Choice of algorithms for reed instruments oscillations: how to solve the equation for the nonlinear characteristics?

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Choice of algorithms for reed instruments oscillations: how to solve the equation for the nonlinear characteristics?

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For the simplest model of clarinet-like instruments, leading to the iterated map scheme, the solving of the nonlinear characteristic equation involves the inversion of the modified characteristic function linking the incoming wave to the outcoming one. The inversion can be problematic when the reed opening is large, i.e. when the associated dimensionless parameter is larger than unity, the inverse function being multi-valued. The problem was already noticed by Gokhstein (1981) concerning oboes or bassoons. To overcome this difficulty, several discretization schemes can be found, e.g. by modeling a reed as a one degree of freedom oscillator, or by modeling the air in the mouthpiece as a simple spring. The use of these schemes at their limits allows studying how the choice between the solutions can be done when using the classical scheme with the multi-valued function for the incoming/outcoming waves. . The case of conical reed instruments is particularly investigated.

1 Introduction

A usual scientific method of investigation is the search for models with a minimum of parameters. This allows an easy understanding of the main phenomena. A famous example was given by the paper by McIntyre et al[1]. For reed instruments, these authors gave a general model, then applied it to clarinet-like instruments. In what follows, we will summarize some general ideas already published, then focus the attention on conical instruments. Obviously at least one supplementary parameter needs to be considered when studying conical instruments instead of cylindrical ones. Three main questions are investigated: i) the needs for discretization; ii) the possibility to keep a short history; iii) the problem of multivalued functions. In particular we will see how the latter problem, already discovered by Gokhstein[2], can be investigated. In practice, this problem seems to be a major attribute of double reed instruments.

2 General model of reed instruments

Since the work by McIntyre et al [1], it is well known that when ignoring the reed dynamics, an elementary model of reed instruments is based upon two equations:

$$p(t) = [h * u](t) \tag{1}$$
$$u(t) = F[p(t)]. \tag{2}$$

The two unknowns are the pressure $p(t)$ and the flow rate $u(t)$ at the entrance of the instrument. In the present paper, we consider dimensionless quantities. The impulse response $h(t)$, inverse FT of the input impedance $Z(\omega)$, is the characteristic of the resonator, assumed to be linear, and the nonlinear function $F(t)$ characterizing the nonlinear excitor is assumed to be quasi-static.

The same authors suggested to slightly modify these equations, in order to reduce the duration of the reflection history, by using the d'Alembert decomposition. If a cylindrical tube is considered at the input of the instrument, the two new unknowns are the incoming $p^-(t)$ and outcoming waves $p^+(t)$ at the input of the tube,

$$p^- = \frac{1}{2}(p-u);\ p^+ = \frac{1}{2}(p+u). \tag{3}$$

Doing that, Eq. (1) is replaced by:

$$p^-(t) = [r * p^+](t), \tag{4}$$

where $r(t)$ is the (plane) reflection function. Two advantages can be found: first, as soon as the length of the cylindrical section is not zero, the incoming wave is 0 at time $t = 0$; moreover, the history of $r(t)$ is in general much shorter than that of $h(t)$. The problem can be solved by solving at each time the following equation:

$$u(t) = p(t) + p_h(t) \tag{5}$$
$$\textit{with}\ p_h(t) = -2 * [r * p^+](t) \tag{6}$$

together with Eq. (2). This can be done by using a graphic method, searching for the intersection between a straight line and the nonlinear function F (see [3]). Some problems can arise when several intersections exist: this is discussed in section 4. Nevertheless this method is very general.

For a purely cylindrical instrument of length ℓ, with no losses and zero radiation impedance, the reflection function is a single delta function,

$$r(t) = -\delta(t - 2\ell/c), \tag{7}$$

and no discretization of the time variable is necessary when the initial condition is given by e.g. a single step of the excitation pressure (c is the sound speed). The search for the intersection of the line with the function F can be replaced by the solving of a modified function G, obtained by a rotation of 45° from the function F, and this leads to the calculation of an iterated map scheme (see [4, 5]). The function G is defined by

$$p_n^+ = G[-p_n^-] \tag{8}$$

and is obtained from Eqs. (2) and (3). For the classical model based upon the Bernoulli equation (and some hypotheses, see Ref. [6]), its analytical expression is given in Ref. [5]. This method can be extended to the case of the so-called Raman model (Ref. [7]), i.e. of a cylindrical tube with losses independent of frequency.

3 Statement of the problem for conical tubes

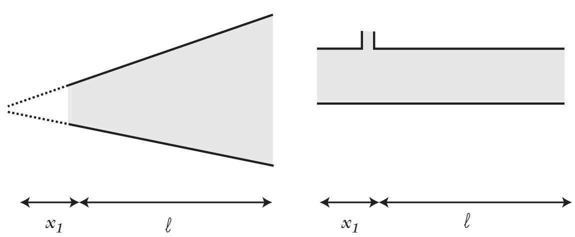

Figure 1: Truncated cone and the corresponding cylindrical saxophone

3.1 The cylindrical saxophone

For truncated cone resonators, we are seeking simple models and calculation methods generalizing as far as possible the iterated map algorithm. In what follows, we limit the discussion to lossless resonators and zero radiation impedance. The first idea is to use the approximation which is called "cylindrical saxophone" (for which a well known possible solution for the steady-state regime is the Helmholtz motion, see Ref. [8]). After some calculations from the input impedance $Z(\omega)$, the Fourier Transform of the global plane reflection function is found to be:

$$\begin{aligned} R(\omega) &= [Z(\omega) - 1] \,/\, [Z(\omega) + 1]\,, \; thus \quad (9) \\ R(\omega) &= 1 - \left[1 - e^{-2jk(x_1+\ell)/c}\right] \sum_{n=0}^{+\infty} \left[\frac{1}{2}\left(e^{-2jkx_1/c} + e^{-2jk\ell/c}\right)\right]^n . \end{aligned}$$

This expression exhibits the successive reflections with the delays $2x_1/c$ and $2\ell/c$, which are the durations of a round trip in the two parts of the tube (in this equation $k = \omega/c$). The plane reflection function remains of infinite extent. In practice its duration is shorter than that of the impulse response, thanks to the factor $(1/2)^n$, but we tried to find expression with a short history. Generalizing Eq. (7), the plane reflection function is used on both sides of the source. The length on the right and on the left are ℓ and x_1, respectively. Using the d'Alembert decomposition and the boundary conditions, the equations for the resonator are:

$$\begin{aligned} p(t) &= p^+(t) - p^+(t - 2x_1/c) = q^+(t) - q^+(t - 2\ell/c); \\ u(t) &= p^+(t) + p^+(t - 2x_1/c) + q^+(t) + q^+(t - 2\ell/c). \end{aligned}$$

Therefore

$$\begin{aligned} u(t) &= 2p(t) + 2p^+(t - 2x_1/c) + 2q^+(t - 2\ell/c) \;\; or \\ u(t) &= 2p(t) - 2p(t - 2L/c) \\ &\quad +u(t - 2x_1/c) + u(t - 2\ell/c) - u(t - 2L/c), \end{aligned}$$

where $L = \ell + x_1$. $q^{\pm}(t)$ are the outcoming and incoming spherical waves, while $p^{\pm}(t)$ are without physical meaning: they are defined in Eqs. (3). These expressions differ from Eq. (5) by the factor 2, but the principle of calculation is the same, and what is essential if the finite duration of the history term. Moreover it is not necessary to discretize the time variable for a simple initial condition.

3.2 Truncated cone without mouthpiece

If the approximation of the cylindrical saxophone is abandoned, the time discretization becomes necessary, because round corners appear in the solutions. The use of the plane reflection function should be possible (see [9, 10]), but it is of infinite extent. Another difficulty exists, but can be overcome: because no cylindrical part is present at the input, the reflection starts at $t = 0$, and the convolution integral giving $p_h(t)$ needs to be calculated until instant t. However, using the rectangle method, the calculation of the integral can be stopped just before this time, and the convergence to the true result is ensured when the discretization path tends to 0.

Therefore the main difficulty of the use of the plane reflection function lies in its duration. Some authors (Ref. almeida [11]) tried to use the spherical reflection function. In the frequency domain, the solution of the acoustic equations in the conical tube can be written as:

$$\begin{aligned} P &= Q^+ + Q^-; U = Q^+ - Q^- + \frac{P}{jkx} \quad (10) \\ where\; Q^{\pm} &= a^{\pm}\exp(\mp jkx)/x. \end{aligned}$$

Capital letters are used for the quantities in the frequency domain. At the input, if the d'Alembert decomposition (Eq. (3)) in planar waves is formally used, the following result is deduced, for $x = x_1$:

$$Q^- - P^- = \frac{P}{2jkx_1};\; Q^+ - P^+ = -\frac{P}{2jkx_1}. \quad (11)$$

Assuming a zero radiation impedance at $x = \ell$, this condition is written as:

$$q^-(t) = -q^+(t - \tau) \quad (12)$$

where $\tau = 2\ell/c$. Finally we get the following expression:

$$p^-(t) = -p^+(t - \tau) - \frac{1}{2}\frac{c}{x_1}\int_{t-\tau}^{t} p(t')dt' \quad (13)$$

As expected, the time variable needs to be discretized for the computing of the integral (the equation could be also written using a finite difference scheme). If the integral is calculated using the top left-corner approximation of the rectangle method, this equation has the form of Eq. (5), if $p_h = -2p^-$. The major point is the finite duration of the integral. This equation can be solved together with Eq. (8) step by step.

The chosen initial condition is the following: the mouthpiece pressure is zero for negative times, then its value jump to a fixed constant. Therefore for $t < 0$, $p = u = p^+ = p^- = 0$, and $p^-(0) = 0$.

For this model, we did not find a correct convergence to a limit cycle. More precisely, the convergence can be very slow, or limit cycles correspond to higher modes of the resonator. Inharmonicity of truncated cone resonators is probably the main explanation. As a matter of fact, the lossless model considers only two parameters for the definition of the truncated cone, the length of the missing part, x_1, and the length of the truncated cone, ℓ. This means that only the ratio of the input and output radii is fixed. When the length ℓ tends to zero, the truncated cone tends to a short cylinder, with a big increase of inharmonicity, the second eigenfrequency tending to 3 times the first one. The reasoning concerning the effect of inharmonicity is confirmed by numerical results: the convergence to a limit cycle becomes very slow or impossible for short lengthes.

3.3 Truncated cone with mouthpiece

As it is well known (see e.g. Ref.[12]), for a good intonation and ease of playing, the mouthpiece needs to have a volume close to that of the missing part of the truncated cone, i.e. $V = x_1 S_1/3$, where S_1 is the input cross section of the truncated cone. With this requirement, we tried two methods of calculation. Notice that this improvement of the model does not add any new geometrical parameter.

3.3.1 First approach

The first idea is to consider a cylindrical mouthpiece with the fixed volume V, the length varying from some centimeters to zero. We do not give the complete algorithm here, because it is based upon the same technique than the previous one, i.e. that of the truncated cone without mouthpiece. With this model, the problems of convergence are almost always solved. It is possible to say that this algorithm calculates the minimum model of reed conical instruments, generalizing the iterated map algorithm for cylindrical instruments, with only one parameter more. Losses independent of frequency can also be easily introduced. A major difference between a cylindrical resonator and a conical one lies in the time discretization, and therefore in the calculation time, but the history duration is the same.

On the other hand, the frequency response is continuous, therefore the resulting sound is a smooth signal, without step. The simplest model for such an instrument allows to listen sounds that are more realistic than those obtained from the iterated map scheme or the scheme yielding to the Helmholtz motion (section 3.1). Notice than because the radiation impedance is assumed to be zero, the output flow rate u_r is given by

$$u_r = 2q^{+}(t - \ell/c). \tag{14}$$

Then assuming the radiation of a monopole, the external pressure is proportional to the derivative of the output flow rate.

3.3.2 Second approach

Another approach of the (short) mouthpiece is the following: if it is short, only two lumped elements are to be taken into account: a shunt compliance, proportional to the volume $V = \ell_m S_m$, which is fixed (ℓ_m and S_m are the length and cross section area of the mouthpiece, respectively), and an acoustic mass in series, $\rho\ell_m/S_m$ (ρ is the air density), proportional to ℓ_m^2/V, which is very small compared to the input impedance of the truncated cone . Therefore it is possible to replace the cylindrical tube of dimensions ℓ_m and S_m by a simple compliance, $V/\rho c^2$.

After some algebra, with this value of the volume, the following equation can be found:

$$\partial_t\left[p(t) - p(t-\tau)\right] = -\frac{6c}{x_1}\left[p^{-}(t) + \frac{c}{2x_1}\int_{t-\tau}^{t} p dt' + p^{+}(t-\tau)\right] \tag{15}$$

Using this equation at time $p(t - t_{sample})$ with an asymmetrical finite difference scheme:

$$\partial_t p(t - t_{sample}) = \left[p(t) - p(t - t_{sample})\right]/t_{sample}\,,$$

an interesting algorithm is found

$$p(t) = \widetilde{p_h}\,, \tag{16}$$

where $\widetilde{p_h}$ is limited to time t, thus $\widetilde{p_h}$ is known. Using the nonlinear function F (Eq. (2)), the problem can be solved *without inversion* of the nonlinear function. It is a very useful tool to study the effect of the nonlinear characteristic on the sound production. In particular, no problem occurs with the difficulty of the multivalued inverse function, when seeking the intersection between the line (5) and the function F. Moreover, when several solutions exist, this method allows to understand how the selection of the solution is done, as explained hereafter.

4 Results and discussion

The chosen nonlinear characteristic is the classical one, based upon Bernoulli law, and is shown in Fig 2. It includes the hypothesis that the flow rate at the input of the instrument can be negative, when the mouthpiece pressure is larger than the mouth one. However, the method presented here would apply to other characteristics such as the one measured in double-reed instruments [13]. Two dimensionless parameters are considered: the mouth pressure, denoted γ, and the maximum flow rate that can enter in the resonator, denoted ζ, related to the reed opening at rest and to the reed stiffness. For $\zeta < 1$, the solution of Eqs. (5) and (2) is unique, while for $\zeta > 1$, there are either 1 or 3 solutions, depending on the value of $p_h = u - p$. Roughly speaking, the first case corresponds to saxophone-like instruments, with single reed, while the second case corresponds to oboe- or bassoon-like instruments, with double reed. The problem of the multivalued function for the case of bassoons was recognized by Gokhstein [2], and it is well known for bowed string instruments. Figure 2 shows this case, with the two limit lines for which the function is multivalued.

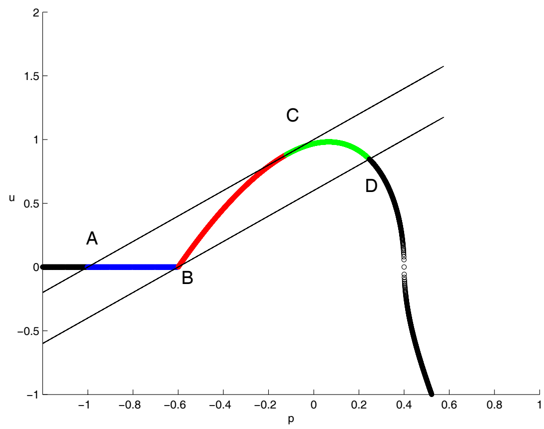

Figure 2: Multivalued nonlinear characteristic $u = F(p)$. For the portions in black color of the curve, there is only one intersection with line $u = p + p_h$. In order to have continuity of the acoustic quantities, the history term p_h needs to have extrema at points A and B.

Figure 3 presents the case of a resonator similar to a baritone saxophone, with $\zeta = 0.55$. Both approaches of computation (cylindrical mouthpiece with fixed volume with length tending to zero, and mouthpiece modeled as a compliance corresponding to the same volume) converge to the same result. It can be noticed that the reed beats (the flow rate vanishes). Moreover, during a significant time, the flow rate becomes negative; this numerical result is usual for conical instruments, while it is very rare for clarinet-like instruments (see Ref. [5]). Nevertheless to our knowledge no experimental evidence of negative pressure difference between mouth and mouthpiece has been published yet.

Figure 4 presents the case of the same resonator with a large reed opening ($\zeta = 2.55$). Notice that ratio of the input and output radii is fixed, but the angle of the cone is free. When several solutions exist, the first approach is not capable to distinguish between the solutions if no selection rule is defined. As opposed to the first approach, the second approach allows to find a result, which seems to be in qualitative accordance with measured signals.

Examining the selection done by this method, it appears that neither jumps nor singular points are allowed. Figure 2 shows the two lines $u = p + p_h$ for the two extreme values of p_h for which several intersections with the nonlinear function exist. It is possible to understand the choice to be done between the different solutions when p_h varies. Starting from $p = -1.2$, i.e. $u - p = p_h = 1.2$, the reed beats, and the lines goes to the right when p_h decreases.

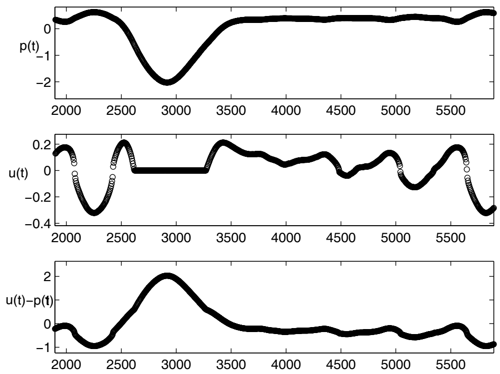

Figure 3: A period of the steady-state regime for a saxophone-like instrument (for clarity, the figure shows a little bit more than one period). $x_1 = 0.3m$; $\ell = 1.5m$. Excitation parameters: excitation pressure $\gamma = 0.4$, "reed opening" $\zeta = 0.55$. Input acoustic pressure p (above), flow rate u (center), difference $u - p$.

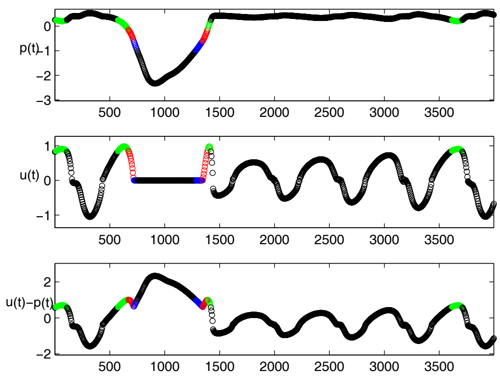

Figure 4: A period of the steady-state regime for a saxophone-like instrument. $x_1 = 0.3m$; $\ell = 1.5m$. Excitation parameters: excitation pressure $\gamma = 0.4$, "reed opening" $\zeta = 2.55$. Input acoustic pressure p (above), flow rate u (center), difference $u - p$. Colors correspond to the regions of the nonlinear characteristic $u = F(p)$, according to Figure 2.

When arriving in the blue part of the curve (point A), the flow rate needs to remain equal to zero in order to avoid jump of the solution. When p_h continues to decrease, the line $u - p = p_h$ reaches point B ($p = 1 - \zeta$), which is the threshold of beating reed: at this point, the jump to point D is not allowed, then, in order to have continuity of the solutions, p_h needs to go back, therefore it needs to increase. This allows for the solution to be on the red part of the curve. Therefore point B needs to correspond to a minimum of p_h. Similarly, point C, where the line is tangent to the nonlinear curve, needs to correspond to a maximum of p_h, then the solution can be located on the green part of the curve. Finally no extremum of p_h is expected at point D. This can be checked on Figure 4. The spectrum of the acoustic quantities is expected to be with more higher components than for small reed opening, because of the "accidents" related to these feature. Otherwise, as expected, a higher amplitude is noticeable for larger reed opening.

5 Conclusion, further work

In conclusion, the present work does not take into account any value of the input and output radii. This could be done by adding a loss parameter. Time discretization is requested for a minimum model of conical reed instruments, but algorithms with short history are perfectly possible. The analysis exhibits the rule to be fulfilled when using a classical method with the search for zeros of multivalued functions. Notice that, for the particular case of the nonlinear characteristic function based upon the Bernoulli law, the problem of parameter ζ larger than unity has been solved previously by using another method: taking the reed dynamics into account, the problem of multivalued function disappears, and a second order equation has to be solved, as shown in Ref. [14, 15]. But, as explained above, the adding of a mouthpiece modeled as a compliance avoids the problem for any choice of the nonlinear characteristic.

Further work remains to be done concerning the calculation time (the previous results have been obtained with a very high sampling frequency, $f_s = 300000Hz$). Moreover with the second approach, we did not observe any problem of convergence when the sampling frequency increases, but it should be interesting to investigate the stability of the two algorithms presented here.

Acknowledgments

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