Weak error in negative Sobolev spaces for the stochastic heat equation

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WEAK ERROR IN NEGATIVE SOBOLEV SPACES FOR THE STOCHASTIC HEAT EQUATION

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Abstract. In this paper, we make another step in the study of weak error of the stochastic heat equation by considering norms as functional.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ a probability space and $T > 0$ a fixed time. $(W(t))_{t\geq 0}$ will be a cylindrical Brownian motion on $L^2(0,1)$. We consider the stochastic heat equation, written in abstract form in $L^2(0,1)$: $X(0) = 0$, for all $t \in [0,T]$ $X(t,0) = X(t,1) = 0$ and

$$dX(t) = \frac{1}{2}\frac{d^2}{dx^2}X(t)dt + dW(t). \tag{1.1}$$

It is well know that this equation admits a unique weak solution (from the analytical point of view).

Let $N \in \mathbb{N}^*$ and $h := T/N$. Consider $(t_k)_{0\leq k\leq N}$ the uniform subdivision of $[0,T]$ defined by $t_k := kh$. We consider the implicit Euler scheme defined as follow:

$$X^N(t_{k+1}) = X^N(t_k) + h\frac{1}{2}\frac{d^2}{dx^2}X^N(t_{k+1}) + \Delta W(k+1), \tag{1.2}$$

where $\Delta W(k+1) = W(t_{k+1}) - W(t_k)$.

Let $f : L^2(0,1) \to \mathbb{R}$ be a functional. The stong error is the study of $E\left|X^N(T) - X(T)\right|^2_{L^2(0,1)}$. The weak error is the study of $\left|Ef\left(X^N(T)\right) - Ef\left(X(T)\right)\right|$ with respect to the time mesh h.

In [6], A. Debussche considers a more general stochastic equation and a more general functional than the one considered here. He obtains a weak error of order 1/2, which is the double of that proved by [15] for the strong speed of convergence. The novelty of this paper his to prove that for the square of the norm the weak error his better than 1/2 in negative Sobolev spaces.

2. Preliminaries and main result

Notations. We collect here some of the notations used through the paper. $< .,. >_{L^2(0,1)}$ is the inner product in $L^2(0,1)$, $H^1_0(0,1)$ is the Sobolev space of functions f in $L^2(0,1)$ vanishing in 0 and 1 with first derivatives in $L^2(0,1)$, $H^2(0,1)$ is the Sobolev space of functions f in $L^2(0,1)$ with first and second derivatives in $L^2(0,1)$. Finally, for $m = 1, 2, \ldots,$ let $(e_m(x) = \sqrt{2}\sin(m\pi x)$ and $\lambda_m = \frac{1}{2}(\pi m)^2$ denote the eigenfunction and eigenvalues of $-\Delta$ with Dirichlet boundary conditions on $(0,1)$.

An $L^2(0,1)$-valued stochastic process $(X(t))_{t\in[0,T]}$ is said to be a solution of (1.1) if: $X(0) = 0$ and for all $g \in H^1_0(0,1) \cap H^2(0,1)$ we have

$$< X(t), g >_{L^2(0,1)} = \int_0^t < X(s), \frac{1}{2}\frac{d^2}{dx^2}g >_{L^2(0,1)} ds + < W(t), g >_{L^2(0,1)} .$$

It is well know that (1.1) admits a unique solution: see [4]. Then $(e_m)_{m\geq 1}$ is a complete orthonormal basis of $L^2(0,1)$. If we denote by $\lambda_m := \frac{1}{2}(\pi m)^2$, $W_{\lambda_m}(t) := \langle W(t), e_m\rangle_H$ and $X_{\lambda_m}(t)$ denote the solution of the evolution equation: $X_{\lambda_m}(0) = 0$ and for $t > 0$:

$$dX_{\lambda_m}(t) = -\lambda_m X_{\lambda_m}(t)dt + dW_{\lambda_m}(t).$$

Then the processes $(X_{\lambda_m}(.))_{m\geq 1}$ are independent and $X(t) = \sum_{m\geq 1} X_{\lambda_m}(t)e_m$ for all $t \geq 0$.

A sequence of $L^2(0,1)$-valued $\left(X^N(t_k)\right)_{k=0,\ldots,N}$ is said to be a solution of (1.2) if: $X^N(t_0) = 0$ and for all $k = 0,\ldots,N-1$ and for all $g \in H_0^1(0,1) \cap H^2(0,1)$ we have

$$\begin{aligned} < X^N(t_{k+1}), g >_{L^2(0,1)} = &< X^N(t_k), g >_{L^2(0,1)} + h < X^N(t_{k+1}), \frac{1}{2}\frac{d^2}{dx^2} g >_{L^2(0,1)} \\ &+ < \Delta W(k+1), g >_{L^2(0,1)} . \end{aligned}$$

It is well know that (1.2) has a unique solution and there exists a constant $C > 0$, independent of N, such that $E\left|X^N(T) - X(T)\right|^2_{L^2(0,1)} \leq Ch^{\frac{1}{2}}$ where $h = T/N$. Now if we denote by $\left(X^N_{\lambda_m}(t_k)\right)_{k=0,\ldots,N}$ the solution of: $X^N_{\lambda_m}(t_0) = 0$ and for $k = 0,\ldots,N-1$

$$X^N_{\lambda_m}(t_{k+1}) = X^N_{\lambda_m}(t_k) - \lambda_m h X^N_{\lambda_m}(t_{k+1}) + W_{\lambda_m}(k+1).$$

The random vectors $(X^N_{\lambda_m}(t_k), k = 0,\ldots,N)_{m=1,2,\ldots}$ are independent and $X^N(t_k) = \sum_{m\geq 1} X^N_{\lambda_m}(t_k)e_m$.

Let $p \geq 0$; we define the spaces H^{-p} as the completion of $L^2(0,1)$ for the topology induced by the norm $|u|^2_{H^{-p}} := \sum_{m\geq 1} \lambda_m^{-p} < u, e_m >^2_H$. The following theorem improves the speed of convergence of X^N to X for negative Sobolev spaces.

Theorem 2.1. *Suppose that $h < 1$ and let $p \in [0, \frac{1}{2})$. There exists a constant $C > 0$, independent of N, such that*

$$\left| E\left|X^N(T)\right|^2_{H^{-p}} - E\left|X(T)\right|^2_{H^{-p}} \right| \leq Ch^{p+\frac{1}{2}}.$$

3. Proof of the theorem 2.1

The proof of the theorem will be done in several steps. First we recall the weak error of the Ornstein-Uhlenbeck process. Secondly we prove some technical lemmas. Then we decompose the weak error and analyse each term of these decomposition.

3.1. Weak error of the Ornstein-Uhlenbeck process.

Let $\lambda > 0$, $(W_\lambda(t))_{t\geq 0}$ be a one dimensional Brownian motion and $(X_\lambda(t))_{t\geq 0}$ be the Ornstein Uhlenbeck process solution of the following stochastic differential equation: $X_\lambda(0) = x \in \mathbb{R}$ and

$$dX_\lambda(t) = -\lambda X_\lambda(t)dt + dW_\lambda(t). \tag{3.1}$$

In this step, we study two properties associated with this process: the Kolmogorov equation and the implicit Euler scheme.

Let $\left(X^{t,x}_\lambda(s)\right)_{t\leq s\leq T}$ be the solution of (3.1) starting from x at time t. It is well know that $X^{t,x}_\lambda(T)$ is a normal random variable:

$$X^{t,x}_\lambda(T) \sim \mathcal{N}\left(e^{-\lambda(T-t)}x, \frac{1-e^{-2\lambda(T-t)}}{2\lambda}\right).$$

For $t \in [0,T]$ and $x \in \mathbb{R}$ set $u_\lambda(t,x) := E\left|X_\lambda^{t,x}(T)\right|^2$. Then u_λ is the solution of the following partial differential equation, called Kolmogorov equation: for all $x \in \mathbb{R}$, $u_\lambda(T,x) = |x|^2$ and for all $(t,x) \in [0,T) \times \mathbb{R}$

$$-\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) - \lambda x\frac{\partial}{\partial x}u(t,x). \tag{3.2}$$

Since $X_\lambda^{t,x}(T)$ has a normal law, we can write u_λ explicitely:

$$u_\lambda(t,x) = \frac{1-e^{-2\lambda(T-t)}}{2\lambda} + e^{-2\lambda(T-t)}x^2. \tag{3.3}$$

With this expression we see that $u_\lambda \in C^{1,2}([0,T]\times\mathbb{R})$ and we have the following derivatives:

$$\frac{\partial}{\partial x}u_\lambda(t,x) = 2e^{-2\lambda(T-t)}x, \tag{3.4}$$

$$\frac{\partial^2}{\partial x^2}u_\lambda(t,x) = 2e^{-2\lambda(T-t)}, \tag{3.5}$$

$$\frac{\partial}{\partial t}u_\lambda(t,x) = -e^{-2\lambda(T-t)} + 2\lambda e^{-2\lambda(T-t)}x^2, \tag{3.6}$$

$$\frac{\partial^2}{\partial t\partial x}u_\lambda(t,x) = 4\lambda e^{-2\lambda(T-t)}x. \tag{3.7}$$

The implicit Euler scheme for the Ornstein-Uhlenbeck equation (3.1) starting from 0 at time t_0, is defined as follow: $X_\lambda^N(t_0) = 0$ and for $k = 0,\dots,N-1$

$$X_\lambda^N(t_{k+1}) = X_\lambda^N(t_k) - \lambda h X_\lambda^N(t_{k+1}) + \Delta W_\lambda(k+1), \tag{3.8}$$

where $\Delta W_\lambda(k+1) = W_\lambda(t_{k+1}) - W_\lambda(t_k)$. Since we have the following equation

$$X_\lambda^N(t_{k+1}) = \frac{1}{1+\lambda h}X_\lambda^N(t_k) + \frac{1}{1+\lambda h}\Delta W_\lambda(k+1), \tag{3.9}$$

we see that the scheme is well defined.

Lemma 3.1. *For $k = 1,\dots,N$ we have $X_\lambda^N(t_k) = \sum_{j=0}^{k-1}\frac{\Delta W_\lambda(k-j)}{(1+\lambda h)^{j+1}}$.*

Proof. We proceed by induction. If $k=1$, we have $X_\lambda^N(t_1) = \frac{1}{1+\lambda h}\Delta W_\lambda(1)$. Suppose the result true until k. Using (3.9), we have

$$\begin{aligned}X_\lambda^N(t_{k+1}) &= \sum_{j=0}^{k-1}\frac{\Delta W_\lambda(k-j)}{(1+\lambda h)^{j+2}} + \frac{1}{1+\lambda h}\Delta W_\lambda(k+1)\\ &= \sum_{l=1}^{k}\frac{\Delta W_\lambda(k+1-l)}{(1+\lambda h)^{l+1}} + \frac{1}{(1+\lambda h)^{0+1}}\Delta W_\lambda(k+1-0),\end{aligned}$$

which concludes the proof. □

Lemma 3.2. *For all $k = 0,\dots,N$, we have the following bound $E\left|X_\lambda^N(t_k)\right|^2 \le \frac{1}{2\lambda}$.*

Proof. Using the independence of the increments of the Brownian motion and Lemma 3.1, we have

$$E\left|X_\lambda^N(t_k)\right|^2 = \sum_{j=0}^{k-1}\frac{1}{(1+\lambda h)^{2(j+1)}}E\left|\Delta W_\lambda(k-j)\right|^2 = h\sum_{j=0}^{k-1}\frac{1}{(1+\lambda h)^{2(j+1)}}.$$

Let $a := 1/(1+\lambda h)^2$; we deduce that $E\left|X_\lambda^N(t_k)\right|^2 = ha\frac{1-a^k}{1-a}$. Simple computations yield $ha/(1-a) = 1/(2\lambda+\lambda^2 h)$, which implies

$$E\left|X_\lambda^N(t_k)\right|^2 = \frac{1}{2\lambda+\lambda^2 h}\left(1-\frac{1}{(1+\lambda h)^{2k}}\right).$$

This concludes the proof. □

For $t \geq 0$, we denote $\mathcal{F}_t^\lambda := \sigma\left(W_\lambda(s), s \leq t\right)$ and $D_\lambda^{1,2}$ the Malliavin Sobolev space with respect to W_λ.

Lemma 3.3. *For all $k = 1, \ldots, N$, we have $X_\lambda^N(t_k) \in D_\lambda^{1,2} \cap L^2\left(\mathcal{F}_{t_k}^\lambda\right)$.*

Proof. This is a consequence of Lemma 3.1, the fact that $L^2\left(\mathcal{F}_{t_k}^\lambda\right)$ and $D_\lambda^{1,2}$ are linear space and for all $j = 0, \ldots, k-1$, $\Delta W_\lambda(k-j) \in D_\lambda^{1,2} \cap L^2\left(\mathcal{F}_{t_k}^\lambda\right)$. □

As usual in the study of weak error, we need to use a continuous process that interpolates the Euler scheme. The interpolation process that we use was introduced in [1]. We recall its construction and prove some of its properties.

Let $k \in \{0, \ldots, N-1\}$ be fixed. In order to interpolate the scheme between the points $\left(t_k, X_\lambda^N(t_k)\right)$ and $\left(t_{k+1}, X_\lambda^N(t_{k+1})\right)$, we define the process as follows: for $t \in [t_k, t_{k+1}]$, set

$$X_\lambda^N(t) := X_\lambda^N(t_k) - \lambda E\left(X_\lambda^N(t_{k+1})|\mathcal{F}_t\right)(t-t_k) + W_\lambda(t) - W_\lambda(t_k). \tag{3.10}$$

In the sequel, we will use the following processes: for $t \in [t_k, t_{k+1}]$

$$\beta_\lambda^{k,N}(t) := -\lambda E\left(X_\lambda^N(t_{k+1})|\mathcal{F}_t\right), \tag{3.11}$$

$$z_\lambda^{k,N}(t) := -\lambda E\left(D_t X_\lambda^N(t_{k+1})|\mathcal{F}_t\right), \tag{3.12}$$

$$\gamma_\lambda^{k,N}(t) := 1 + (t-t_k) z_\lambda^{k,N}(t). \tag{3.13}$$

The next lemma relates the above processes.

Lemma 3.4. *Let $k = 0, \ldots, N-1$. For $t \in [0,T]$, we have*

$$\begin{aligned} d\beta_\lambda^{k,N}(t) &= z_\lambda^{k,N}(t) dW_\lambda(t), \quad z_\lambda^{k,N}(t) = -\frac{\lambda}{1+\lambda h}, \\ \gamma_\lambda^{k,N}(t) &= 1-(t-t_k)\frac{\lambda}{1+\lambda h}, \quad dX_\lambda^N(t) = \beta_\lambda^{k,N}(t)dt + \gamma_\lambda^{k,N}(t) dW_\lambda(t). \end{aligned}$$

Proof. Using the Clark-Ocone formula and Lemma 3.3, we have

$$X_\lambda^N(t_{k+1}) = E\left(X_\lambda^N(t_{k+1})|\mathcal{F}_t\right) + \int_t^{t_{k+1}} E\left(D_s X_\lambda^N(t_{k+1})|\mathcal{F}_s\right) dW_\lambda(s).$$

Multiplying by $(-\lambda)$, we deduce

$$-\lambda X_\lambda^N(t_{k+1}) = \beta_\lambda^{k,N}(t) + \int_t^{t_{k+1}} z_\lambda^{k,N}(s) dW_\lambda(s),$$

which gives the first identity. Applying the Malliavin derivative to (3.9), we have for $s \in [t_k, t_{k+1}]$ $D_s X_\lambda^N(t_{k+1}) = \frac{1}{1+\lambda h}$. Multiplying by $(-\lambda)$, we deduce the second and third equalities.

Finaly, Itô's formula gives us

$$d\Big((t-t_k)\beta_\lambda^{k,N}(t)\Big) = (t-t_k) z_\lambda^{k,N}(t) dW_\lambda(t) + \beta_\lambda^{k,N}(t) dt,$$

which concludes the proof. □

Lemma 3.5. *Let $k \in \{0, \dots, N-1\}$. For any $s \in [t_k, t_{k+1}]$, we have*

$$E\left|\beta_\lambda^{k,N}(s)\right|^2 \le 2\lambda, \quad E\left|X_\lambda^N(s)\right|^2 \le \frac{1}{2\lambda} + h, \quad E\beta_\lambda^{k,N}(s)X_\lambda^N(s) \le 1.$$

Proof. Applying the conditionnal expectation with respect to $\mathcal{F}_s$ on both sides of (3.9) for $s \in [t_k, t_{k+1})$ we have

$$E\left(X_\lambda^N(t_{k+1})|\mathcal{F}_s\right) = \frac{1}{1+\lambda h}\left[X_\lambda^N(t_k) + (W_\lambda(s) - W_\lambda(t_k))\right].$$

Multiplying by $(-\lambda)$ and using (3.11), we obtain

$$\beta_\lambda^{k,N}(s) = -\frac{\lambda}{1+\lambda h}X_\lambda^N(t_k) - \frac{\lambda}{1+\lambda h}\left(W_\lambda(s) - W_\lambda(t_k)\right). \tag{3.14}$$

The independence of $\mathcal{F}_{t_k}$ and $W_\lambda(s) - W_\lambda(t_k)$ yields

$$E\left|\beta_\lambda^{k,N}(s)\right|^2 = \frac{\lambda^2}{(1+\lambda h)^2}E\left|X_\lambda^N(t_k)\right|^2 + \frac{\lambda^2}{(1+\lambda h)^2}(s-t_k).$$

Using Lemma 3.2, we deduce

$$E\left|\beta_\lambda^{k,N}(s)\right|^2 \le \frac{\lambda}{2(1+\lambda h)^2} + \frac{\lambda^2 h}{(1+\lambda h)^2},$$

which proves the first upper estimate.

Using (3.10) and (3.14), we have for $s \in [t_k, t_{k+1}]$

$$X_\lambda^N(s) = \left(1 - \frac{\lambda(s-t_k)}{1+\lambda h}\right)\left[X_\lambda^N(t_k) + (W_\lambda(s) - W_\lambda(t_k))\right]. \tag{3.15}$$

Taking the expectation of the square and using the independence of $\mathcal{F}_{t_k}$ and $W_\lambda(s) - W_\lambda(t_k)$, we have

$$E\left|X_\lambda^N(s)\right|^2 = \left(1 - \frac{\lambda(s-t_k)}{1+\lambda h}\right)^2\left[E\left|X_\lambda^N(t_k)\right|^2 + (s-t_k)\right] \le E\left|X_\lambda^N(t_k)\right|^2 + h \le \frac{1}{2\lambda} + h,$$

where the last upper estimates follows from Lemma 3.2.

Multiplying (3.14) and (3.15), taking expectation we obtain

$$E\left(X_\lambda^N(s)\beta_\lambda^{k,N}(s)\right) = \frac{-\lambda}{1+\lambda h}\left(1 - \frac{\lambda(s-t_k)}{1+\lambda h}\right)\left[E\left|X_\lambda^N(t_k)\right|^2 + (s-t_k)\right].$$

Using Lemma 3.2, we deduce

$$\left|E\left(X_\lambda^N(s)\beta_\lambda^{k,N}(s)\right)\right| \le \frac{\lambda}{1+\lambda h}\frac{1}{2\lambda} + \frac{\lambda h}{1+\lambda h}.$$

This concludes the proof. □

3.2. **Some useful analytical lemmas.** We at first give a precise upper bound of a series defined in terms of the eigenvalues of the Laplace operator with Dirichlet boundary conditions.

Lemma 3.6. *Let $p \in [0, \frac{1}{2})$. There exists a constant $C > 0$, such that for all $\alpha > 0$, we have*

$$\sum_{m\ge1} \lambda_m^{-p} e^{-2\lambda_m \alpha} \le C\alpha^{p-\frac{1}{2}}$$

Proof. The function $(x \in \mathbb{R}_+ \mapsto x^{-2p} e^{-2x^2\alpha})$ is decreasing. So by comparaison, we obtain

$$\sum_{m\geq 1} m^{-2p} e^{-2m^2\alpha} \leq \int_0^\infty x^{-2p} e^{-2x^2\alpha} dx \leq \alpha^{p-\frac{1}{2}} \int_0^\infty y^{-2p} e^{-2y^2} dy = C\alpha^{p-\frac{1}{2}}.$$

Since $\lambda_m = \frac{1}{2}(\pi m)^2$, we deduce the desired upper estimate. □

Lemma 3.7. *Let $q > 0$. There exists a constant $C > 0$, such that for all $\alpha > 0$*

$$\sum_{m\geq 1} \lambda_m^q e^{-\lambda_m \alpha} \leq C\left(1 + \frac{1}{\alpha^{q+\frac{1}{2}}}\right).$$

Proof. Let $f(x) = x^{2q} e^{-x^2\alpha}$. His derivatives is given by $f'(x) = 2x^{2q-1} e^{-x^2\alpha}(q - \alpha x^2)$. *Case 1:* $\alpha > q/4$. Then f is decreasing on $[2, \infty)$ and a standard comparaison argument yields

$$\begin{aligned}
\sum_{m\geq 1} m^{2q} e^{-m^2\alpha} \leq & e^{-\alpha} + 4^q e^{-4\alpha} + \sum_{m\geq 3} \int_{m-1}^m x^{2q} e^{-x^2\alpha} dx \\
\leq & C + \int_0^\infty x^{2q} e^{-x^2\alpha} dx \\
\leq & C + \alpha^{-q-\frac{1}{2}} \int_0^\infty y^{2q} e^{-y^2} dy \\
\leq & C(1 + \alpha^{-q-\frac{1}{2}}).
\end{aligned}$$

Case 2: $\alpha \leq q/4$. The function f is increasing on $[0, \sqrt{q/\alpha}]$. So for each $m = 1, \ldots, [\sqrt{\frac{q}{\alpha}}] - 1$, we have

$$m^{2q} e^{-m^2\alpha} \leq \int_m^{m+1} x^{2q} e^{-x^2\alpha} dx.$$

On the interval $[\sqrt{\frac{q}{\alpha}}, \infty)$, f is decreasing. So for each integer $m \geq [\sqrt{\frac{q}{\alpha}}] + 2$, we have

$$m^{2q} e^{-m^2\alpha} \leq \int_{m-1}^m x^{2q} e^{-x^2\alpha} dx.$$

The above upper estimates yield

$$\begin{aligned}
\sum_{m\geq 1} m^{2q} e^{-m^2\alpha} \leq & \sum_{m\leq [\sqrt{\frac{q}{\alpha}}]-1} \int_m^{m+1} x^{2q} e^{-x^2\alpha} dx + \sum_{m\geq [\sqrt{\frac{q}{\alpha}}]+2} \int_{m-1}^m x^{2q} e^{-x^2\alpha} dx \\
& + \sum_{m\in\{[\sqrt{\frac{q}{\alpha}}],[\sqrt{\frac{q}{\alpha}}]+1\}} m^{2q} e^{-m^2\alpha} \\
\leq & \int_0^\infty x^{2q} e^{-x^2\alpha} dx + \sum_{m\in\{[\sqrt{\frac{q}{\alpha}}],[\sqrt{\frac{q}{\alpha}}]+1\}} m^{2q} e^{-m^2\alpha} \\
\leq & C\alpha^{-q-\frac{1}{2}} + \sum_{m\in\{[\sqrt{\frac{q}{\alpha}}],[\sqrt{\frac{q}{\alpha}}]+1\}} m^{2q} e^{-m^2\alpha}
\end{aligned}$$

Now we study each term of the sum in the right hand side. Since $q \geq \alpha$, we have

$$\left[\sqrt{\frac{q}{\alpha}}\right]^{2q} e^{-\left[\sqrt{\frac{q}{\alpha}}\right]^2 \alpha} \leq \left(\frac{q}{\alpha}\right)^q \leq \left(\frac{q}{\alpha}\right)^{q+\frac{1}{2}} \leq C\alpha^{-q-\frac{1}{2}}.$$

For the second term, we remark that since $q \geq \alpha\left[\sqrt{\frac{q}{\alpha}}\right]+1 \leq 2\left[\sqrt{\frac{q}{\alpha}}\right] \leq 2\sqrt{\frac{q}{\alpha}}$. This implies

$$\left(\left[\sqrt{\frac{q}{\alpha}}\right]+1\right)^{2q} e^{-\left(\left[\sqrt{\frac{q}{\alpha}}\right]+1\right)^2\alpha} \leq \left(2\sqrt{\frac{q}{\alpha}}\right)^{2q} \leq C\alpha^{-q-\frac{1}{2}}.$$

Therefore, in both cases we obtain

$$\sum_{m\geq 1} m^{2q} e^{-m^2\alpha} \leq C\left(1+\frac{1}{\alpha^{q+\frac{1}{2}}}\right).$$

Since $\lambda_m = \frac{1}{2}(\pi m)^2$, the proof is complete. □

Lemma 3.8. *Let $p \in [0, \frac{1}{2})$ and $n \in \mathbb{N}^*$. Let $(v(k,m))_{(k,m)\in\{0,\ldots,N-2\}\times\mathbb{N}^*}$ be a sequence such that for all $k \in \{0,\ldots,N-2\}$ and $m \geq 1$, we have*

$$0 \leq v(k,m) \leq \lambda_m^{n-p} h^{n+1} e^{-2\lambda_m(T-t_{k+1})}.$$

Then, there exists a constant $C > 0$, independent of N, such that

$$\sum_{m\geq 1}\sum_{k=0}^{N-2} v(k,m) \leq Ch^{p+\frac{1}{2}}.$$

Proof. First we remark that $T - t_{k+1} = h(N-k-1)$. Using Lemma 3.7, we deduce the existence of C depending on n and p, but independent of N, such that for $k = 0,\ldots,N-2$:

$$\begin{aligned}\sum_{m\geq 1} v(k,m) \leq& Ch^{n+1}\left(1+\frac{1}{h^{n-p+\frac{1}{2}}(N-k-1)^{n-p+\frac{1}{2}}}\right)\\ \leq& C\left(h^{n+1}+\frac{h^{p+\frac{1}{2}}}{(N-k-1)^{n-p+\frac{1}{2}}}\right).\end{aligned}$$

Therefore, there exists a constant C as above such that

$$\begin{aligned}\sum_{m\geq 1}\sum_{k=0}^{N-2} v(k,m) \leq& C\left(h^n + h^{p+\frac{1}{2}}\sum_{k=0}^{N-2}\frac{1}{(N-k-1)^{n-p+\frac{1}{2}}}\right)\\ \leq& C\left(h^n + h^{p+\frac{1}{2}}\sum_{l=1}^{N-1}\frac{1}{l^{n-p+\frac{1}{2}}}\right) \leq Ch^{p+\frac{1}{2}},\end{aligned}$$

which concludes the proof. □

3.3. **Decomposition of the weak error.** We follow the classical decomposition introduced in [16]. The definition of $u_\lambda(t,x)$ in section 3.1 yields

$$\begin{aligned}E\left|X^N(T)\right|_{H^{-p}}^2 - E\left|X(T)\right|_{H^{-p}}^2 =& \sum_{m\geq 1}\lambda_m^{-p}\left(E\left|X_{\lambda_m}^N(T)\right|^2 - E\left|X_{\lambda_m}(T)\right|^2\right)\\ =& \sum_{m\geq 1}\lambda_m^{-p}\left(Eu_{\lambda_m}\left(T, X_{\lambda_m}^N(T)\right) - u_{\lambda_m}\left(0, X_{\lambda_m}^N(0)\right)\right).\end{aligned}$$

Let $\delta^N(k,m) := \lambda_m^{-p}\left(Eu_{\lambda_m}\left(t_{k+1}, X_{\lambda_m}^N(t_{k+1})\right) - Eu_{\lambda_m}\left(t_k, X_{\lambda_m}^N(t_k)\right)\right)$; then

$$E\left|X^N(T)\right|_{H^{-p}}^2 - E\left|X(T)\right|_{H^{-p}}^2 = \sum_{m\geq 1}\sum_{k=0}^{N-1}\delta^N(k,m).$$

Note that using Lemmas 3.3, 3.4 and (3.4) we deduce that for any $k=0,\dots,N-1$

$$E\int_{t_k}^{t_{k+1}}\left|\gamma_\lambda^{k,N}(t)\frac{\partial u}{\partial x}(t,X_\lambda^N(t))\right|^2 dt<\infty.$$

From now, we do not justify that the stochastic integral are centered. Itô's formula and Lemma 3.4, we imply that for $k=0,\dots,N-1$

$$\begin{aligned}\delta^N(k,m)=&\lambda_m^{-p}E\int_{t_k}^{t_{k+1}}\left\{\frac{\partial}{\partial t}u_{\lambda_m}+\beta_{\lambda_m}^{k,N}(t)\frac{\partial}{\partial x}u_{\lambda_m}+\frac{1}{2}\left|\gamma_{\lambda_m}^{k,N}(t)\right|^2\frac{\partial^2}{\partial x^2}u_{\lambda_m}\right\}\left(t,X_{\lambda_m}^N(t)\right)dt\\ =&\lambda_m^{-p}E\int_{t_k}^{t_{k+1}}\left\{I_{\lambda_m}^{k,N}(t)+\frac{1}{2}J_{\lambda_m}^{k,N}(t)\right\}dt,\end{aligned}$$

where

$$I_{\lambda_m}^{k,N}(t):=\left(\beta_{\lambda_m}^{k,N}(t)+\lambda_m X_{\lambda_m}^N(t)\right)\frac{\partial}{\partial x}u_{\lambda_m}\left(t,X_{\lambda_m}^N(t)\right),\tag{3.16}$$

$$J_{\lambda_m}^{k,N}(t):=\left(\left|\gamma_{\lambda_m}^{k,N}(t)\right|^2-1\right)\frac{\partial^2}{\partial x^2}u_{\lambda_m}\left(t,X_{\lambda_m}^N(t)\right).\tag{3.17}$$

This yields the following decomposition:

$$\begin{aligned}E\left|X^N(T)\right|_{H^{-p}}^2-E\left|X(T)\right|_{H^{-p}}^2=&\sum_{m\geq 1}\delta^N(N-1,m)+\sum_{m\geq1}\sum_{k=0}^{N-2}\lambda_m^{-p}E\int_{t_k}^{t_{k+1}}I_{\lambda_m}^{k,N}(t)dt\\ &+\frac{1}{2}\sum_{m\geq1}\sum_{k=0}^{N-2}\lambda_m^{-p}E\int_{t_k}^{t_{k+1}}J_{\lambda_m}^{k,N}(t)dt.\end{aligned}\tag{3.18}$$

Now we study each term of this decomposition.

Lemma 3.9. *There exists a constant C, independant of N, such that*

$$\sum_{m\geq1}\left|\delta^N(N-1,m)\right|\leq Ch^{p+\frac{1}{2}}.$$

This study is similar to the third step of [6], page 97.

Proof. Using the definition of $u_{\lambda_m}(t,x)$ (3.3) and (3.9), we have

$$u_{\lambda_m}\left(t_N,X_{\lambda_m}^N(t_N)\right)=\left|X_{\lambda_m}^N(t_N)\right|^2=\frac{1}{(1+\lambda_m h)^2}\left|X_{\lambda_m}^N(t_{N-1})+\Delta W_m(N)\right|^2,$$

$$u_{\lambda_m}\left(t_{N-1},X_{\lambda_m}^N(t_{N-1})\right)=\frac{1-e^{-2\lambda_m h}}{2\lambda_m}+e^{-2\lambda_m h}\left|X_{\lambda_m}^N(t_{N-1})\right|^2.$$

By independence between $\Delta W_m(N)$ and $X_{\lambda_m}^N(t_{N-1})$, we have

$$\begin{aligned}\delta^N(N-1,m)=&\lambda_m^{-p}\left\{\frac{1}{(1+\lambda_m h)^2}-e^{-2\lambda_m h}\right\}E\left|X_{\lambda_m}^N(t_{N-1})\right|^2\\ &+\frac{h}{\lambda_m^p(1+\lambda_m h)^2}-\frac{1-e^{-2\lambda_m h}}{2\lambda_m^{1+p}}.\end{aligned}$$

Let $\delta_1(\lambda_m):=\frac{1-2e^{-2\lambda_m h}}{2{\lambda_m}^{1+p}}$, $\delta_2(\lambda_m):=\frac{h}{{\lambda_m}^p(1+\lambda_m h)^2}$, and

$$\delta_3(\lambda_m):=\lambda_m^{-p}\left\{\frac{1}{(1+\lambda_m h)^2}-e^{-2\lambda_m h}\right\}E\left|X_{\lambda_m}^N(t_{N-1})\right|^2.$$

With these notations we have

$$\delta^N(N-1,m) \leq \delta_1(\lambda_m) + \delta_2(\lambda_m) + \delta_3(\lambda_m).$$

First, we study $\delta_1(\lambda_m)$. Since $\frac{1-e^{-2\lambda h}}{2\lambda} = \int_0^h e^{-2\lambda x} dx$, using Lemma 3.6, we obtain

$$\sum_{m\geq 1} \delta_1(\lambda_m) = \int_0^h \sum_{m\geq 1} \lambda_m^{-p} e^{-2\lambda_m x} dx \leq C \int_0^h x^{p-\frac{1}{2}} dx = Ch^{p+\frac{1}{2}}. \tag{3.19}$$

Now we study $\delta_2(\lambda_m)$. Since $\left(x \in [0,\infty) \mapsto x^{-2p}(1+x^2h)^2\right)$ is decreasing, we have for $p \in [0, \frac{1}{2})$

$$\sum_{m\geq 1} \delta_2(\lambda_m) \leq Ch \int_0^\infty \frac{1}{x^{2p}\left(1+x^2h\right)^2} dx \leq Ch^{p+\frac{1}{2}} \int_0^\infty \frac{y^{-2p}}{(1+y^2)^2} dy \leq Ch^{p+\frac{1}{2}}. \tag{3.20}$$

Finally, we study $\delta_3(\lambda_m)$. Using Lemma 3.2, we have

$$\delta_3(\lambda_m) \leq \lambda_m^{-p} \left\{ \frac{1}{(1+\lambda_m h)^2} - e^{-2\lambda_m h} \right\} \frac{1}{2\lambda_m}.$$

Since $\frac{1}{(1+\lambda h)^2} - e^{-2\lambda h} = 2\lambda \int_0^h \left\{ e^{-2\lambda x} - \frac{1}{(1+\lambda x)^3} \right\} dx$, we have

$$\delta_3(\lambda_m) \leq \lambda_m^{-p} \int_0^h \left\{ e^{-2\lambda_m x} + \frac{1}{(1+\lambda_m x)^3} \right\} dx.$$

Using Lemma 3.6, we have for $p \in [0, \frac{1}{2})$

$$\sum_{m\geq 1} \lambda_m^{-p} \int_0^h e^{-2\lambda_m x} dx \leq C \int_0^h x^{p-\frac{1}{2}} dx \leq Ch^{p+\frac{1}{2}}.$$

Now since for $x \geq 0$ the map $\left(y \in \mathbb{R}_+ \mapsto y^{-2p}(1+y^2x)^{-3}\right)$ is decreasing, we have for $p \in [0, \frac{1}{2})$

$$\sum_{m\geq 1} \frac{\lambda_m^{-p}}{(1+\lambda_m x)^3} \leq C \int_0^\infty \frac{1}{y^{2p}\left(1+y^2x\right)} dy \leq Cx^{p-\frac{1}{2}} \int_0^\infty \frac{1}{z^{2p}\left(1+z^2\right)^3} dz \leq Cx^{p-\frac{1}{2}},$$

and hence Fubini's theorem yields

$$\sum_{m\geq 1} \int_0^h \frac{\lambda_m^{-p}}{\left(1+\lambda_m x\right)^3} dx \leq C \int_0^h x^{p-\frac{1}{2}} dx \leq Ch^{p+\frac{1}{2}}.$$

The above inequalities imply $\sum_{m\geq 1} \delta_3(\lambda_m) \leq Ch^{p+\frac{1}{2}}$. This inequality, (3.19) and (3.20) give the stated upper estimate. □

Lemma 3.10. *There exists a constant $C > 0$, independent of N, such that*

$$\sum_{m\geq 1} \sum_{k=0}^{N-1} \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} \left| J_{\lambda_m}^{k,N}(t) \right| dt \leq Ch^{p+\frac{1}{2}}.$$

Proof. Using Lemma 3.4, we have

$$\left| \gamma_{\lambda_m}^{k,N}(t) \right|^2 - 1 = -\frac{2(t-t_k)\lambda_m}{1+\lambda_m h} + \frac{\left|t-t_k\right|^2 {\lambda_m}^2}{(1+\lambda_m h)^2}.$$

Using (3.5) and (3.17), we have

$$\lambda_m^{-p} E \int_{t_k}^{t_{k+1}} \left| J_{\lambda_m}^{k,N}(t) \right| dt \leq C \left(\lambda_m^{1-p} h^2 + \lambda_m^{2-p} h^3 \right) e^{-2\lambda_m (T-t_{k+1})}.$$

Lemma 3.8 concludes the proof. □

Lemma 3.11. *There exists a constant $C>0$, independant of N, such that*

$$\sum_{m\geq 1}\sum_{k=0}^{N-2}\lambda_m^{-p}E\int_{t_k}^{t_{k+1}}\left|I_{\lambda_m}^{k,N}(t)\right|dt\leq Ch^{p+\frac{1}{2}}.$$

Proof. Let $I_{1,\lambda_m}^{k,N}(t):=E\beta_{\lambda_m}^{k,N}(t)\frac{\partial}{\partial x}u_{\lambda_m}\left(t,X_{\lambda_m}^N(t)\right)+E\lambda_m X_{\lambda_m}^N(t_{k+1})\frac{\partial}{\partial x}u_{\lambda_m}\left(t_{k+1},X_{\lambda_m}^N(t_{k+1})\right)$ and $I_{2,\lambda_m}^{k,N}(t):=-\lambda_m EX_{\lambda_m}^N(t_{k+1})\frac{\partial}{\partial x}u_{\lambda_m}\left(t_{k+1},X_{\lambda_m}^N(t_{k+1})\right)+\lambda_m EX_{\lambda_m}^N(t)\frac{\partial}{\partial x}u_{\lambda_m}\left(t,X_{\lambda_m}^N(t)\right)$. Using (3.16), we have

$$EI_{\lambda_m}^{k,N}(t)=I_{1,\lambda_m}^{k,N}(t)+I_{2,\lambda_m}^{k,N}(t). \tag{3.21}$$

First we study $I_{1,\lambda_m}^{k,N}(t)$. Using (3.4), we know that $\frac{\partial}{\partial x}u_{\lambda_m}\in C^{1,2}$. So using Itô's formula and Lemma 3.4, we have

$$\begin{aligned}d\frac{\partial}{\partial x}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)=&\left\{\frac{\partial^2}{\partial t\partial x}u_{\lambda_m}+\beta_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial x^2}u_{\lambda_m}\right\}\left(s,X_{\lambda_m}^N(s)\right)ds\\&+\gamma_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial x^2}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)dW_{\lambda_m}(s)\end{aligned} \tag{3.22}$$

Using this equation, Lemma 3.4 and the Itô formula we deduce

$$\begin{aligned}d\left[\beta_{\lambda_m}^{k,N}(s)\frac{\partial}{\partial x}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)\right]=&\left\{\beta_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial t\partial x}u_{\lambda_m}+\left|\beta_{\lambda_m}^{k,N}(s)\right|^2\frac{\partial^2}{\partial x^2}u_{\lambda_m}\right.\\&\left.+z_{\lambda_m}^{k,N}(s)\gamma_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial x^2}u_{\lambda_m}\right\}\left(s,X_{\lambda_m}^N(s)\right)ds\\&+\left\{\beta_{\lambda_m}^{k,N}(s)\gamma_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial x^2}u_{\lambda_m}+z_{\lambda_m}^{k,N}(s)\frac{\partial}{\partial x}u_{\lambda_m}\right\}\left(s,X_{\lambda_m}^N(s)\right)dW_{\lambda_m}(s).\end{aligned}$$

Integrating between t and t_{k+1}, taking expectation, and using the fact that $\beta_{\lambda_m}^{k,N}(t_{k+1})=-\lambda_m X_{\lambda_m}^N(t_{k+1})$, so that $I_{1,\lambda_m}^{k,N}(t_{k+1})=0$, we obtain

$$\begin{aligned}I_{1,\lambda_m}^{k,N}(t)=-E\int_t^{t_{k+1}}&\left\{\beta_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial t\partial x}u_{\lambda_m}+\left|\beta_{\lambda_m}^{k,N}(s)\right|^2\frac{\partial^2}{\partial x^2}u_{\lambda_m}\right.\\&\left.+z_{\lambda_m}^{k,N}(s)\gamma_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial x^2}u_{\lambda_m}\right\}\left(s,X_{\lambda_m}^N(s)\right)ds.\end{aligned} \tag{3.23}$$

Using (3.7) and Lemma 3.5, we have for $s\in[t,t_{k+1}]$

$$E\beta_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial t\partial x}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)=4\lambda_m e^{-2\lambda_m(T-s)}E\beta_{\lambda_m}^{k,N}(s)X_{\lambda_m}^N(s)\leq C\lambda_m e^{-2\lambda_m(T-t_{k+1})},$$

and hence

$$\lambda_m^{-p}\int_{t_k}^{t_{k+1}}dt\int_t^{t_{k+1}}dsE\beta_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial t\partial x}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)\leq C\lambda_m^{1-p}h^2e^{-\lambda_m(T-t_{k+1})}.$$

Using Lemma 3.8, and the above inequality, we deduce

$$\sum_{m\geq 1}\sum_{k=0}^{N-2}\lambda_m^{-p}\int_{t_k}^{t_{k+1}}dt\int_t^{t_{k+1}}dsE\beta_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial t\partial x}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)\leq Ch^{p+\frac{1}{2}}. \tag{3.24}$$

Using (3.5) and Lemma 3.5, we have for $s\in[t_k,t_{k+1}]$

$$E\left|\beta_{\lambda_m}^{k,N}(s)\right|^2\frac{\partial^2}{\partial x^2}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)=4\lambda_m e^{-2\lambda_m(T-s)}\leq 4\lambda_m e^{-2\lambda_m(T-t_{k+1})},$$

so that

$$\lambda_m^{-p}\int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} dsE\left|\beta_{\lambda_m}^{k,N}(s)\right|^2 \frac{\partial^2}{\partial x^2}u_{\lambda_m}\left(s, X_{\lambda_m}^N(s)\right) \le C\lambda_m^{1-p}h^2 e^{-2\lambda(T-t_{k+1})}.$$

Thus, Lemma 3.8 yields

$$\sum_{m\ge 1}\sum_{k=0}^{N-2}\lambda_m^{-p}\int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} dsE\left|\beta_{\lambda_m}^{k,N}(s)\right|^2 \frac{\partial^2}{\partial x^2}u_{\lambda_m}\left(s, X_{\lambda_m}^N(s)\right) \le Ch^{p+\frac{1}{2}}. \tag{3.25}$$

Using equations (3.5) and Lemma 3.4 we have for all $s\in[t,t_{k+1}]$

$$\begin{aligned}E\left|z_{\lambda_m}^{k,N}(s)\gamma_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial x^2}u_{\lambda_m}\left(s, X_{\lambda_m}^N(s)\right)\right| =&\frac{2\lambda_m}{1+\lambda_m h}\left(1-\frac{(s-t_k)\lambda_m}{1+\lambda_m h}\right)e^{-2\lambda_m(T-s)}\\ \le& C\lambda_m e^{-2\lambda_m(T-t_{k+1})}.\end{aligned}$$

Therefore, we obtain

$$\lambda_m^{-p}\int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} dsE\left|z_{\lambda_m}^{k,N}(s)\gamma_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial x^2}u_{\lambda_m}\left(s, X_{\lambda_m}^N(s)\right)\right| \le C\lambda_m^{1-p}h^2e^{-2\lambda_m(T-t_{k+1})}.$$

Using once more Lemma 3.8, we deduce

$$\sum_{m\ge 1}\sum_{k=0}^{N-2}\lambda_m^{-p}\int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} dsE\left|z_{\lambda_m}^{k,N}(s)\gamma_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial x^2}u_{\lambda_m}\left(s, X_{\lambda_m}^N(s)\right)\right| \le Ch^{p+\frac{1}{2}}.$$

Plugging this inequality together with (3.24) and (3.25) into (3.23) gives us

$$\sum_{m\ge 1}\sum_{k=0}^{N-2}\lambda_m^{-p}E\int_{t_k}^{t_{k+1}}\left|I_{1,\lambda_m}^{k,N}(t)\right|dt \le Ch^{p+\frac{1}{2}}. \tag{3.26}$$

Now we study $I_{2,\lambda_m}^{k,N}(t)$. Using Lemma 3.4, equation (3.22) and the Itô formula we have

$$\begin{aligned}dX_{\lambda_m}^N(s)\frac{\partial}{\partial x}u_{\lambda_m}\left(s, X_{\lambda_m}^N(s)\right) =& \Big\{X_{\lambda_m}^N(s)\frac{\partial^2}{\partial t\partial x}u_{\lambda_m} + X_{\lambda_m}^N(s)\beta_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial x^2}u_{\lambda_m}\\ &+\beta_{\lambda_m}^{k,N}(s)\frac{\partial}{\partial x}u_{\lambda_m}+\left|\gamma_{\lambda_m}^{k,N}(s)\right|^2\frac{\partial^2}{\partial x^2}u_{\lambda_m}\Big\}\left(s, X_{\lambda_m}^N(s)\right)ds\\ &+\Big\{\gamma_{\lambda_m}^{k,N}(s)\frac{\partial}{\partial x}u_{\lambda_m}+X_{\lambda_m}^N(s)\gamma_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial x^2}u_{\lambda_m}\Big\}\left(s, X_{\lambda_m}^N(s)\right)dW_{\lambda_m}(s)\end{aligned}$$

So integrating between t and t_{k+1} and taking expectation, we obtain

$$\begin{aligned}I_{2,\lambda_m}^{k,N}(t) = -\lambda_m E\int_t^{t_{k+1}}&\Big\{X_{\lambda_m}^N(s)\frac{\partial^2}{\partial t\partial x}u_{\lambda_m}+\beta_{\lambda_m}^{k,N}(s)\frac{\partial}{\partial x}u_{\lambda_m}+X_{\lambda_m}^N(s)\beta_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial x^2}u_{\lambda_m}\\ &+\left|\gamma_{\lambda_m}^{k,N}(s)\right|^2\frac{\partial^2}{\partial x^2}u_{\lambda_m}\Big\}\left(s, X_{\lambda_m}^N(s)\right)ds.\end{aligned} \tag{3.27}$$

Using equation (3.7) and Lemma 3.5, we have for all $s\in[t,t_{k+1}]$

$$\begin{aligned}\lambda_m EX_{\lambda_m}^N(s)\frac{\partial^2}{\partial t\partial x}u_{\lambda_m}\left(s, X_{\lambda_m}^N(s)\right) =&4\lambda_m^2 e^{-2\lambda_m(T-s)}E\left|X_{\lambda_m}^N(s)\right|^2\\ \le& C\lambda_m^2\left(\frac{1}{\lambda_m}+h\right)e^{-2\lambda_m(T-t_{k+1})}.\end{aligned}$$

Therefore,

$$\lambda_m^{-p}\int_{t_k}^{t_{k+1}}\int_t^{t_{k+1}}\lambda_m EX_{\lambda_m}^N(s)\frac{\partial^2}{\partial t\partial x}u_{\lambda_m}\left(s, X_{\lambda_m}^N(s)\right) \le C\left(\lambda_m^{1-p}h^2+\lambda_m^{2-p}h^3\right)e^{-2\lambda_m(T-t_{k+1})},$$

and using Lemma 3.8, we deduce

$$\sum_{m\geq 1}\sum_{k=0}^{N-2}\lambda_m^{-p}\int_{t_k}^{t_{k+1}}dt\int_t^{t_{k+1}}ds\lambda_m EX_{\lambda_m}^N(s)\frac{\partial^2}{\partial t\partial x}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)\leq Ch^{p+\frac{1}{2}}. \tag{3.28}$$

The equation (3.4) and Lemma 3.5 yield for all $s\in[t,t_{k+1}]$

$$\lambda_m E\beta_{\lambda_m}^{k,N}(s)\frac{\partial}{\partial x}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)=2\lambda_m e^{-2\lambda_m(T-s)}E\beta_{\lambda_m}^{k,N}(s)X_{\lambda_m}^N(s)\leq C\lambda_m e^{-2\lambda_m(T-t_{k+1})}.$$

This upper estimate implies

$$\lambda_m^{-p}\int_{t_k}^{t_{k+1}}dt\int_t^{t_{k+1}}ds\lambda_m E\beta_{\lambda_m}^{k,N}(s)\frac{\partial}{\partial x}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)\leq C\lambda_m^{1-p}h^2e^{-2\lambda_m(T-t_{k+1})},$$

and Lemma 3.8 yields

$$\sum_{m\geq 1}\sum_{k=0}^{N-2}\lambda_m^{-p}\int_{t_k}^{t_{k+1}}dt\int_t^{t_{k+1}}ds\lambda_m E\beta_{\lambda_m}^{k,N}(s)\frac{\partial}{\partial x}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)\leq Ch^{p+\frac{1}{2}}. \tag{3.29}$$

Using equation (3.5) and Lemma 3.5, we have for all $s\in[t,t_{k+1}]$

$$\lambda_m EX_{\lambda_m}^N(s)\beta_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial x^2}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)\leq C\lambda_m e^{-2\lambda_m(T-t_{k+1})}.$$

Therefore, we obtain

$$\lambda_m^{-p}\int_{t_k}^{t_{k+1}}dt\int_t^{t_{k+1}}ds\lambda_m EX_{\lambda_m}^N(s)\beta_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial x^2}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)\leq C\lambda_m^{1-p}h^2e^{-2\lambda_m(T-t_{k+1})},$$

and Lemma 3.8 implies

$$\sum_{m\geq 1}\sum_{k=0}^{N-2}\lambda_m^{-p}\int_{t_k}^{t_{k+1}}dt\int_t^{t_{k+1}}ds\lambda_m EX_{\lambda_m}^N(s)\beta_{\lambda_m}^{k,N}(s)\frac{\partial^2}{\partial x^2}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)\leq Ch^{p+\frac{1}{2}}. \tag{3.30}$$

Finally, (3.5) and Lemma 3.4 imply that for all $s\in[t,t_{k+1}]$

$$\lambda_m E\left|\gamma_{\lambda_m}^{k,N}(s)\right|^2\frac{\partial^2}{\partial x^2}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)\leq C\lambda_m e^{-2\lambda_m(T-t_{k+1})}.$$

This yields

$$\lambda_m^{-p}\int_{t_k}^{t_{k+1}}dt\int_t^{t_{k+1}}ds\lambda_m E\left|\gamma_{\lambda_m}^{k,N}(s)\right|^2\frac{\partial^2}{\partial x^2}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)\leq C\lambda_m^{1-p}h^2e^{-2\lambda_m(T-t_{k+1})},$$

and Lemma 3.8 implies

$$\sum_{m\geq 1}\sum_{k=0}^{N-2}\lambda_m^{-p}\int_{t_k}^{t_{k+1}}dt\int_t^{t_{k+1}}ds\lambda_m E\left|\gamma_{\lambda_m}^{k,N}(s)\right|^2\frac{\partial^2}{\partial x^2}u_{\lambda_m}\left(s,X_{\lambda_m}^N(s)\right)\leq Ch^{p+\frac{1}{2}}.$$

Plugging this inequality together with (3.28) - (3.30) into (3.27), we deduce

$$\sum_{m\geq 1}\sum_{k=0}^{N-2}\lambda_m^{-p}E\int_{t_k}^{t_{k+1}}\left|I_{2,\lambda_m}^{k,N}(t)\right|dt\leq Ch^{p+\frac{1}{2}}.$$

This equation together with (3.21) and (3.26) conclude the proof. □

Theorem 2.1 is a straightforward consequence of equation (3.18) and Lemmas 3.9-3.11.

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