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Abstract

We consider the feedback stabilization of a simplified 1d model for a fluid–structure interaction system. The fluid equation is the viscous Burgers equation whereas the motion of the particle is given by the Newton's laws. We stabilize this system around a stationary state by using feedbacks located at the exterior boundary of the fluid domain. With one input, we obtain a local stabilizability of the system with an exponential decay rate of order $\sigma < \sigma_0$. An arbitrary order for the exponential decay rate can be proved if a unique continuation result holds true or if two inputs are used to stabilize the system. Our method is based on general arguments for stabilization of nonlinear parabolic systems combined with a change of variables to handle the fact that the fluid domains of the stationary state and of the stabilized solution are different.

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1 Introduction and main result

This article is devoted to the study of the feedback stabilization of a 1d fluid–structure system. This system is a simplified model corresponding to the motion of a rigid body into a viscous incompressible fluid (see [24, 10, 11, 9, 19, 25, 15, 17] for some references). In our case, we replace the Navier–Stokes system by the viscous Burgers equation and the rigid body is reduced to a point particle. To obtain the equation for the particle, we apply the Newton's law by distinguishing the external forces and the force coming from the fluid. This latter force is expressed through the Cauchy tensor when considering the Navier–Stokes equation for the fluid. Here it is written through the jump of the spatial derivative of the velocity of the fluid at the position of the particle. More precisely, the system we consider here can be written as

$$\begin{cases} v_t - v_{xx} + v v_x = f^S \qquad (t \geqslant 0, \quad x \in (-1,1) \setminus \{h(t)\}), \\ v(t, h(t)) = \dot{h}(t), \\ m\ddot{h}(t) = [v_x](t, h(t)) + m\ell^S \qquad (t \geqslant 0), \end{cases} \tag{1}$$

with the boundary conditions

$$v(t,-1) = a^S + u(t), \quad v(t,1) = b^S, \qquad (t \geqslant 0), \tag{2}$$

and with the initial conditions:

$$h(0) = h^0, \quad \dot{h}(0) = \ell^0, \quad v(0,x) = v^0(x) \quad x \in (-1,1) \setminus \{h^0\}. \tag{3}$$

Here $v = v(t,x)$ is the velocity of the fluid and $h = h(t)$ is the position of the particle. We assume that at $x = h(t)$, the velocity of the fluid and the velocity of the particle $\dot{h}$ are equal. This system was introduced

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by Vázquez and Zuazua in [26], where the well-posedness and the large time behavior have been considered (with $(-1,1)$ replaced by $\mathbb{R}$). They also consider the case of several particles in [27] with a result of no collisions between the particles. In (1)–(3), the external forces $f^S = f^S(x)$, ℓ^S, and the boundary conditions a^S, b^S are given and are independent of time. They correspond to a stationary state

$$\left\{\begin{array}{l} -V^S_{yy} + V^S V^S_y = f^S \qquad (y \in (-1,1) \setminus \{H^s\}), \\ V^S(-1) = a^S, \quad V^S(H^S) = 0, \quad V^S(1) = b^S, \\ \qquad\qquad 0 = [V^S_y](H^S) + m\ell^S. \end{array}\right. \tag{4}$$

In the above equations, V^S is the constant velocity of the fluid and, since we consider a stationary solution, we have imposed that the velocity of the particle is 0, so that the particle remains located in H^S. It is worth noting that for every forces f^S (for the fluid) and ℓ^S (for the particle), and for every boundary conditions (a^S, b^S) there are usually no solutions of (4). However, it is possible to construct families of stationary solutions. We discuss this problem in Section 2.

Our aim is to use in (1)–(3) the control $u = u(t)$ at $x = -1$ in order to "reach" the stationary state $(V^S, 0, H^S)$ as $t \to \infty$. More precisely, the control u is searched as a feedback depending on the difference between $(v, \dot{h}, h)$ and $(V^S, 0, H^S)$, so that

$$\|v(t) - V^S\|_{L^2((-1,1))} + |\dot{h}(t)| + |h(t) - H^S| \leqslant Ce^{-\sigma t}\left(\|v^0 - V^S\|_{L^2((-1,1))} + |\ell^0| + |h^0 - H^S|\right) \tag{5}$$

where $\sigma > 0$ is imposed and C is independent of the initial condition (v^0, ℓ^0, h^0). The precise statement of the main result is given in Theorem 2.

Let us point out that the controllability of this simplified 1d fluid–structure systems has already been considered. In the case of exact null controllability, the first result on this system was obtained by Doubova and Fernández-Cara by using two controls, at $x = -1$ and at $x = 1$ (see [12]). Their method is based on Carleman estimates at the left and at the right of the particle. To remove one control on this problem, Liu, Takahashi and Tucsnak developed a new method for the null-controllability in presence of source terms, combined with a spectral method for the linear system (see [22]). Let us note that the two above articles consider only the case $V^S = 0$. To obtain the exact controllability to stationary states V^S (or to trajectories), the method of [22] could be difficult to apply since it is based on a spectral study. The method in [12] may be adapted to obtain such results but in that case, one would need two controls, at $x = -1$ and at $x = 1$. This necessity of two controls is proper to the 1d case, and is a consequence of the fact that the fluid domain is not connected. Therefore, if we only use one input to control or stabilize such a system, it can be seen as a control problem of a coupled system (two Burgers equations and the ODE for the particle) with a control on only one of these equations. In particular, one of the fluid equations is only controlled by the motion of the particle. Here for our stabilization problem, if we use only one input then we obtain (5) for $\sigma < \sigma_0$, $\sigma_0 > 0$. More precisely, to stabilize with a decay rate of order $\sigma > 0$, the following condition has to be satisfied

$$\left\{\begin{array}{l} V^S_y(H^S)\varphi - \varphi_{yy} - V^S\varphi_y = 0, \quad y \in (H^S, 1) \\ \varphi(H^S) = \varphi(1) = 0 \end{array}\right. \qquad \Longrightarrow \qquad \varphi \equiv 0 \quad \text{or} \quad V^S_y(H^S) < -\sigma. \tag{6}$$

We show that if there exists a non trivial solution of the above system, then $V^S_y(H^S) < 0$, and thus we can always stabilize this system with only one input. However, we also prove that the above condition may be false for some σ, and in that case, to obtain a stabilization result with a decay rate of order $-\sigma$, we need two feedback controls. This means that we would have to change the second boundary condition of (2) by

$$v(t,1) = \tilde{u}(t) + b^S,$$

with $\tilde{u}$ a feedback depending on the difference between $(v, \dot{h}, h)$ and $(V^S, 0, H^S)$. In Section 2, we discuss about condition (6), but let us already notice that in the case of [12] or [22], i.e. $V^S \equiv 0$, this condition is trivially satisfied for all $\sigma > 0$. Let us also note that in dimension 2 or 3, for a rigid body immersed into a viscous incompressible fluid (and for a fluid domain that is connected), this problem disappear both for the controllability problem (see [20], [6], [5]) and for the stabilization problem (work in progress).

In order to state our main result, we first need to give a precise definition of weak solutions for the system (1), (2) (3). For that, we denote by $L^2_{\mathrm{loc}}(\mathcal{X})$ [resp. $L^\infty_{\mathrm{loc}}(\mathcal{X})$] the space of functions that belong to $L^2(0,T;\mathcal{X})$

[resp. $L^\infty(0,T;\mathcal{X})$] for all $0<T<+\infty$, and we set for $h\in(-1,1)$:

$$\mathcal{V}^h=\left\{\begin{bmatrix}\phi\\ g\\ k\end{bmatrix}\in H^1_0((-1,1))\times\mathbb{R}^2\ ;\ \phi(h)=g\right\}.$$

Definition 1. *The pair*

$$(v,h)\in L^\infty_{\mathrm{loc}}(L^2((-1,1)))\cap L^2_{\mathrm{loc}}(H^1((-1,1)))\times C^1((0,\infty);\mathbb{R})$$

is a weak solution of (1), (2), (3), *if* (2) *holds in the trace sense, if* $h(0)=h^0$, *if* $v(t,h(t))=\dot{h}$, *and if for any* ${}^t[\phi,g,k]\in C^1([0,\infty);H^1_0((-1,1))\times\mathbb{R}^2)$ *with compact support in time and such that for all* t, ${}^t[\phi(t),g(t),k(t)]\in\mathcal{V}^{h(t)}$,

$$-\int_0^\infty\!\!\int_{-1}^1 v(t,x)\phi_t(t,x)\mathrm{d}x\mathrm{d}t-\int_0^\infty m\dot{h}(t)\dot{g}(t)\mathrm{d}t+\int_0^\infty\!\!\int_{-1}^1\left(v_x\phi_x+vv_x\phi\right)\mathrm{d}x\mathrm{d}t$$
$$=\int_0^\infty\!\!\int_{-1}^1 f^S\phi\,\mathrm{d}x\mathrm{d}t+\int_0^\infty mg(t)\ell^S\mathrm{d}t+\int_{-1}^1 v^0(x)\phi(0,x)\mathrm{d}x+m\ell^0 g(0).\quad(7)$$

Let us notice that, as classical for fluid-structure problems, the test functions in (7) depend on the solutions (on h). We refer the reader to the articles on fluid-structure interaction quoted above for more details.

In the result of stabilization, our feedback control does not depends linearly on the difference of $(v,h,\dot{h})$ and of $(V^S,0,H^S)$. Indeed, one of the main difficulties in this study consists in the fact that the spatial domain is moving. For instance, we see that the velocities of the fluid v and V^S are not defined in the same spatial domains, respectively $(-1,1)\backslash\{H^s\}$ and $(-1,1)\backslash\{h(t)\}$. Moreover the fluid domain in (1)–(3) evolves with the time since the particle is moving. To overcome this difficulty, we consider a change of variables

$$X(t,\cdot):(-1,1)\setminus\{H^s\}\to(-1,1)\setminus\{h(t)\}$$

and our feedback u depends linearly on the difference $v\circ X-V^S$.

A possible construction of such a change of variables (see Section 3) consists in introducing a function $\eta\in C^\infty([-1,1])$ satisfying $\eta(H^S)=1$, $\eta(1)=\eta(-1)=\eta'(H^S)=0$ and to define

$$X(t,y)\stackrel{\mathrm{def}}{=}y+\eta(y)(h(t)-H^S),$$

if

$$\left\|\eta'\right\|_{L^\infty((-1,1))}\left|h(t)-H^S\right|<1\quad\text{for all }t.\quad(8)$$

We are now in position to state our main result that asserts that the system (1)–(3) is locally feedback stabilizable in a neighborhood of a stationary state.

Theorem 2. *Assume* $f^S\in W^{2,\infty}((-1,1))$, *and that* (f^S,ℓ^S,a^S,b^S) *is associated with a stationary solution* $(V^S,0,H^S)$ *of* (4), *with* $V^S\in W^{1,\infty}((-1,1))$. *There exists* $\sigma_0>0$ *such that for all* $\sigma\in(0,\sigma_0)$, *there exist* $\mu,C>0$, *and* $(\widehat{\varphi},\widehat{g},\widehat{k})\in L^2((-1,1))\times\mathbb{R}^2$, *such that, if* $(v^0,\ell^0,h^0)\in L^2((-1,1))\times\mathbb{R}^2$ *satisfies*

$$\|v^0-V^S\|_{L^2((-1,1))}+|\ell^0|+|h^0-H^S|\leqslant\mu\quad(9)$$

then there exists a unique weak solution (v,h) *(in the sense of Definition 1), of* (1), (3) *and of*

$$v(t,-1)=a^S+\int_{-1}^1\Big(v(t,y+\eta(y)(h(t)-H^S))-V^S(y)\Big)\widehat{\varphi}\mathrm{d}y+m\dot{h}(t)\widehat{g}+(h(t)-H^S)\widehat{k},\ v(t,1)=b^S,\quad(t\geqslant0),\quad(10)$$

and satisfying (8). *Moreover* $v\in C([0,+\infty);L^2((-1,1)))$ *and it satisfies*

$$\|v(t)-V^S\|_{L^2((-1,1))}+|\dot{h}(t)|+|h(t)-H^S|\leqslant Ce^{-\sigma t}\left(\|v^0-V^S\|_{L^2((-1,1))}+|\ell^0|+|h^0-H^S|\right).\quad(11)$$

As already explained, in the above result, one may have $\sigma_0<\infty$. We can have $\sigma_0=\infty$ if condition (16) holds true or if we use two inputs.

Corollary 3. *Assume the hypotheses of Theorem 2 and that V^S satisfies one of the two conditions*

$$V_y^S(H^S) \geqslant 0 \tag{12}$$

or

$$\forall y \in (H^S, 1), \quad V_y^S(H^S) + \frac{V_y^S(y)}{2} + \frac{1}{4}(V^S(y))^2 + \left(\frac{\pi}{1-H^S}\right)^2 > 0. \tag{13}$$

Then, we can take $\sigma_0 = \infty$ in Theorem 2.

Remark 4. *Let us give some comments on the above results.*

- *In Theorem 2 the uniqueness of the solutions holds in a stronger sense: any maximal solution*

$$(\tilde{v}, \tilde{h}) \in C(0, T_{\max}; L^2((-1,1))) \cap L^2((0, T_{\max}); H^1((-1,1))) \times C^1(0, T_{\max}; \mathbb{R})$$

  of (1), (3), (10) is a global solution ($T_{\max} = \infty$) and is equal to the solution (v, h) of Theorem 2, see Remark 11 below.

- *Theorem 2 does not deal with the practical construction of the triplet $(\widehat{\varphi}, \widehat{g}, \widehat{k})$. One can compute $(\widehat{\varphi}, \widehat{g}, \widehat{k})$ by solving a finite dimensional Riccati equation as in [1, 23].*

- *If the condition (6) is false for some $\sigma > 0$, then the stabilization result stated in Theorem 2 holds true for this decay rate by changing (10) into*

$$v(t,-1) = a^S + \int_{-1}^{1} \Big(v(t, y + \eta(y)(h(t) - H^S)) - V^S(y)\Big)\, \widehat{\varphi} \mathrm{d}y + m\dot{h}(t)\widehat{g} + (h(t) - H^S)\widehat{k}, \qquad (t \geqslant 0), \tag{14}$$

$$v(t,1) = b^S + \int_{-1}^{1} \Big(v(t, y + \eta(y)(h(t) - H^S)) - V^S(y)\Big)\, \widetilde{\varphi} \mathrm{d}y + m\dot{h}(t)\widetilde{g} + (h(t) - H^S)\widetilde{k}, \qquad (t \geqslant 0), \tag{15}$$

  for two triplets $(\widehat{\varphi}, \widehat{g}, \widehat{k}) \in L^2((-1,1)) \times \mathbb{R}^2$ and $(\widetilde{\varphi}, \widetilde{g}, \widetilde{k}) \in L^2((-1,1)) \times \mathbb{R}^2$. In Corollary 3, we consider a sufficient condition to ensure that (6) holds true for all $\sigma > 0$. Note that this condition is satisfied if V^S is not decreasing too much: if $V_y^S(y) > -2/3(\pi/(1-H^S))^2$ for all $y \in (H^S, 1)$ then (13) holds true.

- *Let us emphasize that in this result, we stabilize around a stationary state which is not necessary equal to zero. To our knowledge, it is one of the first result in this direction.*

The method used to obtain our result of stabilization is a general method based on the stabilization of a linearized system (see, for instance, [1], [2]). Let us remark that in the linear case the stabilization is implied by the approximate controllability. More precisely, the approximate controllability can be obtained through two conditions. The first one is a condition stronger than (6):

$$\left\{ \begin{array}{l} V_y^S(H^S)\varphi - \varphi_{yy} - V^S \varphi_y = 0, \quad y \in (H^S, 1) \\ \varphi(H^S) = \varphi(1) = 0 \end{array} \right. \quad \Longrightarrow \quad \varphi \equiv 0. \tag{16}$$

The second condition is that a family of root vector of the underlying linear operator (see (61)-(62)) is complete. This last property could be deduced by using Keldish's Theorem since the linear operator is a relatively bounded perturbation of a self-adjoint operator (see [2]).

Let us point out that the method used in this paper to obtain the stabilization around a stationary state does not apply if we want to stabilize around a non stationary state. To tackle this question a starting point could be the interesting work [3] of V. Barbu, S. S. Rodrigues and A. Shirikyan concerning the stabilization of the Navier-Stokes equations (without structure interaction) around a non-stationary target state by mean of a time dependent internal feedback law. Let us also note that since we obtain the stabilization by using a linearization process, it leads to a local result for the nonlinear system. There exist few results of global stabilization or of global controllability for such systems. For the Burgers equations alone, without any structure, it is shown in [14] and in [18] that the global controllability does not hold with one or two boundary controls or with an interior control. By using two boundary controls and a particular interior control, Chapouly [7] proves that the Burgers equation is globally controllable. For the system considered here, with a particle, the corresponding results are open. In fact, even the approximate controllability of (1) is an open question.

We underline that in (1) the viscosity of the fluid is assumed to be equal to 1 for simplicity, but there is no difficulty to extend our stabilization result for (1) with terms $(\nu v_x)_x$, $[\nu v_x]$ instead of v_{xx}, $[v_x]$, where ν is a smooth positive function of $x \in (-1,1)$. However, in the case of a viscosity depending on the nonhomogeneous density ρ of the fluid, $\nu \stackrel{\text{def}}{=} \nu(\rho)$, the stabilizability is far more difficult to study. This comes from the fact that the corresponding 1-D fluid-solid interaction system satisfied by (v,ρ) is coupling a parabolic equation for the velocity and a hyperbolic equation for the density and, even for the linearized equations, there are few tools developed to obtain the stabilization of such systems. About the controllability and the stabilizability of such a type of system, but without solid interaction, we shall mention [13, 8].

The outline of the paper is the following. In Section 2 we first discuss the existence of solutions for (4) and on condition (6). In Section 3, we construct a change of variables to reduce the problem (1)–(3) to a cylindrical domain. The Section 4 is devoted to show that the system obtained after the change of variables satisfies the hypotheses of a general framework for feedback stabilization of nonlinear parabolic systems. Finally, Section 5 is devoted to the proof of the main result. We end this paper by giving in Section 6 some generalizations in the case of several particles.

2 Remarks on stationary states and on condition (6)

2.1 Existence of stationary solutions

As explained in the introduction, in general there are no solution to (4). Indeed the quadruplet (f^S, ℓ^S, a^S, b^S) has to satisfy some hypotheses. This can be seen as follows: if we consider for instance $f^S \equiv 0$ and $\ell^S = 0$, then we can see that (4) reduces to

$$\left\{\begin{array}{c} -V^S_{yy} + V^S V^S_y = 0 \qquad (y \in (-1,1)), \\ V^S(H^S) = 0, \\ V^S(-1) = a^S, \quad V^S(1) = b^S. \end{array}\right. \tag{17}$$

Then it is easy to see that the solution of the first two equations is

$$V^S(y) = 2C\tan(C(y-H^S)) \quad \text{or} \quad V^S(y) = -2C\tanh(C(y-H^S)) \quad \text{or} \quad V^S \equiv 0$$

where C is a positive constant to be determined. One has $V^S_y(H^S) = 2C^2$ or $-2C^2$ or 0 depending on the case. One can notice that each of these functions is strictly monotone and in particular, to have a solution to (17), we need that $a^S b^S \leqslant 0$.

In the particular case when $a^S = -b^S$, we see that $H^S = 0$ and that there always exists a solution. Indeed, if $a^S = 0$, then $V^S \equiv 0$. Else if $a^S < 0$ or equivalently $b^S > 0$, there exists a unique $C \in (0, \pi/2)$ such that $2C\tan(C) = b^S$. In the last case, if $a^S > 0$ or equivalently $b^S < 0$, there exists a unique $C > 0$ such that $2C\tanh(C) = a^S$. We have proved the following

Lemma 5. *For any $a^S \in \mathbb{R}$, the problem*

$$\left\{\begin{array}{c} -V^S_{yy} + V^S V^S_y = 0 \qquad (y \in (-1,1)), \\ V^S(0) = 0, \\ V^S(-1) = a^S, \quad V^S(1) = -a^S \end{array}\right. \tag{18}$$

admits a unique solution. Moreover, V^S is either 0 or strictly monotone.

If we allow to add external forces f^S and ℓ^S, then we can easily construct families of stationary solutions. Indeed, for f^S, H^S, a^S and b^S, the two following problems admit at least one solution

$$\left\{\begin{array}{c} -V^{(1)}_{yy} + V^{(1)} V^{(1)}_y = f^S \qquad (y \in (-1,H^S)), \\ V^{(1)}(-1) = a^S, \quad V^{(1)}(H^S) = 0. \end{array}\right. \tag{19}$$

$$\left\{\begin{array}{c} -V^{(2)}_{yy} + V^{(2)} V^{(2)}_y = f^S \qquad (y \in (H^S,1)), \\ V^{(2)}(H^S) = 0, \quad V^{(2)}(1) = b^S. \end{array}\right. \tag{20}$$

This can be proved by following the proof of the existence of the stationary Navier–Stokes system (see, for instance, [16, Theorem 2.3, p. 291]).

Then we can set $V^S = V^{(1)}$ in $(-1, H^S)$ and $V^S = V^{(1)}$ in $(H^S, 1)$, and in that case (H^S, V^S) is solution of (4) provided we take

$$\ell^S := -\frac{1}{m}(V_y^{(2)}(H^S) - V_y^{(1)}(H^S)).$$

2.2 Remarks on condition (6)

In this subsection, we give some conditions that imply (6) and we also prove that there exists V^S satisfying (4) and a non trivial solution φ of

$$\begin{cases} V_y^S(H^S)\varphi - \varphi_{yy} - V^S\varphi_y = 0, \quad y \in (H^S, 1) \\ \varphi(H^S) = \varphi(1) = 0. \end{cases} \tag{21}$$

Let us notice that, if we define the operator $A^S\varphi = \varphi_{yy} + V^S\varphi_y$ with domain $H^2((H^S, 1)) \cap H_0^1((H^S, 1))$ then (6) can be reformulated as follows: $V_y^S(H^S)$ is not an eigenvalue of the operator A^S or $V_y^S(H^S) < -\sigma$.

Let us first state several conditions on V^S or on f^S that ensure that condition (6) holds true.

Proposition 6. *Let us consider a solution $V^S \in W^{1,\infty}((H^S, 1))$ of (4) and assume one of the following conditions:*

$$V_y^S(H^S) \geqslant 0 \tag{22}$$

or

$$\forall y \in (H^S, 1), \quad V_y^S(H^S) + \frac{V_y^S(y)}{2} + \frac{1}{4}(V^S(y))^2 + \left(\frac{\pi}{1 - H^S}\right)^2 > 0 \tag{23}$$

or

$$\sigma \leqslant \frac{1}{3}\left(2\left(\frac{\pi}{1 - H^S}\right)^2 - \|[F^S]^+\|_{L^\infty}\right), \quad \textit{where} \quad F^S(y) = \int_{H^S}^y f^S(s)\mathrm{d}s. \tag{24}$$

Then condition (6) holds true.

Proof. To prove that (22) implies (6), assume $\varphi \in H^2((H^S, 1)) \cap H_0^1((H^S, 1))$ satisfies for some $\lambda \in \mathbb{R}$:

$$\begin{cases} \lambda\varphi - \varphi_{yy} - V^S\varphi_y = 0, \quad y \in (H^S, 1) \\ \varphi(H^S) = \varphi(1) = 0. \end{cases} \tag{25}$$

Since $V^S \in W^{1,\infty}((H^S, 1))$ we have $\varphi_{yy} \in C([H^S, 1])$ and thus $\varphi \in C^2([H^S, 1])$. If φ is not the zero function then φ admits either a minimum or a maximum different from zero at some point $y_0 \in (H^S, 1)$. Assume that $\varphi(y_0) = \max\{\varphi(y) \mid y \in (H^S, 1)\} > 0$. Then we necessarily have $\varphi_y(y_0) = 0$ and $\varphi_{yy}(y_0) \leqslant 0$, and by using the equation we then get $\lambda\varphi(y_0) \leqslant 0$ and $\lambda \leqslant 0$. The other case $\varphi(y_0) = \min\{\varphi(y) \mid y \in (H^S, 1)\} < 0$ (which yields $\varphi_y(y_0) = 0$, $\varphi_{yy}(y_0) \geqslant 0$) leads to the same conclusion. If $\lambda = 0$ then $\varphi_y(y_0) = 0$, $\varphi_{yy}(y_0) = 0$ and φ_y is solution of $\psi_y + V^S\psi = 0$ in $(H^S, 1)$ which leads to $\psi \equiv 0$ and thus $\varphi \equiv 0$. Thus $\lambda < 0$.

In order to prove that (23) implies (6), it is useful to remark that

$$\varphi(y) = z(y)\exp\left(-\int_{H^S}^y \frac{V^S(s)}{2}\mathrm{d}s\right) \tag{26}$$

is solution of (21) if and only if z satisfies

$$z_{yy} = W(V^S)z \quad \Big(y \in (H^S, 1)\Big), \quad z(H^S) = z(1) = 0, \tag{27}$$

where

$$W(V^S) = V_y^S(H^S) + \frac{1}{2}V_y^S + \frac{1}{4}(V^S)^2. \tag{28}$$

Then by multiplying by z the first equation of (27), integrating by parts and using Wirtinger inequality we deduce that

$$\int_{H^S}^1 z^2\left[\left(\frac{\pi}{1 - H^S}\right)^2 + W(V^S)\right] \mathrm{d}y \leqslant 0. \tag{29}$$

In particular, if (23) holds true, we deduce that (16) holds true.

Finally, for the last condition, we integrate the first equation of (4) on (H^S, y) and we deduce

$$\frac{1}{2}V_y^S(y) = \frac{1}{2}V_y^S(H^S) + \frac{1}{4}V^S(y)^2 - \frac{1}{2}F^S(y).$$

Gathering the above equation and (29), we obtain that a necessary condition for the existence of a non trivial solution of (21) is that there exists y such that

$$\frac{3}{2}V_y^S(H^S) + \frac{1}{2}V^S(y)^2 - \frac{1}{2}(F^S)^+(y) + \frac{1}{2}(F^S)^-(y) + \left(\frac{\pi}{1-H^S}\right)^2 \leqslant 0.$$

In particular, since the above condition implies

$$V_y^S(H^S) < \frac{1}{3}\left(\|[F^S]^+\|_{L^\infty} - 2\left(\frac{\pi}{1-H^S}\right)^2\right) \tag{30}$$

then we have $V_y^S(H^S) < -\sigma$ if σ satisfies (24). □

Now let us prove the existence of a couple (V^S, φ) where V^S is solution of (4) and where φ is a non trivial solution of (21). Let us note that any $V^S \in C^\infty([H^S, 1])$ satisfying $V^S(H^S) = 0$ can be extended as a solution of (4) for some (f^S, a^S, b^S, ℓ^S), as explained in Subsection 2.1. Moreover, from Proposition 6, we can limit ourself to functions V^S such that $V_y^S(H^S) < 0$.

Proposition 7. *There exists $V^S \in C^\infty([H^S, 1])$ satisfying $V^S(H^S) = 0$ and $V_y^S(H^S) < 0$ such that equation (21) admits a non trivial solution.*

Proof. In order to construct a solution φ of (21), we use again the change of variables (26) so that we look for a solution z of (27). To obtain such a solution, we first construct a solution $\hat{z}$ of

$$\hat{z}_{\hat{y}\hat{y}} = \hat{W}(\hat{\zeta})\hat{z} \quad (\hat{y} \in (0, \mu)), \quad \hat{z}(0) = \hat{z}(\mu) = 0, \tag{31}$$

where

$$\hat{W}(\hat{\zeta}) = \hat{\zeta}_{\hat{y}}(0) + \frac{1}{2}\hat{\zeta}_{\hat{y}} + \frac{1}{4}(\hat{\zeta})^2, \tag{32}$$

and where $\mu > 0$ will be fixed further.

From $\hat{z}$ and $\hat{\zeta}$ defined as above, one can deduce a solution z of (27) by performing the change of variables

$$z(y) = \frac{\mu}{1-H^S}\hat{z}\left(\frac{\mu}{1-H^S}(y - H^S)\right), \quad V^S(y) = \frac{\mu}{1-H^S}\hat{\zeta}\left(\frac{\mu}{1-H^S}(y - H^S)\right) \quad (y \in (H^S, 1)).$$

Standard calculation shows that in that case, z and V^S satisfy (27) and (28).

We are thus reduced to construct a solution of (31), (32). We start by considering a smooth function $\hat{\zeta} : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\hat{\zeta}(0) = 0, \quad \hat{\zeta}_{\hat{y}}(0) = -1,$$

and

$$\hat{\zeta}(\hat{y}) = -2\varepsilon \tan\left(\varepsilon(\hat{y} - \varepsilon)\right) \quad \left(\hat{y} \in \left[\varepsilon, \varepsilon + \frac{\pi}{4\varepsilon}\right]\right), \tag{33}$$

where ε is a “small” positive constant.

We notice that on the interval $\left[\varepsilon, \varepsilon + \frac{\pi}{4\varepsilon}\right]$, the function $\hat{W}(\hat{\zeta})$ defined by (32) satisfies

$$\hat{W}(\hat{\zeta}) = -(1 + \varepsilon^2). \tag{34}$$

Let us consider a non trivial solution of

$$\hat{z}_{\hat{y}\hat{y}} = \hat{W}(\hat{\zeta})\hat{z} \quad (\hat{y} > 0), \quad \hat{z}(0) = 0. \tag{35}$$

We deduce from (34) that on the interval $\left[\varepsilon, \varepsilon + \frac{\pi}{4\varepsilon}\right]$,

$$\hat{z}(\hat{y}) = a\cos(\sqrt{1+\varepsilon^2}\hat{y} + c),$$

where a and c are some constants. In particular, if $\varepsilon < 1/\sqrt{15}$, we deduce that $\hat{z}$ admits a zero μ in the interval

$$\left[\varepsilon, \varepsilon + \frac{\pi}{\sqrt{1+\varepsilon^2}}\right] \subset \left[\varepsilon, \varepsilon + \frac{\pi}{4\varepsilon}\right].$$

This shows that $\hat{z}$ satisfies (31) and making the two changes of variables described at the beginning of the proof, we deduce the existence of a non trivial solution of (21). □

Let us make some remarks on the above proof. First we can see that, since $\mu \in \left[\varepsilon, \varepsilon + \frac{\pi}{\sqrt{1+\varepsilon^2}}\right]$, the following estimate holds:

$$V_y^S(H^S) \in \left[-\left(\varepsilon + \frac{\pi}{\sqrt{1+\varepsilon^2}}\right)^2 \left(\frac{1}{1-H^S}\right)^2, -\left(\frac{\varepsilon}{1-H^S}\right)^2\right].$$

Second, we construct in this proof of Proposition 7, the function V^S in such a way that $W(V^S)$ is constant in some interval so that we can obtain a simple formula for the solution of (27). Let us remark that we can not have $W(V^S)$ is constant in $(H^S, 1)$. Indeed, in that case, we need to solve $W(V^S) = -\frac{n^2\pi^2}{(1-H^S)^2}$ and $V^S(H^S) = 0$ and this leads to

$$V^S(y) = -\frac{2n\pi}{\sqrt{3}(1-H^S)} \tan\left(\frac{n\pi(y-H^S)}{\sqrt{3}(1-H^S)}\right)$$

which is defined only for $y < (1 - \frac{\sqrt{3}}{2n})H^S + \frac{\sqrt{3}}{2n} < 1$. The next proposition states that, although it is not possible to choose $W(V^S)$ equal to $-\frac{n^2\pi^2}{(1-H^S)^2}$, we can take $W(V^S)$ close to $-\frac{n^2\pi^2}{(1-H^S)^2}$ for some good topology.

Proposition 8. *For all $n \in \mathbb{N}^*$ and all $\epsilon > 0$ there exists $V^S \in C^\infty([H^S, 1])$ satisfying $V^S(H^S) = 0$, $\|W(V^S) + \frac{n^2\pi^2}{(1-H^S)^2}\|_{L^2((H^S,1))} < \epsilon$ and $V_y^S(H^S) \in (-\frac{n^2\pi^2}{(1-H^S)^2} - \epsilon, -\frac{n^2\pi^2}{(1-H^S)^2} + \epsilon)$, such that equation (21) admits a nontrivial solution.*

Proof. The idea here is to use the change of variables (26) and to obtain a non trivial $z \in H^2((H^S, 1))$ solution of (27) by proving that the Schrödinger operator associated to V^S admits 0 for eigenvalue.

Consider, for all $W \in L^2((H^S, 1))$, the corresponding linear operator A_W defined on $L^2((H^S, 1))$ by $A_W z = -z_{yy} + Wz$ and by its domain $D(A_W) = H^2((H^S, 1)) \cap H_0^1((H^S, 1))$. The operators A_W always have a spectrum composed of simple eigenvalues $\lambda_1(W) < \cdots < \lambda_n(W) < \ldots$ and by applying classical results (for instance p.190, p.203 (Theorem 2.14), pp.212-213 (Theorem 3.16) of [21]), we obtain that for each $n \in \mathbb{N}$ the mapping $W \in L^2((H^S, 1)) \mapsto \lambda_n(W) \in \mathbb{R}$ is continuous.

For $\nu > 0$, let us consider a smooth convex function ζ^ν such that

$$\zeta^\nu(y) = \begin{cases} H^S - y & y \in (H^S, H^S + \nu/2) \\ -\nu & y \in (H^S + 2\nu, 1). \end{cases}$$

Using the definition (28) of the operator W, we can see that for all real α

$$W(\alpha\zeta^\nu) \to -\alpha \quad \text{in } L^2((H^S, 1)) \quad \text{as } \nu \to 0.$$

Thus

$$\lambda_n(W(\alpha\zeta^\nu)) \to \lambda_n(-\alpha) = \frac{n^2\pi^2}{(1-H^S)^2} - \alpha. \tag{36}$$

Using this construction with $\alpha = \frac{n^2\pi^2}{(1-H^S)^2} - \frac{\epsilon}{2}$ and with $\alpha = \frac{n^2\pi^2}{(1-H^S)^2} + \frac{\epsilon}{2}$ and for ν small enough we obtain two smooth functions V^1 and V^2 such that

$$V^i(H^S) = 0, \quad V_y^i(H^S) \in \left(-\frac{n^2\pi^2}{(1-H^S)^2} - \epsilon, -\frac{n^2\pi^2}{(1-H^S)^2} + \epsilon\right), \tag{37}$$

$$\left\|W(V^i) + \frac{n^2\pi^2}{(1-H^S)^2}\right\|_{L^2((H^S,1))} < \frac{\epsilon}{2}, \tag{38}$$

$$\left\|V^i\right\|^2_{L^4((H^S,1))} < \frac{\epsilon}{2}, \tag{39}$$

for $i = 1, 2$ and such that

$$\lambda_n(W(V^1)) < 0 < \lambda_n(W(V^2)). \tag{40}$$

In particular, a continuity argument shows that there exists $\theta \in [0,1]$ such that $V^S \stackrel{\text{def}}{=} \theta V^1 + (1-\theta)V^2$ satisfies $V^S(0) = 0$, $V_y(H^S) \in (-\frac{n^2\pi^2}{(1-H^S)^2} - \epsilon, -\frac{n^2\pi^2}{(1-H^S)^2} + \epsilon)$, $\lambda_n(W(V^S)) = 0$. A simple calculation and (39) yield that

$$\begin{aligned}
&\left\| W(V^S) + \frac{n^2\pi^2}{(1-H^S)^2} \right\|_{L^2((H^S,1))} \\
&\qquad \leqslant \theta \left\| W(V^1) + \frac{n^2\pi^2}{(1-H^S)^2} \right\|_{L^2((H^S,1))} + (1-\theta) \left\| W(V^2) + \frac{n^2\pi^2}{(1-H^S)^2} \right\|_{L^2((H^S,1))} \\
&\qquad\qquad + \frac{1}{2} \left\| V^1 \right\|^2_{L^4((H^S,1))} + \frac{1}{2} \left\| V^2 \right\|^2_{L^4((H^S,1))} < \epsilon.
\end{aligned} \tag{41}$$

Now, since $\lambda_n(W(V^S)) = 0$, we deduce that there exists a non trivial $z \in H^2((H^S,1))$ satisfying (27) and using the change of variables (26), we obtain a non trivial solution φ of (21). $\square$

3 Change of variables

As explained in the introduction, a first step to study (1)-(2)-(3) consists in performing a change of variables in order to transform the system into a system written in a cylindrical domain in time and space. In this section, we construct and use a smooth change of variables

$$X(t,\cdot) : (-1,1) \to (-1,1), \quad \text{with} \quad X(t, H^s) = h(t).$$

In particular, $X(t,\cdot)$ transforms $(-1,1) \setminus \{H^s\}$ into $(-1,1) \setminus \{h(t)\}$. We denote by $Y(t,\cdot)$ the inverse of $X(t,\cdot)$ and we set

$$V(t,y) \stackrel{\text{def}}{=} v(t, X(t,y)). \tag{42}$$

Then a short calculation shows that if (v,h) satisfies (1)-(2)-(3) then (V,h) satisfies the following system:

$$\begin{cases}
V_t - (Y_x \circ X)^2 V_{yy} - (Y_{xx} \circ X) V_y + (Y_x \circ X) V V_y + (Y_t \circ X) V_y = f^S \circ X \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t \geqslant 0, \quad y \in (-1,1) \setminus \{H^S\}), \\
V(t, H^S) = \ell(t) \qquad (t \geqslant 0), \\
m\dot{\ell}(t) = [(Y_x \circ X) V_y](t, H^S) + m\ell^S \qquad (t \geqslant 0), \\
\dot{h}(t) = \ell(t) \qquad (t \geqslant 0), \\
V(t,-1) = a^S + u(t), \quad V(t,1) = b^S \qquad (t \geqslant 0), \\
h(0) = h^0, \quad \ell(0) = \ell^0, \\
V(0,y) = V^0(y) \quad y \in (-1,1) \setminus \{H^S\}.
\end{cases} \tag{43}$$

There are many ways to construct the change of variables X (see [12, 22] for another construction). Here, we define it by

$$X(t,y) \stackrel{\text{def}}{=} y + \eta(y)(h(t) - H^S),$$

with $\eta \in C^\infty([-1,1])$ satisfying $\eta(H^S) = 1$, $\eta(1) = \eta(-1) = \eta'(H^S) = 0$. If we assume that

$$\left\| \eta' \right\|_{L^\infty((-1,1))} \left| h(t) - H^S \right| < 1 \quad \text{for all } t \tag{44}$$

then, $X(t,\cdot)$ is a bijection from $(-1,1) \setminus \{H^s\}$ onto $(-1,1) \setminus \{h(t)\}$.

To simplify the calculation and the presentation, in what follows we assume

$$H^S = 0.$$

We also set

$$V = W + V^S$$

where V^S is the solution of (4) associated to the quadruplet (f^S, ℓ^S, a^S, b^S). Developing the first equation of (43), we obtain

$$W_t - (Y_x \circ X)^2 (W_{yy} + V^S_{yy}) - (Y_{xx} \circ X)(W_y + V^S_y) \\ + (Y_x \circ X)(W + V^S)(W_y + V^S_y) + (Y_t \circ X)(W_y + V^S_y) = f^S \circ X. \quad (45)$$

From the definitions of X and Y we have the following relations

$$X_y = 1 + \eta' h, \quad X_{yy} = \eta'' h, \quad X_t = \eta h', \quad (46)$$

$$(Y_x \circ X) = \frac{1}{1 + \eta' h} = 1 - \eta' h + \varepsilon_1, \quad \text{with} \quad \varepsilon_1 = \frac{(\eta' h)^2}{1 + \eta' h} \quad (47)$$

$$(Y_x \circ X)^2 = \frac{1}{(1 + \eta' h)^2} = 1 - 2\eta' h + \varepsilon_2, \quad \text{with} \quad \varepsilon_2 = \frac{3(\eta' h)^2 + 2(\eta' h)^3}{(1 + \eta' h)^2} \quad (48)$$

$$(Y_t \circ X) = \frac{-\eta \ell}{1 + \eta' h} = -\eta \ell + \varepsilon_3, \quad \text{with} \quad \varepsilon_3 = \frac{(\eta \eta') h \ell}{1 + \eta' h} \quad (49)$$

$$(Y_{xx} \circ X) = \frac{-\eta'' h}{(1 + \eta' h)^3} = -\eta'' h + \varepsilon_4 \quad \text{with} \quad \varepsilon_4 = \eta'' h \frac{3(\eta' h) + 3(\eta' h)^2 + (\eta' h)^3}{(1 + \eta' h)^3}. \quad (50)$$

The above relations and (45) yield

$$W_t - W_{yy} - V^S_{yy} + 2\eta' h V^S_{yy} + \eta'' h V^S_y + (V^S W)_y + V^S V^S_y - \eta' h V^S V^S_y - \eta h' V^S_y \\ = f^S \circ X + [(Y_x \circ X)^2 - 1] W_{yy} + \varepsilon_2 V^S_{yy} + (Y_{xx} \circ X) W_y + \varepsilon_4 V^S_y \\ - (Y_x \circ X) W W_y + (1 - (Y_x \circ X))(V^S W)_y - \varepsilon_1 V^S V^S_y - (Y_t \circ X) W_y - \varepsilon_3 V^S_y. \quad (51)$$

Using the definition of X we have the following relation

$$(f^S \circ X)(t, y) = f^S(y + \eta(y) h(t)) = f^S(y) + \eta(y) h(t) f^S_y(y) + \eta^2(y) h^2(t) \varepsilon_5(y, h(t)), \quad (52)$$

with

$$\varepsilon_5(y, h) = \int_0^1 f^S_{yy}(y + \theta \eta(y) h)(1 - \theta) \mathrm{d}\theta.$$

In particular, since $f^S \in W^{2,\infty}(-1, 1)$, then we see that ε_5 is bounded.

Combining (51) with (46)–(50), (52) and (4), we obtain

$$W_t - W_{yy} + h \left(\eta' V^S_y \right)_y + (V^S W)_y - \eta \ell V^S_y - \eta' h f^S - \eta h f^S_y \\ = \eta^2 h^2 \varepsilon_5 + [(Y_x \circ X)^2 - 1] W_{yy} + \varepsilon_2 V^S_{yy} + (Y_{xx} \circ X) W_y + \varepsilon_4 V^S_y \\ - (Y_x \circ X) W W_y + (1 - (Y_x \circ X))(V^S W)_y - \varepsilon_1 V^S V^S_y - (Y_t \circ X) W_y - \varepsilon_3 V^S_y. \quad (53)$$

Finally, by setting

$$\mathbf{X} \stackrel{\text{def}}{=} \begin{bmatrix} W \\ \ell \\ h \end{bmatrix},$$

and

$$F(\mathbf{X}) \stackrel{\text{def}}{=} \eta^2 h^2 \varepsilon_5 + [(Y_x \circ X)^2 - 1] W_{yy} + \varepsilon_2 V^S_{yy} + (Y_{xx} \circ X) W_y + \varepsilon_4 V^S_y \\ - (Y_x \circ X) W W_y + (1 - (Y_x \circ X))(V^S W)_y - \varepsilon_1 V^S V^S_y - (Y_t \circ X) W_y - \varepsilon_3 V^S_y \quad (54)$$

i.e.

$$F(\mathbf{X}) \stackrel{\text{def}}{=} \eta^2 h^2 \int_0^1 f^S_{yy}(y + \theta \eta(y) h)(1 - \theta) \mathrm{d}\theta - \frac{(\eta' h)^2 + 2\eta' h}{(1 + \eta' h)^2} W_{yy} + \frac{3(\eta' h)^2 + 2(\eta' h)^3}{(1 + \eta' h)^2} V^S_{yy} \\ - \frac{\eta'' h}{(1 + \eta' h)^3} W_y + \eta'' h \frac{3(\eta' h) + 3(\eta' h)^2 + (\eta' h)^3}{(1 + \eta' h)^3} V^S_y \\ - \frac{1}{1 + \eta' h} W W_y + \frac{\eta' h}{1 + \eta' h} (V^S W)_y - \frac{(\eta' h)^2}{1 + \eta' h} V^S V^S_y + \frac{\eta \ell}{1 + \eta' h} W_y - \frac{(\eta \eta') h \ell}{1 + \eta' h} V^S_y \quad (55)$$

we can see that (W, ℓ, h) is solution of

$$\left\{\begin{array}{l} W_t - W_{yy} + h\left(\eta' V_y^S - \eta f^S\right)_y + (V^S W)_y - \eta \ell V_y^S = F(\mathbf{X}) \qquad (t \geqslant 0, \quad y \in (-1,1) \setminus \{0\}), \\ W(t,0) = \ell(t) \qquad (t \geqslant 0), \\ m\dot{\ell}(t) = [W_y](t,0) \qquad (t \geqslant 0), \\ \dot{h}(t) = \ell(t) \qquad (t \geqslant 0), \\ W(t,-1) = u(t), \quad W(t,1) = 0 \qquad (t \geqslant 0), \\ h(0) = h^0, \quad \ell(0) = \ell^0, \\ W(0,y) = V^0(y) - V^S(y) \quad y \in (-1,1) \setminus \{0\}. \end{array}\right. \tag{56}$$

4 Feedback stabilization of the system in the new variables

In this section we show that the system (56) can be written as a system of the form

$$\mathbf{X}' = A\mathbf{X} + N(\mathbf{X}) + Bu \quad \text{in } [\mathcal{D}(A^*)]', \quad \mathbf{X}(0) = \mathbf{X}^0, \tag{57}$$

where

- $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is a closed linear operator with compact resolvent in the real Hilbert space $\mathcal{H}$ and is the infinitesimal generator of an analytic semigroup on $\mathcal{H}$;
- $B : \mathbb{R} \to [\mathcal{D}(A^*)]'$ is strictly relatively bounded: for some $0 \leqslant \gamma < 1$ and for $\lambda_0 > 0$ large enough, $(\lambda_0 - A)^{-\gamma} B : \mathbb{R} \to \mathcal{H}$ is bounded;
- N is a nonlinear operator which can be "unbounded" (see below for the precise statement).

The first hypothesis implies in particular that the number of "unstable" modes is finite: for any prescribed $\sigma > 0$, there are only N eigenvalues of A with real part greater than $-\sigma$: λ_k, $k = 1, \ldots, N$ (N depending on σ).

Systems of the form (57) can be stabilized by a feedback operator provided a unique continuation property holds true. More precisely we have the following result (see, for instance, [1], [2]).

Theorem 9. *Assume the above properties on A and B and let us consider $\sigma > 0$ and λ_k, $k = 1, \ldots, N$ as above. Assume the following unique continuation property*

$$\forall \varepsilon \in \mathcal{D}(A^*), \quad \lambda \in \{\lambda_k \mid k = 1, \ldots, N\} \quad A^* \varepsilon = \bar{\lambda} \varepsilon \quad \textit{and} \quad B^* \varepsilon = 0 \implies \varepsilon = 0. \tag{58}$$

Then

- *there exists $K \in \mathcal{L}(\mathcal{H}; \mathbb{R})$ such that $A + BK$ is exponentially stable of order $-\sigma$.*
- *Let us set $\mathcal{V}_K \stackrel{def}{=} [\mathcal{D}(A + BK), \mathcal{H}]_{1/2}$ and $B_{\mathcal{H}}(0,R) \stackrel{def}{=} \{\mathbf{X} \in \mathcal{H} \mid \|\mathbf{X}\|_{\mathcal{H}} < R\}$ for $R > 0$ and assume that $N : B_{\mathcal{H}}(0,R) \cap \mathcal{V}_K \to \mathcal{V}'$ satisfies the following relations*

$$\|N(\mathbf{X})\|_{\mathcal{V}'} \leqslant C \|\mathbf{X}\|_{\mathcal{H}} \|\mathbf{X}\|_{\mathcal{V}_K}, \quad (\mathbf{X} \in B_{\mathcal{H}}(0,R)), \tag{59}$$

$$\|N(\mathbf{X}_1) - N(\mathbf{X}_2)\|_{\mathcal{V}'} \leqslant C \left(\|\mathbf{X}_1 - \mathbf{X}_2\|_{\mathcal{H}} \|\mathbf{X}_1\|_{\mathcal{V}_{\mathcal{K}}} + \|\mathbf{X}_2\|_{\mathcal{H}} \|\mathbf{X}_1 - \mathbf{X}_2\|_{\mathcal{V}_K} \right) \quad (\mathbf{X}_1, \mathbf{X}_2 \in B_{\mathcal{H}}(0,R)). \tag{60}$$

Then there exist $c, C > 0$ such that for all $\|\mathbf{X}^0\|_{\mathcal{H}} < c$, there exists a unique solution $\mathbf{X} \in L^\infty_{\mathrm{loc}}(\mathcal{H}) \cap L^2_{\mathrm{loc}}(\mathcal{V}_K)$ of (57) with $u = K\mathbf{X}$ which satisfies $\mathbf{X}(t) \in B_{\mathcal{H}}(0,R)$ for all $t \geqslant 0$. Moreover, this solutions satisfies

$$\|\mathbf{X}\|_{W_\sigma(\mathcal{V}_K, \mathcal{V}')} \leqslant C \|\mathbf{X}^0\|_{\mathcal{H}}, \quad \|\mathbf{X}(t)\|_{\mathcal{H}} \leqslant C \|\mathbf{X}^0\|_{\mathcal{H}} e^{-\sigma t} \quad (t \geqslant 0).$$

In the above result and in what follows, we set

$$W_\sigma(\mathcal{X}, \mathcal{Y}) = \left\{ z \,;\, e^{\sigma(\cdot)} z \in L^2(0, +\infty; \mathcal{X}) \cap H^1(0, +\infty; \mathcal{Y}) \right\}.$$

Remark 10. *Let us note that since $B^* : \mathbb{R} \to D(A^*)$, condition (58) implies that the eigenvalues λ_k, $k = 1, \ldots, N$ are simple. The geometrical multiplicity of the unstable modes plays an important role to calculate the minimal number of controllers to stabilize such systems (see [1], [2] for more details).*

Remark 11. *Note that in Theorem 9 the uniqueness of solutions holds in a class of functions satisfying $\mathbf{X}(t) \in B_{\mathcal{H}}(0,R)$ in order to have $N(\mathbf{X}(t))$ well defined at each time $t \geqslant 0$. However, it can be checked that the uniqueness of the solutions holds in a stronger sense: any maximal solution $\tilde{\mathbf{X}} \in C(0,T_{\max};\mathcal{H}) \cap L^2(0,T_{\max};\mathcal{V}_K)$ of (57) with $u = K\mathbf{X}$ is a global solution ($T_{\max} = +\infty$) and is equal to the solution $\mathbf{X}$ of Theorem 9. Indeed, since a continuity argument guarantees that $T_{\max} > 0$, this can be obtained by performing a priori estimate as in [1, Thm.15].*

The remaining part of this section is devoted to prove that (56) can be written in the form (57) with $\mathbf{X} := {}^t[W,\ell,h]$. We also show that (58) holds true and that N satisfies (59), (60).

For this purpose, we introduce the Hilbert space $\mathcal{H} \stackrel{\text{def}}{=} L^2((-1,1)) \times \mathbb{R}^2$ equipped with the inner product

$$\left(\begin{bmatrix} W \\ \ell \\ h \end{bmatrix}, \begin{bmatrix} \varphi \\ g \\ k \end{bmatrix} \right)_{\mathcal{H}} \stackrel{\text{def}}{=} \int_{-1}^{1} W\varphi \mathrm{d}y + m\ell g + hk.$$

Then we set

$$A \begin{bmatrix} W \\ \ell \\ h \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} W_{yy} - h\left(\eta' V_y^S - \eta f^S\right)_y - (V^S W)_y + \eta \ell V_y^S \\ \dfrac{1}{m}[W_y](0) \\ \ell \end{bmatrix} \tag{61}$$

$$\mathcal{D}(A) \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} W \\ \ell \\ h \end{bmatrix} \in \mathcal{H} \ ; \ W \in H^2((-1,1)\setminus\{0\}) \cap H_0^1((-1,1)), \quad W(0) = \ell \right\}. \tag{62}$$

It is clear that A is densely defined with compact resolvent. Let us check that A is the infinitesimal generator of an analytic semigroup on $\mathcal{H}$. We introduce the following dense subset of $\mathcal{H}$:

$$\mathcal{V} \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} W \\ \ell \\ h \end{bmatrix} \in \mathcal{H} \ ; \ W \in H_0^1((-1,1)), \quad W(0) = \ell \right\}$$

and we define the continuous bilinear form $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ given by

$$a\left(\begin{bmatrix} W \\ \ell \\ h \end{bmatrix}, \begin{bmatrix} \varphi \\ g \\ k \end{bmatrix} \right) \stackrel{\text{def}}{=} -\int_{-1}^{1} \left(W_y\varphi_y + h\left(\eta' V_y^S - \eta f^S\right)_y \varphi + (V^S W)_y \varphi - \eta \ell V_y^S \varphi \right) \mathrm{d}y + \ell k. \tag{63}$$

With an integration by parts, it is easily seen that

$$\left(A \begin{bmatrix} W \\ \ell \\ h \end{bmatrix}, \begin{bmatrix} \varphi \\ g \\ k \end{bmatrix} \right)_{\mathcal{H}} = a\left(\begin{bmatrix} W \\ \ell \\ h \end{bmatrix}, \begin{bmatrix} \varphi \\ g \\ k \end{bmatrix} \right) \quad \forall \begin{bmatrix} W \\ \ell \\ h \end{bmatrix} \in \mathcal{D}(A), \ \begin{bmatrix} \varphi \\ g \\ k \end{bmatrix} \in \mathcal{V} \tag{64}$$

and that $\mathcal{D}(A)$ defined in (62) is exactly the set of element $\mathbf{X}$ such that $\mathbf{Y} \mapsto a(\mathbf{X},\mathbf{Y})$ is continuous for the topology of $\mathcal{H}$. Then since $-a(\cdot,\cdot)$ is regularly accretive, i.e there exist $\alpha > 0$ and $\beta > 0$ such that $\beta\|\mathbf{X}\|_{\mathcal{H}}^2 - a(\mathbf{X},\mathbf{X}) \geqslant \alpha\|\mathbf{X}\|_{\mathcal{V}}^2$, we deduce that A is the infinitesimal generator of an analytic semigroup on $\mathcal{H}$ (see [4, Thm. 2.12]).

Moreover, we can verify that the adjoint of A is given by

$$A^* \begin{bmatrix} \varphi \\ g \\ k \end{bmatrix} = \begin{bmatrix} \varphi_{yy} + V^S \varphi_y \\ \dfrac{1}{m}[\varphi_y](0) + \dfrac{k}{m} + \dfrac{1}{m}\displaystyle\int_{-1}^{1} \eta V_y^S \varphi \mathrm{d}y \\ \displaystyle\int_{-1}^{1} \left(-\eta' V_y^S + \eta f^S\right)_y \varphi \mathrm{d}y \end{bmatrix}, \quad \mathcal{D}(A^*) = \mathcal{D}(A),$$

and since the maximal accretivity of $\beta - A$ ensures that $\mathcal{D}((\beta - A)^\theta) = [\mathcal{D}(A),\mathcal{H}]_{1-\theta}$ and $[\mathcal{D}(A^*),\mathcal{H}]_{1-\theta} = \mathcal{D}((\beta - A^*)^\theta)$ for $\theta \in [0,1]$ ([4, Chap. 2, Prop. 6.1]), an interpolation argument yields that $\mathcal{D}((\beta - A)^\theta) = \mathcal{D}((\beta - A^*)^\theta)$ is composed with ${}^t[W,\ell,h] \in H^{2\theta}((-1,1)\setminus\{0\}) \times \mathbb{R}^2$ such that $W \in H_0^{\min(2\theta,1)}((-1,1))$, $W(0) = \ell$ if $\theta > 1/4$.

Next, we define $B : \mathbb{R} \to [\mathcal{D}(A^*)]'$ by

$$B^* \begin{bmatrix} \varphi \\ g \\ k \end{bmatrix} \stackrel{\text{def}}{=} \varphi_y(-1) \tag{65}$$

and the above characterization of $\mathcal{D}((\beta - A)^\theta)$ guarantees $(\beta - A)^{-\gamma} B \in \mathcal{L}(\mathcal{H})$ for $\gamma \in (3/4, 1)$.

Next, we want to show the uniqueness property (58) in Theorem 9. More precisely, let us assume that $\lambda \in \mathbb{C}$,

$$\Re\lambda \geqslant -\sigma \tag{66}$$

and let us prove the following implication:

$$\begin{cases} \lambda\varphi - \varphi_{yy} - V^S \varphi_y = 0, \quad y \in (-1,1) \setminus \{0\}, \\ \varphi(0) = g, \\ m\lambda g = [\varphi_y](0) + k + \displaystyle\int_{-1}^{1} \eta V_y^S \varphi \mathrm{d}y, \\ \lambda k = \displaystyle\int_{-1}^{1} \left(-\eta' V_y^S + \eta f^S\right)_y \varphi \mathrm{d}y, \\ \varphi(-1) = \varphi(1) = 0, \end{cases} \quad \text{and} \quad \varphi_y(-1) = 0 \implies \varphi \equiv 0 \quad \text{in} \quad (-1,0). \tag{67}$$

Assume that

$$\begin{cases} \lambda\varphi - \varphi_{yy} - V^S \varphi_y = 0, \quad y \in (-1,1) \setminus \{0\}, & \text{(68a)} \\ \varphi(0) = g, & \text{(68b)} \\ m\lambda g = [\varphi_y](0) + k + \displaystyle\int_{-1}^{1} \eta V_y^S \varphi \mathrm{d}y, & \text{(68c)} \\ \lambda k = \displaystyle\int_{-1}^{1} \left(-\eta' V_y^S + \eta f^S\right)_y \varphi \mathrm{d}y, & \text{(68d)} \\ \varphi(-1) = \varphi_y(-1) = 0, & \text{(68e)} \\ \varphi(1) = 0. & \text{(68f)} \end{cases}$$

Combining classical result on linear differential equations and (68a) and (68e) we obtain $\varphi \equiv 0$ on $[-1, 0]$. Consequently, $\varphi(0) = \varphi_y(0^-) = g = 0$. Therefore, (68a) – (68f) can be reduced to

$$\begin{cases} \lambda\varphi - \varphi_{yy} - V^S \varphi_y = 0, \quad y \in (0,1), & \text{(69a)} \\ \varphi(0) = \varphi(1) = 0, & \text{(69b)} \\ \lambda\varphi_y(0) = \displaystyle\int_{0}^{1} \left(\eta' V_y^S - \eta f^S\right)_y \varphi \mathrm{d}y - \int_0^1 \lambda\eta V_y^S \varphi \mathrm{d}y & \text{(69c)} \end{cases}$$

Multiplying (69a) by ηV_y^S and using (4), we obtain

$$\int_0^1 \lambda\varphi\eta V_y^S \mathrm{d}y - \int_0^1 (\varphi_{yy}\eta V_y^S + \varphi_y \eta V_{yy}^S + f^S \varphi_y \eta) \mathrm{d}y = 0.$$

Integrating by parts the above equation yields

$$\int_0^1 \lambda\varphi\eta V_y^S \mathrm{d}y + \varphi_y(0) V_y^S(0) - \int_0^1 \varphi (V_y^S \eta')_y \mathrm{d}y + \int_0^1 (f^S \eta)_y \varphi \mathrm{d}y = 0.$$

Comparing the above equation with (69c) gives

$$0 = (\lambda - V_y^S(0))\varphi_y(0). \tag{70}$$

From the above equality $\lambda = V_y^S(0)$ or $\varphi_y(0) = 0$. In the first case, we deduce from (66) that $V_y^S(0) \geqslant -\sigma$. Consequently, we deduce from (69a)–(69b) and from (6) that $\varphi \equiv 0$ on $[0, 1]$. In the second case, $\varphi_y(0) = 0$ and we deduce directly from (69a)–(69b) that $\varphi \equiv 0$ on $[0, 1]$. Combining both cases ends the proof of (58).

Applying the first part of Theorem 9, we deduce the existence of $K \in \mathcal{L}(\mathcal{H}, \mathbb{R})$ such that $A+\sigma+BK$ is the infinitesimal generator of an analytic and exponentially stable semigroup on $\mathcal{H}$. There exists ${}^t[\widehat{\varphi}, \widehat{g}, \widehat{k}] \in \mathcal{H}$ such that

$$K\left(\begin{bmatrix} W \\ \ell \\ h \end{bmatrix}\right) = \int_{-1}^{1} W\widehat{\varphi}\mathrm{d}y + m\ell\,\widehat{g} + h\,\widehat{k}, \quad ({}^t[W, \ell, h] \in \mathcal{H}). \tag{71}$$

Moreover, one verifies that $\mathcal{D}(A+BK)$ is the subspace of $H^2((-1,1)\setminus\{0\}) \times \mathbb{R}^2$ composed with ${}^t[W, \ell, h]$ such that

$$W(-1) = \int_{-1}^{1} W\widehat{\varphi}\mathrm{d}y + m\ell\,\widehat{g} + h\,\widehat{k}, \quad W(0) = \ell, \quad W(1) = 0, \tag{72}$$

and an interpolation argument yields that $\mathcal{V}_K \stackrel{\text{def}}{=} [\mathcal{D}(A+BK), \mathcal{H}]_{1/2}$ is the closed subspace of $H^1((-1,1))\times\mathbb{R}^2$ composed with ${}^t[W, \ell, h]$ satisfying (72) (see [1] for similar arguments in the case of Navier–Stokes equations).

The next step consists in defining the nonlinear map N. First, recalling (55), we notice that

$$\begin{aligned} F\left(\mathbf{X}\right) = \eta^2 h^2 \int_0^1 f^S_{yy}(y+\theta\eta(y)h)(1-\theta)\mathrm{d}\theta - \left(\frac{(\eta'h)^2 + 2\eta' h}{(1+\eta'h)^2} W_y\right)_y + \left(\frac{(\eta'h)^2 + 2\eta' h}{(1+\eta'h)^2}\right)_y W_y \\ + \left(\frac{3(\eta'h)^2 + 2(\eta'h)^3}{(1+\eta'h)^2} V^S_y\right)_y - \left(\frac{3(\eta'h)^2 + 2(\eta'h)^3}{(1+\eta'h)^2}\right)_y V^S_y \\ - \frac{\eta'' h}{(1+\eta'h)^3} W_y + \eta'' h \frac{3(\eta'h) + 3(\eta'h)^2 + (\eta'h)^3}{(1+\eta'h)^3} V^S_y \\ - \frac{1}{1+\eta'h} W W_y + \frac{\eta'h}{1+\eta'h}(V^S W)_y - \frac{(\eta'h)^2}{1+\eta'h} V^S V^S_y + \frac{\eta\ell}{1+\eta'h} W_y - \frac{\eta\eta' h\ell}{1+\eta'h} V^S_y \end{aligned} \tag{73}$$

so that

$$F\left(\mathbf{X}\right) = F_1\left(\mathbf{X}\right) + \left(F_2\left(\mathbf{X}\right)\right)_y,$$

with

$$F_2\left(\mathbf{X}\right) = -\frac{(\eta'h)^2 + 2\eta' h}{(1+\eta'h)^2} W_y + \frac{3(\eta'h)^2 + 2(\eta'h)^3}{(1+\eta'h)^2} V^S_y$$

Note that F is well defined if the following condition holds

$$\|\eta'\|_{L^\infty((-1,1))} |h| < 1. \tag{74}$$

(See (44)).

Then, we define N by the formula

$$\left\langle N\left(\mathbf{X}\right), \begin{bmatrix} \varphi \\ g \\ k \end{bmatrix} \right\rangle_{\mathcal{V}', \mathcal{V}} = \int_{-1}^{1} F_1\left(\mathbf{X}\right)\varphi \,\mathrm{d}y - \int_{-1}^{1} F_2\left(\mathbf{X}\right)\varphi_y \,\mathrm{d}y, \tag{75}$$

for ${}^t[\varphi, g, k] \in \mathcal{V}$. Note that if $\mathbf{X} \in \mathcal{D}(A)$, then $N(\mathbf{X}) = \begin{bmatrix} F\left(\mathbf{X}\right) \\ 0 \\ 0 \end{bmatrix}$. This comes from the fact that $\eta'(0) = 0$.

Then we have the following result

Lemma 12. *Let $R >$ be small enough so that* (74) *is satisfied for all $\mathbf{X} = {}^t[W, \ell, h] \in B_{\mathcal{H}}(0, R)$. Then the nonlinear operator N defined by* (75) *satisfies* (59), (60).

Proof. With this choice of R inequality (74) holds and combining it with the regularity of V^S we deduce that

$$\|F_2\left(\mathbf{X}\right)\|_{L^2((-1,1))} \leqslant C|h| \|W_y\|_{L^2((-1,1))} + C|h|^2 \leqslant C\|\mathbf{X}\|_{\mathcal{H}} \|\mathbf{X}\|_{\mathcal{V}_K} \quad (\|\mathbf{X}\|_{\mathcal{H}} \leqslant R).$$

The estimate for F_1 can be done similarly, as (60).

To be more precise, let us remark that in (73), one can write the first term as

$$r(h) \stackrel{\text{def}}{=} \eta^2 h^2 \int_0^1 f^S_{yy}(y+\theta\eta(y)h)(1-\theta)\mathrm{d}\theta = \int_y^{y+\eta(y)h} f^S_{yy}(z)(y+\eta(y)h - z)\mathrm{d}z.$$

In particular,

$$|r(h_1)-r(h_2)| \leqslant \left|\int_{y+\eta(y)h_2}^{y+\eta(y)h_1} f_{yy}^S(z)(y+\eta(y)h_1-z)\mathrm{d}z\right| + \left|\int_{y}^{y+\eta(y)h_2} f_{yy}^S(z)\eta(y)(h_1-h_2)\mathrm{d}z\right|$$
$$\leqslant \|\eta\|_{L^\infty((-1,1))}^2 \|f_{yy}^S\|_{L^\infty((-1,1))}|h_1-h_2|^2 + \|\eta\|_{L^\infty((-1,1))}^2 |h_2| \|f_{yy}^S\|_{L^\infty((-1,1))}|h_1-h_2|.$$

□

The above lemma permits to apply the second part of Theorem 9. In particular, there exist $c, C > 0$ such that if

$$\|W^0\|_{L^2((-1,1))} + |\ell^0| + |h^0| \leqslant c, \tag{76}$$

then there exists a unique solution $\mathbf{X} = {}^t[W,\ell,h] \in L^\infty_{\mathrm{loc}}(\mathcal{H}) \cap L^2_{\mathrm{loc}}(\mathcal{V}_K)$ of (57) with $u = K\mathbf{X}$ satisfying (44) (with $H^S = 0$), with $\mathbf{X}^0 = {}^t[W^0,\ell^0,h^0]$, with A defined by (61), (62), B defined by (65), N defined by (75) and K given by (71). Moreover, this solution satisfies (72) and

$$\|{}^t[W,\ell,h]\|_{W_\sigma(\mathcal{V}_K,\mathcal{V}')} \leqslant C\left(\|W^0\|_{L^2((-1,1))} + |\ell^0| + |h^0|\right), \tag{77}$$

$$\|W(t)\|_{L^2((-1,1))} + |\ell(t)| + |h(t)| \leqslant Ce^{-\sigma t}\left(\|W^0\|_{L^2((-1,1))} + |\ell^0| + |h^0|\right). \tag{78}$$

5 Proof of the main result

Finally, to obtain Theorem 2 it remains to go back to the original variables and thus to obtain the existence and uniqueness of a solution (1), (10), (3) satisfying (11). First, let us check that if

$$\|v^0 - V^S\|_{L^2((-1,1))} + |\ell^0| + |h^0| \leqslant \mu \tag{79}$$

for μ small enough and if $W^0 := v^0(X(0,\cdot)) - V^S$, then (W^0,ℓ^0,h^0) satisfies (76). Indeed, for μ small enough, relation (44) holds for $t=0$, and thus, we can apply a change of variables and use the fact that $V^S \in W^{1,\infty}((-1,1))$ to deduce the result. Applying the results of the previous section, we deduce the existence of ${}^t[W,\ell,h] \in W_\sigma(\mathcal{V}_K,\mathcal{V}')$ of (57) satisfying (44) (with $H^S=0$), with A defined by (61), (62), B defined by (65), N defined by (75) and K given by (71). This solution satisfies (72), (77) and (78). In particular for all ${}^t[\varphi,g,k] \in \mathcal{V}$,

$$\left\langle \begin{bmatrix} W \\ \ell \\ h \end{bmatrix}', \begin{bmatrix} \varphi \\ g \\ k \end{bmatrix} \right\rangle_{\mathcal{V}',\mathcal{V}} + \int_{-1}^1 \left(W_y\varphi_y + h\left(\eta' V_y^S - \eta f^S\right)_y \varphi + (V^S W)_y\varphi - \eta\ell V_y^S\varphi \right)\mathrm{d}y - \ell k$$
$$= \int_{-1}^1 F_1(\mathbf{X})\,\varphi\ \mathrm{d}y - \int_{-1}^1 F_2(\mathbf{X})\,\varphi_y\ \mathrm{d}y$$

and thus, writing $V = W + V^S$,

$$\left\langle \begin{bmatrix} V \\ \ell \\ h \end{bmatrix}', \begin{bmatrix} \varphi \\ g \\ k \end{bmatrix} \right\rangle_{\mathcal{V}',\mathcal{V}} + \int_{-1}^1 (Y_x\circ X)^2 V_y\varphi_y + (Y_{xx}\circ X)V_y\varphi\ \mathrm{d}y - \ell k$$
$$+ \int_{-1}^1 \left[(Y_x\circ X)VV_y + (Y_t\circ X)V_y\right]\varphi\ \mathrm{d}y = \int_{-1}^1 (f^S\circ X)\varphi\ \mathrm{d}y + mg\ell^S. \tag{80}$$

It is classical to see that the above relation holds true for all ${}^t[\varphi,g,k] \in C^1([0,+\infty);\mathcal{V})$ with compact support in $[0,\infty)$. In that case, integrating in time the above relation yields

$$-\int_0^\infty \left\langle \begin{bmatrix} V \\ \ell \\ h \end{bmatrix}, \begin{bmatrix} \varphi \\ g \\ k \end{bmatrix}' \right\rangle_{\mathcal{H}} \mathrm{d}t - \left\langle \begin{bmatrix} V(0) \\ \ell(0) \\ h(0) \end{bmatrix}, \begin{bmatrix} \varphi(0) \\ g(0) \\ k(0) \end{bmatrix} \right\rangle_{\mathcal{H}} + \int_0^\infty \int_{-1}^1 \left[(Y_x\circ X)^2 V_y\varphi_y + (Y_{xx}\circ X)V_y\varphi\right]\ \mathrm{d}y\ \mathrm{d}t$$
$$-\int_0^\infty \ell k\ \mathrm{d}t + \int_0^\infty \int_{-1}^1 \left[(Y_x\circ X)VV_y + (Y_t\circ X)V_y\right]\varphi\ \mathrm{d}y\ \mathrm{d}t = \int_0^\infty \int_{-1}^1 (f^S\circ X)\varphi\ \mathrm{d}y\ \mathrm{d}t + \int_0^\infty mg\ell^S\ \mathrm{d}t. \tag{81}$$

Here, we have used the classical relation

$$\frac{d}{dt}\langle u,v\rangle_{\mathcal{H}} = \langle u',v\rangle_{\mathcal{V}',\mathcal{V}} + \langle u,v'\rangle_{\mathcal{V},\mathcal{V}'}$$

for all $u,v \in L^2(\mathcal{H}) \cap H^1(\mathcal{V}')$.

Let us consider $(\phi,g,k) \in C^1([0,\infty);H_0^1((-1,1))\times\mathbb{R}^2)$ with compact support in time and such that for all t, ${}^t[\phi(t),g(t),k(t)] \in \mathcal{V}^{h(t)}$, We now define

$$v(t,x) \stackrel{\text{def}}{=} V(t,Y(t,x)), \quad \text{and} \quad \varphi(t,y) \stackrel{\text{def}}{=} X_y(t,y)\phi(t,X(t,y)). \tag{82}$$

Since $h \in C^1([0,\infty);\mathbb{R})$, and since $\varphi(t,0) = X_y(t,0)\phi(t,h(t)) = g(t)$, we deduce that ${}^t[\varphi,g,k]$ belongs to $C^1([0,+\infty);\mathcal{V})$ and has a compact support in $[0,\infty)$. Moreover, a short calculation (note that $X_y(y) = Y_x(X(y))^{-1}$) shows that

$$\int_{-1}^{1}\big((Y_x\circ X)^2V_y\varphi_y + (Y_{xx}\circ X)V_y\varphi + (Y_x\circ X)VV_y\varphi\big)\,\mathrm{d}y = \int_{-1}^{1}(v_x\phi_x + vv_x\phi)\,\mathrm{d}x, \tag{83}$$

and

$$-\int_0^\infty \left\langle \begin{bmatrix} V\\ \ell\\ h\end{bmatrix}, \begin{bmatrix}\varphi\\ g\\ k\end{bmatrix}'\right\rangle_{\mathcal{H}} \mathrm{d}t + \int_0^\infty\int_{-1}^{1}(Y_t\circ X)V_y\varphi\ \mathrm{d}y\ \mathrm{d}t$$
$$= -\int_0^\infty\int_{-1}^{1} v\phi_t\ \mathrm{d}x\ \mathrm{d}t - \int_0^\infty m\ell\dot{g}\ \mathrm{d}t - \int_0^\infty h\dot{k}\ \mathrm{d}t. \tag{84}$$

Let us prove the last inequality (the proof of (83) is similar). Integrating by parts, we obtain

$$-\int_0^\infty \left\langle \begin{bmatrix} V\\ \ell\\ h\end{bmatrix}, \begin{bmatrix}\varphi\\ g\\ k\end{bmatrix}'\right\rangle_{\mathcal{H}} \mathrm{d}t + \int_0^\infty\int_{-1}^{1}(Y_t\circ X)V_y\varphi\ \mathrm{d}y\ \mathrm{d}t$$
$$= -\int_0^\infty\int_{-1}^{1} V\varphi_t\ \mathrm{d}y\ \mathrm{d}t - \int_0^\infty m\ell\dot{g}\ \mathrm{d}t - \int_0^\infty h\dot{k}\ \mathrm{d}t$$
$$-\int_0^\infty\int_{-1}^{1}(Y_t\circ X)\varphi_yV\ \mathrm{d}y\ \mathrm{d}t - \int_0^\infty\int_{-1}^{1}(Y_{xt}\circ X)X_y\varphi V\ \mathrm{d}y\ \mathrm{d}t. \tag{85}$$

On the other hand, (82) yields

$$\phi_t(t,x) = Y_{xt}(t,x)\varphi(t,Y(t,x)) + Y_x(t,x)\varphi_t(t,Y(t,x)) + Y_t(t,x)Y_x(t,x)\varphi_y(t,Y(t,x)).$$

The above relation combined with (85) implies (84)

Combining (83) and (84) with (81) gives

$$-\int_0^\infty\int_{-1}^{1} v\phi_t\ \mathrm{d}x\ \mathrm{d}t - \int_0^\infty m\ell\dot{g}\ \mathrm{d}t - \int_0^\infty h\dot{k}\ \mathrm{d}t - \int_{-1}^{1} v(0,x)\phi(0,x)\ \mathrm{d}x - m\ell(0)g(0) - h(0)k(0)$$
$$+\int_0^\infty\int_{-1}^{1}(v_x\phi_x + vv_x\phi)\ \mathrm{d}x\ \mathrm{d}t - \int_0^\infty \ell k\ \mathrm{d}t$$
$$= \int_0^\infty\int_{-1}^{1} f^S\phi\ \mathrm{d}x\ \mathrm{d}t + \int_0^\infty mg\ell^S\ \mathrm{d}t. \tag{86}$$

The above relation implies $\ell = \dot{h}$ and (7).

Moreover, we deduce from (72) that

$$V(-1) = a^S + \int_{-1}^{1}(V - V^S)\widehat{\varphi}\mathrm{d}y + m\ell\,\widehat{g} + h\,\widehat{k}, \quad V(0) = \ell, \quad V(1) = b_S$$

and thus that (10) holds true. This gives the existence of a solution of (1), (10), (3).

Finally, we deduce from (77), (78) that

$$\|v(t,X(t,\cdot))-V^S\|_{L^2((-1,1))}+|\ell(t)|+|h(t)|\leqslant C\left(\|v^0\circ X(0,\cdot)-V^S\|_{L^2((-1,1))}+|\ell^0|+|h^0|\right)e^{-\sigma t} \tag{87}$$

and

$$\int_0^\infty e^{\sigma t}\left(\|v(t,X(t,\cdot))-V^S\|_{H^1((-1,1))}+|\ell(t)|+|h(t)|\right)\,\mathrm{d}t \\ \leqslant C\left(\|v^0\circ X(0,\cdot)-V^S\|_{L^2((-1,1))}+|\ell^0|+|h^0|\right). \tag{88}$$

Applying a change of variables and using the regularity of V^S, we easily deduce

$$\|v(t,\cdot)-V^S\|_{L^2((-1,1))}+|\ell(t)|+|h(t)|\leqslant C\left(\|v^0-V^S\|_{L^2((-1,1))}+|\ell^0|+|h^0|\right)e^{-\sigma t}. \tag{89}$$

Moreover, one can notice that if $V^S\in W^{2,\infty}((-1,1))$, then we obtain

$$\int_0^\infty e^{\sigma t}\left(\|v(t,\cdot)-V^S\|_{H^1((-1,1))}+|\ell(t)|+|h(t)|\right)\,\mathrm{d}t\leqslant C\left(\|v^0-V^S\|_{L^2((-1,1))}+|\ell^0|+|h^0|\right). \tag{90}$$

In the general case, (88) and (44) yield $v\in L^2_{\mathrm{loc}}(H^1((-1,1)))$. Note that since $W_\sigma(\mathcal{V}_K,\mathcal{V})$ is continuously embedded in $C([0,+\infty);\mathcal{H})$ we have $V\in C([0,+\infty);L^2((-1,1)))$, $\ell=\dot h\in C([0,+\infty);\mathbb{R})$ and then $h\in C^1([0,+\infty);\mathbb{R})$, which with $v=V\circ Y$ yields $v\in C([0,+\infty);L^2((-1,1)))$.

To end the proof of the main result, let us show the uniqueness of (v,h).

Assume $(\tilde v,\tilde h)\in\left(L^\infty_{\mathrm{loc}}(L^2((-1,1)))\cap L^2_{\mathrm{loc}}(H^1((-1,1)))\right)\times C^1([0,\infty);\mathbb{R})$ is a weak solution of (1), (3), (10) satisfying (44) (with $H^S=0$). Then, setting $\tilde\ell=\dot{\tilde h}$ and applying the change of variables (82) corresponding to $\tilde h$, we deduce that $(\tilde V,\tilde\ell,\tilde h)$ satisfies (81) for ${}^t[\varphi,g,k]\in C^1([0,+\infty);\mathcal{V})$. This yields that ${}^t[\tilde V,\tilde\ell,\tilde h]\in H^1_{\mathrm{loc}}(\mathcal{V}')$ and (80). Setting $\tilde W=\tilde V-V^S$, and since (76) holds, we deduce that ${}^t[\tilde W,\tilde\ell,\tilde h]\in L^\infty_{\mathrm{loc}}(\mathcal{H})\cap L^2_{\mathrm{loc}}(\mathcal{V}_K)$ is the unique solution of (57) with $u=K\mathbf{X}$ satisfying (44) (with $H^S=0$), with $\mathbf{X}^0={}^t[W^0,\ell^0,h^0]$, with A defined by (61), (62), B defined by (65), N defined by (75) and K given by (71), namely ${}^t[\tilde W,\tilde\ell,\tilde h]={}^t[W,\ell,h]$.

Then by going back to the original variables, it gives the uniqueness stated in the theorem and it concludes the proof.

6 Case of several particles

We can extend our problem to the case of n particles, $n\in\mathbb{N}^*$. In that case, the system we need to consider is

$$\left\{\begin{array}{l} v_t-v_{xx}+vv_x=f^S \qquad (t\geqslant 0,\quad x\in(-1,1)\setminus\{h_1(t),\dots,h_n(t)\}),\\ v(t,h_i(t))=\dot h_i(t) \qquad (t\geqslant 0,\ i\in\{1,\dots,n\}),\\ m_i\ddot h_i(t)=[v_x](t,h_i(t))+m_i\ell_i^S \qquad (t\geqslant 0,\ i\in\{1,\dots,n\}),\end{array}\right. \tag{91}$$

with the boundary conditions

$$v(t,-1)=a^S+u(t),\quad v(t,1)=b^S,\qquad (t\geqslant 0), \tag{92}$$

with the initial conditions

$$h_i(0)=h_i^0,\quad \dot h_i(0)=\ell_i^0\quad i\in\{1,\dots,n\}, \tag{93}$$

$$v(0,x)=v^0(x)\quad x\in(-1,1)\setminus\{h_1^0,\dots,h_n^0\}. \tag{94}$$

We also consider a stationary state

$$\left\{\begin{array}{c} -V^S_{yy}+V^SV^S_y=f^S \qquad (y\in(-1,1)\setminus\{H_1^s,\dots,H_n^s\}),\\ V^S(-1)=a^S,\quad V^S(1)=b^S,\\ V^S(H_i^S)=0\quad i\in\{1,\dots,n\},\\ 0=[V^S_y](H_i^S)+m_i\ell_i^S\quad i\in\{1,\dots,n\}.\end{array}\right. \tag{95}$$

It is possible to construct a change of variables

$$X(t,\cdot):(-1,1)\setminus\{H_1^s,\dots,H_n^s\}\to(-1,1)\setminus\{h_1(t),\dots,h_n(t)\}$$

in a similar way as in the case $n = 1$. This is done for $(h_1(t), \ldots, h_n(t))$ such that for $c > 0$ small enough

$$\sup_{i=1,\ldots,n} \left| h_i(t) - H_i^S \right| < c \quad \text{for all } t. \tag{96}$$

The condition (6) becomes

$$\left\{ \begin{array}{l} V_y^S(H_i^S)\varphi - \varphi_{yy} - V^S\varphi_y = 0, \quad y \in (H_i^S, H_{i+1}^S) \\ \varphi(H_i^S) = \varphi(H_{i+1}^S) = 0 \end{array} \right. \quad \Longrightarrow \quad \varphi \equiv 0 \quad \text{or} \quad V_y^S(H_i^S) < -\sigma, \tag{97}$$

for all $i = 1, \ldots, n$ and where we have written $H_{n+1}^S = 1$ for convenience.

Then we can generalize Theorem 2 as follows:

Theorem 13. *Assume $f^S \in W^{2,\infty}(-1,1)$, and that $(f^S, \ell_1^S, \ldots, \ell_n^S, a^S, b^S)$ is associated with a stationary solution $(V^S, 0, H_1^S, \ldots, H_n^S)$ of (95), with $V^S \in W^{1,\infty}((-1,1))$. There exists $\sigma_0 > 0$ such that for all $\sigma \in (0, \sigma_0)$, there exist μ, $C > 0$ and $(\widehat{\varphi}, \widehat{g}_1, \ldots, \widehat{g}_n, \widehat{k}_1, \ldots, \widehat{k}_n,) \in L^2((-1,1)) \times \mathbb{R}^{2n}$, such that, if $(v^0, \ell_1^0, \ldots, \ell_n^0, h_1^0, \ldots, h_n^0) \in L^2((-1,1)) \times \mathbb{R}^{2n}$ satisfies*

$$\|v^0 - V^S\|_{L^2((-1,1))} + \sum_{i=1}^n \left(|\ell_i^0| + |h_i^0 - H_i^S| \right) \leqslant \mu$$

then there exists a unique weak solution

$$(v, h_1, \ldots, h_n) \in L^\infty_{\text{loc}}(L^2((-1,1))) \cap L^2_{\text{loc}}(H^1((-1,1))) \times C^1((0,\infty); \mathbb{R}^n),$$

of (91), (93) and of

$$v(t,-1) = a^S + \int_{-1}^1 \Big(v(t, X(t,y)) - V^S(y) \Big) \widehat{\varphi} \mathrm{d}y + \sum_{i=1}^n \Big(m_i \dot{h}_i(t) \widehat{g}_i + (h_i(t) - H_i^S)\widehat{k}_i \Big), \ v(t,1) = b^S, \quad (t \geqslant 0) \tag{98}$$

and satisfying (96). Moreover $v \in C([0,+\infty); L^2((-1,1)))$ and it satisfies:

$$\|v(t) - V^S\|_{L^2((-1,1))} + \sum_{i=1}^n \left(|\dot{h}_i(t)| + |h_i(t) - H_i^S| \right) \leqslant Ce^{-\sigma t} \left(\|v^0 - V^S\|_{L^2((-1,1))} + \sum_{i=1}^n \left(|\ell_i^0| + |h_i^0 - H_i^S| \right) \right). \tag{99}$$

The definition of weak solutions in that case is similar to the Definition 1 in the case of one particle.

As in Theorem 2, the constant $\sigma_0 > 0$ depends on (97) and can be infinite if (97) holds true for all $\sigma > 0$.

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