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***Abstract*—There is an extensive literature dedicated to the exact recovery of a given subset by Orthogonal Matching Pursuit (OMP) and Basis Pursuit (BP). We consider Tropp's exact recovery condition (ERC) [1] for OMP and the null-space property [2] for BP. Under these conditions, any sparse representation indexed by the subset can be recovered. We address the bad recovery issue, *i.e.*, the opposite extreme situation where the subset cannot be recovered *for any* amplitude values. We elaborate the bad recovery conditions (BRC) recently sketched in [3]. The BRC dedicated to BP is a direct consequence of the null-space property. It does not depend on the amplitudes, but only on the sign pattern. For OMP, this is not the case, and the BRC condition is not directly related to the ERC. The BRC conditions are tested for deterministic dictionaries corresponding to low pass filtering operators. We stress that the BRC of OMP may be frequently met for supports of low cardinality.**

I. Context

Among the pioneering recovery analyses of OMP, the ERC condition proposed by Tropp [1] is a key tool. Indeed, it is not only a sufficient but also a necessary condition for the exact recovery of a given support using OMP. On the one hand, the ERC is a worst-case condition applicable to any sparse representation indexed by the given support. On the other hand, when the ERC is not met, there exists a sparse representation for which OMP is guaranteed to select a wrong atom during the first iteration. Our bad recovery analysis differs from the necessary part of the exact recovery analysis: while a necessary exact recovery condition guarantees that *some* k-sparse representation cannot be recovered, bad recovery conditions ensure that *any* k-sparse representation will not be recovered. This behavior is also referred to as the "non-reachability" of a subset [3]. The main motivation in studying non-reachable subsets is to sharpen the current knowledge regarding the subsets that are not systematically recovered using OMP and BP.

The starting point of our bad recovery analysis of OMP is the "partial" exact recovery analysis when $q < k$ iterations have already been performed and q true atoms have been selected. In [3], we have extended Tropp's ERC (corresponding to $q = 0$) to a weaker exact recovery condition at the q-th iteration. This also led us to design a sufficient bad recovery condition (BRC-OMP) at iteration $q = k - 1$ [3, Th. 7]. Under this condition, a wrong atom selection is guaranteed to occur at the k-th iteration when the first $k-1$ iterations have all succeeded to select true atoms.

Independently, we elaborated a sufficient and worst case necessary bad recovery condition for basis pursuit (BRC-BP), defined as the intersection of as many bad conditions as possibilities for the "sign pattern", *i.e.*, the signs of the nonzero amplitudes in the k-sparse representation [3, Prop. 2]. This result is closely connected to the null-space property [2]. It is also consistent with the exact recovery analysis of BP for a given sign pattern, because this analysis does not depend on the amplitude values [4, 5].

In [3], both BRCs were illustrated on simple toy examples and for random dictionaries. Here, we elaborate their applicability for deterministic dictionaries corresponding to a spike deconvolution problem from low pass filtered data. Our goal is to understand whether the BRCs may be frequently met for some typical inverse problems involving ill-conditioned or correlated dictionaries.

II. Sparse deconvolution with a low pass filter

We consider the sparse spike train deconvolution problem $\boldsymbol{y} = \boldsymbol{h} * \boldsymbol{x}$ with a low pass impulse response filter $\boldsymbol{h}$. Exact recovery studies show that the ERC condition is usually not satisfied unless there is some minimal spacing between two consecutive spikes, which is related to the cutoff frequency of the filter $\boldsymbol{h}$ [6, 7].

We address the opposite case where consecutive spikes are very close and show that the BRC of OMP is frequently met. Specifically, we carry out a closed form calculation showing the inability of OMP to recover two very close spikes. We further extend this study by addressing the continuous case as the limit discrete case where the sampling step tends towards 0. We observe that the BRC-OMP condition is always fulfilled when the sampling step is small enough, and that the selected wrong atoms can be predicted from the knowledge of the impulse response.

The non-reachability of certain vectors using ℓ_1 minimization for ill-conditioned inverse problems has been pointed out in the recent literature, and adaptations of BP algorithms were proposed to track these sparse representations [8]. We further show that the bad recovery condition of BP may be met for spike deconvolution depending on the support size and the cut-off frequency. Contrary to the OMP behavior, guaranteed bad recovery using BP only relies on the cut-off frequency of $\boldsymbol{h}$ and essentially depends on the size of the support $\mathcal{Q}$, but not on the distance between the nonzero spikes.

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