Risk indicators with several lines of business: comparison, asymptotic behavior and applications to optimal reserve allocation

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RISK INDICATORS WITH SEVERAL LINES OF BUSINESS: COMPARISON, ASYMPTOTIC BEHAVIOR AND APPLICATIONS TO OPTIMAL RESERVE ALLOCATION

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Abstract. In a multi-dimensional risk model with dependent lines of business, we propose to allocate capital with respect to the minimization of some risk indicators. These indicators are sums of expected penalties due to the insolvency of a branch while the global reserve is either positive or negative. Explicit formulas in the case of two branches are obtained for several models (independent exponential, correlated Pareto). The asymptotic behavior (as the initial capital goes to infinity) is studied. For higher dimension and several periods, no explicit expression is available. Using a stochastic algorithm, we get estimations of the allocation, compare the different allocations and study the impact of dependence.

Introduction

The current change of regulation leads the insurance industry to address new questions regarding solvency. In Europe, insurance groups will soon have to comply with new rules, namely Solvency II. In comparison to the previous regulation system, Solvency II aims at defining solvency requirements that are better adjusted to the underlying risks. Solvency requirements may either be computed thanks to a standard formula, or with internal models that companies are encouraged to develop.

While a bottom-up approach is used in the standard formula to aggregate risks (one first studies each small risk separately and then aggregates them thanks to a kind of correlation matrix), a top-down approach may be used in some internal or partial internal models to allocate economic capital: once the main risk drivers for the overall company have been identified and the global solvency capital requirement has been computed, it is necessary to split this overall buffer capital into marginal solvency capitals for each line of business, in order to penalize as equitably as possible lines of business and customers according to the share of the overall risk they represent after diversification. Once this is done, one may also want to allocate some free additional surplus in some different zones in order to avoid as far as possible that some lines of business become insolvent too often. Capital fungibility between lines of business or between entities of a large insurance group that lie in different countries is indeed limited by different entity-specific or country-specific solvency constraints. For example, when AIG experienced problems in the USA, it was impossible for the group to transfer some funds from some European branches to the ones in distress. Some

European branches were able to continue business almost as usual and were not impacted, while some other lines would have to get up to 37.8 billion of fresh liquidity.

One possible way to define optimality of the global reserve allocation is to minimize the expected sum of penalties that each line of business would have to pay due to its temporary potential insolvency, following ideas presented in [11]. If one neglects discounting factors, a first approximation of this penalty is given by the time-integrated expected negative part of the surplus process. Closed-form formulas were available in the classical risk model for exponentially distributed claim amounts, which led to a semi-explicit optimal reserve allocation.
However, this approach does not take into account the dependence structure. Also, one could object that if the company is ruined at the group level (*i.e.* the sum of the surpluses is negative), then the allocation does not change anything. Following [4], we define and study the expected *orange area*: the reserve of line k at time j is denoted by R_j^k (defined as u^k plus the aggregate premium minus the aggregate claim of the k^{th} branch over the first j periods). We look for the optimal allocation $(u^1, \ldots, u^d)$ - where u^k stands for the part of capital allocated to the k^{th} branch - that minimizes

$$I_1 = \sum_{j=1}^{p\wedge\tau} \sum_{k=1}^{d} \mathbb{E}\left(|R_j^k| \mathbb{1}_{\left\{R_j^k<0\right\}} \mathbb{1}_{\left\{R_j^1+\cdots+R_j^d\geq 0\right\}} \right)$$

under the constraint that $u^1 + \cdots + u^d = u$, where

$$\tau = \inf\left\{ j \in \mathbb{N}^*,\, R_j^1 + \cdots + R_j^d < 0 \right\}.$$

The I_1 indicator is the *stopped* version of the indicator considered in [4]:

$$I_2 = \sum_{j=1}^{p} \sum_{k=1}^{d} \mathbb{E}\left(|R_j^k| \mathbb{1}_{\{R_j^k<0\}} \mathbb{1}_{\{R_j^1+\cdots+R_j^d\geq 0\}} \right).$$

Remark that in the case $p = 1$, $I_1 = I_2$. Below, I_1 is referred to as the *stopped orange area* and I_2 is referred to as the *orange area*. In the present paper, we also consider the alternative risk indicator, called expected *violet area*:

$$J = \sum_{j=1}^{p} \sum_{k=1}^{d} \mathbb{E}\left(|R_j^k| \mathbb{1}_{\left\{R_j^k<0\right\}} \mathbb{1}_{\left\{R_j^1+\cdots+R_j^d\leq 0\right\}} \right).$$

The three studied cases are represented in the figure below. In the case when $p = 1$, these three indicators may be considered as particular cases of *Dhaene et al.*'s [6] general framework which resumes as the minimization of

$$\sum_{j=1}^{d} \mathbb{E}\left[\xi_j D\left(\frac{X_j - u^j}{v_j} \right) \right]$$

where X_j stands for the loss of the j^{th} branch, (v_j) for a weight sequence and D is a deviation function.
Our proposal corresponds to take $\xi_j = \mathbb{1}_{\{S\leq u\}}$ or $\xi_j = \mathbb{1}_{\{S\geq u\}}$ (where S is the aggregate loss) and $D(x) = x^+$ (see equations (31), (69), (72) and (74)

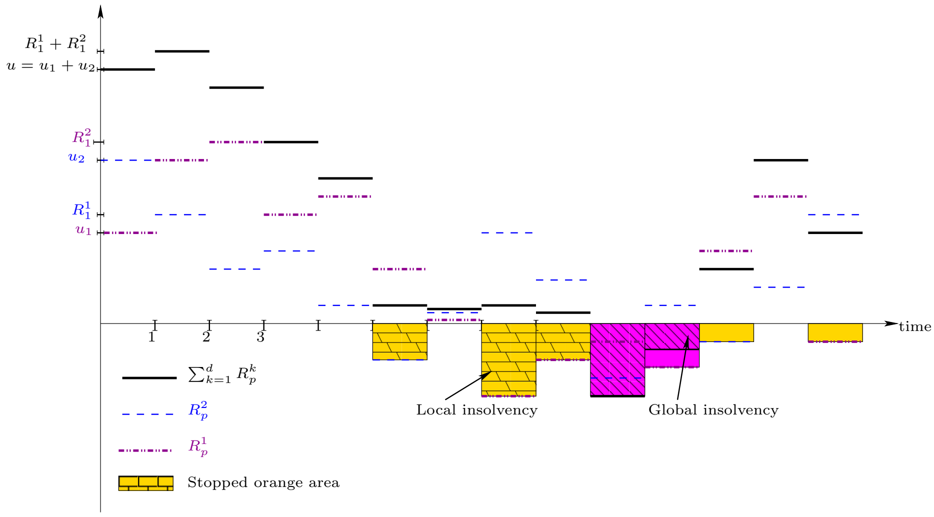


FIGURE 1. Representation of the three indicators

in [6]).

The number p of periods could also correspond to one year in the Solvency II Solvency Capital Requirement computation problem, or to 3 to 5 years in the Own Risk and Solvency Assessment (ORSA) framework. ORSA is defined as the process undertaken by an insurer or an insurance group to assess the adequacy of its risk management and current and future solvency position; in that setting, time is often discrete to limit the complexity and one is interested in ruin at yearly inventories. The Article 45 of the Solvency II directive defines the ORSA as a part of the risk-management system of an insurance or reinsurance undertaking to conduct its own risk and solvency assessment (see [7] for example). The National (U.S) Association of Insurance Commissioner (NAIC) had also initiated reflections and proposed principles on ORSA, a review may be found in [12].
Here we consider very basic insurance discrete time models for different lines, and we try to study the optimal reserve allocation. We shall see that considering the *orange area* (I_1 or I_2 indicator) is a completely different approach from the economic capital allocation problem, where one tries to penalize contributions to the overall risk and to diversification effect.

As noticed in [6], *different capital allocations must in some sense correspond to different questions that can be asked within the context of risk management.* A review on various ways to allocate capital may be found in [5]. Also J.-P. Laurent [10] gives a global view of capital allocation and exhibits connections between several theories (including Euler, Shapley-Aumann and Pareto optimal allocations). Let us remark that even if our purpose implies a *risk driven* allocation of capital, we do not refer explicitly to risk measure theory (see [3, 8, 9]). An example is the TVaR allocation principle (which appears also as a particular case of [6], see equation (54)), according to which each line of business k has to pay for cost of capital corresponding to average marginal loss for line k given that things go wrong for the global company (the VaR of the sum is overshot). This allocation principle is similar in spirit to the minimization of the indicator J (*violet area*), while the *orange area* is placed in a different context and relevant when the global company is safe. We are more interested in the stability and liquidity issues with limited capital fungibility. When the global company is ruined, this allocation has no importance (even if in practice, one may break the mutual solidarity principle and decide to run separately healthy lines). Finally, note that we do not take into account the penalties in the overall reserve process; in theory the line of business or the global company would have to pay the penalty upfront, and this could accelerate ruin.

The present article is an attempt to explore the quantitative behavior of the allocation procedure, based on the minimization of a risk indicator. While Dhaene et al. (see Theorem 3 in [6]) show that the optimization problem leads to compute quantile functions and comonotonic sums related to each risk and to the specific form of the indicator (see Theorem 3), we propose explicit computations in dimension 2 and provide a simulation study in higher dimension and for multi-periodic models. The simulation study is made possible thanks to the algorithmic procedure proposed in [4]. Let us emphasize that in [6], the question of the effective computation of the solution to the optimization problem is not addressed.

Our paper is organized as follows. In Section 1, we tackle the case when there are two lines of business ($d = 2$) and one period ($p = 1$). We derive semi-explicit formulas and/or asymptotics in the cases where aggregate claim amounts are independent exponentials, independent Generalized Pareto Distribution (GPD), correlated GPD, for the minimization of the *orange area*. In Section 2, we present the same computations for the minimization of the *violet area*, and compare both minimization strategies. Section 3 consists in a simulation study for the case treated in Section 1 and Section 2, using the algorithm developed in [4]. This section helps for benchmarking the parameters of the algorithm. In Section 4, models in higher dimension ($d = 10$), still for one period, are considered. In Section 5, we investigate time horizons $p > 1$ and $d = 2, \dots, 5$ lines of business. For the models considered in these two last sections, semi-explicit formulas are generally not available.

1. Time horizon $p = 1$, $d = 2$ lines

In this section, we consider the one-period problem with two lines of business. Let

$$R_1^k = u^k + c^k - X^k,$$

where for line $k = 1$, 2, u^k is the initial reserve, c^k is the premium and X^k is the aggregate claim amount during the first period. We start with a basic example with a discrete distribution. It illustrates the fact that the use of the *orange area* may lead to inconsistent allocation, for example, if the total capital u is too small. Then we consider more meaningful models for the claims, for which we may derive explicit solutions.

1.1. **A basic example.** The following example is very basic but a caricature. It was chosen to highlight the fact that the expected *orange area* must not be used to penalize lines of business according to their risk contributions. Imagine that $c^1 = c^2 = 10$, that $X^1 = 1000Y^1$ and $X^2 = 20Y^2$, where Y^1 and Y^2 are independent Bernoulli random variables with parameter 1/2, and that $u = 10$. The expected *orange area* is equal to $\frac{10-u^2}{4}$ for $0 \leq u^2 \leq u$, because the aggregate reserves are negative if $X^1 = 1000$. Consequently, the optimal reserve allocation would be to allocate all the capital to the less risky line of business (line 2). This decision corresponds to minimizing penalties in case of temporary insolvency of one line, but is in complete disagreement with any reasonable economic capital allocation principle. Let us remark that in this example, the *violet area* equals $\frac{1}{4}(1980 - u^1)$, so that it is minimized by allocating all the capital to the most risky branch, as expected.
We remark that in this example, u and the premiums c^1 and c^2 are not consistent with the claims X^1 and X^2. Below[a], we have taken $u = 960$, $c^1 = 50$ and $c^2 = 1$. We have considered two cases: firstly Y^1 and Y^2 are independent Bernoulli variables with parameter $\frac{1}{2}$. Secondly, they are still independent Bernoulli variables but Y^1 has parameter $\frac{1}{2}$ and Y^2 has parameter $\frac{1}{3}$.

1.1.1. *$Y^1 \rightsquigarrow \mathcal{B}(\frac{1}{2})$, $Y^2 \rightsquigarrow \mathcal{B}(\frac{1}{2})$.* A direct computation gives:

$$I_1 = I_2 = J = \frac{1}{4}\left((950 - u^1)\mathbb{1}_{\{u^1<950\}} + (19 - u^2)\mathbb{1}_{\{u^2<19\}}\right).$$

The *violet* and the *orange area* are the same. Under the constraint $u^1+u^2 = u$, it is minimized for $10 \leq u^2 \leq 19$ (in this case, there is not a unique minimum).

1.1.2. *$Y^1 \rightsquigarrow \mathcal{B}(\frac{1}{2})$, $Y^2 \rightsquigarrow \mathcal{B}(\frac{1}{3})$.* A direct computation gives:

$$\begin{aligned} I_1 = I_2 &= \frac{1}{6}\left(2 * (950 - u^1)\mathbb{1}_{\{u^1<950\}} + (19 - u^2)\mathbb{1}_{\{u^2<19\}}\right), \\ J &= \frac{1}{6}\left((950 - u^1)\mathbb{1}_{\{u^1<950\}\}} + (19 - u^2)\mathbb{1}_{\{u^2<19\}}\right). \end{aligned}$$

There is a unique minimum for $I_1 = I_2$, which is reached for $u^2 = 10$ and $u^1 = 950$.
In the case of the *violet area* J, the minimum is reached for $u^2 \in [10; 19]$, the minimum is not unique.

Let us now assume that the insurance company has previously correctly penalized lines of business according to their risks, and then wants to allocate some additional safety capital u in order to avoid temporary insolvencies of some lines. The *orange area* has been designed to answer this question.

1.2. **General remarks.** We now consider more general cases, still for the period $p = 1$ and focus on the *orange area.* For the sake of simplicity, in the remainder of this section, we will consider u^k instead of $u^k + c^k$. The premium is thus included in the allocation here. Remark that for $p > 1$ this would be a non sense to include the premium in the allocation.

[a]The premiums have been taken equal to 10% of the expected value of the claims.

The gradient with respect to (u^1, u^2) of the average *orange area* I_2 is given by

$$\left(\mathbb{P}(R_1^1 < 0, R_1^1 + R_1^2 \geq 0), \quad \mathbb{P}(R_1^2 < 0, R_1^1 + R_1^2 \geq 0)\right).$$

We are looking for non degenerate solutions (*i.e.* each line receives a positive initial reserve u^k in the optimal strategy), an easy calculation with Lagrange's multipliers with the constraint $u^1 + u^2 = u$ leads to an optimal allocation when

$$\mathbb{P}\left(R_1^1 < 0, R_1^1 + R_1^2 \geq 0\right) = \mathbb{P}\left(R_1^2 < 0, R_1^1 + R_1^2 \geq 0\right),$$

which can be rewritten as

$$\mathbb{P}\left(X^1 > u^1, X^1 + X^2 \leq u\right) = \mathbb{P}\left(X^2 > u^2, X^1 + X^2 \leq u\right). \tag{1.1}$$

We shall consider several probabilistic models for the claims X^1 and X^2: independent exponentials (Section 1.3), independent GPD (Section 1.4), as well as conditionally independent exponentials (Section 1.5) which are correlated Pareto distributed (GPD).

1.3. **Independent exponentials.** Assume X^1 and X^2 are independent exponential random variables with respective parameters μ^1 and μ^2. In the particular case where $\mu^1 = \mu^2$, the optimal allocation is $u^1 = u^2 = u/2$. From now on, assume that $\mu^2 > \mu^1$.

1.3.1. *Optimal allocation.*
Let us denote α and β the real numbers such that $\mu^2 = \alpha\mu^1$ and $u^1 = \beta u$, $\alpha > 1$ and $0 \leq \beta \leq 1$. Equation (1.1) leads to the optimal allocation

$$(\alpha-1)\left(h\left(\beta\right) - h\left(\alpha(1-\beta)\right)\right) - \alpha h(1) + (1+\alpha) h\left(\beta + \alpha - \alpha\beta\right) = h(\alpha) \tag{1.2}$$

where h is the function defined by $h(x) = \exp\left(-u\mu^1 x\right)$.

Proposition 1.1. *There exists a unique $\beta = \psi(\alpha, u, \mu^1) \in [0,1]$ satisfying Equation (1.2). Moreover, $0 < \psi(\alpha, u, \mu^1) \leq \frac{\alpha}{\alpha+1}$ and $\alpha \mapsto \psi(\alpha, u, \mu^1)$ is non decreasing.*

Proof. The proof is a straightforward application of the convexity of h and the implicit function theorem. □

Remark 1.2. *The fact that $\psi(\alpha, u, \mu^1)$ (and thus u^1) increases with α is consistent with the fact that if α increases then the first branch becomes riskier.*

1.3.2. *Asymptotic as the capital u goes to infinity.* Studying the asymptotic as $u \to \infty$ is useful in order to make comparison between models easier.

Proposition 1.3. *For $\alpha > 1$,*

$$\lim_{u\to\infty} \psi(\alpha, u, \mu^1) = \frac{\alpha}{\alpha+1}.$$

Proof. Multiplying (1.2) by $h(-\beta)$, we get

$$(\alpha-1) - \alpha h(1-\beta) + (1+\alpha) h\left(\alpha(1-\beta)\right) - (\alpha-1) h\left(\alpha(1-\beta) - \beta\right) - h(\alpha-\beta) = 0.$$

Remark that $\lim_{u\to\infty} \psi(\alpha, u, \mu^1) \neq 1$ since the left hand side of the previous equation has to go to 0 when u tends to ∞. Then, one has

$$\lim_{u\to\infty} -\alpha h\left(1 - \psi(\alpha, u, \mu^1)\right) + (1+\alpha) h\left(\alpha\left(1 - \psi(\alpha, u, \mu^1)\right)\right) - h\left(\alpha - \psi(\alpha, u, \mu^1)\right) = 0,$$

and consequently $h\left(\alpha\left(1-\psi(\alpha,u,\mu^1)\right)-\psi(\alpha,u,\mu^1)\right) \underset{u\to\infty}{\longrightarrow} 1$, which leads to

$$\lim_{u\to\infty} \alpha\left(1-\psi(\alpha,u,\mu^1)\right)-\psi(\alpha,u,\mu^1)=0$$

and Proposition 1.3 follows. □

1.4. **Independent Pareto.** We consider two independent Generalized Pareto Distribution (GPD) denoted by X^1 and X^2 with respective parameters $\left(\frac{1}{a},\frac{b}{a}\right)$ and $\left(\frac{1}{a},\frac{b}{\alpha a}\right)$, $a,b>0$, $\alpha\geq 1$. The density and survival functions are given by:

$$f_{X^1}(x)=\frac{a}{b}\left(1+\frac{x}{b}\right)^{-a-1},\quad f_{X^2}(x)=\frac{\alpha a}{b}\left(1+\frac{\alpha x}{b}\right)^{-a-1}$$

and

$$\overline{F}_{X^1}(x)=\left(1+\frac{x}{b}\right)^{-a},\quad \overline{F}_{X^2}(x)=\left(1+\frac{\alpha x}{b}\right)^{-a}.$$

Equation (1.1) yields in this case

$$\begin{aligned}
&\overline{F}_{X^1}(u^1) \quad - \quad \frac{a}{b}\int_{u^1}^{u}\left(1+\frac{x}{b}\right)^{-a-1}\left(1+\frac{\alpha(u-x)}{b}\right)^{-a}dx\\
&=\overline{F}_{X^2}(u^2) \quad - \quad \frac{a}{b}\int_{u^2}^{u}\left(1+\frac{\alpha x}{b}\right)^{-a-1}\left(1+\frac{(u-x)}{b}\right)^{-a}dx.
\end{aligned}\tag{1.3}$$

We do not know how to derive an explicit optimal allocation from this equation. Nevertheless, an asymptotic as $u\to\infty$ is reachable. We begin with a general result for independent variables.

Theorem 1.4. *Let X^1 and X^2 be two independent, continuous and non negative random variables such that*[b] $\overline{F}_{X^1}(x)=\Theta(\overline{F}_{X^2}(x))$ *as $x\to\infty$. The solution (u^1,u^2) to Equation (1.1), under the constraint $u^1+u^2=u$ is equivalent, as $u\to\infty$ to*

$$\overline{F}_{X^1}(u^1)=\overline{F}_{X^2}(u^2).$$

Proof. First, we start with the proof of the existence of two real numbers κ_1 and κ_2 such that $0<\kappa_1<\kappa_2<1$ and for u large enough,

$$\kappa_1\leq\frac{u^1}{u}\leq\kappa_2.\tag{1.4}$$

Let $\beta(u)=\frac{u^1}{u}$. If (1.4) was not satisfied then taking if necessary a sequence $u_k\to\infty$, we could assume that either $\beta\to 0$ or $\beta\to 1$. If $\beta\to 0$ then $\overline{F}_{X^1}(u^1)\to 0$ and $\overline{F}_{X^2}(u^2)\to 1$. Remark that

$$\mathbb{P}(X^1>u^1,\ X^1+X^2\leq u)=\overline{F}_{X^1}(u^1)-\mathbb{P}(X^1>u^1,\ X^1+X^2>u),$$

so that (1.1) rewrites:

$$\overline{F}_{X^1}(u^1)-\mathbb{P}(X^1>u^1,\ X^1+X^2>u)=\overline{F}_{X^2}(u^2)-\mathbb{P}(X^2>u^2,\ X^1+X^2>u).$$

Both terms $\mathbb{P}(X^1>u^1,\ X^1+X^2>u)$ and $\mathbb{P}(X^2>u^2,\ X^1+X^2>u)$ go to 0 as $u\to\infty$, so that we can not have $\overline{F}_{X^1}(u^1)\to 0$ and $\overline{F}_{X^2}(u^2)\to 1$.

[b] Recall that $f(x)$ is said to be $\Theta(g(x))$ as $x\to\infty$ if the ratio $\frac{f(x)}{g(x)}$ is bounded (from below and from above) as $x\to\infty$.

and thus β can not go to 0. The same reasoning proves that β cannot go to 1 and thus (1.4) is satisfied.
Now,

$$\begin{aligned}\mathbb{P}(X^1 > u^1,\ X^1+X^2 > u) &= \underbrace{\mathbb{P}(X^1 > u^1,\ X^2 > \sqrt{u},\ X^1+X^2 > u)}_{(1)} \\ &\quad + \underbrace{\mathbb{P}(X^1 > u^1,\ X^2 < \sqrt{u},\ X^1+X^2 > u)}_{(2)}.\end{aligned}$$

The term (1) is less than $\mathbb{P}(X^1 > u^1)\mathbb{P}(X^2 > \sqrt{u}) = o(\overline{F}_{X^1}(u^1))$. The term (2) is less than $\overline{F}_{X^2}(\sqrt{u})\overline{F}_{X^1}(u-\sqrt{u})$. Thus, (1.4) leads to

$$\overline{F}_{X^1}(u-\sqrt{u}) \leq \overline{F}_{X^1}(u^1)$$

for u large enough, so that

$$\mathbb{P}(X^1 > u^1,\ X^1+X^2 > u) = o\left(\overline{F}_{X^1}(u^1)\right)$$

and

$$\mathbb{P}(X^2 > u^2,\ X^1+X^2 > u) = o\left(\overline{F}_{X^2}(u^2)\right).$$

The fact that $\overline{F}_{X^1}(x) = \Theta(\overline{F}_{X^2}(x))$ yields the desired result. □

Proposition 1.5. *The unique solution to (1.3) satisfies*

$$\lim_{u\to\infty} \beta(u) = \frac{\alpha}{\alpha+1},$$

where $\beta(u) = \frac{u^1}{u}$.

Proof. The proof is a straightforward consequence of Theorem 1.4 together with the resolution of $\overline{F}_{X^1}(u^1) = \overline{F}_{X^2}(u^2)$. □

1.5. **Correlated Pareto.** In this section, we consider a model inspired from [2]. Let X be a random variable with exponential distribution $\mathcal{E}(\Theta)$, where the parameter Θ is $\Gamma(a,b)$ distributed. Recall that the resulting mixed survival function of X is given by

$$1 - F_X(x) = \int_0^\infty e^{-\theta x} f_\Theta(\theta)d\theta = \left(1+\frac{x}{b}\right)^{-a}, \quad x \geq 0.$$

In other words, X has a Generalized Pareto distribution (GPD).

1.5.1. *Explicit solution for fixed u.* Let X^1 and X^2 be two GPD according to the above model. Conditionally to μ^1, we assume that $X^1 \sim \mathcal{E}(\mu^1)$, $X^2 \sim \mathcal{E}(\alpha\,\mu^1)$ with $1 < \alpha$, X^1 and X^2 are independent. We assume moreover that $\mu^1 \sim \Gamma(a,b)$, $a, b > 0$. Following [2], we have that the dependence structure of the X^i's is given by a survival Clayton copula.
Conditionally to μ^1 equation (1.1) leads to:

$$(\alpha-1)h(\beta) - \alpha h(1) + h(\alpha-\alpha\beta+\beta) = (\alpha-1)h(\alpha(1-\beta)) - \alpha h(\alpha-\alpha\beta+\beta) + h(\alpha).$$

We now integrate with respect to μ^1 and get the same equation replacing the function h by $s(x) = (1+x\frac{u}{b})^{-a}$. Hence this equation can be rewritten as $f(\alpha,\beta,u) = 0$ with

$$\begin{aligned} f(\alpha,\beta,u) &= (\alpha-1)\left(s(\beta) - s\left(\alpha(1-\beta)\right)\right) \\ &\quad -\alpha s(1) + (\alpha+1)s\left(\alpha-\alpha\beta+\beta\right) - s(\alpha). \end{aligned} \tag{1.5}$$

Proposition 1.6. *There exists a unique* $\beta = \Phi(\alpha, u, a, b) \in [0, 1]$ *satisfying equation (1.5). Moreover,* $0 < \Phi(\alpha, u, a, b) \leq \frac{\alpha}{\alpha+1}$.

Proof. We remark that $f(\alpha, 0, u) = -f(\alpha, 1, u) = (\alpha - 1) + s(\alpha) - \alpha s(1) \geq 0$ as by convexity of s, one has

$$\frac{s(\alpha) - s(1)}{\alpha - 1} \geq s(1) - s(0)\,.$$

Moreover one has

$$\begin{aligned} \frac{b}{a(\alpha-1)u}\frac{\partial f}{\partial \beta} &= -g(\beta) + (\alpha+1)g(\alpha - \alpha\beta + \beta) - \alpha g(\alpha(1-\beta)) \\ &= -g(\beta) + g(\alpha - \alpha\beta + \beta) + \alpha g(\alpha - \alpha\beta + \beta) - \alpha g(\alpha(1-\beta)) \qquad (1.6) \\ &\leq 0 \end{aligned}$$

with $g(x) = (1 + x\frac{u}{b})^{-(a+1)}$, as g is decreasing. Hence there exists a unique $\beta_0 \in [0, 1]$ such that $f(\alpha, \beta_0, u) = 0$. We also have

$$f(\alpha, \frac{\alpha}{\alpha+1}, u) = s(\gamma) - s(\alpha) + \alpha s(\gamma) - \alpha s(1) \tag{1.7}$$

with $\gamma = \frac{2\alpha}{\alpha+1}$. We remark that $1 < \gamma < \alpha$, hence using the convexity of s we have

$$s(\gamma) - s(\alpha) \leq (s(1) - s(\gamma))\frac{\alpha - \gamma}{\gamma - 1}$$

which implies that $f(\alpha, \frac{\alpha}{\alpha+1}, u) \leq 0$ hence $\beta_0 \leq \frac{\alpha}{\alpha+1}$. □

1.5.2. *Asymptotic as the capital u goes to infinity.* One gets the following result, considering the asymptotic as u goes to ∞ :

Proposition 1.7. *For* $\alpha > 1$, $\lim_{u\to\infty} \Phi(\alpha, u, \mu^1) = \Phi_0$ *is the solution to:*

$$(\alpha-1)\left(\Phi_0^{-a} - (\alpha(1-\Phi_0))^{-a}\right) - \alpha + (\alpha+1)(\alpha - \alpha\Phi_0 + \Phi_0)^{-a} - \alpha^{-a} = 0. \tag{1.8}$$

We have $\Phi_0 < \frac{\alpha}{\alpha+1}$.

Proof. Equation (1.5) is equivalent, when $u \to \infty$ to

$$(\alpha-1)\left(\beta^{-a} - (\alpha(1-\beta))^{-a}\right) - \alpha + (\alpha+1)\,(\alpha - \alpha\beta + \beta)^{-a} - \alpha^{-a} = 0. \tag{1.9}$$

One wants to prove that the solution of (1.9) above satisfies $\beta < \frac{\alpha}{\alpha+1}$ when $\alpha > 1$. Indeed, define

$$g(\beta) = (\alpha-1)\beta^{-a} - \alpha + (\alpha+1)(\alpha - \alpha\beta + \beta)^{-a} - (\alpha-1)\alpha^{-a}(1-\beta)^{-a} - \alpha^{-a}.$$

One has $g(0^+) = +\infty$ and $g(1^-) = -\infty$. Straightforward computations prove that g is decreasing in β and that $g(\frac{\alpha}{\alpha+1}) < 0$. This proves that the solution Φ_0 of (1.9) satisfies $\Phi_0 < \frac{\alpha}{\alpha+1}$. □

2. Comparison with an indicator designed for crisis

Recall that the *violet area* is given by the indicator

$$J(u^1, \ldots, u^d) = \sum_{j=1}^{p}\sum_{k=1}^{d} E\left(|R_j^k|\mathbb{1}_{\{R_j^k<0\}}\mathbb{1}_{\{R_j^1+\cdots+R_j^d\leq 0\}}\right)$$

under the constraint $u^1 + \cdots + u^d = u$. As mentioned in the introduction, this corresponds to a more classical economic capital allocation principle, closely related, in spirit, to the TVaR based allocation.

In this section, we consider the capital allocation that minimizes this *violet area*. As the computations are similar to the ones corresponding to the *orange area*, we do not go into the details. Firstly, we focus on the *independent exponential distributions* case and on the *conditionally exponential independent distributions* one. Then, we give asymptotic results for the *independent GPD* case. For this last purpose, we study the asymptotic behavior of probabilities for subexponential distributions. An easy calculation leads to the convexity of the function J since

$$(u^1,\ldots,u^d)\mapsto \left|R_j^k\right| \mathbb{1}_{\{R_j^k<0\}}\mathbb{1}_{\{\sum_{k=1}^d R_j^k\le 0\}}$$

is a convex function on the set $\{(v_1,\ldots,v_d)\in(\mathbb{R}^+)^d\,,\ v_1+\ldots+v_d=u\}$ and the sum and expectation of convex functions is convex. Moreover, since $x\mapsto|x|$ is strictly convex so does the function J. Thus there exists a unique minimum u^* to the optimal allocation problem with the indicator J. It is reached for $u^1+u^2=u$ with

$$\mathbb{P}(X^1\ge u^1,\ X^1+X^2\ge u)=\mathbb{P}(X^2\ge u^2,\ X^1+X^2\ge u). \tag{2.1}$$

We have the following implicit solutions to the optimal allocation issue (recall that $u^1=\beta u$):

- **Independent exponential distributions:**

$$(\alpha+1)e^{-u\mu^1(\alpha-\alpha\beta+\beta)}-\alpha e^{-u\mu^1}-e^{-\alpha u\mu^1}=0. \tag{2.2}$$

- **Conditionally exponential independent distributions:**

$$(\alpha+1)\left(1+\frac{u(\alpha-\alpha\beta+\beta)}{b}\right)^{-a}-\alpha\left(1+\frac{u}{b}\right)^{-a}-\left(1+\frac{u\alpha}{b}\right)^{-a}=0. \tag{2.3}$$

Going through the limit $u\to\infty$ in equation (2.2) leads to $\beta\to1$ while in equation (2.3) it leads to

$$\alpha+\alpha^{-a}-(\alpha+1)(\alpha-\alpha\beta+\beta)^{-a}=0.$$

These asymptotic behaviors are illustrated on Figure 2 below. The bold lines are for the independent exponential model, with parameter $\mu^1=\frac{1}{20}$, the simple lines are for the conditionally independent model with parameter $a=1$ and $b=20$, in both cases, $\alpha=5$. The violet lines are for the J indicator and the orange ones for the I_2 indicator.

2.1. **Some general results for independent and subexponential distributions.** In this section, we give some results for the asymptotic behavior of the allocation with respect to the violet area, in the case where the two distributions are subexponential and independent. For results on subexponential distributions, we refer to Asmussen [1].

Recall that a distribution is subexponential if it is concentrated on $[0,\infty[$ and its distribution function F satisfies:

$$\frac{\overline{F}^{*2}(x)}{\overline{F}(x)}\overset{x\to\infty}{\longrightarrow}2, \tag{2.4}$$

where F^{*2} is the convolution square, that is, the distribution function of two independent variables with distribution function F. The class of subexponential distributions is denoted by $\mathcal{S}$. In [1] (Chapter IX, Proposition 1.4),

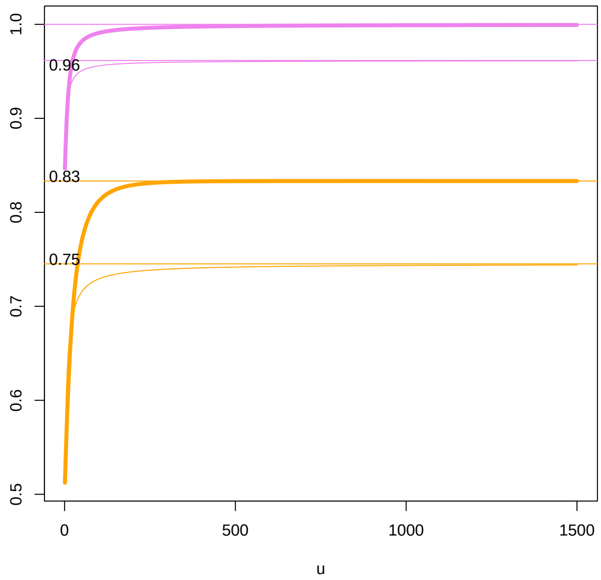


FIGURE 2. Asymptotic behavior for the independent exponential and conditionally independent exponential models

it is proven that regularly varying distributions are subexponential and the following result.

Proposition 2.1 ([1]). *Let $F \in \mathcal{S}$ then for any $y_0 \in \mathbb{R}^+$,*

$$\frac{\overline{F}(x-y)}{\overline{F}(x)} \overset{x\to\infty}{\longrightarrow} 1$$

uniformly in $y \in]0, y_0]$.

We derive the following result from the properties of the subexponential class.

Theorem 2.2. *Let X and Y be two independent random variables, concentrated on $\mathbb{R}^+$. Let F_X be the distribution function of X, $u, v \in \mathbb{R}+$, $v \leq u$. Assume that*

(1) $\frac{v}{u} \to \beta$ *with* $0 < \beta < 1$,
(2) $F_X \in \mathcal{S}$,
(3) for any $t > 0$, $\frac{\overline{F}_X(tu)}{\overline{F}_X(u)} \overset{u\to\infty}{=} O(1)$.

Then

$$\lim_{u\to\infty} \frac{\mathbb{P}(X \geq v,\ X+Y \geq u)}{\overline{F}_X(u)} = 1. \tag{2.5}$$

Proof. We closely follow ideas from [1], Chapter IX. The fact that

$$\liminf_{u\to\infty} \frac{\mathbb{P}(X \geq v,\ X+Y \geq u)}{\overline{F}_X(u)} \geq 1$$

follows from

$$\mathbb{P}(X \geq v,\ X+Y \geq u) \geq \mathbb{P}(X \geq v,\ \max(X,Y) \geq u)$$

and

$$\begin{aligned}\mathbb{P}(X \geq v,\ \max(X,Y) \geq u) &= \mathbb{P}(X \geq u) + \mathbb{P}(X \geq v,\ Y \geq u) - \mathbb{P}(X \geq u,\ Y \geq u)\\ &= \overline{F}_X(u) + \overline{F}_X(v)\overline{F}_Y(u) - \overline{F}_X(u)\overline{F}_Y(u).\end{aligned}$$

Now, we shall prove that

$$\limsup_{u\to\infty} \frac{\mathbb{P}(X \geq v,\ X+Y \geq u)}{\overline{F}_X(u)} \leq 1.$$

Let $w \in \mathbb{R}^+$ be fixed. We have

$$\mathbb{P}(X \geq v,\ X+Y \geq u) = \mathbb{P}(X \geq v, Y \leq w,\ X+Y \geq u) + \mathbb{P}(X \geq v, Y \geq w,\ X+Y \geq u).$$

For u large enough,

$$\begin{aligned}\mathbb{P}(X \geq v, Y \leq w,\ X+Y \geq u) &= \int_0^w F_Y(dy)\overline{F}_X(\max(v, u-y))\\ &= \int_0^w F_Y(dy)\overline{F}_X(u-y).\end{aligned}$$

Now, Proposition 2.1 gives

$$\lim_{u\to\infty} \frac{\mathbb{P}(X \geq v, Y \leq w,\ X+Y \geq u)}{\overline{F}_X(u)} = F_Y(w).$$

On the other hand,

$$\begin{aligned}\mathbb{P}(X \geq v, Y \geq w,\ X+Y \geq u) &= \int_w^{u-v} F_Y(dy)\overline{F}_X(u-y)\\ &\quad + \int_{u-v}^{\infty} F_Y(dy)\overline{F}_X(v)\\ &\leq \overline{F}_X(u-w)\overline{F}_Y(w) + \overline{F}_X(v)\overline{F}_Y(u-v).\end{aligned}$$

So that

$$\limsup_{u\to\infty} \frac{\mathbb{P}(X \geq v, Y \geq w,\ X+Y \geq u)}{\overline{F}_X(u)} \leq \overline{F}_Y(w).$$

□

We deduce the following result for independent GPD.

Corollary 2.3. *If X^1 and X^2 are like in Section 1.4, then the unique solution to (2.1) satisfies*

$$\beta(u) \overset{u\to\infty}{\longrightarrow} 1$$

where $\beta(u) = \frac{u^1}{u}$.

Proof. We apply Theorem 2.2 to X^1 and X^2 independent GPD with respective parameters $(\frac{1}{a}, \frac{b}{a})$ and $(\frac{1}{a}, \frac{b}{\alpha a})$, $a, b > 0$, $\alpha > 1$. We consider the quantities

$$\ell_1 = \frac{\mathbb{P}(X^1 \geq u^1,\ X^1 + X^2 \geq u)}{\mathbb{P}(X^1 \geq u^1)} \text{ and } \ell_2 = \frac{\mathbb{P}(X^2 \geq u^2,\ X^1 + X^2 \geq u)}{\mathbb{P}(X^1 \geq u^1)}.$$

If $u^1 + u^2 = u$ and (2.1) is satisfied then $\ell_1 = \ell_2$.
If $\beta(u) = \frac{u^1}{u}$ is bounded away from 0 and 1, then up to considering a sequence of real numbers u going to infinity, we may assume that

$$\frac{u^1}{u} \to \beta \text{ with } 0 < \beta < 1.$$

In that case, applying Theorem 2.2 gives

$$\lim_{u \to \infty} \ell_1 = \beta^a \text{ and } \lim_{u \to \infty} \ell_2 = \left(\frac{\beta}{\alpha}\right)^a$$

which is contradictory with the fact that $\ell_1 = \ell_2$.
So that, up to considering a sequence of real numbers u going to infinity, we have that either $\frac{u^1}{u} \to 0$ or $\frac{u^1}{u} \to 1$.
Let us assume that $\frac{u^1}{u} \to 0$. Following the lines of the proof of Theorem 2.2, we prove that

$$\lim_{u \to \infty} \frac{\mathbb{P}(X^1 \geq u^1,\ X^1 + X^2 \geq u)}{\overline{F}_{X^1}(u)} = 1 + \frac{1}{\alpha^a}$$

and

$$\lim_{u \to \infty} \frac{\mathbb{P}(X^2 \geq u^2,\ X^1 + X^2 \geq u)}{\overline{F}_{X^1}(u)} = \frac{1}{\alpha^a}.$$

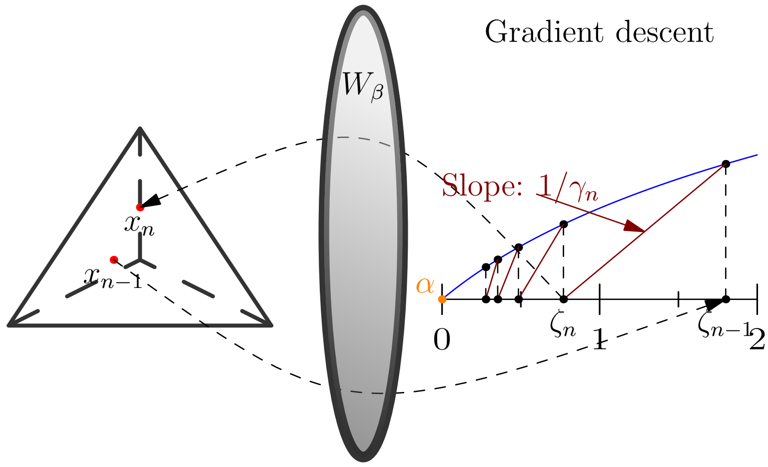
This is in contradiction with the fact that the two above expressions are equal. We conclude that $\beta(u) \to 1$ as $u \to \infty$. □

2.2. **Summary of comparisons.** Below is summarized the asymptotic behavior for the three models: (M1) independent exponential, (M2) independent GPD, (M3) conditionally independent exponential and the two indicators (I) *orange area*, (J) *violet area*. The asymptotic of β is expressed with respect to α and to the parameters of the model.

	M1	M2	M3
I	$\frac{\alpha}{\alpha+1}$	$\frac{\alpha}{\alpha+1}$	$(\alpha-1)\beta^{-a} - \alpha + (\alpha+1)(\alpha - \alpha\beta + \beta)^{-a} - (\alpha - 1)(\alpha(1-\beta))^{-a} - \alpha^{-a} = 0$
J	1	1	$\alpha + \alpha^{-a} - (\alpha+1)(\alpha - \alpha\beta + \beta)^{-a} = 0$

The algorithm given in [4] can be easily adapted to the estimation of allocation minimizing the *violet area*. The keystone for the proof of convergence of this algorithm relies in the strong convexity of the auxiliary function used in the mirror algorithm, in the properties of the Legendre transform and in martingales techniques. The *violet area* as well as the *stopped orange area* satisfy the hypothesis of convergence of the algorithm (in [4], it has been verified for the *orange area*). The stochastic algorithm is developed in order to estimate the optimal solution $(u^1, \ldots, u^d)$ under the constraint $u^1 + \cdots + u^d = u$. This algorithm is *distribution free* and is efficient even when d is quite large ($d = 30$ or $d = 50$). This is a Kiefer-Wolfowitz version of the stochastic mirror algorithm.

2.3. **Rapid description of the algorithm.** The procedure may be summarized in the following picture:



The algorithm constructs two random sequences:

- ▷ (χ_n) in $C = \{x \in \mathbb{R}^d,\ x^i \geq 0,\ x^1 + \cdots + x^d = u\}$,
- ▷ (ξ_n) in the dual space E^* of $\mathbb{R}^d$

Algorithm 2.4.

- ▷ *Initialization:* $\xi_0 = 0 \in E^*$, $\chi_0 \in C$
- ▷ *Update: for* $n = 1$ *to* N *do*
  - $\xi_n = \xi_{n-1} - \gamma_n \psi(\chi_{n-1})$
  - $\chi_n = \nabla W_{\beta_n}(\xi_n)$
- ▷ *Output:*

$$S^N = \frac{\sum_{n=1}^N \gamma_n \chi_{n-1}}{\sum_{n=1}^N \gamma_n}$$

where ψ is the discrete gradient of the vector $\mathcal{I} \in \mathbb{R}^d$ whose kth coordinate is

$$\mathcal{I}_k = \left(\sum_{j=1}^p g_k(R_j^k) \mathbb{1}_{\{R_j^k < 0\}} \mathbb{1}_{\{\sum_{i=1}^d R_j^i \geq 0\}} \right)$$

and respectively of the vector $\mathcal{J} \in \mathbb{R}^d$ whose kth coordinate is

$$\mathcal{J}_k = \left(\sum_{j=1}^p g_k(R_j^k) \mathbb{1}_{\{R_j^k < 0\}} \mathbb{1}_{\{\sum_{i=1}^d R_j^i \leq 0\}} \right),$$

W stands for an auxiliary function whose gradient maps E^* into C (see [4] for details as well as the proof of the consistency of the estimator S^N of the argmin of I_1, I_2 or J).

3. Simulations for $p = 1$, $d = 2$

We shall perform simulations for $p = 1$ and $d = 2$ for the models described above (exponential independent = IE, conditionally exponential= CIE, independent GPD = IGPD). Recall that the IGPD and CIE models have the same margins. For the IE and CIE models, we compare the simulated results with the theoretical ones. The simulations are done for the *non crisis indicator* (*orange area*) as well as for the *crisis indicator* (violet area). Then, we have performed simulations in higher dimension ($d > 2$).

3.1. A comparison of our three models.

We have chosen $\mu^1 = \frac{1}{20}$ for the IE model, $a = 3$ and $b = 60$ for the CIE model and $\xi = \frac{1}{a}$, $\sigma = \frac{b}{a}$ for the IGPD model. In order to get the estimation of the minimum, we have performed 10 times the stochastic algorithm on data of length 15 000. The mean and the standard deviation over the 10 estimations are given below. For $\alpha = 5$ and $\alpha = 10$, we have taken $u = 50$, we compare with the theoretical value using the relative mean squared error (rmse) and the mean squared error (mse) for the IE and CIE models. Recall that if $\widehat{u}^i_k$ is the estimated value of u^i on the k^{th} sample, then

$$rmse(u^i) = \frac{1}{k}\sum_{j=1}^{k}\left(\frac{\widehat{u}^i_k - u^i}{u^i}\right)^2,$$

and

$$mse(u^i) = \frac{1}{k}\sum_{j=1}^{k}\left(\widehat{u}^i_k - u^i\right)^2.$$

3.2. The indicator I_2: the *orange area*.

	$\alpha = 5$								
	IE model			CIE model			IGPD model		
	$\widehat{u}^1$	$\widehat{u}^2$	β	$\widehat{u}^1$	$\widehat{u}^2$	β	$\widehat{u}^1$	$\widehat{u}^2$	β
mean	38.37	11.63	0.767	36.8	13.2	0.735	35.83	14.17	0.717
sd dev	0.085	0.085	...	0.115	0.115	...	0.133	0.133	...
th.	38.46	11.54	0.769	36.84	13.16	0.737	non available		
$\sqrt{\text{rmse}}$	0.0032	0.0105	...	0.0032	0.009	...	non available		
$\sqrt{\text{mse}}$	0.121	0.121	...	0.119	0.119	...	non available		
	$\alpha = 10$								
mean	42.64	7.36	0.852	40.95	9.05	0.819	40.05	9.48	0.801
sd dev	0.161	0.161	...	0.138	0.138	...	0.164	0.164	...
th.	42.96	7.04	0.859	41.22	8.78	0.824	non available		
$\sqrt{\text{rmse}}$	0.0085	0.0516	...	0.0073	0.0342	...	non available		
$\sqrt{\text{mse}}$	0.364	0.363	...	0.301	0.3	...	non available		

This table shows that, for the three models, the more the first branch is risky, the more we allocate capital to it. Also, we recall that, asymptotically, the capital allocated to the riskiest branch for the IGPD model is higher than for the CIE model (see Figure 2) . In the table above, the values of u are (relatively) small, we conclude that the asymptotic has not been reached.

3.3. **The indicator J: the *violet area*.**

	$\alpha = 5$								
	IE model			CIE model			IGPD model		
	$\widehat{u}^1$	$\widehat{u}^2$	β	$\widehat{u}^1$	$\widehat{u}^2$	β	$\widehat{u}^1$	$\widehat{u}^2$	β
mean	46.83	3.17	0.94	47.18	2.81	0.94	48.13	1.87	0.96
sd dev	0.35	0.35	...	0.136	0.136	...	0.114	0.114	...
th.	49.08	0.92	0.98	48.36	1.64	0.967	non available		
$\sqrt{\text{rmse}}$	0.046	2.49	...	0.025	0.722	...	non available		
$\sqrt{\text{mse}}$	2.28	2.28	...	1.19	1.19	...	non available		
	$\alpha = 10$								
mean	47.62	2.38	0.95	48.68	1.32	0.819	49.16	0.84	0.98
sd dev	0.43	0.43	...	0.083	0.083	...	0.063	0.063	...
th.	49.77	0.23	0.995	49.6	0.39	0.99	non available		
$\sqrt{\text{rmse}}$	0.044	9.7	...	0.019	2.32	...	non available		
$\sqrt{\text{mse}}$	2.19	2.19	...	0.92	0.92	...	non available		

The fact that the *rmse* and *mse* are larger than in the case of the *orange area* is due to the fact that less data satisfy the condition $\sum_{k=1}^{d} R_j^k \leq 0$ and thus the estimation is less accurate.

4. Large number of lines of business

We consider 10 lines, with different situations. We begin with some generalization of the previous models and then propose a model for which we have a block of 5 correlated lines of business, and a block of lines which are mutually independent.

4.1. **Generalizations of the previous models in dimension 10.**

- Independent exponential: $X_1^1 \rightsquigarrow \mathcal{E}(\mu^1)$, for $i = 2, \ldots, 7$, $X_1^i \rightsquigarrow \mathcal{E}(\alpha^1 \mu^1)$ and for $i = 8, \ldots, 10$, $X_1^i \rightsquigarrow \mathcal{E}(\alpha^2 \mu^1)$.
- Conditional exponential: $X_1^1 \rightsquigarrow \mathcal{E}(\Theta)$, for $i = 2, \ldots, 7$, $X_1^i \rightsquigarrow \mathcal{E}(\alpha^1 \Theta)$ and for $i = 8, \ldots, 10$, $X_1^i \rightsquigarrow \mathcal{E}(\alpha^2 \Theta)$, where $\Theta \rightsquigarrow \Gamma(\alpha, b)$.
- Independent GPD: $X_1^1 \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{a})$, for $i = 2, \ldots, 7$, $X_1^i \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{\alpha_1 a})$ and for $i = 8, \ldots, 10$, $X_1^i \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{\alpha_2 a})$.

The last two models have the same margins. We have chosen $\alpha^1 = 5$ and $\alpha^2 = 8$ and $u = 80$. We have performed our stochastic algorithm 10 times on data sets of length 20 000 for the *orange area*.

	IE			**CIE**			**IGPD**		
	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$
u^1	27.04	0.095	0.338	25.68	0.194	0.321	22.38	0.178	0.28
u^2	6.7	0.031	0.084	6.85	0.05	0.086	7.17	0.044	0.09
u^3	6.69	0.039	0.084	6.83	0.083	0.085	7.26	0.06	0.091
u^4	6.69	0.037	0.084	6.87	0.054	0.086	7.24	0.054	0.09
u^5	6.7	0.051	0.084	6.88	0.079	0.086	7.18	0.061	0.09
u^6	6.71	0.078	0.084	6.85	0.066	0.086	7.21	0.087	0.09
u^7	6.72	0.021	0.084	6.86	0.063	0.086	7.19	0.081	0.09
u^8	4.23	0.026	0.053	4.41	0.028	0.055	4.79	0.034	0.06
u^9	4.26	0.032	0.053	4.4	0.057	0.055	4.77	0.059	0.06
u^{10}	4.26	0.028	0.053	4.38	0.057	0.055	4.81	0.036	0.06

The same observations as in dimension 2 can be done. Moreover unsurprisingly, the allocation is identical for the branches 2 to 7 and for the branches 8 to 10.

4.2. **A block of correlated GPD and a block of independent GDP, in dimension** 10**.** In this section, we consider a block of 5 conditionally exponential variables and an independent block of 5 independent GPD variables, with the same margins as the 5 correlated GPD variables of the first block. We refer this model to the mixed model (MM).

- Conditional exponential: $X_1^1 \rightsquigarrow \mathcal{E}(\Theta)$ and for $i = 2, \dots, 5$, $X_1^i \rightsquigarrow \mathcal{E}(\alpha^1\Theta)$, where $\Theta \rightsquigarrow \Gamma(\alpha, b)$.
- Independent GPD: $X_1^1 \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{a})$ and for $i = 6, \dots, 10$, $X_1^i \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{\alpha^1 a})$.

The MM model has to be compared with the IGPD and CIE models with the same margins. We have chosen $\alpha^1 = 5$ and $u = 80$. We have performed our stochastic algorithm 10 times on data sets of length 20 000.

	MM			**CIE**			**IGPD**		
	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$
u^1	20.56	0.15	0.257	21.48	0.23	0.269	18.42	0.219	0.23
u^2	5.78	0.043	0.072	6.49	0.123	0.081	6.88	0.138	0.086
u^3	5.77	0.05	0.072	6.52	0.133	0.082	6.88	0.151	0.086
u^4	5.8	0.059	0.072	6.49	0.116	0.081	6.82	0.16	0.085
u^5	5.79	0.052	0.072	6.52	0.116	0.082	6.82	0.132	0.085
u^6	7.25	0.059	0.091	6.5	0.11	0.081	6.81	0.113	0.085
u^7	7.25	0.071	0.091	6.5	0.085	0.081	6.89	0.154	0.086
u^8	7.31	0.071	0.091	6.5	0.13	0.081	6.82	0.14	0.085
u^9	7.25	0.066	0.091	6.49	0.121	0.081	6.82	0.143	0.085
u^{10}	7.26	0.078	0.091	6.49	0.121	0.081	6.83	0.153	0.085

This example illustrates the fact that for the expected *orange area*, lines correlated to the most dangerous one receive less capital that lines with the same marginal distributions but independent from the riskiest line. This is in accordance with the basic example presented in the introduction. If line 1 is insolvent, then it is likely that the group is ruined (the aggregate reserves are likely to be negative). Consequently, a greater part of the insolvencies of lines 2 to 5 than the ones of lines 6 to 10 will not contribute to the *orange area* because the sum of reserves is negative. Once again, the *orange area* should not be used for economic capital allocation, as it encourages to take additional risks that are correlated to the main source of risk. Logically, the effect is reversed if one uses the expected *violet area*. In that case, a greater part of insolvencies of lines 6 to 10 will not contribute, and consequently the required capital for lines 2 to 5 will be higher than for lines 6 to 10, as in the TVaR allocation principle.

Below are the results for the same models as above, in the case of the *violet area* (indicator J). We have performed 10 simulations of length 23 000.

	MM			CIE			IGPD		
	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$
u^1	41.4	0.23	0.52	37.68	0.31	0.47	41.99	0.254	0.52
u^2	5.2	0.07	0.065	4.68	0.068	0.058	4.19	0.044	0.052
u^3	5.19	0.08	0.065	4.72	0.076	0.059	4.23	0.046	0.053
u^4	5.18	0.07	0.065	4.67	0.069	0.058	4.2	0.073	0.053
u^5	5.16	0.07	0.065	4.73	0.063	0.059	4.2	0.094	0.052
u^6	3.57	0.08	0.045	4.68	0.053	0.059	4.22	0.048	0.053
u^7	3.56	0.06	0.044	4.72	0.094	0.059	4.23	0.069	0.053
u^8	3.58	0.05	0.045	4.69	0.046	0.058	4.24	0.07	0.053
u^9	3.56	0.05	0.045	4.69	0.04	0.059	4.25	0.064	0.053
u^{10}	3.59	0.06	0.045	4.72	0.069	0.059	4.24	0.051	0.053

As expected, the optimal allocation with respect to the *violet area* leads to penalize all the branches correlated to the more risky one; the independent branches are less penalized.

5. Time horizon $p > 1$, d lines of business

We finish our simulation series with several multi-periodic simulations. We shall consider $p = 2$, $p = 3$, $p = 4$ and $d = 3$ lines of business. We have performed simulation only for models with independence in time. As already mentioned, for multi-periodic cases, the premium c has to be taken into account. We fix it for each branch as 5% of the expectation of the branch. We have performed 10 simulations of length 15 000, for our three models, with $\alpha = 5$, $u = 30$, $a = 3$, $b = 60$. The simulations are done for the *orange area* (indicator I_2) then for the *orange area* with the stopping time (indicator I_1) and finally for the *violet area* (indicator J). It is expected that the behavior of the stopped and the non stopped *orange area* are quite similar because in both cases, the indicator does not take into the event which lead to global insolvency.

5.1. The *orange area*, the I_2 indicator.

	IE			CIE			IGPD		
	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$
					$p = 2$				
u^1	17.06	0.11	0.586	13.67	0.26	0.456	11.87	0.19	0.396
u^2	6.49	0.1	0.216	8.16	0.13	0.272	9.1	0.22	0.303
u^3	6.45	0.07	0.215	8.16	0.24	0.272	9.03	0.18	0.301
					$p = 3$				
u^1	16.5	0.1	0.55	12.85	0.18	0.428	10.52	0.22	0.351
u^2	6.75	0.08	0.225	8.56	0.15	0.285	9.76	0.15	0.325
u^3	6.74	0.11	0.225	8.58	0.1	0.286	9.72	0.15	0.324
					$p = 4$				
u^1	15.99	0.12	0.533	11.8	0.16	0.393	9.48	0.16	0.316
u^2	7.01	0.08	0.234	9.1	0.12	0.303	10.27	0.16	0.342
u^3	6.99	0.09	0.233	9.1	0.08	0.303	10.25	0.11	0.342

These multi-periodic simulations show the same kind of behavior than for $p = 1$. Nevertheless, we remark that for the IGPD model, with $p = 4$, the more risky branch becomes less allocated. This is a quite surprising result which may be explained by the fact that $u = 30$ in this context is quite small (recall that the premiums are added at each time $n = 1, \ldots, p$). For larger values of u, this phenomenon does not appear anymore as it can be seen below ($u = 80$).

	IE			**CIE**			**IGPD**		
	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$
u^1	53.77	0.17	0.672	46.02	0.24	0.575	43.98	0.29	0.55
u^2	13.12	0.12	0.164	16.99	0.11	0.212	18.03	0.26	0.225
u^3	13.11	0.09	0.164	16.98	0.23	0.212	17.99	0.21	0.225

As in dimension 10, the case of mixed models is more interesting. We have done a last simulation with $p = 4$, $d = 5$, $u = 80$. The first 3 lines are conditionally exponential and the last two lines are independent GPD (parameters are the same as above).

	mean	sd dev.	$\frac{u^i}{u}$
u^1	31.33	0.36	0.392
u^2	10.63	0.16	0.133
u^3	10.6	0.24	0.133
u^4	13.74	0.27	0.172
u^5	13.7	0.27	0.171

It has to be compared with the correlated Pareto and independent GPD models with the same margin. This is summarized below.

	IE		**CIE**		**IGPD**	
	mean	sd dev.	mean	sd dev.	mean	sd dev.
u^1	41.1	0.23	32.68	0.28	30.1	0.31
u^2	9.69	0.09	11.86	0.14	12.49	0.17
u^3	9.71	0.08	11.78	0.24	12.51	0.23
u^4	9.78	0.12	11.83	0.15	12.4	0.14
u^5	9.72	0.09	11.85	0.12	12.5	0.2

5.2. **The stopped *orange area*, the I_1 indicator.** If there is only one period ($p = 1$), there is no difference between the I_2 and the I_1 indicators. In order to compare the optimization with the I_1 indicator with respect to the I_2, we have performed simulation with $p = 5$ and $d = 5$ on the same models as above (independent exponential, conditionally independent exponential, independent GPD, mixed GPD).

	IE		**CIE**		**IGPD**		**MM**	
	mean	sd dev.	mean	sd dev.	mean	sd dev.	mean	sd dev.
u^1	41.13	0.22	32.9	0.3	30.12	0.22	27.34	0.38
u^2	9.73	0.1	11.79	0.15	12.44	0.14	11.25	0.22
u^3	9.72	0.08	11.79	0.15	12.51	0.21	11.15	0.15
u^4	9.72	0.1	11.79	0.19	12.52	0.2	15.1	0.26
u^5	9.71	0.12	11.72	0.19	12.42	0.17	15.16	0.23

We remark that the only notable difference is observed for the mixed model.

5.3. **The *violet area*, the J indicator.** As already noticed, for the *violet area*, more data is needed because for less of the aggregate loss go over u. The simulation below are done on data of length 16 000, 10 times.

	IE			**CIE**			**IGPD**		
	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$
	$p=3$								
u^1	45.35	2.91	0.567	70.2	1.2	0.877	77.81	0.7	0.972
u^2	8.54	0.82	0.107	2.42	0.34	0.03	0.53	0.19	0.007
u^3	8.72	0.59	0.109	2.52	0.39	0.032	0.56	0.21	0.007
u^4	8.65	0.92	0.108	2.34	0.29	0.029	0.54	0.19	0.007
u^5	8.74	0.68	0.109	2.52	0.35	0.032	0.56	0.17	0.007
	$p=4$								
u^1	62.63	2.37	0.78	73.63	0.326	0.92	79.6	0.157	0.995
u^2	4.31	0.6	0.053	1.62	0.119	0.02	0.1	0.041	0.001
u^3	4.34	0.6	0.054	1.52	0.194	0.019	0.11	0.042	0.001
u^4	4.41	0.68	0.055	1.58	0.327	0.02	0.09	0.033	0.001
u^5	4.3	0.59	0.054	1.65	0.203	0.021	0.11	0.045	0.002
	$p=5$								
u^1	55.12	3.26	0.689	71.61	0.79	0.895	78.41	0.52	0.98
u^2	6.23	0.83	0.078	2.06	0.3	0.026	0.37	0.12	0.005
u^3	6.29	0.95	0.079	1.96	0.41	0.024	0.41	0.15	0.005
u^4	6.18	0.91	0.077	2.24	0.24	0.028	0.42	0.17	0.005
u^5	6.18	0.89	0.077	2.13	0.17	0.027	0.4	0.14	0.005

We have also performed simulations for the mixed model (the first 3 lines are conditionally exponential and the last two lines are independent GPD with the parameters are the same as above).

	$p=3$			$p=4$			$p=5$		
	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$
u^1	74.74	0.76	0.934	77.64	0.24	0.971	76.3	0.85	0.954
u^2	2.32	0.34	0.029	1.11	0.11	0.014	1.74	0.36	0.022
u^3	2.18	0.47	0.027	1.1	0.23	0.014	1.56	0.49	0.019
u^4	0.38	0.12	0.005	0.066	0.02	0.001	0.21	0.07	0.003
u^5	0.37	0.09	0.005	0.072	0.02	0.001	0.2	0.08	0.003

6. Conclusion

In this article, we have presented some results for the allocation of capital with respect to the minimization of some risk indicators. In dimension $d = 2$ for one period ($p = 1$), explicit formulas and some asymptotics are given for specific models, a simulation study is done, it allows to benchmark the parameters of the optimization algorithm. Then, simulations are done in higher dimension and for several periods ($p > 1$). It emphasizes the fact that minimization driven allocations require to chose the right risk indicator. As an example, we would recommend to use the orange area for free capital allocation, while the violet area is more consistent for economic capital allocation.

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