tag{35}$$

where $\gamma_k \to 0, k \to \infty$ (see in Lemma 13 the relation (22)).

Now we choose an arbitrary large T (we will later pass to the limit $T \to \infty$). For this T we choose large k such that $e^{\varepsilon_0 T}\alpha_k \leq \varepsilon_1$ and $\|P^{(k)} - \bar{P}^{(k)}\| \leq \varepsilon_1$ for some ε_1.

Using this and (35) we obtain

$$\|e^{\mathcal{A}T}P^{(k)}\bar{w}^k\| \geq \|e^{\bar{\mathcal{A}}T}\bar{P}^{(k)}\bar{w}^k\| - \varepsilon_1 \geq T - 1 - \varepsilon_1. \tag{36}$$

On the other hand, $\|P^{(k)} - \bar{P}^{(k)}\| \leq \varepsilon_1$ and $\|\bar{w}^k\| = 1$ give $\|P^{(k)}\bar{w}^k - \bar{P}^{(k)}\bar{w}^k\| \leq \varepsilon_1$ which implies

$$\|P^{(k)}\bar{w}^k\| \leq 1 + \varepsilon_1. \tag{37}$$

Collecting (36) and (37), we arrive to

$$\frac{\|e^{\mathcal{A}T}P^{(k)}\bar{w}^k\|}{\|P^{(k)}\bar{w}^k\|} \geq \frac{T - 1 - \varepsilon_1}{1 + \varepsilon_1} \to +\infty \quad \text{as} \quad T \to +\infty.$$

Hence $e^{\mathcal{A}t}$ is unstable by the Banach-Steinhaus theorem. The proof of Theorem 24 is complete. $\square$

3.2.2 Dilemma: stable or unstable (the case p3)

In this section we prove that in the case $\sigma_1 = \sigma(A_{-1}) \cap \{|\lambda| = 1\}$ is not simple, system (2) can be either stable or unstable. We give (see Theorem 29) two examples of system (2) (one stable and one unstable) for $z \in \mathbb{R}^2$ and $A_{-1} = -I$, i.e. $\sigma_1 = \{-1\}$ and there are two eigenvectors. In these examples both systems (one stable and one unstable) have the same spectrum located in the open left-half plane.

General auxiliary considerations

We will consider the following particular case of the system (2) with a control

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \bar{\mathcal{A}}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} + \mathcal{B}u(t), \tag{38}$$

where $\mathcal{D}(\bar{\mathcal{A}}) = \{(y, z(\cdot)) : z \in H^1(-1, 0; \mathbb{C}^n), y = z(0) - A_{-1}z(-1)\} \subset M_2$, the control u is m dimensional and

$$\mathcal{B}u = \begin{pmatrix} B \\ 0 \end{pmatrix},$$

where B is a $n \times m$ matrix.

Let us consider $\mathcal{A} = \bar{\mathcal{A}} + \mathcal{B}Q$, where Q is a linear bounded operator and assume that $\sigma(\bar{\mathcal{A}} + \mathcal{B}Q) \cap \sigma(\bar{\mathcal{A}}) = \emptyset$.

The equation for an eigenvector and eigenvalue of $\mathcal{A}$ is $(\bar{\mathcal{A}} + \mathcal{B}Q)x = \lambda x$ or equivalently $(\bar{\mathcal{A}} - \lambda I)x + \mathcal{B}Qx = 0$. Apply the resolvent $R_\lambda = R(\bar{\mathcal{A}}, \lambda)$ to the last equation $x + R(\bar{\mathcal{A}}, \lambda)\mathcal{B}Qx = 0$. Hence there exists a vector $c_x \in \mathbb{C}^m$, such that

$$x = R(\bar{\mathcal{A}}, \lambda)\mathcal{B}c_x \tag{39}$$

Then c_x must satisfy

$$R(\bar{\mathcal{A}}, \lambda)\mathcal{B}c_x + R(\bar{\mathcal{A}}, \lambda)\mathcal{B}QR(\bar{\mathcal{A}}, \lambda)\mathcal{B}c_x = 0. \tag{40}$$

Remark 25 *Without loss of generality we can assume that the column of B, say $b_1, .., b_m$, are linearly independent, and then we have* $\operatorname{Ker}\mathcal{B} = \{0\}$. *Hence there exists $\mathcal{B}^{-1}_{\text{left}}$, the left-inverse of $\mathcal{B}$.*

The last equation (40) gives $c_x + QR(\bar{\mathcal{A}}, \lambda)\mathcal{B}c_x = 0$ or equivalently

$$(I + QR(\bar{\mathcal{A}}, \lambda)\mathcal{B})c_x = 0, \tag{41}$$

where $I + QR(\bar{\mathcal{A}}, \lambda)\mathcal{B}$ is $m \times m$. The characteristic equation, for the eigenvalues of $\mathcal{A}$ is then given by $\det(I + QR(\bar{\mathcal{A}}, \lambda)\mathcal{B}) = 0$.

This result is quite general and may be formulated in the following Lemma.

Lemma 26 *Let $\bar{\mathcal{A}}$ be a linear operator with $\rho(\bar{\mathcal{A}}) \neq \emptyset$ defined in the Hilbert space X and $\mathcal{A} = \bar{\mathcal{A}} + \mathcal{B}Q$, where $\mathcal{B}$ are linear bounded operator from $\mathbb{C}^m$ to X and from X to $\mathbb{C}^m$ respectively, $\mathcal{B}$ being left-invertible. Then $\lambda \notin \sigma(\bar{\mathcal{A}})$ is an eigenvalue of $\mathcal{A}$ if and only if* $\det(I + QR(\bar{\mathcal{A}}, \lambda)\mathcal{B}) = 0$ *and the corresponding eigenvector is $x = R(\bar{\mathcal{A}}, \lambda)\mathcal{B}c_x$, where $c_x \in \mathrm{Ker}(I + QR(\bar{\mathcal{A}}, \lambda)\mathcal{B})$.*

Particular auxiliary case $n = 1$

We need this scalar case to study the properties of the characteristic equation (see (44)) which will be also the characteristic equation of the 2-dimensional system (46) in the next paragraph. Our purpose is to design a 2-dimensional system, which, depending on the feedback, may be asymptotically stable or unstable.

Consider the scalar equation

$$\dot{z}(t) = -\dot{z}(t-1) + u \tag{42}$$

which is a particular case of the system (38) with $n = 1$, $A_{-1} = -1$ and the operator is defined as

$$\mathcal{B}u = b \cdot u, \quad b = (1; \varphi)^T \in M_2(-1, 0; \mathbb{C}), \quad \varphi(\theta) = 0, \tag{43}$$

where $\varphi(\theta), \theta \in [-1, 0]$ is a scalar function.

From Theorem 2 the spectrum is $\sigma(\bar{\mathcal{A}}) = \{\lambda^0 = 0\} \cup \{\lambda_k = \mathrm{i} \cdot (2\pi k + \pi)\}_{k \in \mathbb{Z}}$, and eigenvectors

$$v^{00} = (1; 1/2)^T \in M_2, \quad v^k = \left(0; \mathrm{e}^{\lambda_k \theta}\right)^T \in M_2$$

form an orthogonal basis of M_2 with the norm

$$\|(y, z(\cdot))\|_1^2 = \|y\|^2 + \int_1^0 \|\frac{1}{2}y - z(\theta)\|^2 \mathrm{d}\theta.$$

It is easy to verify that system (42) satisfies all assumptions of [28], more precisely (see [28] for more details):

i) $\bar{\mathcal{A}}$ is a skew-adjoint (in $\|\cdot\|_1$-norm) unbounded operator with discrete spectrum consisting of simple eigenvalues $\{\lambda_k\}$,

ii) there exists a constant $C_\sigma = \frac{1}{2} \min\limits_{i \neq j} |\lambda_i - \lambda_j| = \pi/2 > 0$,

iii) the operator $\mathcal{B}$ is associated with the vector $b \in M_2$; besides, if $\{v^k\}$ is an orthogonal eigenbasis of $\bar{\mathcal{A}}$, i.e. $\bar{\mathcal{A}}v^k = \lambda_k v^k$, then $b_k = \langle b, v^k \rangle_1 \neq 0$ for all k.

If we are interested in a bounded control for system (42), i.e. $u(x) = q^* x \equiv \langle x, q \rangle_1$ for some $q \in M_2$, then (see (41)) the characteristic equation for the eigenvalues $\widetilde{\lambda}_n$ of the operator $\mathcal{A} = \bar{\mathcal{A}} + bq^*$ is

$$1 + \frac{q^{00} b^{00}}{\lambda} + \sum_k \frac{q^k b^k}{\widetilde{\lambda}_n - \lambda_k} = 0, \tag{44}$$

where b_k is defined in iii) and $q_k = \langle q, v^k \rangle_1$.

To study the properties of equation (44) we apply the following result.

Theorem 27 *[28, Theorem 4] Let $\{\widetilde{\lambda}_n\}$ be any set of complex numbers such that*

a) $|\lambda_n - \widetilde{\lambda}_n| < C_\sigma$ *for all* n;

b) $\sum_n \frac{|\lambda_n - \widetilde{\lambda}_n|^2}{|b_n|^2} < \frac{C_\sigma}{\|b\|_1^2}$, *where* $\|b\|_1 \cdot \|q\|_1 < C_\sigma/2 = \pi/4$.
There then exists a unique control $u(x) = q^* x$ such that the spectrum $\sigma(\mathcal{A})$ of the operator $\mathcal{A} = \bar{\mathcal{A}} + bq^*$ is $\{\widetilde{\lambda}_n\}$ and, moreover, the corresponding eigenvectors constitute a Riesz basis.

We will also need the following

Corollary 28 *Fix any sequence $\{b_k\} \subset \ell_2$, $b_k \neq 0$. Then for any set of complex numbers such that*

c1) $|\lambda_n - \widetilde{\lambda}_n| < \pi/2$ *for all* n;

c2) $\sum_n \frac{|\lambda_n - \widetilde{\lambda}_n|^2}{|b_n|^2} < \frac{\pi}{2} \left(\sum_i |b_i|^2\right)^{-1}$
there exists a unique sequence $\{q_k\} \subset \ell_2$, such that $\{\widetilde{\lambda}_n\}$ are all the roots of the equation (44) and $(\sum_i |q_i|^2) \cdot (\sum_i |b_i|^2) < \pi^2/16$.

Particular case $n = 2$

Consider the system

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + Bu, \quad A_{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{45}$$

The operator form (38) is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \bar{\mathcal{A}}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} + \mathcal{B}u, \qquad \bar{\mathcal{A}}\begin{pmatrix} y(t) \\ z(\cdot) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{d}z(\theta)/\mathrm{d}\theta \end{pmatrix}, \tag{46}$$

where the domain of $\bar{\mathcal{A}}$ is given by $\mathcal{D}(\bar{\mathcal{A}}) = \{(y, z(\cdot)) : z \in H^1(-1, 0; \mathbb{C}^2), y = z(0) - A_{-1}z(-1)\} \subset M_2$.

The operator $\mathcal{B}$ associated with the matrix B, is defined as

$$\mathcal{B}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = b_1u_1 + b_2u_2, \quad b_1 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \varphi\right)^T, \quad b_2 = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}; \varphi\right)^T, \tag{47}$$

where $u_i \in \mathbb{C}, \varphi(\theta) : [-1, 0] \to \mathbb{C}^2,\ \varphi(\theta) = 0$, and $b_i \in M_2$.

From Theorem 2 the spectrum is $\sigma(\bar{\mathcal{A}}) = \{\lambda^0 = 0\} \cup \{\lambda_k = \mathrm{i} \cdot (2\pi k + \pi)\}_{k\in\mathbb{Z}}$, there are no rootvectors and eigenvectors are

$$v_1^{00} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}\right)^T, \quad v_2^{00} = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}\right)^T, \tag{48}$$

$$v_1^{k} = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}; \mathrm{e}^{\lambda_k\theta} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)^T, \quad v_2^{k} = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}; \mathrm{e}^{\lambda_k\theta} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)^T. \tag{49}$$

Theorems 15 and 16 (see also [24]) give that the two-dimensional subspaces $\bar{V}^{(k)} = Lin\,\{v_1^k, v_1^k\}$ and $\bar{V}^{(00)} = Lin\,\{v_1^{00}, v_2^{00}\}$ form an orthogonal basis of subspaces in M_2. Moreover eigenvectors form an orthogonal basis in the norm

$$\|(y, z(\cdot))\|_1^2 = \|y\|^2 + \int_1^0 \|\frac{1}{2}y - z(\theta)\|^2 \mathrm{d}\theta.$$

As in [24] (see also Theorem 15), we define in M_2 the eigenprojectors $\bar{P}^{(k)} = (\bar{P}^{(k)})^2 : M_2 \to \bar{V}^{(k)}$.

As $\sum_k \bar{P}^{(k)} = I$ and $P^{(k)}$ are orthogonal to each other, we can write the operator in (41) as

$$I + QR(\bar{\mathcal{A}}, \lambda)\mathcal{B} = I + (\sum_k Q\bar{P}^{(k)}) \cdot R(\bar{\mathcal{A}}, \lambda) \cdot (\sum_k \bar{P}^{(k)}\mathcal{B}) = I + \sum_k Q\bar{P}^{(k)}R(\bar{\mathcal{A}}, \lambda)\mathcal{B}.$$

Since $\bar{V}^{(k)}$ are invariant for $\bar{\mathcal{A}}$ we have $R(\bar{\mathcal{A}}, \lambda)|_{\bar{V}^{(k)}} = (\lambda - \lambda_k)^{-1} \cdot I$, which gives

$$I + QR(\bar{\mathcal{A}}, \lambda)\mathcal{B} = I + \frac{1}{\lambda}Q\bar{P}^{(00)}\mathcal{B} + \sum_k \frac{1}{\lambda - \lambda_k}Q\bar{P}^{(k)}\mathcal{B}. \tag{50}$$

Let us study in details the operator $Q\bar{P}^{(k)}\mathcal{B} : \mathbb{C}^2 \to \mathbb{C}^2$. First we consider the operator $\bar{P}^{(k)}\mathcal{B} : \mathbb{C}^2 \to \bar{V}^{(k)}$, and use ((47) - (49):

$$\begin{aligned}\bar{P}^{(k)}\mathcal{B}(u_1, u_2) &= v_1^k \langle b_1 u_1 + b_2 u_2, v_1^k \rangle_1 + v_2^k \langle b_1 u_1 + b_2 u_2, v_2^k \rangle_1 \\ &= v_1^k \left(\langle b_1, v_1^k \rangle_1 \cdot u_1 + \langle b_2, v_1^k \rangle_1 \cdot u_2 \right) \\ &\quad + v_2^k \left(\langle b_1, v_2^k \rangle_1 \cdot u_1 + \langle b_2, v_2^k \rangle_1 \cdot u_2 \right) \\ &= (v_1^k, v_1^k) \begin{pmatrix} \langle b_1, v_1^k \rangle_1 & \langle b_2, v_1^k \rangle_1 \\ \langle b_1, v_2^k \rangle_1 & \langle b_2, v_2^k \rangle_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.\end{aligned}$$

Hence

$$Q\bar{P}^{(k)}\mathcal{B}(u_1, u_2) = \begin{pmatrix} q_1(\bar{P}^{(k)}\mathcal{B}(u_1, u_2)) \\ q_2(\bar{P}^{(k)}\mathcal{B}(u_1, u_2)) \end{pmatrix}$$

$$= \begin{pmatrix} q_1(v_1^k) & q_1(v_2^k) \\ q_2(v_1^k) & q_2(v_2^k) \end{pmatrix} \begin{pmatrix} \langle b_1, v_1^k \rangle_1 & \langle b_2, v_1^k \rangle_1 \\ \langle b_1, v_2^k \rangle_1 & \langle b_2, v_2^k \rangle_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{51}$$

Using the explicit form of b_1, b_2 and v_1^k, v_2^k we see that $\langle b_1, v_1^k \rangle_1 = \langle b_2, v_2^k \rangle_1 = b^k \neq 0$ and $\langle b_1, v_2^k \rangle_1 = \langle b_2, v_1^k \rangle_1 = 0$. We conclude

$$Q\bar{P}^{(k)}\mathcal{B}(u_1, u_2) = \begin{pmatrix} q_1(v_1^k) & q_1(v_2^k) \\ q_2(v_1^k) & q_2(v_2^k) \end{pmatrix} \begin{pmatrix} b^k & 0 \\ 0 & b^k \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{52}$$

Now we prove the main result of this section (cf. item **p3)** of Theorem 22).

Theorem 29 *Consider the system (46) (see also (45)). For any sequence of complex numbers* $\{\widetilde{\lambda}_k\} \subset \{\lambda : \operatorname{Re}\lambda < 0\}$ *such that*

c1) $|\lambda_n - \widetilde{\lambda}_n| < \pi/2$ *for all* n;

c2) $\sum_n \frac{|\lambda_n - \widetilde{\lambda}_n|^2}{|b^n|^2} < \frac{\pi}{2} \left(\sum_i |b^i|^2\right)^{-1}$;

c3) $\frac{1}{k} \cdot \frac{|\widetilde{\lambda}_k - \lambda_k|}{(-\operatorname{Re}\widetilde{\lambda}_k)} \to \infty, \quad k \to \infty$

there exist two bounded linear feedback controls $Q_i : M_2 \to \mathbb{C}^2$, $i = 1, 2$, *such that the system (46) with both controls, i.e.* $\dot{x} = (\bar{\mathcal{A}} + \mathcal{B}Q_i)x$, $i = 1, 2$, *has the same spectrum* $\sigma(\bar{\mathcal{A}} + \mathcal{B}Q_1) = \sigma(\bar{\mathcal{A}} + \mathcal{B}Q_2) = \{\widetilde{\lambda}_k\}$ *and the corresponding semigroup* $e^{t(\bar{\mathcal{A}} + \mathcal{B}Q_1)}$ *is asymptotically stable while the semigroup* $e^{t(\bar{\mathcal{A}} + \mathcal{B}Q_2)}$ *is unstable.*

PROOF: The proof consists of two parts: stable and unstable.

We will show that there exists Q, such that $Q\bar{P}^{(k)}\mathcal{B} = \widetilde{q}^k b^k \cdot I$ for some $\widetilde{q}^k \in \mathbb{C}$. Using (52), we set

$$q_1(v_1^k) = q_2(v_2^k) = \widetilde{q}^k, \qquad q_1(v_2^k) = q_2(v_1^k) = 0. \tag{53}$$

Using (52) and (53), we get from (50):

$$I + QR(\bar{\mathcal{A}}, \lambda)\mathcal{B} = \left(1 + \frac{\widetilde{q}^{00}b^{00}}{\lambda} + \sum_k \frac{\widetilde{q}^k b^k}{\lambda - \lambda_k}\right) \cdot I. \tag{54}$$

The characteristic equation is

$$1 + \frac{\widetilde{q}^{00}b^{00}}{\lambda} + \sum_k \frac{\widetilde{q}^k b^k}{\lambda - \lambda_k} = 0. \tag{55}$$

Since the characteristic equation (55) coincides with (44), we can apply Corollary 28 to get the existence of $\{\widetilde{q}^{00}, \widetilde{q}^1, \widetilde{q}^2, \ldots\} \subset \ell_2$, such that the roots of (55) are $\{\widetilde{\lambda}_k\}$. Since eigenvectors ($\{v_i^k\}$) form an orthogonal basis and $\{\widetilde{q}^{00}, \widetilde{q}^1, \widetilde{q}^2, \ldots\} \subset \ell_2$, then the control Q defined by (53) is bounded.

Let us find the eigenvectors of $\mathcal{A}$ (see (39), (41) and (54)) to show that the system is stable. In this case we can take $c_x = c_x^1 = \binom{1}{0}$ and $c_x = c_x^2 = \binom{0}{1}$. Using (47), we get $\mathcal{B}c_x^1 = b_1$ and $\mathcal{B}c_x^2 = b_2$. These together with the explicit form of the resolvent $R(\bar{\mathcal{A}}, \lambda)$ (see Proposition 1) and (39) give two eigenvectors of $\mathcal{A}$:

$$\begin{aligned} \xi_1^k &= -\tfrac{1}{\widetilde{\lambda}_k}\left(\tbinom{1}{0}; \mathrm{e}^{\widetilde{\lambda}_k\theta}(1 + \mathrm{e}^{-\widetilde{\lambda}_k}) \cdot \tbinom{1}{0}\right)^T \\ \xi_2^k &= -\tfrac{1}{\widetilde{\lambda}_k}\left(\tbinom{0}{1}; \mathrm{e}^{\widetilde{\lambda}_k\theta}(1 + \mathrm{e}^{-\widetilde{\lambda}_k}) \cdot \tbinom{0}{1}\right)^T. \end{aligned} \tag{56}$$

Theorem 15 gives that the subspaces $V^{(k)} = Lin\{\xi_1^k, \xi_2^k\}$ form a Riesz basis of their linear span. The explicit form (56) shows that $\langle \xi_1^k, \xi_2^k \rangle_1 = 0$, hence the eigenvectors form a Riesz basis of their linear span. Finally, the condition $\{\widetilde{\lambda}_k\} \subset \{\lambda : \mathrm{Re}\,\lambda < 0\}$ allows us to apply the general Theorem 20 to get the stability of the system.

B. Unstable part of Theorem 29.

Using (52), we set

$$q_1(v_1^k) = q_1(v_2^k) = q_2(v_2^k) = \widetilde{q}^k, \qquad q_2(v_1^k) = 0. \tag{57}$$

Using (52) and (57), we get from (50):

$$I + QR(\bar{\mathcal{A}}, \lambda)\mathcal{B} = I + \left(\frac{\widetilde{q}^{00} b^{00}}{\lambda} + \sum_k \frac{\widetilde{q}^k b^k}{\lambda - \lambda_k} \right) \cdot D, \quad D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{58}$$

The characteristic equation is again (55) so, as in the stable part of the Theorem, we can apply Corollary 28 to get the existence of $\{\widetilde{q}^{00}, \widetilde{q}^1, \widetilde{q}^2, \ldots\} \subset \ell_2$. This implies that the corresponding operator Q (defined by (57)) is bounded.

Since $c_x = c_x^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of D (the eigenspace is one-dimensional), we get from (39), (41) and (58) that $x = \widetilde{v} = R(\bar{\mathcal{A}}, \lambda) b_1$ is an eigenvector of $\mathcal{A}$.

Let us find the rootvector $\widetilde{w}$ of $\mathcal{A} : (\bar{\mathcal{A}} + \mathcal{B}Q)\widetilde{w} = \lambda \widetilde{w} + \widetilde{v}$ or equivalently $(\bar{\mathcal{A}} - \lambda I)\widetilde{w} + \mathcal{B}Q\widetilde{w} = \widetilde{v}$. We apply the resolvent $R(\bar{\mathcal{A}}, \lambda)$ to get

$$\widetilde{w} + R(\bar{\mathcal{A}}, \lambda)\mathcal{B}Q\widetilde{w} = R(\bar{\mathcal{A}}, \lambda)\widetilde{v} = R_\lambda^2(\bar{\mathcal{A}}, \lambda) b_1. \tag{59}$$

If we set $d = Q\widetilde{w}$, then we obtain

$$\widetilde{w} = R_\lambda^2(\bar{\mathcal{A}}, \lambda) b_1 - R(\bar{\mathcal{A}}, \lambda)\mathcal{B} d. \tag{60}$$

Replacing this into (59):

$$R_\lambda^2(\bar{\mathcal{A}}, \lambda) b_1 - R(\bar{\mathcal{A}}, \lambda)\mathcal{B}d + R(\bar{\mathcal{A}}, \lambda)\mathcal{B}Q \left(R_\lambda^2(\bar{\mathcal{A}}, \lambda) b_1 - R(\bar{\mathcal{A}}, \lambda)\mathcal{B}d \right) = R_\lambda^2(\bar{\mathcal{A}}, \lambda) b_1.$$

We now apply the left inverse operator $(R(\bar{\mathcal{A}}, \lambda)\mathcal{B})_{\text{left}}^{-1}$ to obtain $-d + QR_\lambda^2(\bar{\mathcal{A}}, \lambda) b_1 - QR(\bar{\mathcal{A}}, \lambda)\mathcal{B}d = 0$ or equivalently $(I + QR(\bar{\mathcal{A}}, \lambda)\mathcal{B})d = QR_\lambda^2(\bar{\mathcal{A}}, \lambda) b_1$.

Using (58) and $R(\bar{\mathcal{A}}, \lambda)|_{\bar{V}^{(k)}} = (\lambda - \lambda_k)^{-1} \cdot I$, (see also (50)), we can write

$$\left[I + \left(\frac{\widetilde{q}^{00} b^{00}}{\lambda} + \sum_k \frac{\widetilde{q}^k b^k}{\lambda - \lambda_k} \right) \cdot D \right] d = \left(\frac{\widetilde{q}^{00} b^{00}}{\lambda^2} + \sum_k \frac{\widetilde{q}^k b^k}{(\lambda - \lambda_k)^2} \right) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{61}$$

Since λ is a root of the characteristic equation (55), we deduce that

$$\left[I + \left(\frac{\widetilde{q}^{00} b^{00}}{\lambda} + \sum_k \frac{\widetilde{q}^k b^k}{\lambda - \lambda_k} \right) \cdot D \right] = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

Hence (61) gives

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} d = \begin{pmatrix} \frac{\widetilde{q}^{00} b^{00}}{\lambda^2} + \sum_k \frac{\widetilde{q}^k b^k}{(\lambda - \lambda_k)^2} \\ 0 \end{pmatrix}.$$

Finally, we obtain

$$d = \begin{pmatrix} p \\ \gamma \end{pmatrix}, \qquad p \in \mathbb{C}, \quad \gamma = \frac{\widetilde{q}^{00} b^{00}}{\lambda^2} + \sum_k \frac{\widetilde{q}^k b^k}{(\lambda - \lambda_k)^2}. \tag{62}$$

We substitute this d into (60) and arrive to the formula for a rootvector $\widetilde{w} = \widetilde{w}_k$, corresponding to eigenvalue $\widetilde{\lambda}_k \in \sigma(\mathcal{A})$

$$\widetilde{w}_k = \sum_i \frac{1}{(\widetilde{\lambda}_k - \lambda_i)^2} \bar{P}^{(i)} b_1 - \frac{1}{\widetilde{\lambda}_k - \lambda_i} \bar{P}^{(i)} \mathcal{B} \begin{pmatrix} p_i \\ \gamma_i \end{pmatrix}, \tag{63}$$

where γ_i is defined in (62) for $\lambda = \widetilde{\lambda}_k$. Let us take $p_i = (\widetilde{\lambda}_k - \lambda_i)^{-1}$. Then (47) and (63) give

$$\widetilde{w}_k = \sum_i \frac{-\gamma_i}{\widetilde{\lambda}_k - \lambda_i} \bar{P}^{(i)} b_2. \tag{64}$$

The eigenvector $\widetilde{v} = \widetilde{v}_k$, corresponding to eigenvalue $\widetilde{\lambda}_k \in \sigma(\mathcal{A})$ (see (39)) is given by

$$\widetilde{v}_k = \sum_i \frac{1}{\widetilde{\lambda}_k - \lambda_i} \bar{P}^{(i)} b_1. \tag{65}$$

Lemma 30 *The norms of the eigen- and rootvectors satisfy*

$$\lim_{|k| \to \infty} \frac{\|\widetilde{v}_k\|_1}{\|\widetilde{w}_k\|_1} \cdot \frac{1}{|\widetilde{\lambda}_k - \lambda_k|} = 1. \tag{66}$$

Proof of the Lemma 30: First, we prove (see (65), (64)), that

$$\lim_{|k| \to \infty} \frac{\|\widetilde{v}_k\|_1}{\|\bar{P}^{(k)} b_1\|_1 \cdot |\widetilde{\lambda}_k - \lambda_k|^{-1}} = 1; \quad \lim_{|k| \to \infty} \frac{\|\widetilde{w}_k\|_1}{\|\bar{P}^{(k)} b_2\|_1 \cdot |\widetilde{\lambda}_k - \lambda_k|^{-2}} = 1. \tag{67}$$

To prove the first estimate in (67) we use (65) and the orthogonality of the projectors $\bar{P}^{(k)}$ in $\|\cdot\|_1$-norm:

$$\|\widetilde{v}_k\|_1^2 = \frac{\|\bar{P}^{(k)} b_1\|_1^2}{|\widetilde{\lambda}_k - \lambda_k|^2} + \sum_{i\neq k} \frac{\|\bar{P}^{(i)} b_1\|_1^2}{|\widetilde{\lambda}_k - \lambda_i|^2}. \tag{68}$$

Property c1) (see the statement of Theorem) and the location of λ_k gives $|\widetilde{\lambda}_k - \lambda_i| > \pi/2$, hence $|\widetilde{\lambda}_k - \lambda_i|^{-1} < 2/\pi$. This implies $\sum_{i\neq k} \frac{\|\bar{P}^{(i)} b_1\|_1^2}{|\widetilde{\lambda}_k - \lambda_i|^2} < 2\|b_1\|_1^2/\pi$. The last estimate and (68) give

$$1 \leq \frac{\|\widetilde{v}_k\|_1^2}{\|\bar{P}^{(k)} b_1\|_1^2 \cdot |\widetilde{\lambda}_k - \lambda_k|^{-2}} \leq 1 + \frac{|\widetilde{\lambda}_k - \lambda_k|^2}{\|\bar{P}^{(k)} b_1\|_1^2} \cdot \frac{2\|b_1\|_1^2}{\pi}. \tag{69}$$

The property c2) (see the statement of Theorem) implies $\frac{|\widetilde{\lambda}_k - \lambda_k|^2}{\|\bar{P}^{(k)} b_1\|_1^2} \to 0$ as $k \to \infty$, hence from (69) to get the first estimate in (67).

The second estimate in (67) is proved in the same manner using $|\gamma_k| \approx |\widetilde{\lambda}_k - \lambda_k|^{-1}$, which follows from the fact that $\widetilde{\lambda}_k$ is a root of the characteristic equation (55).

Using the explicit form of b_1, b_2 and v_1^k, v_2^k we see that $\|\bar{P}^{(k)} b_1\| = \|\bar{P}^{(k)} b_2\|$. Hence (67) completes the proof of Lemma 30.

As we have shown, one can choose Q in such a way that for each $\widetilde{\lambda}_k \in \sigma(\mathcal{A})$, there exist an eigenvector $\widetilde{v}_k$ and a rootvector $\widetilde{w}_k$ such that $(\mathcal{A} - \widetilde{\lambda}_k I)\widetilde{w}_k = \widetilde{v}_k$.

Consider $\mathcal{L}_k = Lin\,\{\widetilde{v}_k, \widetilde{w}_k\}$ which is an invariant subspace for $\mathcal{A}$. Hence

$$\mathrm{e}^{t\mathcal{A}}|_{\mathcal{L}_k} = \mathrm{e}^{t\mathcal{A}|_{\mathcal{L}_k}} = \mathrm{e}^{\widetilde{\lambda}_k t} \mathrm{e}^{t(\mathcal{A} - \widetilde{\lambda}_k I)|_{\mathcal{L}_k}} = \mathrm{e}^{\widetilde{\lambda}_k t} \sum_i [(\mathcal{A} - \widetilde{\lambda}_k I)|_{\mathcal{L}_k} \cdot t]^i / i!.$$

Here $(\mathcal{A} - \widetilde{\lambda}_k I)|_{\mathcal{L}_k}$ is 2×2-matrix. We also used $\mathrm{e}^{t\mathcal{A}} = \mathrm{e}^{\widetilde{\lambda}_k t} \mathrm{e}^{t(\mathcal{A} - \widetilde{\lambda}_k I)}$.

Since $(\mathcal{A} - \widetilde{\lambda}_k I)\widetilde{w}_k = \widetilde{v}_k$, we get $[(\mathcal{A} - \widetilde{\lambda}_k I)|_{\mathcal{L}_k}]^i = 0$ for all $i > 1$. Hence

$$\mathrm{e}^{t\mathcal{A}} \widetilde{w}_k = \mathrm{e}^{t\mathcal{A}}|_{\mathcal{L}_k} \widetilde{w}_k = \mathrm{e}^{\widetilde{\lambda}_k t} (\widetilde{w}_k + t \cdot \widetilde{v}_k). \tag{70}$$

Since $\mathcal{L}_k = V_k$, where $\{V_k\}$ form a Riesz basis of subspaces on M_2 Theorem 16 (see Theorem 6 [24]), there then exists a bounded operator $\widetilde{R} : M_2 \to M_2$ with bounded inverse, such that the subspaces $\{\mathcal{L}_k\}$ are orthogonal to each other in the norm $\|\widetilde{R} \cdot \|$.

Let us show that there exist $x \in M_2$ and $\{t_k\}_1^\infty$, $t_k \to \infty$, such that $\|\mathrm{e}^{t_k \mathcal{A}} x\| \to \infty$, as $k \to \infty$.

Consider $x = \sum_k \alpha_k \widetilde{v}_k + \beta_k \widetilde{w}_k \in M_2$, where $\alpha_k, \beta_k \in \mathbb{C}$ will be chosen later.

Using (70), we consider

$$
\begin{aligned}
\|\widetilde{R} \mathrm{e}^{t_k \mathcal{A}} x\|^2 &= \sum_k \mathrm{e}^{2t \mathrm{Re}\, \widetilde{\lambda}_k} \|(\alpha_k + t\beta_k) \widetilde{R} \widetilde{v}_k + \beta_k \widetilde{R} \widetilde{w}_k\|^2 \\
&\geq \mathrm{e}^{2t \mathrm{Re}\, \widetilde{\lambda}_{k_0}} \|(\alpha_{k_0} + t\beta_{k_0}) \widetilde{R} \widetilde{v}_{k_0} + \beta_{k_0} \widetilde{R} \widetilde{w}_{k_0}\|^2.
\end{aligned}
$$

In the sequel, $k = k_0$. It gives

$$
\begin{aligned}
\|\widetilde{R} \mathrm{e}^{t \mathcal{A}} x\| &\geq \mathrm{e}^{t \mathrm{Re}\, \widetilde{\lambda}_k} \|(\alpha_k + t\beta_k) \widetilde{R} \widetilde{v}_k + \beta_k \widetilde{R} \widetilde{w}_k\| \\
&\geq \mathrm{e}^{t \mathrm{Re}\, \widetilde{\lambda}_k} \left[|\alpha_k + t\beta_k| \|\widetilde{R} \widetilde{v}_k\| - |\beta_k| \|\widetilde{R} \widetilde{w}_k\| \right].
\end{aligned}
\tag{71}
$$

To choose α_k and β_k we first notice that $x \in M_2$ iff $\|\widetilde{R} x\| \leq \infty$, that is possible if, for example, $\sum_i |\alpha_i|^2 \|\widetilde{R} \widetilde{v}_i\|^2 + |\beta_i|^2 \|\widetilde{R} \widetilde{w}_i\|^2 \leq \infty$. Taking this into account, we set $\alpha_i = (i \cdot \|\widetilde{R} \widetilde{v}_i\|)^{-1}$ and $\beta_i = \xi_i \cdot \|\widetilde{R} \widetilde{w}_i\|^{-1}$, where $\sum_i \xi_i^2 \leq \infty, \xi_i \geq 0$.

Then $\|\widetilde{R} x\|^2 = \sum_i i^{-2} + \xi_i^2 \leq \infty$.

Now we set $t_k = (-\mathrm{Re}\, \widetilde{\lambda}_k)^{-1}$ and get from (71)

$$
\|\widetilde{R} \mathrm{e}^{t_k \mathcal{A}} x\| \geq \mathrm{e}^{-1} \left[\frac{1}{k} + (-\mathrm{Re}\, \widetilde{\lambda}_k)^{-1} \xi_k \frac{\|\widetilde{R} \widetilde{v}_k\|}{\|\widetilde{R} \widetilde{w}_k\|} - \xi_k \right].
$$

Since $\frac{1}{k} + \xi_k$ is bounded, we get a sufficient condition for $\|\mathrm{e}^{t_k \mathcal{A}} x\| \to \infty$:

$$
\xi_k \cdot \frac{1}{(-\mathrm{Re}\, \widetilde{\lambda}_k)} \cdot \frac{\|\widetilde{R} \widetilde{v}_k\|}{\|\widetilde{R} \widetilde{w}_k\|} \to \infty. \tag{72}
$$

Due to the equivalence of the norms $\|\cdot\|$ and $\|\widetilde{R} \cdot\|$, we get the condition similar to (72) with the initial norm $\|\cdot\|$ in M_2.

Let us take, for example, $\xi_k = 1/k$. Then Lemma 30 gives a sufficient condition on the location of the spectrum $\{\widetilde{\lambda}_k\} = \sigma(\mathcal{A})$ for $\mathrm{e}^{\mathcal{A}t}$ to be an unstable semigroup (see c3) in the statement of the theorem):

$$
\frac{1}{k} \cdot \frac{|\widetilde{\lambda}_k - \lambda_k|}{(-\mathrm{Re}\, \widetilde{\lambda}_k)} \to \infty, \quad k \to \infty. \tag{73}
$$

The proof of Theorem 29 is complete. □

4 Conclusion

It is well known that for the systems of neutral type the analysis of stability conditions is more complicated than for a system with simple delays. We have shown that the condition of asymptotic stability may first be analyzed by means of the principal neutral term (the matrix A_{-1}). If the part of the spectrum of this matrix, which lies on the unit circle is simple (distinct eigenvalues), or if there is a Jordan chain, then the condition of asymptotic stability can be easily characterized. In the case of multiple eigenvalues without Jordan chain, the analysis of non-exponential asymptotic stability is still an open problem in the sense that the system may be stable or unstable according to other additional terms in the system.

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