Considerations on travelling waves in the horn equation and energetic aspects

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The digital waveguide synthesis of wind resonators and of the vocal tract is based on decompositions into travelling waves. Typical ones are planar waves in straight pipes and spherical waves in conical pipes. However, approximating a bore by cascading such basic segments introduce unrealistic discontinuities on the radius R or the slope R' (with acoustic consequences). It also can generate artificial instabilities in time-domain simulations, e.g. for non convex junctions of cones.

In this paper, we investigate the case of the "conservative curvilinear horn equation" for segments such that the flaring parameter R"/R is constant, with which smooth profiles can be built. First, acoustic states that generalize planar waves and spherical waves are studied. Examining the energy balance and the passivity for these travelling waves allows to characterize stability domains. Second, two other definitions of travelling waves are studied: (a) one locally diagonalizes the wave propagation operator, (b) one diagonalizes the transfer matrix of a segment. The propagators obtained for (a) are known to efficiently factorizes computations in simulations but are not stable if the flaring parameter is negative. A study in the Laplace domain reveals that propagators (b) are stable for physically meaningful configurations.

1 Outline

This paper is organized as follows. Section 2 gives some recalls on standard results of linear acoustics for conservative propagation. Section 3 deals with the passivity of dynamical systems and that of the 1D horn model which governs the acoustic Kirchhoff variables. Section 4 establishes the energy and examines the passivity for planar travelling waves and spherical travelling waves. For the spherical waves, the passivity criterion leads to a new interpretation of the instability of digital waveguides for convex junctions of conical pipes. In order to cope with the instability of systems that govern such travelling waves in the convex cases, section 5 investigates on a new definition of decoupled travelling waves such that they are governed by stable propagation operators.

2 Standard results of linear acoustics

2.1 Wave equation and energy

In an ideal fluid, the linear propagation of acoustic waves of pressure p and of particle velocity $\vec{v}$ is governed by [1]

$$\partial_t p + \rho c^2 \mathrm{div}\,\vec{v} = 0, \qquad (1)$$

$$\partial_t \vec{v} + \frac{1}{\rho}\vec{\mathrm{grad}}\,p = 0, \qquad (2)$$

where ρ is the mass density and c is the speed of the sound. Deriving $-\frac{1}{c^2}\partial_t(1) + \rho\,\mathrm{div}\,(2)$ yields the wave equation:

$$\Delta p - \frac{1}{c^2}\partial_t^2 p = 0. \qquad (3)$$

The acoustic energy localized in a domain Ω is

$$E(t) = \int_\Omega e(\vec{x}, t)\,\mathrm{d}\omega(\vec{x}) \;\; \text{where} \;\; e = \frac{1}{2}\rho\,\vec{v}.\vec{v} + \frac{1}{2\rho c^2}p^2. \qquad (4)$$

From (1-2), the power balance is found to be

$$E'(t) = \int_\Omega \left(\rho\,\vec{v}.\partial_t\vec{v} + \frac{p\partial_t p}{\rho c^2}\right)\mathrm{d}\omega = -\int_\Omega \left(\vec{v}.\vec{\mathrm{grad}}\,p + p\,\mathrm{div}\vec{v}\right)\mathrm{d}\omega$$

$$= -\int_\Omega \mathrm{div}\left(p\,\vec{v}\right)\mathrm{d}\omega = -\int_{\partial\Omega} p\,\vec{v}.\mathrm{d}\vec{S}, \qquad (5)$$

where $\mathrm{d}\vec{S}$ is the infinitesimal surface vector pointing outwards to the boundary $\partial\Omega$, meaning that *the increase in energy equals the external power supplied to the domain Ω.*

2.2 Horn equation

A uni-dimensional model of propagation in horns with a smooth profile $x \mapsto R(x)$ has been established by Lagrange [2] and Bernoulli [3]. It assumes that: (i) the acoustic fields only depend on the space variable x and the time t so that $\vec{\mathrm{grad}}\,p = \partial_x p\,.\,\vec{e_x}$; (ii) $\mathrm{div}\vec{v} \approx \frac{1}{S}\partial_x(S\,v)$, denoting $v = \vec{v}.\vec{e_x}$ and $S(x) = \pi R(x)^2$ (local cross-section area of the bore). Introducing the acoustic state composed of the Kirchhoff variables

$$X(x,t) = \begin{bmatrix} p(x,t) \\ u(x,t) \end{bmatrix} \;\; \text{with} \;\; u = S\,v \;\; \text{(airflow)}, \qquad (6)$$

the model is given by the modified version of (1-2)

$$\partial_t X + M\,\partial_x X = 0, \;\; \text{where} \;\; M = \begin{bmatrix} 0 & \frac{\rho c^2}{S} \\ \frac{S}{\rho} & 0 \end{bmatrix} \qquad (7)$$

is an invertible matrix with eigenvalues $\pm c$. The corresponding wave equation is obtained by computing $[-\frac{1}{c^2}\partial_t, \frac{\rho}{S}\partial_x].(7)$, which leads to the so-called "Webster equation" [4]

$$\left(\partial_x^2 + 2\frac{R'(x)}{R(x)}\partial_x - \frac{1}{c^2}\partial_t^2\right)p(x,t) = 0,$$

which is equivalent to, for C^2-regular profiles,

$$\left(\partial_x^2 - \Upsilon(x) - \frac{1}{c^2}\partial_t^2\right)\left[R(x)\,p(x,t)\right] = 0, \;\; \text{with} \;\; \Upsilon = \frac{R''}{R}. \qquad (8)$$

The energy localized in the bore for $x \in (0, L)$ is

$$E(t) = \int_0^L e(x,t)\,S(x)\,\mathrm{d}x = \frac{1}{2}\int_0^L X(x,t)^T W(x)\,X(x,t)\,\mathrm{d}x,$$

$$\text{with} \;\; W = W^T = \mathrm{diag}\left(\frac{S}{\rho c^2}, \frac{\rho}{S}\right). \qquad (9)$$

Denoting $Q = W(x)\,M(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = Q^T$, the power balance is

$$E'(t) = \int_0^L X(x,t)^T Q\,\partial_x X(x,t)\,\mathrm{d}x = \left[\frac{1}{2}X(x,t)^T Q\,X(x,t)\right]_{x=0}^{x=L}$$

$$= p(L,t)\,u(L,t) - p(0,t)\,u(0,t), \qquad (10)$$

which has the same physical meaning as (5).

2.3 Exact 1D problems and curvilinear abscissa

Planar waves. The horn equation is an approximate model which restores exact results for straight pipes: (7-10) are all exact 1D versions of (1-5) for the case of planar waves, where R, S, M, W are constant with respect to the axial variable x.

Spherical waves. This is not the case for spherical waves in conical pipes. To be exact, the space variable $x \in [0, L]$ should be replaced in (6-10) by the spherical coordinate $r \in [r_{min}, r_{max}]$, which measures the distance between a point and the apex ($r = 0$) of the conical pipe described by $R(r) = r \sin \theta_0$, with solid angle $\widehat{\Omega} = 2\pi(1 - \cos \theta_0)$.

Notice that, in this case, u defined with $S(r) = S_{disk}(r) = \pi R(r)^2$ corresponds to a normalized version $u = \frac{S_{disk}}{S_{sph}} U_{spher} = \frac{\pi}{\widehat{\Omega}} U_{spher} = \frac{U_{spher}}{2(1 - \cos \theta_0)}$ of the airflow U_{spher} that crosses the portions of the spheres matching with the spherical wavefronts.

Horn equation with curvilinear abscissa A refinement of (7-10) is obtained by replacing the axial abscissa x by the curvilinear abscissa ℓ, which measures the length of the wall. More precisely, for a bore with smooth profile $x \mapsto r(x)$, function R is defined by

$$R(\ell) = r\left(\mathcal{L}^{-1}(\ell)\right) \text{ where } \ell = \mathcal{L}(x) = \int_0^x \sqrt{1 + \left[r'(z)\right]^2} \, dz.$$

This model is established in [5]. It is based on the exact derivation of (3) in a rectified isobar map and the assumption of the quasi-sphericity of isobars near the wall at order 2. Its relevance has been confirmed in [6] for a trombone bell, by comparing modelled and measured input impedances. Moreover, it restores the following exact results.

Remark 1 (Exact 1D models) *The horn equation with curvilinear abscissa is an exact model for both straight pipes ($\ell = x$, $R(x) = R_0$) and conical pipes ($\ell = r$ and $R(r) = r \sin \theta_0$), that is, for profiles such that Υ is zero (see (8)).*

This 1D model is the one that is considered in the following, but the variable label x is conserved for sake of legibility and consistency with formula (6-10).

2.4 Scattering matrix for Υ-constant profiles

Assume $X(x, t)$ to be zero for $t < 0$. Denote $\widehat{X}(x, s)$ its (mono-lateral) Laplace transform w.r.t. t, so that (7-8) become

$$\partial_x \widehat{X}(x, s) + M(x)^{-1} s \widehat{X}(x, s) = 0, \qquad (11)$$

$$\left(\partial_x^2 - \left[\frac{s^2}{c^2} + \Upsilon(x)\right]\right)\left[R(x)\,\widehat{p}(x, s)\right] = 0, \qquad (12)$$

respectively. These equations admit analytic solutions for Υ-constant profiles, which include

- $\Upsilon < 0$: convex pipes, $\quad R(x) = A \cos(\sqrt{-\Upsilon}x) + B \sin(\sqrt{-\Upsilon}x)$,
- $\Upsilon = 0$: cylinders, cones, $R(x) = A + Bx$,
- $\Upsilon > 0$: flared pipes $\quad R(x) = A \cosh(\sqrt{\Upsilon}x) + B \sinh(\sqrt{\Upsilon}x)$.

The solution is given by, for all $s \in \mathbb{C}_0^+ = \left\{s \in \mathbb{C}, \Re e(s) > 0\right\}$,

$$\widehat{X}(L, s) = \mathbf{T}(L, s) X(0, s), \qquad (13)$$

where the transfer matrix $\mathbf{T}$ has determinant one and can be written under the following factorized way

$$\mathbf{T}(L, s) = \Lambda(L, s)\, \mathbf{M}(L, s)\, \Lambda(0, s)^{-1}, \text{ with } \Lambda(x, s) = \text{diag}\left(\frac{L}{R(x)}, \frac{\pi R(x)}{\rho s}\right).$$

The coefficients of matrix $\mathbf{M}$ can be written under the "compact form" $[\mathbf{M}(L, s)]_{i,j} = \left(V_{i,j}(s)\right)^T \Phi\left(L\Gamma(s)\right)$ with

$$V_{1,1} = \begin{bmatrix} 1 \\ \sigma_0 \end{bmatrix}, \quad V_{1,2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad V_{2,1} = \begin{bmatrix} \sigma_L - \sigma_0 \\ \sigma_0 \sigma_L - (L\Gamma)^2 \end{bmatrix}, \quad V_{2,2} = \begin{bmatrix} 1 \\ -\sigma_L \end{bmatrix},$$

$\Phi(z) = \left[\cosh z, \frac{\sinh z}{z}\right]^T$, where Γ is the analytic continuation over $\mathbb{C}_0^+$ of the positive square-root on $\mathbb{R}^+$ of

$$\Gamma(s)^2 = \left(\frac{s}{c}\right)^2 + \Upsilon, \qquad (14)$$

and where the dimensionless quantity $\sigma = \frac{R'}{R/L}$ can be interpreted as a ratio of slopes.

Remark 2 (Wavenumber) *The wavenumber k is such that $ik = \Gamma(i\omega)$ (the Laplace variable is chosen on the Fourier axis $s = i\omega$ where ω denote the angular frequency). For straight and conical pipes ($\Upsilon = 0$), the wavenumber simply becomes $k = \omega/c$.*

3 Passivity for the Kirchhoff's variables

In the sense of dynamical systems theory, a passive system is defined as follows (see e.g. [7, def. 6.3]).

Definition 1 (Passivity) *A dynamical system with input U, output Y (with dim U = dim Y) and state Z is passive, if there exists a differentiable storage function $\mathcal{V}$ such that*

$$\frac{d}{dt}\mathcal{V}(Z) \leq U^T Y, \quad \text{where } \mathcal{V} \text{ is positive semidefinite,} \quad (15)$$

that is, where $\mathcal{V}(Z) \geq 0$ for all Z and $\mathcal{V}(0) = 0$. Moreover, it is said lossless if $\frac{d}{dt}\mathcal{V}(Z) = U^T Y$.

Result 1 (1D horn model for the Kirchhoff's variables)
It is clear that the system defined by (7) and with (e.g.)

input: $\quad U(t) = \left[u(0, t), -u(L, t)\right]^T$, $\qquad$ *(incoming airflows)*

output: $\quad Y(t) = \left[p(0, t), p(L, t)\right]^T$, $\qquad$ *(pressure at boundaries)*

state: $\quad Z \; : t \longmapsto X(\cdot, t) \in L^2(0, L)$, *(Kirchhoff variables (6))*
is passive and lossless, because of (10).

Indeed, in this case, a storage function is defined from the energy by (see (9))

$$\mathcal{V}\left(X(\cdot, t)\right) = \frac{1}{2}\int_0^L X(x, t)\, W(x)\, X(x, t)\, dx. \qquad (16)$$

It is positive definite on $L^2(0, L)$ since for all $x \in (0, L)$, $W(x)$ is a positive definite matrix. Moreover, according to remark 1, this result is also available for the exact models of planar waves in straight pipes and spherical waves in conical pipes.

Link between passivity and stability. The passivity property (15) guarantees that, when the input is zero, the quantity $\mathcal{V}(X)$ stored by the system (here, the energy) cannot increase. Moreover, the lossless property guarantees that the energy is time-invariant when the input is zero. This ensures a "system stability" in the sense that the trajectories of the state Z live in a bounded domain (characterized by elements whose $L^2(0, L; W)$-norm with weight W, namely, the square-root of (16), is smaller than a finite value $\sqrt{E_{max}}$).

Preserving this property and this behaviour in simulations is an issue which is relevant from both the physical and the numerical points of view. Such considerations and dedicated methods have been extensively used for sound synthesis purposes (see [8] and references therein). Detailed results and

theorems on the so-called "Lyapunov stability" for passive systems can also be found in [7].

The issue addressed in the next section is concerned with the passivity of digital waveguides, which are efficient computational models for cascaded straight or conical pipes [9, 10] as well as pipes with Υ-constant profiles [11, 12]. As digital waveguides do not govern the Kirchhoff's variables but some "travelling waves", the passivity of these systems must be established with respect to these new variables.

4 Passivity for basic travelling waves

4.1 "Planar-like" travelling waves

Consider the change of variables $X_p = [p^+, p^-]^T = P_p^{-1} X$ given by

$$\begin{bmatrix} p^+(x,t) \\ p^-(x,t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & \frac{\rho c}{S(x)} \\ 1 & -\frac{\rho c}{S(x)} \end{bmatrix} \begin{bmatrix} p(x,t) \\ u(x,t) \end{bmatrix}. \quad (17)$$

According to (7) and since $\partial_x(P_p X_p) = (\partial_x P_p) X_p + P_p (\partial_x X_p)$, it follows that X_p is governed by

$$\left(\partial_t \pm c\partial_x \right) p^\pm(x,t) = c\zeta(x) \left(p^-(x,t) - p^+(x,t) \right) \text{ where } \zeta = \frac{R'}{R}. \quad (18)$$

In straight pipes ($\zeta = 0$), $p^\pm$ define decoupled travelling waves, which are functions of $t \mp x/c$. In this case, the change of variables (17) diagonalizes M into $M_p = P_p^{-1} M P_p^{-1} = \text{diag}(c, -c)$ (see (7)), as well as transfer matrix T (see (13)) into $T_p(L,s) = P_p^{-1} T(L,s) P_p = \text{diag}(e^{-\frac{s}{c}L}, e^{+\frac{s}{c}L})$, where $e^{-\frac{s}{c}L}$ corresponds to a delay operator of time L/c.

In the general case, $p^\pm$ define locally-travelling waves that are coupled through the right hand-side of (18).

Moreover, the (lossless) passivity still holds for X_p in the sense of definition 1, as stated below.

Result 2 (Passivity for X_p) *The acoustic energy is a positive semidefinite function of the state X_p. It is given by*

$$\mathcal{V}_p\left(X_p(\cdot, t) \right) = \mathcal{V}\left(P_p(\cdot) X_p(\cdot, t) \right),$$

which leads to (16) where W is replaced by the definite positive matrix $W_p(x) = P_p(x)^T W(x) P_p(x) = \frac{S(x)}{\rho c^2} I_2$.

A consequence is that the digital waveguide decompositions obtained by simulating X_p for some cascaded straight pipes lead to stable simulations.

The detailed formula of the energy is

$$E(t) = \mathcal{V}_p\left(X_p(\cdot, t) \right) = \int_0^L \frac{p^+(x,t)^2 + p^-(x,t)^2}{\rho c^2} S(x) \, dx,$$

and the power supplied to the pipe (at boundaries) is

$$E'(t) = U^T(t)Y(t) = \left[S(x) \frac{p^-(x,t)^2 - p^+(x,t)^2}{\rho c} \right]_{x=0}^{x=L}.$$

4.2 "Spherical-like" travelling waves

Now, consider the change of variables $X_{sph} = [\psi^+, \psi^-]^T = P_{sph}^{-1} X$ defined in the Laplace domain by (see [11, 13, 14])

$$\begin{bmatrix} \widehat{\psi^+}(x,s) \\ \widehat{\psi^-}(x,s) \end{bmatrix} = \frac{R(x)}{2} \begin{bmatrix} 1 - \frac{c}{s}\zeta(x) & \frac{\rho c}{S(x)} \\ 1 + \frac{c}{s}\zeta(x) & -\frac{\rho c}{S(x)} \end{bmatrix} \begin{bmatrix} p(x,t) \\ u(x,t) \end{bmatrix}. \quad (19)$$

Notice that, in the time domain, P_{sph} and P_{sph}^{-1} are time operators where $\frac{1}{s}$ corresponds to a time integration ∂_t^{-1}. According to (11), it follows that, in the time domain, X_{sph} is governed by

$$\left(\partial_t \pm c\partial_x \right) p^\pm(x,t) = -\frac{c^2 \Upsilon(x)}{2} \partial_t^{-1} \left(\psi^-(x,t) - \psi^+(x,t) \right). \quad (20)$$

In straight and conical pipes ($\Upsilon = 0$), $\psi^\pm$ define decoupled travelling waves, which are functions of $t \mp x/c$. Similarly to section 4.1, this leads to diagonal forms of matrix M and transfer matrix T. Hence, X_{sph} introduces a class of decoupled travelling waves which is larger than X_p.

However, contrarily to X_p, the passivity for X_{sph} can be lost for some geometrical configurations. Indeed, as detailed in [14], an exact derivation of the acoustic energy as a function of X_{sph} yields (variables x and t are omitted for sake of conciseness)

$$E = \frac{\pi}{\rho c^2} \int_0^L \left[\psi^{+2} + \psi^{-2} + \frac{c^2 \Upsilon}{2} \left(\partial_t^{-1}(\psi^+ + \psi^-) \right)^2 \right] dx$$
$$- \frac{c^2}{2\rho} \left[\zeta \left(\partial_t^{-1}(\psi^+ + \psi^-) \right)^2 \right]_{x=0}^{x=L}. \quad (21)$$

To our knowledge, this formula is new. It shows that no obvious positive semidefinite function $\mathcal{V}_{sph}$ of $X_{sph}(\cdot, t)$ appears, to describe the energy, because the additional quantity $\partial_t^{-1}(\psi^+ + \psi^-)$ is involved in the formula. Moreover, even considering an "extended variable" composed of this quantity and $X_{sph}(\cdot, t)$, (21) is not guaranteed to be positive semidefinite, if Υ is negative (locally convex pipe).

For straight pipes (Υ and ζ are zero), the passivity for X_p is obvious, choosing the weight $W_{sph} = \frac{\pi}{\rho c^2} I_2$ to define the storage function.

For conical pipes, the passivity is not guaranteed because of the last term in (21). However, examining the junction of two conical pipes provides a new interpretation, based on energy and passivity, of a well-known phenomena: the instability in simulations with X_{sph} of conical digital waveguides which are connected with a convex junction (see e.g. [15]).

Result 3 (Passivity at junctions of conical pipes for X_{sph}) *Consider two conical segments ⓐ and ⓑ, continuously connected at $x = L_a$ (see Figure 1). Since $\psi^+ + \psi^- = R p$, this quantity is continuous at the junction $x = L_a$, as well as its time integration. Using (21), the total energy $E_{tot} = E_a + E_b$ is given by,*

$$E_{tot} = E\Big|_{L=L_a+L_b} + \frac{\pi}{\rho c^2}\left((\zeta_b - \zeta_a)\left(\partial_t^{-1}(\psi^+ + \psi^-) \right)^2 \right)\Big|_{x=L_a}$$

Hence, the junction at $x = L_a$ is passive if the following condition is fulfilled

$$\zeta_b(L_a) - \zeta_a(L_a) \geq 0, \quad (22)$$

that is, for a non convex junction.

The case of convex junctions requires a special treatment to remove the instability: a possible solution consists in deriving a so-called "minimal realization" (in the sense of automatic control theory), as detailed in [16].

4.3 Summary

The passivity of planar-like travelling waves holds even for non straight pipes. That of spherical-like travelling waves
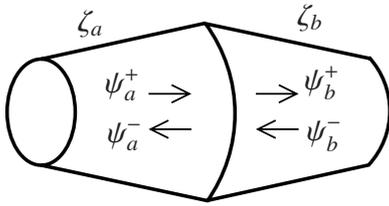
Figure 1: Convex junction of cones

holds for straight, conical and flared profiles, if there is no convex junction of these segments.

Except for straight and conical pipes, these travelling waves are coupled. The next section is concerned with decoupled travelling waves in Υ-constant segments leading to stable systems for all physically meaningful geometries.

5 Decoupled travelling waves in Υ-constant segments

Consider a Υ-constant segment and define Γ as in (14).

5.1 Locally decoupled waves

The change of variables $X_q = [q^+, q^-]^T = P_q^{-1} X$ proposed in [11] and defined in the Laplace domain by

$$\begin{bmatrix} q^+(x,s) \\ q^-(x,s) \end{bmatrix} = \frac{R(x)}{2} \begin{bmatrix} 1 - \frac{\zeta(x)}{\Gamma(s)} & \frac{\rho c}{S(x)} \\ 1 + \frac{\zeta(x)}{\Gamma(s)} & -\frac{\rho c}{S(x)} \end{bmatrix} \begin{bmatrix} p(x,t) \\ u(x,t) \end{bmatrix}, \qquad (23)$$

provides locally-in-space decoupled waves that are governed by

$$\left(\Gamma(s) \pm \partial_x\right) q^{\pm}(x,t) = 0. \qquad (24)$$

The associated transfer matrix is

$$T_q = \mathrm{diag}\left(\exp(-\Gamma(s)L), \exp(\Gamma(s)L) \right).$$

A complex analysis of $\exp(-\Gamma(s)L)$ proves that it is the transfer function of a stable causal delayed operator, if $\Upsilon \geq 0$. The smooth connection of Υ-constant pipes leads to a digital waveguide framework for which efficient simulations are available (see [11] for a detailed study).

However, for convex profiles (resp. $\Upsilon < 0$), this definition of waves leads to unstable systems. A special treatment based on a signal processing approach can restore stable systems and simulations (see [11, chap. 4.3]), but these methods have no clear physical foundations.

However, the study of the transfer matrix T (see (13)) reveals that it defines a stable system for $\Upsilon < 0$ as far as the bore profile corresponds to a physically meaningful geometry, for which $R(x) > 0$. This gives a clue to understand why T_q yields unstable systems: it models the propagation for a finite length L but in an unbounded domain (e.g. $x \in \mathbb{R}$). Now, this unbounded domain does not corresponds to a physically meaningful geometry for $\Upsilon < 0$ since it defines a medium with infinitely many parts on which $R(x)$ is negative.

In order to cope with this problem, a change of variables has been looked for, such that it diagonalizes the transfer matrix T for a (finite) segment with a positive profile, rather than locally decoupling waves as (23) do.

5.2 Positive profiles

The three profile models described in section 2.4 can be unified by $R(x) = R(0) S_{L^2\Upsilon}\left(1 - \frac{x}{L}\right) + R(L) S_{L^2\Upsilon}\left(\frac{x}{L}\right)$, where $S_\nu(\xi) = \frac{\sinh(\sqrt{\nu}\xi)}{\sinh \sqrt{\nu}}$ is a C^∞-regular positive function for all $(\nu, \xi) \in \mathcal{D} = (-\pi^2, +\infty) \times (0,1)$. Such a profile is positive on $(0, L)$ iff

$$L^2\Upsilon > -\pi^2 \text{ and } R(0), R(L) \text{ are positive.}$$

In this case, we introduce the coefficient of asymmetry $\theta = \ln \frac{R(L)}{R(0)}$ and the geometrical average $\rho = \sqrt{R(0)R(L)}$, so that $R(0) = \rho e^{-\theta/2}$ and $R(L) = \rho e^{\theta/2}$. A basic study gives the nature of the profiles w.r.t. $L^2\Upsilon$ and θ^2, which is summarized in Table 1 and Figure 2.

$L^2\Upsilon$	$-\pi^2$	0	$+\infty$
$\theta^2 < L^2\Upsilon$	$\times$	$\times$	cosh-type
$\theta^2 = L^2\Upsilon$	$\times$	straight pipe	exp-type
$\theta^2 > L^2\Upsilon$	convex	conical	sinh-type

Table 1: Categorization of profile types w.r.t. θ and $L^2\Upsilon$.
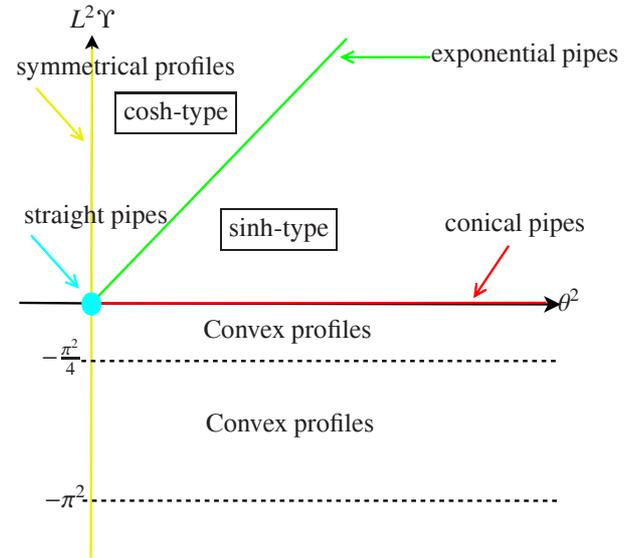
Figure 2: Categorization of profile types w.r.t. θ and $L^2\Upsilon$.

5.3 Diagonalization of the scattering matrix T

As detailed in [17], the eigenvalues of T (see (13)) are the roots $(\lambda(s), \lambda(s)^{-1})$ of the characteristic polynomial

$$P(\lambda) = \lambda^2 - 2 b' \lambda + 1 \text{ with } b' = \phi_1\left((L\Gamma)^2\right) + F(\theta, L^2\Upsilon)\,\phi_2\left((L\Gamma)^2\right),$$

$$F(\theta, L^2\Upsilon) = \frac{\phi_1(\theta^2) - \phi_1(L^2\Upsilon)}{\phi_2(L^2\Upsilon)}, \text{ and } \phi_1(z) = \cosh\sqrt{z},\ \phi_2(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}}.$$

A complex analysis of the solution $\lambda = b'\left(1 - \sqrt{\frac{b'^2-1}{b'^2}}\right)$ w.r.t. s reveals that λ has no singularities in $\mathbb{C}_0^+$ for all positive profiles and that it corresponds to a delayed causal stable operator. When F is zero ($\theta = L^2\Upsilon$, red line in Figure 3), the profiles are of exponential types and $\lambda(s) = \exp\left(-L\Gamma(s)\right)$, as for locally decoupled waves. This is not the case otherwise.

A sequence (w.r.t. $L^2\Upsilon$) of Bode diagrams are plotted in Figure 3, for the case of symmetrical profiles ($\theta = 0$).
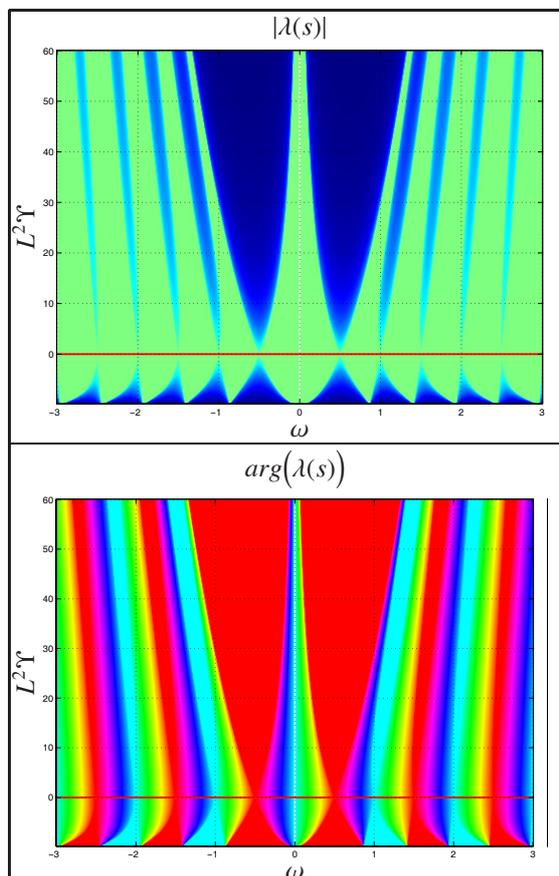
Figure 3: Bode diagrams w.r.t. ω and $L^2\Upsilon$, for $\theta = 0$. Top: modulus (green:1, dark blue:0). Bottom: phase (circular color map). The red line displays the reference $L^2\Upsilon = \theta^2$.

At the reference $L^2\Upsilon = \theta^2 = 0$ (straight pipe), $\lambda(s) = e^{-sL/c}$ is an ideal delay. When $L^2\Upsilon$ moves away from 0, propagator λ restrains some energy on a set of frequency ranges (dark blue areas on the modulus). It does not amplifies any area. Moreover, this feature can be observed for both flared and convex pipes, as expected. A next step could address the complex analysis of λ and passivity.

6 Conclusion and perspectives

In this paper, the passivity of the acoustic propagation in a horn has been investigated for planar-like ($p^\pm$) and spherical-like ($\psi^\pm$) travelling waves. The acoustic energy, derived as a function of $\psi^\pm$, leads to a new interpretation of the instability which happens when simulating a cascade of conical pipes including convex junctions, with digital waveguides. This problem is also known to happen with (more refined) locally-decoupled travelling waves in smooth convex pipes. To cope with this problem, a new definition of waves that are globally decoupled for a segment is looked for, by diagonalizing the acoustic transfer matrix in the Laplace domain. As a main result, the eigenvalues define stable delayed propagation operators for all positive Υ-constant profiles.

Future work will consist in: defining the globally decoupled waves by choosing adapted candidates for the eigenvectors; building the storage function associated with these waves to examine the passivity; and building accurate approximations that preserve passivity in order to derive an efficient digital waveguide framework.

References

[1] A. Chaigne and J. Kergomard. *Acoustique des instruments de musique*. Belin, 2008.

[2] J. L. Lagrange. Nouvelles recherches sur la nature et la propagation du son. Misc. Taurinensia (Mélanges Phil. Math., Soc. Roy. Turin), 1760-1761.

[3] D. Bernoulli. Physical, mechanical and analytical researches on sound and on the tones of differently constructed organ pipes. Mém. Acad. Sci. (Paris), 1762.

[4] A. G. Webster. Acoustical impedance, and the theory of horns and of the phonograph. *Proc. Nat. Acad. Sci. U.S.*, 5:275–282, 1919. Errata, ibid. **6**, p.320 (1920).

[5] T. Hélie. Mono-dimensional models of the acoustic propagation in axisymmetric waveguides. *J. Acoust. Soc. Am.*, 114:2633–2647, 2003.

[6] T. Hélie, T. Hézard, R. Mignot, and D. Matignon. On the 1d wave propagation in wind instruments with a smooth profile. In *Forum Acusticum*, volume 6, pages 1–6, Aalborg, Danemark, Juillet 2011.

[7] H. K. Khalil. *Nonlinear systems*. Prentice Hall, Upper Saddle River, NJ 07458, 3rd ed. edition, 2002.

[8] S. Bilbao. *Numerical sound synthesis*. Wiley, 2009.

[9] J. O. Smith. Physical modeling using digital waveguides. *Computer Music Journal*, 16:74–91, 1992.

[10] J. O. Smith. *Applications of Digital Signal Processing to Audio and Acoustics*, chapter Principles of Digital Waveguide Models of Musical Instruments, pages 417–466. Kluwer Academic Publishers, 1998.

[11] R. Mignot. *Réalisation en guides d ondes numériques stables d un modèle acoustique réaliste pour la simulation en temps-réel d instruments à vent*. Thèse de doctorat, Edite de Paris - Telecom ParisTech, Paris, 2009.

[12] R. Mignot, T. Hélie, and D. Matignon. From a model of lossy flared pipes to a general framework for simulation of waveguides. *Acta Acustica*, 97:477–491, 2011.

[13] T. Hélie. Ondes découplées et ondes progressives pour les problčmes mono-dimensionnels d'acoustique linéaire. In *CFA*, volume 8, pages 1–4, Tours, France, Avril 2006.

[14] L. Delebecque. Etude de la passivité dans les pavillons acoustiques pour la simulation stable en guides d'ondes. Master's thesis, Univ. Paris 6, 2011. ATIAM.

[15] J. Gilbert, J. Kergomard, and J. D. Polack. On the reection functions associated with discontinuities in conical bores. *J. Acoust. Soc. Am.*, 87:1773–1780, 1990.

[16] R. Mignot, T. Hélie, and D. Matignon. Stable realization of a delay system modeling a convergent acoustic cone. In *Mediterranean Conference on Control and Automation*, pages 1574–1579, Ajaccio, France, 2008.

[17] T. Hézard. Ondes découplées et ondes progressives dans les tubes acoustiques f́ section variable pour la représentation en guides d'ondes. Master's thesis, Univ. Paris 6, 2010. ATIAM.