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Absorbing boundary conditions for anisotropic
elastodynamic media

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1 Introduction

Reverse Time Migration (RTM) technique produces images of the subsurface thanks to the propagation of waves. We focus on Tilted Transverse Isotropic (TTI) media which come into the category of anisotropic media. The finite element problem of this solution requires the design of efficient boundary conditions that are able to attenuate possible artificial reflected waves generated by the boundaries of the computational domain. In the case of elastic waves, Perfectly Matched Layers (PMLs) are widely used. However, instabilities may appear in the layers, either due to the numerical discretization or to the continuous PML problem as for the TTI case. But RTM is based on the cross-correlation of the source propagation. Thus artificial reflected waves should not be taken into account by the imaging condition. In this work, we propose to consider an Absorbing Boundary Condition (ABC) and to study how it performs in the RTM framework. Moreover, it is easily included in a high-order Discontinuous Galerkin Method (DGM) formulation. Herein we limit our study to a low-order ABC and numerical experiments with a DGM scheme illustrate the good performance of this condition, which is new for the TTI case.

2 Anisotropic elastodynamics

Let $\Omega$ be an open bounded domain of $\mathbb{R}^2$, its boundary is denoted by $\Gamma = \partial \Omega$ with the exterior normal $\nu_r$. Let $x = (x, z) \in \Omega$ and $t \in [0, T]$ be the space and time variables, the velocity-stress formulation\(^\text{1}\) of the elastic wave equation is

$$
\begin{align}
\rho(x)\partial_t v(x, t) &= \nabla \sigma(x, t) \\
\partial_t \sigma(x, t) &= C(x) : \varepsilon(v(x, t))
\end{align}
$$

with $\rho > 0$ the volumic mass, $v \in H^1(\Omega \times [0, T])$ the unknown velocity field, $\sigma$ such that $[\sigma_{ij}] \in H^1(\Omega \times [0, T])$ the stress tensor, $C$ the stiffness tensor (elasticity coefficients) and $\varepsilon(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$ the strain tensor (infinitesimal strain theory) with $\nabla$ the divergence and $\nabla$ the gradient.

In geophysics, Earth’s crust (geological layers of rocks) is assumed to be locally polar anisotropic, also called transversely isotropic\(^\text{2}\). The difference with full anisotropy lies in the elastic tensor of the medium. This stiffness tensor is fourth-rank, i.e. composed of elements $C_{ijkl}$ with $i, j, k, l \in \{1, 2\}$ meaning $[x, z]$. Since it is symmetric in $(i, j)$ and $(k, l)$, it can be viewed as a matrix in 2D Voigt notation\(^\text{3}\),

$$
C = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{12} & C_{22} & C_{23} \\
C_{13} & C_{23} & C_{33}
\end{bmatrix}.
$$

Wave equation is usually studied in isotropic media to simplify the computations. In these cases, materials are characterized by the two Lamé coefficients $\lambda, \mu$ and

$$
C_{11} = \lambda + 2\mu, \ C_{22} = \lambda + 2\mu, \ C_{33} = \mu, \ C_{12} = \lambda.
$$

This can be written with the P-waves and S-waves velocities $V_p$ and $V_s$,

$$
C_{11} = \rho V_p^2, \ C_{22} = \rho V_s^2, \ C_{33} = \rho V_s^2, \ C_{12} = \rho (V_p^2 - 2V_s^2).
$$

Geophysicists are interested in transversely isotropic media to improve the accuracy of the results in the wave simulation step of the RTM. When the symmetry axis is vertical (resp. horizontal), it is called Vertical (resp. Horizontal)

\(^\text{1}\)the velocity-stress formulation is the first-order hyperbolic system of the wave equation defined in Virieux \cite{10} (instead of second-order terms in the displacement-stress formulation)

\(^\text{2}\)transversely isotropic material has identical properties along any direction perpendicular to a special one, which is called the symmetry axis

\(^\text{3}\)2D Voigt notation $ij(v)$: $xx(1)$, $zz(2)$, $xz(3)$
Transverse Isotropy (VTI, resp. HTI). When symmetry axis is away from the vertical one, with a measured angle $\theta$, it is called Tilted Transverse Isotropy (TTI). Different behaviours of the wavefronts in TI media are represented in Figure 1.

![Wavefronts for isotropic and transversely isotropic (vertical and tilted) media](image)

**Figure 1:** Wavefronts for isotropic and transversely isotropic (vertical and tilted) media

In order to characterize VTI, Thomsen [7] defined three medium parameters: $\epsilon$, $\delta$, $\gamma$ (the last concerns 3D case only) and the tensor reads as

$$C_{11} = \rho V_p^2 (1 + 2\epsilon), \quad C_{22} = \rho V_p^2, \quad C_{33} = \rho V_s^2,$$

$$C_{12} = \rho \left( \sqrt{V_p^2 - V_s^2} + 2\delta V_p^2 - V_s^2 \right).$$

TTI media are characterized by the VTI coefficients and the angle $\theta$. The TTI 2D tensor is computed from the VTI one by the rotation

$$C^{TTI}_{ijkl} = \sum_{p=1}^{2}\sum_{q=1}^{2}\sum_{s=1}^{2}\sum_{t=1}^{2} R_{pi} R_{pj} R_{qk} R_{sl} C^{VTI}_{pqrs} \forall \{i,j,k,l\} \quad (3)$$

with

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$ 

In this case, the elastic tensor becomes fully non-zero (but still symmetric).

### 3 Discontinuous Galerkin Method

Let $\Omega$ be a polygonal mesh (composed of triangles $K$ in 2D) forming a partition of $\Omega$. Contrary to FEM, the principle of DGM consists in approximating the unknowns $v$ and $\sigma$ by discontinuous functions $v_h$ and $\sigma_h$ such that $\{v_h, \sigma_h\} \in L^2(\Omega_h \times [0, T])$, $v_h|_K \in H^1(K \times [0, T])$ and $\sigma_h|_K \in H^1(K \times [0, T])$. The integration against discontinuous test functions $w$ and $\xi$ is performed locally on each triangle,

$$\sum_k \int_K \rho_h \partial_t w v_h d\Omega = \sum_k \left( \int_{\Gamma_h} [\sigma_h] [v] d\Gamma - \int_K \sigma : \nabla w_{\Omega} d\Omega \right) \quad (4)$$

$$\sum_k \int_{\Gamma_h} \sigma : \xi d\Gamma = \sum_k \left( \int_{\Gamma_h} [C : \xi] n_{\Omega} [v] d\Gamma - \int_K \nabla (C : \xi) w_{\Omega} d\Omega \right),$$

with the centered fluxes formulation (i.e. the jump and mean formula chosen such that: $[u] = \frac{u_{K_1} + u_{K_2}}{2}$, $[u] = u_{K_i} - u_{K_j}$).

These equations can be written as a linear system,

$$\begin{align*}
\rho M_e \partial_t v_h + R_e \sigma_h &= 0 \\
M_e \partial_t \sigma_h + R_e v_h &= 0 \quad (5)
\end{align*}$$

with $M_e$, $M_e$ block-diagonal mass matrices and $R_e$, $R_e$ stiffness matrices.

The time discretization is done with the classical leapfrog time scheme. It is an explicit scheme which is stable under a CFL condition\(^4\) available in [4].

Time domain $[0, T]$ is divided into time steps $\Delta t$. Let $v_h^n$ be the approximation of the velocity $v_h(t)$ at the discrete time $t = n\Delta t$ and $\sigma_h^{n+1/2}$ the approximation of the stress tensor $\sigma_h(t)$ at the discrete time $t = (n + \frac{1}{2})\Delta t$. Semi-discrete linear system (5) becomes

$$\begin{align*}
\rho M_e \frac{v_h^{n+1} - v_h^n}{\Delta t} + R_e \sigma_h^{n+1/2} &= 0 \\
M_e \frac{\sigma_h^{n+1/2} - \sigma_h^{n}}{\Delta t} + R_e v_h^{n+1} &= 0
\end{align*} \quad (6)$$

Since $M_e$ and $M_e$ are block-diagonal due to DGM, this scheme is quasi-explicit.

### 4 VTI ABC

In order to simplify the presentation of the results, we only consider two cases, a rectangle mesh and a rhombus mesh (actually a rectangle mesh rotated by an angle $\pi/6$). We will formulate low-order ABCs on these meshes.

![Rectangle and rhombus meshes](image)

**Figure 2:** Rectangle and rhombus meshes

We begin with the rectangular mesh because it is easier to work on vertical and horizontal boundaries. To formulate VTI ABCs we decompose problem (1) in P-waves (resp. S-waves) by setting the velocity $V_p = 0$ (resp. $V_p = 0$). It forms two anisotropic acoustic wave equations for which we can easily construct ABCs by applying Enquist-Majda’s methodology [5, 6]. For the P-waves, we obtain a condition on $\sigma_{xx}$ on vertical boundaries and a condition on $\sigma_{zz}$ on horizontal boundaries. For S-waves, we obtain a condition on $\sigma_{zz}$ on all boundaries. These VTI ABCs are presented in Table 1, with $C_p_{\text{max}} = -\sqrt{\rho C_{11}}$, $C_p_{\text{min}} = -\sqrt{\rho C_{33}}$ and $C_S = -\sqrt{\rho C_{33}}$. In the isotropic case, we recover the low-order ABCs proposed by Tsokga [8] (called order 1/2).

Let us introduce $\mathbf{u} = (u_x, u_z)$ the normal vector exterior of each face of $\Gamma$ (i.e. $\mathbf{n}$ varies depending on the face) and the associated basis ($\mathbf{n}, \mathbf{t}$), as shown in Figure 2.

\(^{4}\)Courant-Friedrichs-Lewy condition is an inequality concerning time and space discretization steps. It has to be respected in order to assure numerically convergence.
The boundary conditions are localized on the boundary of the mesh $\Omega_b$. In the 2D case, it concerns the edges $\Gamma$ of the triangles $K$ in contact with the mesh boundary. Thus, in the numerical scheme the boundary conditions appear in the surface integral term, i.e. in the integral on $\Gamma$. Note that the DGM formulation (4) contains two surface integral terms. One is in the first equation, reading as an integral of $\sigma_{nn}$ and one is in the second equation, reading as an integral of $\nu$. The ABC can be included in the formulation by plugging the equation of Table 1 into one of these terms. We arbitrarily decided to consider the first one:

$$\int_{\Gamma} \sigma_{nn} \nu \cdot w d\mathbf{x}. \quad (7)$$

In order to obtained ABCs for an arbitrary face of normal $\mathbf{n}$, it is useful to decompose $\sigma$ in the $(\mathbf{n}, \tau)$ basis:

$$\sigma \cdot \mathbf{n} = \begin{pmatrix} \sigma_{nn} \\ \sigma_{tn} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \sigma_{nn} \\ \sigma_{tn} \end{pmatrix}. \quad \text{So it is sufficient to consider } \sigma_{nn} \text{ and } \sigma_{tn} \text{ to formulate an ABC available for any faces of the mesh. The formulation of Table 1 in the \((\mathbf{n}, \tau)\) basis can be rewritten as}

$$\begin{cases}
\sigma_{nn} = C_{p_{\text{max}}} \mathbf{V} \cdot \mathbf{n}, \text{ on boundaries } \{2, 4\}, \\
\sigma_{nn} = C_{p_{\text{min}}} \mathbf{V} \cdot \mathbf{n}, \text{ on boundaries } \{1, 3\}, \\
\sigma_{tn} = C_{s} \mathbf{V} \cdot \tau, \text{ on boundaries } \{1, 2, 3, 4\}.
\end{cases} \quad (8)$$

Let us remark that in the VTI ABCs (8) we use $C_{p_{\text{max}}}$ on the boundaries $\{2, 4\}$ and $C_{p_{\text{min}}}$ on the boundaries $\{1, 3\}$. In order to obtain VTI ABCs on an arbitrary face, it seems natural to defined a velocity $C_{p_{n}}$ depending on the face exterior normal direction $\mathbf{n}$. As TI media form ellipsoidal wavefronts (see Figure 1) we propose to consider

$$C_{p_{n}} = -\sqrt{C_{p_{\text{max}}}^{2} n_{x}^{2} + C_{p_{\text{min}}}^{2} n_{z}^{2}}. \quad (9)$$

And we formulate the prototype VTI ABC:

$$\begin{cases}
\sigma_{nn} = C_{p_{n}} \mathbf{V} \cdot \mathbf{n} \\
\sigma_{tn} = C_{s} \mathbf{V} \cdot \tau
\end{cases} \quad (10)$$

They are then incorporated in the surface integral term (7),

$$\int_{\Gamma} \begin{pmatrix} \sigma_{nn} \\ \sigma_{tn} \end{pmatrix} \cdot \begin{pmatrix} w_{n} \\ w_{t} \end{pmatrix} = \int_{\Gamma} \begin{pmatrix} \sigma_{nn} \\ \sigma_{tn} \end{pmatrix} \cdot \begin{pmatrix} w_{n} n_{x} + w_{z} n_{z} \\ -w_{n} n_{x} + w_{z} n_{z} \end{pmatrix} = \int_{\Gamma} \left[ (C_{p_{n}} \mathbf{V} \cdot \mathbf{n} n_{x} - C_{s} \mathbf{V} \cdot \tau n_{z}) w_{x} + (C_{p_{n}} \mathbf{V} \cdot \mathbf{n} n_{z} + C_{s} \mathbf{V} \cdot \tau n_{x}) w_{z} \right] \quad (11)$$

### Table 1: VTI ABCs on the rectangle boundaries

<table>
<thead>
<tr>
<th>Boundary</th>
<th>P-waves</th>
<th>S-waves</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\sigma_{zz} = -C_{p_{\text{min}}} v_{z}$</td>
<td>$\sigma_{xz} = -C_{s} v_{x}$</td>
</tr>
<tr>
<td>2</td>
<td>$\sigma_{xx} = +C_{p_{\text{max}}} v_{x}$</td>
<td>$\sigma_{xz} = +C_{s} v_{z}$</td>
</tr>
<tr>
<td>3</td>
<td>$\sigma_{zz} = +C_{p_{\text{min}}} v_{z}$</td>
<td>$\sigma_{xz} = +C_{s} v_{x}$</td>
</tr>
<tr>
<td>4</td>
<td>$\sigma_{xx} = -C_{p_{\text{max}}} v_{x}$</td>
<td>$\sigma_{xz} = -C_{s} v_{z}$</td>
</tr>
</tbody>
</table>

### Numerical results of prototype VTI ABC

It is well-known that low-order ABCs generate spurious reflections. Our performance criterion to validate the VTI ABC formulation (10) is to not generate more reflected waves than those of the isotropic case, which will serve as reference case. Figure 3 depicts the evolution of the magnitude of $v$ at two time steps ($t = 1s$ and $t = 5s$) in an isotropic medium. It shows that the propagated wave is absorbed and the magnitude of the reflection waves is one power of ten less.
to S-waves. Indeed a P-waves source generates S-waves in a TI medium.

On the contrary, prototype VTI ABC induces additional reflected S-waves in the rhombus mesh, polluting thus the image, see Figure 5. Actually it is important that the VTI ABC performs as well in the rectangle case as in the rhombus case. Indeed VTI on a rhombus mesh is equivalent to TTI on a rectangular mesh with the corresponding TTI angle.

Therefore, we have to improve the prototype VTI ABC. To this aim, we rewrite the term (11) and we replace the coefficient \( C_p \) by \( C_{p_{\text{max}}} \) in the first part and by \( C_{p_{\text{min}}} \) in the second part:

\[
\int_\Gamma \left[ (C_{p_{\text{max}}} V \cdot n_x - C_s V \cdot \tau_x) w_x 
+ (C_{p_{\text{min}}} V \cdot n_z + C_s V \cdot \tau_z) w_z \right] \, dS
\]

(12)

This can be obtained from the new ABC VTI formulation:

\[
\begin{align*}
\sigma_{mn} &= (C_{p_{\text{max}}} n_j^2 + C_{p_{\text{min}}} n_j^2) V \cdot n + C_s V \cdot \tau \\
\sigma_{tn} &= -((C_{p_{\text{max}}} - C_{p_{\text{min}}}) n_j n_j) V \cdot n + C_s V \cdot \tau
\end{align*}
\]

(13)

### Numerical results of VTI ABC

Let us remark that the new VTI ABC coincides with the prototype one for horizontal and vertical boundaries. The results are thus the same in the rectangle mesh while the new VTI ABC performs well now in the rhombus mesh: additional S-wave reflections have disappeared, see Figure 6.

### 5 TTI ABC

As explained above, TTI simulation in a rectangular mesh (which is the interesting case for geophysics), is similar to VTI simulation in a rotated mesh. Thus, to extend the above VTI ABC to TTI, we define \( n' \) by (see Figure 7)

\[
\begin{pmatrix}
    n'_x \\
    n'_z
\end{pmatrix}
= \begin{pmatrix}
    \cos \theta \\
    \sin \theta
\end{pmatrix}
\]

We underline that \( n' \) is independent of the boundary (contrary to \( n \), the face exterior normal).

In the new VTI ABC (13), \((n_x, n_z)\) are the coordinates of \( n \) in the \((n', \tau')\) basis, which reduces in the VTI case to \((e_x, e_z)\). Let us introduce for TTI the vector \((l_x, l_z)\): the coordinates of \( n \) in the \((n, \tau)\) basis. Using the TTI angle \( \theta \) and the boundary angle \( \alpha \), defined in Figure 7, \((l_x, l_z)\) reads as

\[
\begin{pmatrix}
l_x \\
l_z
\end{pmatrix}
= \begin{pmatrix}
\cos(\alpha - \theta) \\
\sin(\alpha - \theta)
\end{pmatrix}
\]

Thus, following the same reasoning, the TTI ABC can be formulated as:

\[
\begin{align*}
\sigma_{mn} &= (C_{p_{\text{max}}} l_j^2 + C_{p_{\text{min}}} l_j^2) V \cdot n + C_s V \cdot \tau \\
\sigma_{tn} &= -((C_{p_{\text{max}}} - C_{p_{\text{min}}}) l_j l_j) V \cdot n + C_s V \cdot \tau
\end{align*}
\]

(14)

We specify that the coefficients \( C_{p_{\text{max}}}, C_{p_{\text{min}}}, C_s \) of the TTI ABC come from the VTI tensor. Thereby, they do not vary with the TTI angle \( \theta \) (contrary to the tensor in the TTI equation).

### Numerical results of TTI ABC

The TTI ABC performs well (as well as in the VTI and the isotropic cases) in the rectangular mesh, see Figure 8. Moreover, other tests show that it gives similar results on arbitrary meshes and for any \( \theta \).
6 Conclusion

We have proposed a new Absorbing Boundary Condition for Tilted Transverse Isotropic media which can be easily included in a Discontinuous Galerkin Method formulation with few additional computational costs. As all the low-order ABCs, it generates spurious reflections, but their amplitudes are comparable to the reflections produced by low-order ABCs in isotropic media. Since low-order ABCs do not impact the accuracy of Reverse Time Migration images in isotropic media, we expect the new ABC to be efficient in a RTM framework. This should be confirmed by the numerical tests we are now performing. In a future work, we will extend the methodology in order to obtain higher (and therefore more accurate) ABCs.

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References


