Geometric Optics and Boundary Layers for Nonlinear Schrödinger Equations

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Abstract

We justify supercritical geometric optics in small time for the defocusing semiclassical Nonlinear Schrödinger Equation for a large class of non-necessarily homogeneous nonlinearities. The case of a half-space with Neumann boundary condition is also studied.

1 Introduction

We consider the nonlinear Schrödinger equation in $\Omega \subset \mathbb{R}^d$

$$i\varepsilon \frac{\partial \Psi^\varepsilon}{\partial t} + \frac{\varepsilon^2}{2} \Delta \Psi^\varepsilon - \Psi^\varepsilon f(|\Psi^\varepsilon|^2) = 0, \qquad \Psi^\varepsilon : \mathbb{R}^+ \times \Omega \to \mathbb{C} \tag{1}$$

with an highly oscillating initial datum under the form

$$\Psi^\varepsilon_{|t=0} = \Psi^\varepsilon_0 = a^\varepsilon_0 \exp\left(\frac{i}{\varepsilon}\varphi^\varepsilon_0\right), \tag{2}$$

where φ^ε_0 is real-valued. We are interested in the semiclassical limit $\varepsilon \to 0$. The nonlinear Schrödinger equation (1) appears, for instance, in optics, and also as a model for Bose-Einstein condensates, with $f(\rho) = \rho - 1$, and the equation is termed Gross-Pitaevskii equation, or also with $f(\rho) = \rho^2$ (see [13]). Some more complicated nonlinearities are also used especially in low dimensions, see [12].

At first, let us focus on the case $\Omega = \mathbb{R}^d$. To guess the formal limit, when ε goes to zero, it is classical to use the *Madelung transform*, i.e to seek for a solution of (1) under the form

$$\Psi^\varepsilon = \sqrt{\rho^\varepsilon} \exp\left(\frac{i}{\varepsilon}\varphi^\varepsilon\right).$$

By separating real and imaginary parts an by introducing $u^\varepsilon \equiv \nabla \varphi^\varepsilon$, this allows to rewrite (1) as an hydrodynamical system

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot \big(\rho^\varepsilon u^\varepsilon\big) = 0 \\ \partial_t u^\varepsilon + \big(u^\varepsilon \cdot \nabla\big) u^\varepsilon + \nabla\big(f(\rho^\varepsilon)\big) = \dfrac{\varepsilon^2}{2} \nabla \left(\dfrac{\Delta \sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}} \right). \end{cases} \tag{3}$$

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The system (3) is a compressible Euler equation with an additional term in the right-hand side called *quantum pressure*. As ε tends to 0, the quantum pressure is formally negligible and (3) reduces to the (compressible) Euler equation

$$\begin{cases} \partial_t \rho + \nabla \cdot \big(\rho u\big) = 0 \\ \\ \partial_t u + \big(u \cdot \nabla\big) u + \nabla\big(f(\rho)\big) = 0. \end{cases} \tag{4}$$

The justification of this formal computation has received much interest recently. The case of analytic data was solved in [7]. Then for data with Sobolev regularity and a defocusing nonlinearity, so that (4) is hyperbolic, it was noticed by Grenier, [9], that it is more convenient to use the transformation

$$\Psi^\varepsilon = a^\varepsilon \exp\big(i \frac{\varphi^\varepsilon}{\varepsilon}\big) \tag{5}$$

and to allow the amplitude a^ε to be complex. By using an identification between $\mathbb{C}$ and $\mathbb{R}^2$, this allows to rewrite (1) as

$$\begin{cases} \partial_t a^\varepsilon + u^\varepsilon \cdot \nabla a^\varepsilon + \dfrac{a^\varepsilon}{2} \nabla \cdot u^\varepsilon = \dfrac{\varepsilon}{2} J\, \Delta a^\varepsilon \\ \\ \partial_t u^\varepsilon + \big(u^\varepsilon \cdot \nabla\big) u^\varepsilon + \nabla\big(f(|a^\varepsilon|^2)\big) = 0, \end{cases} \tag{6}$$

where J is the matrix of complex multiplication by i:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

When $\varepsilon = 0$, we find the system

$$\begin{cases} \partial_t a + u \cdot \nabla a + \dfrac{a}{2} \nabla \cdot u = 0 \\ \\ \partial_t u + \big(u \cdot \nabla\big) u + \nabla\big(f(|a|^2)\big) = 0, \end{cases} \tag{7}$$

which is another form of (4), since then $(\rho \equiv |a|^2, u)$ solves (4). The rigorous convergence of (6) towards (7) provided the initial conditions suitably converge was rigorously performed by Grenier [9] in the case $f(\rho) = \rho$ (which corresponds to the cubic defocusing NLS). More precisely, it was proven in [9] that there exists $T > 0$ independent of ε such that the solution of (6) is uniformly bounded in H^s on $[0, T]$. In terms of the unknown Ψ^ε of (1), this gives that

$$\sup_{\varepsilon \in (0,1]} \sup_{[0,T]} \big\| \Psi^\varepsilon \exp\big(-i \frac{\varphi}{\varepsilon}\big) \big\|_{H^s} < +\infty$$

for every s where $(a, u = \nabla \varphi)$ is the solution of (7). Furthermore, the justification of WKB expansions under the form

$$\Psi^\varepsilon - \Big(\sum_{k=0}^{m} \varepsilon^k a^k\Big) e^{\frac{i\varphi}{\varepsilon}} = \mathcal{O}(\varepsilon^m) e^{\frac{i\varphi}{\varepsilon}}$$

for every m was performed in [9]. The main idea in the work of Grenier [9] is to use the symmetrizer

$$S \equiv \ \text{diag}\, \Big(1, 1, \frac{1}{4f'(|a|^2)}, \cdots, \frac{1}{4f'(|a|^2)}\Big)$$

of the hyperbolic system (7) to get H^s energy estimates which are uniform in ε for the singularly perturbed system (6). The case of nonlinearities for which f' vanishes at zero (for instance the case $f(\rho)=\rho^2$) was left opened in [9]. The additional difficulty is that for such nonlinearities, the system (7) is only weakly hyperbolic at $a=0$ and in particular the symmetrizer S becomes singular at $a=0$.

In more recent works, see [19], [14], [1] it was proven that for every weak solution of (1) with $f(\rho)=\rho-1$ or $f(\rho)=\rho$, the limits as $\varepsilon\to 0$

$$|\Psi^\varepsilon|^2-\rho\to 0 \quad \text{in } L^\infty([0,T],L^2) \qquad \varepsilon Im\left(\bar\Psi^\varepsilon\nabla\Psi^\varepsilon\right)-\rho u\to 0 \quad \text{in } L^\infty([0,T],L^1_{loc}) \tag{8}$$

hold under some suitable assumption on the initial data. The approach used in these papers is completely different, and relies on the modulated energy method introduced in [4]. The advantage of this powerfull approach is that it allows to describe the limit of weak solutions and to handle general nonlinearities once the existence of a global weak solution in the energy space for (1) is known. Nevertheless, it does not give precise qualitative information on the solution of (1), for example, it does not allow to prove that the solution remains smooth on an interval of time independent of ε if the initial data are smooth or to justify WKB expansion up to arbitrary orders in smooth norms.

In the work [2], the possibility of getting the same result as in [9] for pure power nonlinearities $f(\rho)=\rho^\sigma$ in the case $\Omega=\mathbb{R}^d$ was studied. It was first noticed that, thanks to the result of [15], the system

$$\begin{cases} \partial_t a+\nabla\varphi\cdot\nabla a+\dfrac{a}{2}\,\Delta\varphi &=0\\ \partial_t\varphi+\dfrac12|\nabla\varphi|^2+f(|a|^2) &=0,\end{cases} \tag{9}$$

with the initial condition $\big(a,\varphi\big)_{/t=0}=\big(a_0,\varphi_0\big)\in H^\infty$ has a unique smooth maximal solution $(a,\varphi)\in\mathcal{C}\big([0,T^*[,H^s(\mathbb{R}^d)\times H^{s-1}(\mathbb{R}^d)\big)$ for every s. It was then established:

Theorem 1 ([2]) *Let $d\le 3$, $\sigma\in\mathbb{N}^*$ and initial data a_0^ε, $\varphi_0^\varepsilon\equiv\varphi_0$ in H^∞ such that, for some functions $(\varphi_0,a_0)\in H^\infty$,*

$$\left\|a_0^\varepsilon-a_0\right\|_{H^s}=\mathcal{O}(\varepsilon),$$

for every $s\ge 0$. Then, there exists $T^*>0$ such that (9) with $f(\rho)=\rho^\sigma$ has a smooth maximal solution $(a,\varphi)\in\mathcal{C}([0,T^*[,H^\infty\times H^\infty)$. Moreover, there exists $T\in(0,T^*)$ independent of ε, such that the solution of (1), (2) remains smooth on $[0,T]$ and verifies the estimate

$$\sup_{\varepsilon\in(0,1]}\left\|\Psi^\varepsilon\exp\big(-i\frac{\varphi}{\varepsilon}\big)\right\|_{L^\infty([0,T],H^s)}<+\infty, \tag{10}$$

where

- *if $\sigma=1$, then $s\in\mathbb{N}$ is arbitrary,*
- *if $\sigma=2$ and $d=1$, then one can take $s=2$,*
- *if $\sigma=2$ and $2\le d\le 3$, then one can take $s=1$,*
- *if $\sigma\ge 3$ then one can take $s=\sigma$.*

As emphasized in [2], in some cases, the global existence of smooth solutions is already known for (1). For example, in the quintic case, $\sigma = 2$, global existence is known for $d \leq 3$ (see [6] for the difficult critical case $d = 3$), so that only the bound (10) is interesting. Nevertheless, Theorem 1 may be also applied to cases where (1) is H^1 super-critical ($\sigma \geq 3$, $d = 3$ for example) and hence the fact that it is possible to construct a smooth solution on a time interval independent of ε is already interesting. The main ingredient used in [2] is a subtle transformation of (1) into a perturbation of a quasilinear symmetric hyperbolic system with non smooth coefficients when $\sigma \geq 2$.

The first aim of this paper is to prove that the estimate (10) holds true for every s, every dimension d and every nonlinearity f which satisfies the following assumption:

$$(\mathcal{A}) \qquad f \in \mathcal{C}^\infty\big([0,+\infty)\big), \qquad f(0) = 0, \qquad f' > 0 \text{ on } (0,+\infty), \qquad \exists n \in \mathbb{N}^*, \ \ f^{(n)}(0) \neq 0.$$

Note that we allow f' to vanish at the origin. The assumption $(\mathcal{A})$ takes into account in particular all the homogeneous polynomial nonlinearities $f(\rho) = \rho^\sigma$ but also nonlinearities under the form $f(\rho) = \rho^{\sigma_1} + \rho^{\sigma_2}$ or $\frac{\rho^\sigma}{1+\rho}$ for example. Our result reads:

Theorem 2 *We assume $(\mathcal{A})$, and consider an initial data (2) with φ_0^ε real-valued, a_0^ε, φ_0^ε in H^∞ such that, for some real-valued functions $(\varphi_0, a_0) \in H^\infty$, we have for every s,*

$$\big\|a_0^\varepsilon - a_0\big\|_{H^s} = \mathcal{O}(\varepsilon) \qquad \text{and} \qquad \big\|\varphi_0^\varepsilon - \varphi_0\big\|_{H^s} = \mathcal{O}(\varepsilon).$$

Then, there exists $T^* > 0$ such that (7) with initial value (a_0, φ_0) has a unique smooth maximal solution $(a, \varphi) \in \mathcal{C}([0, T^*[, H^\infty \times H^\infty)$. Moreover, there exists $T \in (0, T^*]$ such that for every $\varepsilon \in (0,1)$, the solution Ψ^ε to (1)-(2) exists at least on $[0,T]$ and satisfies for every s

$$\sup_{\varepsilon \in (0,1]} \Big\|\Psi^\varepsilon \exp\big(-\frac{i}{\varepsilon}\varphi\big)\Big\|_{L^\infty([0,T],H^s)} < +\infty.$$

More precisely, there exists $\varphi^\varepsilon = \varphi + \mathcal{O}_{H^\infty}(\varepsilon)$ such that, for every s,

$$\Big\|\Psi^\varepsilon \exp\big(-\frac{i}{\varepsilon}\varphi^\varepsilon\big) - a\Big\|_{L^\infty([0,T],H^s)} = \mathcal{O}(\varepsilon). \tag{11}$$

Let us give a few comments on the statement of Theorem 2.

At first, note that Theorem 2 contains a result of local existence of smooth solutions for (9) in the case of non necessarily homogeneous nonlinearities satisfying $(\mathcal{A})$. Since $(a, \nabla\varphi)$ solves a compressible type Euler equation, the case of a homogeneous nonlinearity was studied in [15], and we thus give an extension of this result to smooth non-linearities satisfying assumption $(\mathcal{A})$. A precise statement of our result with the required regularity of the initial data is given in Theorem 4 below. The new difficulty when f is not homogeneous is that the nonlinear symmetrization does not seem to allow to transform the problem into a classical symmetric or symmetrizable hyperbolic system with smooth coefficients.

The correction of order ε that we have to add to the phase to get the estimate (11) is expected. Indeed, a perturbation of order ε in the phase modifies the amplitude at the leading order.

Our approach to prove Theorem 2 is completely different from the one of [2] and [9]. We do not work any more on the system (6) or any reformulation of (1) into a perturbation of a quasilinear symmetric hyperbolic system, but directly on the NLS equation (1). Basically, we first prove the linear stability for (1) in arbitrary Sobolev norms of highly oscillating solution of the form $ae^{i\varphi/\varepsilon}$ and then use a fixed point argument to prove the nonlinear stability. The crucial estimate of linear stability of highly oscillating solution is given in Lemma 1 and Theorem 3.

This actually allows to justify WKB expansions up to arbitrary orders (see Theorem 5). Since we deal in this paper with sufficiently smooth and in particular bounded solutions, the assumption $(\mathcal{A})$ can be replaced by a local version where we assume that $f' > 0$ on $(0, \beta)$ with β independent of ε if the initial datum verifies $|a_0|^2 < \beta$. Indeed, since a^0 takes it values in the (weak) hyperbolic region of the limit system (7), there still exists a local smooth solution of (7) defined on $[0, T]$ for some $T > 0$ and the stability argument leading to Theorem 2 still holds. Consequently, our result can also be applied to nonlinearities like $f(\rho) = \rho^{\sigma_1} - \rho^{\sigma_2}$ for every $\sigma_2 > \sigma_1$ provided $|a_0|^2 \leq \beta < 1$. Note that when σ_2 is too large, the classical global existence result of weak solutions (see [8]) for (1) is not valid and hence it does not seem possible to use the modulated energy method of [1], [14] to investigate the semi-classical limit.

Finally, the last advantage of our approach is that it can be easily generalized to the case of a domain with boundary and to non-zero condition at infinity. This will be the aim of the second part of the paper. We shall restrict ourself to a physical case, the Gross-Pitaevskii equation, i.e. $f(\rho) = \rho - 1$. The generalization to more general nonlinearities satisfying an assumption like $(\mathcal{A})$ is rather straightforward. This simplifying assumption is only made to avoid the multiplication of difficulties. Again to avoid too many technicalities, we restrict ourselves to the simplest domain $\Omega = \mathbb{R}^d_+ = \mathbb{R}^{d-1} \times (0, +\infty)$. For $x \in \mathbb{R}^d_+$, we shall use the notation $x = (y, z)$, $y \in \mathbb{R}^{d-1}$, $z > 0$. We add to (1) the Neumann boundary condition

$$\partial_z \Psi^\varepsilon(t, y, 0) = 0. \tag{12}$$

We also impose the following condition at infinity

$$\Psi^\varepsilon(t, x) \sim \exp\Big(-i\, t\, \frac{|u^\infty|^2}{2\varepsilon} + i\, \frac{u^\infty \cdot x}{\varepsilon} \Big), \qquad |x| \to +\infty, \tag{13}$$

that we can write in hydrodynamical variables

$$\big|\Psi^\varepsilon(t, x)\big|^2 \to 1, \qquad u^\varepsilon(t, x) \to u^\infty, \qquad |x| \to +\infty,$$

where u^∞ is a constant vector. This condition appears naturally when we study a moving obstacle in the fluid. Indeed, if we start from (1) with the Neumann boundary condition on an obstacle moving at constant velocity and fluid at rest at infinity, then we can use the Galilean invariance of (1) to transform the problem into the study of (1) in a fixed domain but with the condition (13) at infinity.

This problem with such boundary conditions is physically meaningfull since it can be used to describe superfluids past an obstacle (we refer to [16] for example). The semiclassical limit ε tends to zero was already studied in [14] by using the modulated energy method. The limit (8) was proven with (ρ, u) the solution of the compressible Euler equation with boundary condition $u \cdot n_{/\partial\Omega} = 0$, n being the normal to the boundary. Note that the result of [14] is restricted to the two-dimensional case only in order to have a global solution in the energy space of (1). By using more recent results on the Cauchy problem, [3], one can also get the result in the three-dimensional case at least when $u^\infty = 0$. Our aim here is to give a more precise description of the convergence which takes into account boundary layers. More precisely, since the solution of the Euler system (9) cannot match the Neumann boundary condition $\partial_z a(t,y,0) = 0$, a boundary layer of weak amplitude ε and of size ε appears. They are formally described for example in [16]. WKB expansions $\Psi^\varepsilon = a^\varepsilon e^{i\frac{\varphi^\varepsilon}{\varepsilon}}$ are thus to be seek under the form

$$a^\varepsilon = a^0 + \sum_{k=1}^m \varepsilon^k \Big(a^k(t,x) + A^k(t,y,\frac{z}{\varepsilon})\Big), \quad \varphi^\varepsilon = \varphi^0 + \sum_{k=1}^m \varepsilon^k \Big(\varphi^k(t,x) + \Phi^k(t,y,\frac{z}{\varepsilon})\Big) \tag{14}$$

where the profiles $A^k(t,y,Z)$, $\Phi^k(t,y,Z)$ are exponentially decreasing in the Z variable and are chosen such that

$$\partial_z a^k(t,y,0) + \partial_Z A^{k+1}(t,y,0) = 0, \quad \partial_z \varphi^k(t,y,0) + \partial_Z \Phi^{k+1}(t,y,0) = 0$$

so that the approximate WKB expansion $\Psi^{WKB} = a^\varepsilon \exp\big(\frac{i}{\varepsilon}\varphi^\varepsilon\big)$ matches the Neumann boundary condition (12). Our result (Theorem 6) is that under suitable assumptions on the initial conditions, we have the nonlinear stability of WKB expansions: in particular we have the existence of a smooth solution for (1), (12), (13) on a time interval independent of ε and the estimate

$$\big\|\Psi^\varepsilon e^{-i\frac{\varphi^\varepsilon}{\varepsilon}} - a^\varepsilon\big\|_{W^{1,\infty}} \lesssim \varepsilon. \tag{15}$$

Note that it is necessary to incorporate the boundary layer εA^1 in order to get (15) since its gradient has amplitude one in L^∞. The case of Dirichlet boundary condition which is also physically meaningfull, we again refer to [16], seems more complicated to handle as often in boundary layer theory in fluid mechanics since the boundary layers involved have amplitude one. This is left for future work.

The paper is organized as follows. In section 2, we prove the linear stability in H^s of an approximate WKB solution of (1) under the form $a^\varepsilon \exp\big(i\frac{\varphi^\varepsilon}{\varepsilon}\big)$ in the case $\Omega = \mathbb{R}^d$. This is the crucial part towards the proof of Theorem 2. Next in section 3, we give the construction of a WKB expansion up to arbitrary order and give the proof of the local existence of smooth solution for the compressible Euler equation with a pressure law satisfying $(\mathcal{A})$. In section 4, we give the justification of WKB expansions at every order and recover Theorem 2 as a particular case. This part uses in a classical way the linear stability result and a fixed point argument. Finally, in section 5, we study the problem in the half-space with Neumann boundary condition.

2 Linear Stability

In this section, we consider a smooth WKB approximate solution $\Psi^a = a^\varepsilon \exp\big(i\frac{\varphi^\varepsilon}{\varepsilon}\big)$ of (1) such that

$$NLS(\Psi^a) = R^\varepsilon \exp\big(i\frac{\varphi^\varepsilon}{\varepsilon}\big), \tag{16}$$

where

$$NLS(\Psi) \equiv i\varepsilon\partial_t\Psi + \frac{\varepsilon^2}{2}\Delta\Psi - \Psi f(|\Psi|^2).$$

Moreover, we also set

$$R_\varphi \equiv \partial_t \varphi^\varepsilon + \frac{1}{2}\,|\nabla \varphi^\varepsilon|^2 + f(|a^\varepsilon|^2), \tag{17}$$

$$R_a \equiv \partial_t a^\varepsilon + \nabla\varphi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2}\,a^\varepsilon \Delta\varphi^\varepsilon, \tag{18}$$

so that

$$R^\varepsilon = -a^\varepsilon R_\varphi + i\varepsilon R_a + \frac{\varepsilon^2}{2}\,\Delta a^\varepsilon.$$

Looking for an exact solution of (1) under the form

$$\Psi^\varepsilon = \Psi^a + w\, e^{i\frac{\varphi^\varepsilon}{\varepsilon}} = (a^\varepsilon + w)e^{i\frac{\varphi^\varepsilon}{\varepsilon}},$$

we find that w solves the nonlinear Schrödinger equation

$$i\varepsilon\Big(\partial_t w + u^\varepsilon \cdot \nabla w + \frac{1}{2}\,w\,\nabla\cdot u^\varepsilon\Big) + \frac{\varepsilon^2}{2}\Delta w - 2(w, a^\varepsilon) f'(|a^\varepsilon|^2)a^\varepsilon = R_\varphi w - R^\varepsilon + Q^\varepsilon(w), \tag{19}$$

where $(\cdot,\cdot)$ stands for the real scalar product in $\mathbb{C} \simeq \mathbb{R}^2$, with

$$u^\varepsilon \equiv \nabla\varphi^\varepsilon$$

and the nonlinear term $Q^\varepsilon(w)$ is defined by

$$Q^\varepsilon(w) \equiv (a^\varepsilon + w)\Big(f(|a^\varepsilon + w|^2) - f(|a^\varepsilon|^2)\Big) - 2(w, a^\varepsilon) f'(|a^\varepsilon|^2)a^\varepsilon. \tag{20}$$

Of course, R^ε will be very small and R_φ (and R_a) are to be thought small (at least $\mathcal{O}(\varepsilon)$) for applications to nonlinear stability results. Nevertheless, in this section the exact form of these terms is not important. The way to construct an accurate WKB solution Ψ^a will be explained in the next section.

Remark 1 If we work with a non-linearity f such that $f(A^2) = 0$ for some $A \in \mathbb{R}$, we can impose a non-zero condition at infinity such as $a_0 \in A + H^\infty$ and $\nabla\varphi_0 \in U^\infty + H^\infty$ for some constant vector $U^\infty \in \mathbb{R}^d$. Since we can still look for the perturbation w in H^s, this does not affect the proofs.

Since we expect the correction term w to be small, we shall only consider in this section the linearized equation

$$i\varepsilon\frac{\partial w}{\partial t} + \mathcal{L}^\varepsilon\, w = R_\varphi w + F^\varepsilon, \quad x \in \mathbb{R}^d, \tag{21}$$

where the linear operator $\mathcal{L}^\varepsilon$ is defined as

$$\mathcal{L}^\varepsilon(w) \equiv \frac{\varepsilon^2}{2}\Delta w + i\varepsilon\, u^\varepsilon \cdot \nabla w + \frac{i\varepsilon}{2}\, w\,\nabla\cdot u^\varepsilon - 2f'(|a^\varepsilon|^2)(w, a^\varepsilon)a^\varepsilon.$$

In this section, F^ε is considered as a given source term. Of course, for the proof of Theorem 2, we shall apply the result of this section to

$$F^\varepsilon = -R^\varepsilon + Q^\varepsilon(w). \tag{22}$$

Furthermore, let us emphasize that at this stage, R_φ is seen as a multiplicative operator with no link with the vector field u^ε appearing in $\mathcal{L}^\varepsilon$, even though we will use this lemma with $u^\varepsilon = \nabla\varphi^\varepsilon$. We notice that $\mathcal{L}^\varepsilon$ is formally self-adjoint, but only the first and last term give rise to a nonnegative quadratic functional. Indeed, the quadratic form (in H^1) associated to the operator

$$\mathcal{S}^\varepsilon\, w \equiv -\frac{\varepsilon^2}{2}\Delta w + 2f'(|a^\varepsilon|^2)(w,a^\varepsilon)a^\varepsilon$$

is, since $f' \geq 0$,

$$\int_{\mathbb{R}^d} \big(w, \mathcal{S}^\varepsilon\, w\big) = \frac{1}{2}\int_{\mathbb{R}^d} \varepsilon^2|\nabla w|^2 + 4f'(|a^\varepsilon|^2)(w,a^\varepsilon)^2 \geq 0.$$

It is then natural to consider the (squared) norm $\displaystyle\int_{\mathbb{R}^d} \big(w, \mathcal{S}^\varepsilon(w)\big)$ as a good energy for the linearized equation (21). Consequently, we introduce the weighted norm

$$N^\varepsilon(w) \equiv \frac{1}{2}\int_{\mathbb{R}^d} \varepsilon^2|\nabla w|^2 + 4f'(|a^\varepsilon|^2)(w,a^\varepsilon)^2 + K\varepsilon^2|w|^2$$

for every $K > 0$ (K will be chosen sufficiently large only in the next subsection).

Our first result of this section is a linear stability result in the energy norm $N^\varepsilon(w)$.

Lemma 1 *Assume that $u^\varepsilon : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ and $a^\varepsilon : [0,T]\times\mathbb{R}^d \to \mathbb{C}$ are smooth and such that*

$$M \equiv \big\|\nabla_x u^\varepsilon\big\|_{L^\infty([0,T]\times\mathbb{R}^d)} + \big\|\nabla_x(\nabla\cdot u^\varepsilon)\big\|_{L^\infty([0,T]\times\mathbb{R}^d)} + \big\||a^\varepsilon|^2\big\|_{L^\infty([0,T]\times\mathbb{R}^d)} < +\infty.$$

Let $w \in \mathcal{C}^1([0,T],H^2)$ be a solution of (21). Then, there exists C_M depending only on d, f and M such that for every $\varepsilon \in (0,1]$, the solution w of (21) satisfies the energy estimate

$$\begin{aligned}\frac{d}{dt}N^\varepsilon\big(w(t)\big) \leq C_M\Big(1 + \frac{1}{\varepsilon}\big\|R_a(t)\big\|_{L^\infty} + \frac{1}{\varepsilon}\big\|R_\varphi(t)\big\|_{W^{1,\infty}} + \frac{1}{\varepsilon^2}\big\|R_\varphi(t)\big\|_{L^\infty}\Big)N^\varepsilon\big(w(t)\big) \\ + \big\|F^\varepsilon(t)\big\|_{L^2}^2 - \int_{\mathbb{R}^d}\frac{4}{\varepsilon}f'(|a^\varepsilon|^2)(w,a^\varepsilon)(a^\varepsilon,iF^\varepsilon) + \int_{\mathbb{R}^d}(\varepsilon\Delta w, iF^\varepsilon).\end{aligned} \tag{23}$$

Note that it is very easy to get from (23) and the Gronwall inequality a classical estimate of linear stability. Indeed, assuming that $R_a = \mathcal{O}_{L^\infty([0,T],L^\infty)}(\varepsilon)$ and $R_\varphi = \mathcal{O}_{L^\infty([0,T],W^{1,\infty})}(\varepsilon^2)$ (which is true if $(a^\varepsilon,\varphi^\varepsilon)$ come from the WKB method), we infer from a crude estimate for the two last terms in (23) that for $0 \leq t \leq T$,

$$\frac{d}{dt}N^\varepsilon\big(w(t)\big) \leq CN^\varepsilon\big(w(t)\big) + \frac{1}{\varepsilon^2}\big\|F^\varepsilon(t)\big\|_{H^1}^2,$$

which gives for $0 \leq t \leq T$

$$N^\varepsilon\big(w(t)\big) \leq e^{Ct}\Big(N^\varepsilon\big(w(0)\big) + \frac{1}{\varepsilon^2}\int_0^t \big\|F^\varepsilon(\tau)\big\|_{H^1}^2\, d\tau\Big),$$

which is a more classical result of linear stability in the energy norm $N^\varepsilon(w)$ since the amplification rate C is independent of ε. Nevertheless, to get H^s estimates and the best nonlinear results as possible, it is important to have the special structure of the two last terms in (23).

Modulated linearized functionals like N^ε were also used in asymptotic problems in fluid mechanics, see [10] for example.

2.1 Proof of Lemma 1

The norms L^∞, $W^{1,\infty}$, L^2 ... always stand for the norms in the x variable. At first, since $\mathcal{S}^\varepsilon$ is self adjoint, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d}\big(\mathcal{S}^\varepsilon w, w\big) = \int_{\mathbb{R}^d} 2\big(\mathcal{S}^\varepsilon w, \partial_t w\big) + 2\partial_t\big[f'(|a^\varepsilon|^2)\big](w,a^\varepsilon)^2 + 4f'(|a^\varepsilon|^2)(w,a^\varepsilon)(w,\partial_t a^\varepsilon). \tag{24}$$

Next, we use (21) to express $\partial_t w$ as

$$\partial_t w = -\frac{i}{\varepsilon}\,\mathcal{S}^\varepsilon w - \Big(u^\varepsilon\cdot\nabla w + \frac{1}{2}\,w\,\nabla\cdot u^\varepsilon\Big) - \frac{i}{\varepsilon}R_\varphi w - \frac{i}{\varepsilon}F^\varepsilon$$

to get

$$2\int_{\mathbb{R}^d}\big(\mathcal{S}^\varepsilon w, \partial_t w\big) = 2\int_{\mathbb{R}^d}\Big(\frac{\varepsilon^2}{2}\Delta w - 2f'(|a^\varepsilon|^2)(w,a^\varepsilon)a^\varepsilon, u^\varepsilon\cdot\nabla w + \frac{1}{2}\,w\,\nabla\cdot u^\varepsilon + \frac{i}{\varepsilon}R_\varphi w + \frac{i}{\varepsilon}F^\varepsilon\Big). \tag{25}$$

We shall now estimate the various terms in the right-hand side of (25). Integrating by parts, we get

$$\begin{aligned}\int_{\mathbb{R}^d}\big(\varepsilon^2\,\Delta w, \frac{i}{\varepsilon}R_\varphi w\big) &= -\varepsilon\int_{\mathbb{R}^d}\big(\nabla w, iw\nabla R_\varphi\big)\\ &\le \varepsilon\,\|\nabla R_\varphi\|_{L^\infty}\|w\|_{L^2}\|\nabla w\|_{L^2}\\ &\le \frac{1}{\varepsilon}\,\|R_\varphi\|_{W^{1,\infty}}N^\varepsilon(w).\end{aligned}$$

Note that we have used that R_φ is real-valued and thus that

$$(\nabla w, iR_\varphi\nabla w) = 0$$

for the first equality. We also easily obtain by integration by parts that

$$\begin{aligned}\int_{\mathbb{R}^d}\Big(\varepsilon^2\Delta w, w\,\nabla\cdot u^\varepsilon\Big) &\le C\Big(\big\|\nabla\cdot u^\varepsilon\big\|_{L^\infty} + \big\|\nabla(\nabla\cdot u^\varepsilon)\big\|_{L^\infty}\Big)\Big(\varepsilon^2\big\|\nabla w\big\|_{L^2}^2 + \varepsilon^2\big\|w\big\|_{L^2}^2\Big)\\ &\le C_M N^\varepsilon(w).\end{aligned}$$

In the proof, C_M is a harmless number which changes from line to line and which depends only on M. In particular, it is independent of ε. Moreover, we can also write for $k = 1,\cdots,d$,

$$\begin{aligned}\int_{\mathbb{R}^d}\big(\partial^2_{kk}w, u^\varepsilon\cdot\nabla w\big) &= -\int_{\mathbb{R}^d} u^\varepsilon\cdot\nabla\frac{|\partial_k w|^2}{2} - \int\big(\partial_k w, \partial_k u^\varepsilon\cdot\nabla w\big)\\ &= \int_{\mathbb{R}^d}\frac{|\partial_k w|^2}{2}\,\nabla\cdot u^\varepsilon - \int_{\mathbb{R}^d}\big(\partial_k w, \partial_k u^\varepsilon\cdot\nabla w\big)\end{aligned}$$

and hence, we immediately infer

$$\int_{\mathbb{R}^d}\Big(\varepsilon^2\Delta w, u^\varepsilon\cdot\nabla w\Big) \le C_M N^\varepsilon(w).$$

Furthermore, from the inequality $2ab \le a^2+b^2$, there holds

$$
\begin{aligned}
-\frac{4}{\varepsilon}\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(w,a^\varepsilon)\big(a^\varepsilon, iR_\varphi w\big) &\le \frac{C_M}{\varepsilon^2}\|R_\varphi\|_{L^\infty}\int_{\mathbb{R}^d}\big(f'(|a^\varepsilon|^2)\big)^{\frac12}\big|(w,a^\varepsilon)\big|\,\varepsilon|w| \\
&\le \frac{C_M}{\varepsilon^2}\|R_\varphi\|_{L^\infty}\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(w,a^\varepsilon)^2+\varepsilon^2|w|^2 \\
&\le \frac{C_M}{\varepsilon^2}\|R_\varphi\|_{L^\infty}N^\varepsilon(w).
\end{aligned}
\tag{26}
$$

Consequently, we can replace (25) in (24) and use the above estimates to get

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d}\big(\mathcal{S}^\varepsilon w,w\big) &= \int_{\mathbb{R}^d} 4f'(|a^\varepsilon|^2)(w,a^\varepsilon)\Big((w,\partial_t a^\varepsilon)-\big(u^\varepsilon\cdot\nabla w+\frac12\,w\,\nabla\cdot u^\varepsilon, a^\varepsilon\big)\Big) \\
&\quad +2\int_{\mathbb{R}^d}\partial_t\big[f'(|a^\varepsilon|^2)\big](w,a^\varepsilon)^2+E_1,
\end{aligned}
\tag{27}
$$

where E_1 satisfies the estimate

$$
\begin{aligned}
E_1 &\le C_M\Big(1+\frac{1}{\varepsilon}\big\|R_\varphi\big\|_{W^{1,\infty}}+\frac{1}{\varepsilon^2}\big\|R_\varphi\big\|_{L^\infty}\Big)N^\varepsilon(w) \\
&\quad -\frac{4}{\varepsilon}\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(w,a^\varepsilon)(a^\varepsilon,iF^\varepsilon)+\int_{\mathbb{R}^d}(\varepsilon\Delta w, iF^\varepsilon).
\end{aligned}
\tag{28}
$$

To estimate the first integral in the right hand side of (27), we use the equation (18) to get

$$
\begin{aligned}
&4\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(w,a^\varepsilon)\Big((w,\partial_t a^\varepsilon)-\big(u^\varepsilon\cdot\nabla w+\frac12\,w\,\nabla\cdot u^\varepsilon,a^\varepsilon\big)\Big) \\
&\quad = 4\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(w,a^\varepsilon)\Big((w,R_a)-u^\varepsilon\cdot\nabla(w,a^\varepsilon)-(w,a^\varepsilon)\nabla\cdot u^\varepsilon\Big) \\
&\quad = 4\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(w,a^\varepsilon)(w,R_a)-2\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)\,u^\varepsilon\cdot\nabla\big((w,a^\varepsilon)^2\big)-4\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(w,a^\varepsilon)^2\,\nabla\cdot u^\varepsilon \\
&\quad = 4\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(w,a^\varepsilon)(w,R_a)+2\int_{\mathbb{R}^d}(w,a^\varepsilon)^2\,u^\varepsilon\cdot\nabla\big[f'(|a^\varepsilon|^2)\big]-2\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(w,a^\varepsilon)^2\,\nabla\cdot u^\varepsilon.
\end{aligned}
$$

To get the last line, we have integrated by parts the second integral. Note that the last term is bounded by $C_M N^\varepsilon(w)$, and, as for (26), that the first integral is bounded by $\dfrac{C_M}{\varepsilon}\|R_a\|_{L^\infty}N^\varepsilon(w)$. Consequently, we can replace the above identity in (27) to get

$$
\frac{d}{dt}\int_{\mathbb{R}^d}\big(\mathcal{S}^\varepsilon w,w\big)=\int_{\mathbb{R}^d}2(w,a^\varepsilon)^2\Big(\partial_t+u^\varepsilon\cdot\nabla\Big)f'(|a^\varepsilon|^2)+E_1+E_2=:I+E_1+E_2,
\tag{29}
$$

where E_2 is such that

$$
E_2\le C_M\Big(1+\frac{1}{\varepsilon}\big\|R_a\big\|_{L^\infty}\Big)N^\varepsilon(w).
\tag{30}
$$

To estimate I, we use again the equation (18) which gives

$$
\Big(\partial_t+u^\varepsilon\cdot\nabla\Big)f'(|a^\varepsilon|^2)=2f''(|a^\varepsilon|^2)\big(a^\varepsilon,\partial_t a^\varepsilon+u^\varepsilon\cdot\nabla a^\varepsilon\big)=2f''(|a^\varepsilon|^2)\Big(R_a-\frac12\,a^\varepsilon\,\nabla\cdot u^\varepsilon,a^\varepsilon\Big)
$$

and hence we find

$$I \leq C \int_{\mathbb{R}^d} |a^\varepsilon|^2 \left|f''(|a^\varepsilon|^2)\right| (w, a^\varepsilon)^2 + 4 \int_{\mathbb{R}^d} |a^\varepsilon| \, |f''(|a^\varepsilon|^2)| \, (w, a^\varepsilon)^2 \, |R_a|.$$

To conclude, we shall use the assumption $(\mathcal{A})$. By defining $n \in \mathbb{N}^*$ the first integer such that $f^{(n)}(0) \neq 0$, we see from Taylor expansion that

$$f'(\rho) = \rho^{n-1} q(\rho) \tag{31}$$

for some smooth positive function q on $[0, +\infty)$. In particular, since $q > 0$, we have

$$\rho \mapsto \frac{\rho f''(\rho)}{f'(\rho)} = n - 1 + \rho \frac{q'(\rho)}{q(\rho)} \in \mathcal{C}^\infty\big([0, +\infty)\big),$$

which implies

$$\left|\rho f''(\rho)\right| \leq C_M f'(\rho) \qquad \text{for } \ 0 \leq \rho \leq M. \tag{32}$$

This yields

$$\int_{\mathbb{R}^d} |a^\varepsilon|^2 \, |f''(|a^\varepsilon|^2)| (w, a^\varepsilon)^2 \leq C_M \int_{\mathbb{R}^d} (w, a^\varepsilon)^2 f'(|a^\varepsilon|^2) \leq C_M N^\varepsilon(w),$$

where, again, C_M depends only on M. In a similar way, we also obtain

$$\begin{aligned} \int_{\mathbb{R}^d} (w, a^\varepsilon)^2 |a^\varepsilon| \, |f''(|a^\varepsilon|^2)| \, |R_a| &\leq \left\|R_a\right\|_{L^\infty} \int_{\mathbb{R}^d} |w| \cdot \left|(w, a^\varepsilon)\right| \cdot |a^\varepsilon|^2 \left|f''(|a^\varepsilon|^2)\right| \\ &\leq \frac{C_M}{\varepsilon} \left\|R_a\right\|_{L^\infty} \int_{\mathbb{R}^d} \left(\varepsilon |w|\right) \left|(w, a^\varepsilon) \sqrt{f'(|a^\varepsilon|^2)}\right| \\ &\leq \frac{C_M}{\varepsilon} \left\|R_a\right\|_{L^\infty} N^\varepsilon(w). \end{aligned}$$

Consequently, we have proven that

$$I \leq C_M \Big(1 + \frac{1}{\varepsilon} \left\|R_a\right\|_{L^\infty}\Big) N^\varepsilon(w). \tag{33}$$

To get the result of Lemma 1, it remains to perform the L^2 estimate. Taking the L^2 scalar product of (21) with iw and using that

$$(w, u^\varepsilon \cdot \nabla w + \frac{1}{2} w \, \nabla \cdot u^\varepsilon) = \frac{1}{2} \nabla \cdot \left(|w|^2 u^\varepsilon\right),$$

we get

$$\frac{d}{dt}\Big(\frac{\varepsilon^2}{2} \|w\|_{L^2}^2\Big) = \int_{\mathbb{R}^d} \varepsilon (F^\varepsilon, iw) + 2\varepsilon \int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(w, a^\varepsilon)(a^\varepsilon, iw).$$

Note that we have once again used that R_φ is real-valued and hence that $(R_\varphi w, iw) = 0$. The first integral is clearly bounded by $N^\varepsilon(w) + \left\|F^\varepsilon\right\|_{L^2}$ whereas for the second one, we have

$$\int_{\mathbb{R}^d} 2\varepsilon f'(|a^\varepsilon|^2)(w, a^\varepsilon)(a^\varepsilon, iw) \leq C_M \int_{\mathbb{R}^d} \Big(f'(|a^\varepsilon|^2)(w, a^\varepsilon)^2 + \varepsilon^2 |w|^2\Big) \leq C_M N^\varepsilon(w).$$

As a consequence, we get

$$\frac{d}{dt}\Big(\frac{\varepsilon^2}{2} \left\|w\right\|_{L^2}^2\Big) \leq C_M N^\varepsilon(w) + \left\|F^\varepsilon\right\|_{L^2}^2. \tag{34}$$

Finally, we can collect (28), (29), (30), (33) and (34) to get (23). This completes the proof. $\square$

2.2 Higher order estimates

Since our final aim is to prove Theorem 2 by a fixed point argument, we also need to have H^s estimates for s sufficiently large for the solution of the linear equation (21). This is the aim of the following. Note that the term $-2(w,a^\varepsilon)f'(|a^\varepsilon|^2)a^\varepsilon$ in (19) can be seen as a singular term with variable coefficients. Consequently, a crude way to get H^s estimates is to apply $\varepsilon^{|\alpha|}\partial^\alpha$ to the equation, the weight $\varepsilon^{|\alpha|}$ being used to compensate the singular commutator when we take the derivative of (19), and then to apply Lemma 1 to the resulting equation. Nevertheless, it is possible to avoid the loss of $\varepsilon^{|\alpha|}$ with more work by using more clever higher order modulated functionals. We set $N_1^\varepsilon \equiv N^\varepsilon$ and, if $s\in\mathbb{N}$, $s\geq 2$, we define the following weighted norm, where $\alpha\in\mathbb{N}^d$ are multi-indices

$$\begin{aligned} N_s^\varepsilon(w) &\equiv \sum_{|\alpha|\leq s-1} N^\varepsilon(\partial^\alpha w) + K\|\mathrm{Re}\; w\|^2_{H^{s-2}} \\ &= \frac{1}{2}\,\varepsilon^2\|\nabla w\|^2_{H^{s-1}} + 2\sum_{|\alpha|\leq s-1}\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(\partial^\alpha w, a^\varepsilon)^2 + K\big(\varepsilon^2\|w\|^2_{H^{s-1}} + \|\mathrm{Re}\; w\|^2_{H^{s-2}}\big). \end{aligned} \tag{35}$$

In this section, we shall use that

$$a^\varepsilon = a^0 + \varepsilon a^r$$

with a^0 real-valued and

$$\sup_{\varepsilon\in(0,1]} \|a^r\|_{L^\infty([0,T],W^{s,\infty})} \leq C.$$

Note that this allows to write

$$\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(\partial^\alpha w, a^\varepsilon)^2 \geq \frac{1}{2}\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(a^0)^2|\mathrm{Re}\;\partial^\alpha w|^2 - C\varepsilon^2\|\mathrm{Re}\;\partial^\alpha w\|^2_{L^2}$$

and hence by choosing K sufficiently large ($K>C$) we get the lower bound

$$N_s^\varepsilon(w) \geq \frac{1}{2}\sum_{|\alpha|\leq s-1} N^\varepsilon\big(\partial_x^\alpha w\big) + \sum_{|\alpha|\leq s-1}\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(a^0)^2|\mathrm{Re}\;\partial^\alpha w|^2\,dx \tag{36}$$

Note that we also have the equivalence of norms:

$$\big\|w\big\|^2_{H^s} \leq \frac{2}{\varepsilon^2}\,N_s^\varepsilon(w), \quad N_s^\varepsilon(w) \leq C(|a^\varepsilon|_{W^{s-1,\infty}})\big\|w\big\|^2_{H^s} + \|\mathrm{Re}\; w\|^2_{H^{s-2}}. \tag{37}$$

The main result of this section is:

Theorem 3 *Let $0<T<\infty$, $s\in\mathbb{N}^*$, f satisfying $(\mathcal{A})$ and $w\in\mathcal{C}^1([0,T],H^s)$ a solution of* (21) *with $u^\varepsilon:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ and $a^\varepsilon:[0,T]\times\mathbb{R}^d\to\mathbb{C}$ such that*

$$M \equiv \sup_{0<\varepsilon<1}\Big(\big\|u^\varepsilon\big\|_{L^\infty([0,T],W^{s+1,\infty}(\mathbb{R}^d))} + \big\|a^\varepsilon\big\|_{L^\infty([0,T],W^{s,\infty}(\mathbb{R}^d))}\Big) < +\infty.$$

Assume finally that, for some $a^0\in L^\infty([0,T],W^{s,\infty}(\mathbb{R}^d))$ real-valued, *a^ε writes*

$$a^\varepsilon = a^0 + \mathcal{O}_{W^{s,\infty}}(\varepsilon) \tag{38}$$

uniformly on $[0,T]$. Then, there exists C, depending only on d, f and M, such that

$$\frac{d}{dt}N_s^\varepsilon\big(w(t)\big) \leq C\Big(1+\frac{1}{\varepsilon}\big\|R_a(t)\big\|_{L^\infty}+\frac{1}{\varepsilon^2}\big\|R_\varphi(t)\big\|_{W^{s-1,\infty}}\Big)N_s^\varepsilon\big(w(t)\big)+C\big\|F^\varepsilon(t)\big\|^2_{H^s}+\frac{C}{\varepsilon^2}\big\|\mathrm{Im}\,F^\varepsilon(t)\big\|^2_{H^{s-1}}.$$

Remark 2 In view of (38), a^ε is real up to $\mathcal{O}(\varepsilon)$, hence, in the integral in the right-hand side of (23), the real and imaginary parts of F^ε do not play the same role. This explains that the estimate is better for $\operatorname{Re} F^\varepsilon$ than for $\operatorname{Im} F^\varepsilon$. As a matter of fact, for $s = 1$, Theorem 3 follows immediately from Lemma 1 and (38).

2.3 Proof of Theorem 3

We estimate separately the two terms in $N_s^\varepsilon(w)$, when $s \geq 2$ (otherwise, the result follows from Lemma 1 as we have seen). Let us set

$$\Sigma(w) \equiv \|\text{Re } w\|^2_{H^{s-2}}.$$

Note that we have

$$\Sigma(w) \leq N_s^\varepsilon(w). \tag{39}$$

In the proof, C is a constant depending only on d, f and M.

We shall first prove that

$$\frac{d}{dt}\Sigma(w) \leq C\Big(1 + \frac{1}{\varepsilon^2}\big\|R_\varphi\big\|_{W^{s-2,\infty}}\Big)N_s^\varepsilon(w) + C\big\|F^\varepsilon\big\|^2_{H^{s-2}} + \frac{C}{\varepsilon^2}\big\|\operatorname{Im} F^\varepsilon\big\|^2_{H^{s-2}}. \tag{40}$$

For $\alpha \in \mathbb{N}^d$, we have

$$\begin{aligned}\partial_t\big(\partial^\alpha w\big) + u^\varepsilon \cdot \nabla\big(\partial^\alpha w\big) = \quad & \frac{i\varepsilon}{2}\Delta(\partial^\alpha w) - \frac{i}{\varepsilon}\partial^\alpha F^\varepsilon - \frac{i}{\varepsilon}\partial^\alpha\big(R_\varphi w\big) \\ & -\frac{2i}{\varepsilon}\partial^\alpha\big(f'(|a^\varepsilon|^2)(a^\varepsilon, w)a^\varepsilon\big) - \big[\partial^\alpha, u^\varepsilon\cdot\nabla\big]w - \frac{1}{2}\,\partial^\alpha\big(w\nabla\cdot u^\varepsilon\big).\end{aligned} \tag{41}$$

Next, by taking the real part of (41), we get

$$\partial_t\big(\partial^\alpha \text{Re } w\big) + u^\varepsilon \cdot \nabla\big(\partial^\alpha \text{Re } w\big) \;=\; -\big[\partial^\alpha, u^\varepsilon\cdot\nabla\big]\text{Re } w - \frac{1}{2}\,\partial^\alpha\big(\text{Re } w\,\nabla\cdot u^\varepsilon\big) + \mathcal{R}^\varepsilon$$

where

$$\mathcal{R}^\varepsilon = \text{Re}\Big(\frac{i\varepsilon}{2}\Delta(\partial^\alpha w) - \frac{i}{\varepsilon}\partial^\alpha F^\varepsilon - \frac{i}{\varepsilon}\partial^\alpha\big(R_\varphi w\big) - \frac{2i}{\varepsilon}\partial^\alpha\big(f'(|a^\varepsilon|^2)(a^\varepsilon, w)a^\varepsilon\big)\Big). \tag{42}$$

By using (38), we have

$$\text{Im } \partial^\gamma a^\varepsilon = \mathcal{O}(\varepsilon), \quad \forall\gamma,\ |\gamma| \leq |\alpha|$$

and

$$|(\partial^\beta a^\varepsilon, \partial^\gamma w)| \leq C_{\beta,\gamma}\Big(|\text{Re } \partial^\gamma w| + \varepsilon|\partial^\gamma w|\Big) \tag{43}$$

for every β, γ. Consequently, we immediately obtain for every α, $|\alpha| \leq s-2$,

$$\begin{aligned}\|\mathcal{R}^\varepsilon\|_{L^2} \;&\leq\; C\Big(\varepsilon\|w\|_{H^s} + \frac{\|R_\varphi\|_{W^{s-1,\infty}}}{\varepsilon^2}\|w\|_{H^{s-2}} + \|\text{Re } w\|_{H^{s-2}} + \varepsilon\|w\|_{H^{s-2}}\Big) + \frac{1}{\varepsilon}\|\text{Im } F^\varepsilon\|_{H^{s-2}} \\ &\leq\; C\Big(1 + \frac{\|R_\varphi\|_{W^{s-2,\infty}}}{\varepsilon^2}\Big)N_s^\varepsilon(w)^{\frac{1}{2}} + \frac{1}{\varepsilon}\,\|\text{Im } F^\varepsilon\|_{H^{s-2}}.\end{aligned}$$

Consequently, the standard L^2 energy estimate for (42) gives

$$\frac{d}{dt}\|\mathrm{Re}\ \partial^\alpha w\|_{L^2}^2 \leq C\Big(1+\frac{\|R_\varphi\|_{W^{s-1,\infty}}}{\varepsilon^2}\Big)N_s^\varepsilon(w)+\frac{1}{\varepsilon^2}\|\mathrm{Im}F^\varepsilon\|_{H^{s-2}}^2.$$

Note that we have used that

$$\int_{\mathbb{R}^d}\Big(u^\varepsilon\cdot\nabla(\partial^\alpha\mathrm{Re}\ w),\partial^\alpha\mathrm{Re}\ w\Big)=-\frac{1}{2}\int_{\mathbb{R}^d}(\nabla\cdot u^\varepsilon)|\partial^\alpha\mathrm{Re}\ w|^2.$$

Consequently, (40) is proven.

The next step is to estimate $N^\varepsilon(\partial^\alpha w)$ for $|\alpha|\leq s-1$. By applying ∂^α to (21), we get

$$i\varepsilon\frac{\partial(\partial^\alpha w)}{\partial t}+\mathcal{L}^\varepsilon\big(\partial^\alpha w\big)=R_\varphi\partial^\alpha w+\tilde{F}^\varepsilon, \tag{44}$$

where

$$\tilde{F}^\varepsilon\equiv\mathcal{C}^\alpha+\mathcal{D}^\alpha+\partial^\alpha F^\varepsilon+[\partial^\alpha,R_\varphi]w,$$

with

$$\begin{aligned}\mathcal{C}^\alpha &\equiv 2\,\partial^\alpha\Big(f'(|a^\varepsilon|^2)a^\varepsilon(w,a^\varepsilon)\Big)-2f'(|a^\varepsilon|^2)(\partial^\alpha w,a^\varepsilon)a^\varepsilon,\\ \mathcal{D}^\alpha &\equiv -i\varepsilon\big[\partial^\alpha,u^\varepsilon\cdot\nabla\big]w-\frac{i\varepsilon}{2}\big[\partial^\alpha,\nabla\cdot u^\varepsilon\big]w.\end{aligned}$$

To estimate $N^\varepsilon(\partial^\alpha w)$, we shall use Lemma 1. Towards this, we need to estimate the commutators in the right hand side of (44). For $|\alpha|\leq s-1$, the following estimates hold for $\mathcal{C}^\alpha$ and $\mathcal{D}^\alpha$:

$$\big\|[\partial^\alpha,R_\varphi]w\big\|_{H^1}^2 \leq C\big\|R_\varphi\big\|_{W^{s,\infty}}^2\big\|w\big\|_{H^s}^2\leq\frac{C}{\varepsilon^2}\big\|R_\varphi\big\|_{W^{s,\infty}}^2N_s^\varepsilon(w), \tag{45}$$

$$\big\|\mathcal{D}^\alpha\big\|_{H^1}^2 \leq C\,\varepsilon^2\big\|w\big\|_{H^s}^2\leq CN_s^\varepsilon(w), \tag{46}$$

$$\big\|\big(if'(|a^\varepsilon|^2)^{\frac{1}{2}}a^\varepsilon,\mathcal{D}^\alpha\big)\big\|_{L^2}^2 \leq C\varepsilon^2N_s^\varepsilon(w), \tag{47}$$

$$\big\|\mathcal{C}^\alpha\big\|_{H^1}^2 \leq CN_s^\varepsilon(w), \tag{48}$$

$$\big\|(ia^\varepsilon,\mathcal{C}^\alpha)\big\|_{L^2}^2 \leq C\varepsilon^2N_s^\varepsilon(w). \tag{49}$$

The estimates (45) and (46) follow easily from (37). For (47), we note that

$$\begin{aligned}\frac{1}{\varepsilon}\big(ia^\varepsilon,\mathcal{D}^\alpha\big) &= -\big(a^\varepsilon,[\partial^\alpha,u^\varepsilon\cdot\nabla]w\big)-\frac{1}{2}\big(a^\varepsilon,[\partial^\alpha,\nabla\cdot u^\varepsilon]w\big)\\ &= -\sum_{\gamma<\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\big(\partial^{\alpha-\gamma}u^\varepsilon\big)\cdot\big(a^\varepsilon,\nabla\partial^\gamma w\big)-\frac{1}{2}\sum_{\gamma<\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\partial^{\alpha-\gamma}\big(\nabla\cdot u^\varepsilon\big)\big(a^\varepsilon,\partial^\gamma w\big)\end{aligned}$$

since u^ε is real. Next, we can use (38) and (43) again. In particular, in the above expansion, the terms $\big(a^\varepsilon,\partial^\gamma w\big)$ are bounded in L^2 by $\Sigma(w)+\varepsilon^2\|w\|_{H^{s-2}}^2$ and thus by $N_s^\varepsilon(w)$. Similarly, the terms $\big(a^\varepsilon,\nabla\partial^\gamma w\big)$ are bounded in L^2 by $N_s^\varepsilon(w)$ if $|\gamma|\leq s-3$. Consequently, we get

$$\big\|\big(if'(|a^\varepsilon|^2)^{\frac{1}{2}}a^\varepsilon,\mathcal{D}^\alpha\big)\big\|_{L^2}^2\leq C\Big(\sum_{|\beta|=s-1}\int_{\mathbb{R}^d}f'(|a^\varepsilon|^2)(\partial^\beta w,a^\varepsilon)^2+N_s^\varepsilon(w)\Big)\leq CN_s^\varepsilon(w),$$

which yields (47). Next, we turn to $\mathcal{C}^\alpha$. The Leibnitz formula gives

$$\mathcal{C}^\alpha = \sum_{\substack{\tilde\alpha<\alpha,\\ \tilde\alpha+\beta+\lambda+\mu=\alpha}} * \,\partial^\lambda\big[f'(|a^\varepsilon|^2)\big]\big(\partial^{\tilde\alpha} w, \partial^\beta a^\varepsilon\big)\partial^\mu a^\varepsilon, \tag{50}$$

where $*$ is a real coefficient depending only on $\tilde\alpha$, β, λ and μ. Since $|\tilde\alpha| \le |\alpha|-1 \le s-2$, we can use again (38) through (43) to get that

$$\|\mathcal{C}^\alpha\|_{L^2}^2 \le C\Big(\Sigma(w) + \varepsilon^2\|w\|_{H^s}^2\Big) \le CN_s^\varepsilon(w).$$

Since $(ia^\varepsilon, \partial^\mu a^\varepsilon) = \mathcal{O}(\varepsilon)$ thanks to (38), we also get (49). For the H^1 norm, the same argument yields

$$\|\mathcal{C}^\alpha\|_{H^1}^2 \le C\Big(\Sigma(w) + \varepsilon^2\|w\|_{H^s}^2 + \sum_{\substack{|\gamma|=s-1,\\ |\beta+\lambda+\mu|=1}} \int_{\mathbb{R}^d} \big|\partial^\lambda\big[f'(|a^\varepsilon|^2)\big]\big(\partial^\gamma w, \partial^\beta a^\varepsilon\big)\partial^\mu a^\varepsilon\big|^2\Big).$$

To estimate the last sum, we first consider the terms with $\beta = 0$. They are always bounded by

$$C\int_{\mathbb{R}^d} \big[f'(|a^\varepsilon|^2) + |a^\varepsilon|^2|f''(|a^\varepsilon|^2)|\big]\big(\partial^\gamma w, a^\varepsilon\big)^2$$

with $|\gamma| = s-1$ and hence, thanks to (32), they are bounded by

$$C\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)\big(\partial^\gamma w, a^\varepsilon\big)^2$$

and hence by $N_s^\varepsilon(w)$. Next, we consider the terms with $|\beta| = 1$. Since then $\lambda = \mu = 0$, we have to estimate terms like

$$\mathcal{T} = \int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)\big(\partial^\gamma w, \partial^\beta a^\varepsilon\big)^2|a^\varepsilon|^2.$$

By using again (38) and (43), we get

$$\mathcal{T} \le C\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)|a^0|^2|\mathrm{Re}\,\partial^\gamma w|^2 + C\varepsilon^2\|w\|_{H^{s-1}}^2$$

and hence, by using (36), we finally obain

$$\mathcal{T} \le CN_s^\varepsilon(w).$$

Consequently, (48) is proven. This ends the estimates of the commutators.

We are now able to establish:

$$\begin{aligned}\frac{d}{dt}N^\varepsilon\big(\partial^\alpha w\big) \le C\,\Big(1 + \frac{1}{\varepsilon^2}\big\|R_\varphi\big\|_{W^{s-1,\infty}} + \frac{1}{\varepsilon}\big\|R_a\big\|_{L^\infty}\Big)N_s^\varepsilon(w)\\ + \big\|F^\varepsilon\big\|_{H^s}^2 + \frac{C}{\varepsilon^2}\big\|\mathrm{Im}\,F^\varepsilon\big\|_{H^{s-1}}^2.\end{aligned} \tag{51}$$

Indeed, from Lemma 1, we deduce

$$\frac{d}{dt}N^\varepsilon(\partial^\alpha w) \leq C\ \Big(1+\frac{1}{\varepsilon}\|R_\varphi\|_{W^{1,\infty}}+\frac{1}{\varepsilon}\|R_a\|_{L^\infty}+\frac{1}{\varepsilon^2}\|R_\varphi\|_{L^\infty}\Big)N^\varepsilon(\partial^\alpha w)$$
$$+\|\tilde F^\varepsilon\|^2_{L^2}+\frac{4}{\varepsilon}\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(\partial^\alpha w, a^\varepsilon)\big(ia^\varepsilon,\tilde F^\varepsilon\big)-\int_{\mathbb{R}^d}(i\varepsilon\Delta\partial^\alpha w,\tilde F^\varepsilon). \tag{52}$$

To estimate the right-hand side of (52), we first estimate $\|\tilde F^\varepsilon\|^2_{L^2}$. Combining (45) and (46) with (48), we infer

$$\|\tilde F^\varepsilon\|^2_{L^2} \leq \|F^\varepsilon\|^2_{H^{s-1}}+C\Big(1+\frac{1}{\varepsilon^2}\|R_\varphi\|^2_{W^{s-1,\infty}}\Big)N^\varepsilon_s(w). \tag{53}$$

Next, we turn to the term

$$\frac{4}{\varepsilon}\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(\partial^\alpha w, a^\varepsilon)\big(ia^\varepsilon,\tilde F^\varepsilon\big)=\frac{4}{\varepsilon}\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(\partial^\alpha w, a^\varepsilon)\big(ia^\varepsilon,\mathcal{C}^\alpha+\mathcal{D}^\alpha+\partial^\alpha F^\varepsilon+[\partial^\alpha,R_\varphi]w\big),$$

which splits as four integrals. For the first one, by (49) and Cauchy-Schwarz:

$$\frac{4}{\varepsilon}\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(\partial^\alpha w, a^\varepsilon)\big(ia^\varepsilon,\mathcal{C}^\alpha\big)\leq C\Big(\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)\big(\partial^\alpha w, a^\varepsilon\big)^2\Big)^{\frac12}N^\varepsilon_s(w)^{\frac12}\leq CN^\varepsilon_s(w).$$

For the second one, we use (47) and Cauchy-Schwarz, which gives

$$\frac{4}{\varepsilon}\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)^{\frac12}(\partial^\alpha w, a^\varepsilon)\big(if'(|a^\varepsilon|^2)^{\frac12}a^\varepsilon,\mathcal{D}^\alpha\big)\leq CN^\varepsilon_s(w).$$

For the third integral, we simply write, using once again (38)

$$\frac{1}{\varepsilon}\big\|\big(ia^\varepsilon,\partial^\alpha F^\varepsilon\big)\big\|_{L^2}\leq C\|F^\varepsilon\|_{H^{s-1}}+\frac{C}{\varepsilon}\|\operatorname{Im} F^\varepsilon\|_{H^{s-1}},$$

which yields by Cauchy-Schwarz

$$\frac{4}{\varepsilon}\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(\partial^\alpha w, a^\varepsilon)\big(ia^\varepsilon,\partial^\alpha F^\varepsilon\big)\leq CN^\varepsilon_s(w)+C\|F^\varepsilon\|^2_{H^{s-1}}+\frac{C}{\varepsilon^2}\|\operatorname{Im} F^\varepsilon\|^2_{H^{s-1}}.$$

Finally, for the fourth integral, we have by (45)

$$\frac{4}{\varepsilon}\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(\partial^\alpha w, a^\varepsilon)\big(ia^\varepsilon,[\partial^\alpha,R_\varphi]w\big)\leq\frac{C}{\varepsilon}\|R_\varphi\|_{W^{s-1,\infty}}N^\varepsilon(w).$$

By summing these estimates, we find

$$\frac{4}{\varepsilon}\int_{\mathbb{R}^d} f'(|a^\varepsilon|^2)(\partial^\alpha w, a^\varepsilon)\big(ia^\varepsilon,\tilde F^\varepsilon\big)\leq C\Big(1+\frac{1}{\varepsilon}\|R_\varphi\|_{W^{s-1,\infty}}\Big)N^\varepsilon_s(w)+C\|F^\varepsilon\|^2_{H^{s-1}}+\frac{C}{\varepsilon^2}\|\operatorname{Im} F^\varepsilon\|^2_{H^{s-1}}. \tag{54}$$

Finally, we handle the term

$$-\int_{\mathbb{R}^d}\big(i\varepsilon\Delta\partial^\alpha w,\tilde F^\varepsilon\big)=-\int_{\mathbb{R}^d}\big(i\varepsilon\Delta\partial^\alpha w,\mathcal{C}^\alpha+\mathcal{D}^\alpha+\partial^\alpha F^\varepsilon+[\partial^\alpha,R_\varphi]w\big).$$

By using an integration by parts, we have

$$-\int_{\mathbb{R}^d}\big(i\varepsilon\Delta\partial^\alpha w,\tilde F^\varepsilon\big)\leq\|\mathcal{C}^\alpha\|^2_{H^1}+\|\mathcal{D}^\alpha\|^2_{H^1}+\|[\partial^\alpha,R_\varphi]w\|^2_{H^1}+\|F^\varepsilon\|^2_{H^s}+CN^\varepsilon_s(w)$$
$$\leq\|F^\varepsilon\|^2_{H^s}+C\Big(1+\frac{1}{\varepsilon^2}\|R_\varphi\|_{W^{s-1,\infty}}\Big)N^\varepsilon_s(w)$$

thanks to (45), (46) and (48). Consequently, we can collect the last estimate and (52), (53), (54) to get (51). This ends the proof of Theorem 3.

3 Construction of WKB expansions

In this section, we construct an approximate solution of (1) using a WKB expansion. The first step is to prove the local existence of smooth solutions of the limit hydrodynamical system.

3.1 Well-posedness of the limit system

We consider the system

$$\begin{cases} \partial_t a + u \cdot \nabla a + \dfrac{1}{2}\, a\, \nabla \cdot u = 0 \\ \\ \partial_t u + u \cdot \nabla u + \nabla\big(f(a^2)\big) = 0, \end{cases} \tag{55}$$

which is only weakly hyperbolic, with the pressure law f satisfying assumption $(\mathcal{A})$ and the initial condition $(a, u)_{|t=0} = (a_0, u_0)$.

Theorem 4 *Assume that f satisfies $(\mathcal{A})$ and let $s > 2 + d/2$. Then, for every initial conditions $(a_0, u_0) \in H^s \times H^s$ with $a_0 \in \mathbb{R}$, there exists $T > 0$ and a unique solution (a, u) of (55) such that $(a, u) \in \mathcal{C}([0, T], H^{s-1} \times H^s) \cap \mathcal{C}^1([0, T], H^{s-2} \times H^{s-1})$.*

Let us remark that if $n = 1$, then $f'(0) > 0$ and thus $f' > 0$ in $[0, +\infty)$ (by $(\mathcal{A})$). In this case, (55) is symmetrizable (with the symmetrizer $S = \mathrm{diag}\big(1, \frac{1}{4f'(a^2)}, ..., \frac{1}{4f'(a^2)}\big)$ used in [9]) and the local existence and uniqueness for (55) follows easily.

Proof of Theorem 4.

The first step is to rewrite the system by using more convenient unknowns. At first, we notice that thanks to $(\mathcal{A})$, we can write f under the form

$$f(\rho) = \rho^n \tilde{f}(\rho),$$

with $\tilde{f}$ smooth on $[0, +\infty)$ and such that $\tilde{f}(0) \neq 0$. Next, since we have by assumption $f(0) = 0$ and $f'(\rho) > 0$ for $\rho \neq 0$, we also have that $f(\rho) > 0$ for $\rho > 0$. This implies that $\tilde{f}(\rho) > 0$ for $\rho \geq 0$. This allows to define a smooth function h on $\mathbb{R}$ by

$$\mathrm{h}(a) \equiv a\big[\tilde{f}(a^2)\big]^{\frac{1}{2n}}. \tag{56}$$

Note that $\mathrm{h}(a) \neq 0$ for $a \neq 0$. It is usefull to notice that we can also write h under the form

$$\mathrm{h}(a) = \mathrm{sgn}(a)\, f(a^2)^{\frac{1}{2n}}$$

and hence that we have

$$\mathrm{h}(a)^{2n} = f(a^2), \qquad a \in \mathbb{R}.$$

Furthermore, since $f' > 0$ and $\tilde{f}(0) > 0$ in $(0, +\infty)$, we deduce that $\mathrm{h}'(a) > 0$ for $a \neq 0$ and that $\mathrm{h}'(0) = \big[\tilde{f}(0)\big]^{\frac{1}{2n}} > 0$, so that $\mathrm{h}' > 0$ on $\mathbb{R}$. Thus h is a smooth diffeomorphism from $\mathbb{R}$ to $\mathrm{h}\big(\mathbb{R}\big)$. In particular, this allows to define a smooth positive function c on $\mathrm{h}\big(\mathbb{R}\big)$ such that

$$\frac{1}{2}\, a\mathrm{h}'(a) = \mathrm{h}(a)\, c\big(\mathrm{h}(a)\big), \quad \forall a \in \mathbb{R}.$$

With this definition, (h, u), with $h \equiv \mathrm{h}(a)$, solves the system

$$\begin{cases} \partial_t h + u \cdot \nabla h + hc(h)\, \nabla \cdot u = 0 \\ \\ \partial_t u + u \cdot \nabla u + \nabla\big(h^{2n}\big) = 0. \end{cases} \tag{57}$$

Since a is in H^s if and only if h is in H^s, we shall prove local existence of smooth solution for the weakly hyperbolic system (57). As we shall see below, the nonlinear symmetrization method of [15] does not allow to reduce (57) to a symmetric or symmetrizable system with smooth coefficients except in the case where $c(h) = \tilde{c}(h^n)$ for some smooth map $\tilde{c}$. Nevertheless, it will be still possible to use the same idea to prove the existence of an energy estimate with loss for the system (57). When we are in such a situation, the simplest way to construct a solution is to use the vanishing viscosity method. Indeed, this approximation method allows to preserve the nonlinear energy estimate verified by (57). We thus consider for $\epsilon > 0$ the system

$$\begin{cases} \partial_t h_\epsilon + u_\epsilon \cdot \nabla h_\epsilon + h_\epsilon c(h_\epsilon) \nabla \cdot u_\epsilon = \epsilon\, \Delta h_\epsilon \\ \\ \partial_t u_\epsilon + u_\epsilon \cdot \nabla u_\epsilon + \nabla\big(h_\epsilon^{2n}\big) = \epsilon\, \Delta u_\epsilon. \end{cases} \tag{58}$$

The local existence of smooth solutions for this parabolic system is very easy to obtain. Moreover, we note that h_ϵ remains nonnegative if the initial datum $(h_\epsilon)_{|t=0}$ is nonnegative. In the following, we shall only prove an H^s energy estimate independent of ϵ for this system which ensures that the solution remains smooth on an interval of time independent of ϵ. The final step which consists in using the uniform bounds to pass to the limit when ϵ goes to zero to get a solution of (57) is very classical and hence will not be detailled. In the proof of the energy estimates, we shall omit the subscript ϵ for notational convenience.

As in the work of [15], we introduce the unknown $H \equiv h^n = a^n \tilde{f}(a^2)^{\frac{1}{2}}$. Note that by definition of h, H is in H^s as soon as a is in H^s. We get for (H, u) the system

$$\begin{cases} \partial_t H + u \cdot \nabla H + nHc(h)\, \nabla \cdot u = \epsilon\, n h^{n-1} \Delta h = \epsilon \Big(\Delta H - n(n-1) h^{n-2} |\nabla h|^2 \Big) \\ \\ \partial_t u \ + u \cdot \nabla u \ + \ 2H \nabla H = \epsilon\, \Delta u. \end{cases} \tag{59}$$

Note that it does not seem possible to get a classical hyperbolic symmetric system (in the case $\epsilon = 0$) involving only H and u as in the case of homogeneous pressure laws considered in [15]. Indeed, the coefficient $c(h) = c(H^{\frac{1}{n}})$ is not (in general) a smooth function of H. Nevertheless, it will be possible to prove that the system with unknowns (h, H, u) though only weakly hyperbolic (when $\epsilon = 0$) satisfies an energy estimate. We notice that the symmetrizer

$$S \equiv \mathrm{diag}\Big(1, \frac{n}{2} c(h) I_d\Big),$$

which is positive since $c(h)$ is positive, symmetrizes the first order part of (59). We shall first perform an H^s energy estimate ($s > 2 + d/2$) on (59) but we have to track carefully the dependence on h in the energy estimates.

To prove our H^s energy estimate, we shall make an extensive use of the following classical (see

[18] for example) tame estimates

$$\|fg\|_{H^k} \le C_k\Big(\|f\|_{L^\infty}\|g\|_{H^k} + \|f\|_{H^k}\|g\|_{L^\infty}\Big), \tag{60}$$

$$\|\partial^\alpha(fg) - f\partial^\alpha g\|_{L^2} \le C_k\Big(\|f\|_{H^k}\|g\|_{L^\infty} + \|\nabla f\|_{L^\infty}\|g\|_{H^{k-1}}\Big), \quad |\alpha| \le k, \tag{61}$$

$$\|F(u)\|_{H^k} \le C(\|u\|_{L^\infty})(1 + \|u\|_{H^k}) \tag{62}$$

if F is smooth and such that $F(0) = 0$.

At first, we notice that $(\partial^\alpha H, \partial^\alpha u)$ for $|\alpha| \le s$ solves the system

$$\begin{cases} \partial_t\partial^\alpha H + u\cdot\nabla\partial^\alpha H + nc(h)\big(\nabla\cdot u\big)\partial^\alpha H &= \epsilon\left(\Delta\partial^\alpha H - n(n-1)\partial^\alpha(h^{n-2}|\nabla h|^2)\right) \\ &\quad -[\partial^\alpha, u]\cdot\nabla H - n[\partial^\alpha, Hc(h)]\nabla\cdot u \\ \partial_t\partial^\alpha u \ + u\cdot\nabla\partial^\alpha u \ + \ 2H\nabla\partial^\alpha H &= \epsilon\,\Delta\partial^\alpha u - [\partial^\alpha, u]\cdot\nabla u - [\partial^\alpha, 2H]\nabla H. \end{cases}$$

By using (61) to estimate in L^2 the commutators in the right hand-side, we get in a classical way by integration by parts

$$\begin{aligned} &\frac{d}{dt}\Big[\frac{1}{2}\int_{\mathbb{R}^d}|\partial^\alpha H|^2 + \frac{n}{2}c(h)|\partial^\alpha u|^2\Big] + \epsilon\int_{\mathbb{R}^d}|\nabla\partial^\alpha H|^2 + \frac{n}{2}c(h)|\nabla\partial^\alpha u|^2 \\ &\le C_0\Big(\|(h,u)\|_{W^{1,\infty}}\Big)\|V\|_{H^s}^2 + \mathcal{C}^\alpha + \epsilon\,\mathcal{D}^\alpha + \mathcal{R}^\alpha, \end{aligned} \tag{63}$$

where $V \equiv (H, u)$, C_0 is a non-decreasing function depending only on f, s and d, and

$$\mathcal{C}^\alpha \equiv -n\int_{\mathbb{R}^d}(\partial^\alpha H)\,[\partial^\alpha, Hc(h)](\nabla\cdot u),$$

$$\mathcal{D}^\alpha \equiv -\frac{n}{2}\int_{\mathbb{R}^d}c'(h)\big((\nabla h\cdot\nabla)\partial^\alpha u\big)\cdot\partial^\alpha u - n(n-1)\int_{\mathbb{R}^d}\partial^\alpha\big(h^{n-2}|\nabla h|^2\big)\,\partial^\alpha H,$$

$$\mathcal{R}^\alpha \equiv \frac{n}{4}\int_{\mathbb{R}^d}c'(h)\partial_t h|\partial^\alpha u|^2.$$

We have singled out the three terms above since they are the ones involving h which must be estimated with care. Note that the estimate of $\mathcal{C}^\alpha$ will be crucial since this term involves high order derivatives of h. Next, we can integrate (63) in time, sum the estimates for $|\alpha| \le s$ and use that $c(h) > 0$, hence $nc(h)/2 \ge \frac{1}{C_1(\|h\|_{L^\infty})}$ to obtain

$$\begin{aligned} &\|V(t)\|_{H^s}^2 + \epsilon\int_0^t\|\nabla V(\tau)\|_{H^s}^2\,d\tau \\ &\le C_1\big(\|h\|_{L^\infty}\big)\Big(\|V(0)\|_{H^s}^2 + \int_0^t C_0\big(\|(h,u)(\tau)\|_{W^{1,\infty}}\big)\|V(\tau)\|_{H^s}^2 + \mathcal{C}(\tau) + \epsilon\mathcal{D}(\tau) + \mathcal{R}(\tau)\,d\tau\Big), \end{aligned} \tag{64}$$

with

$$\mathcal{C} \equiv \sum_{|\alpha|\le s}\mathcal{C}^\alpha, \quad \mathcal{D} \equiv \sum_{|\alpha|\le s}\mathcal{D}^\alpha, \quad \mathcal{R} \equiv \sum_{|\alpha|\le s}\mathcal{R}^\alpha.$$

Estimate for $\mathcal{C}$. We claim that

$$\mathcal{C} \le C_0(\|(h,u)\|_{W^{1,\infty}})\Big(\|V\|_{H^s}^2 + \|h\|_{H^{s-1}}^2\Big). \tag{65}$$

The crucial point is that this estimate only involves the H^{s-1} norm of h. This will allow to conclude by using that for the first equation in (59), the H^{s-1} norm of h is controlled by the H^s norm of u.

By using the commutator estimate (61), we have

$$\begin{aligned}\mathcal{C} &\leq C\|H\|_{H^s}\Big(\|Hc(h)\|_{H^s}\,\|\nabla\cdot u\|_{L^\infty}+\|\nabla\big(Hc(h)\big)\|_{L^\infty}\,\|\nabla\cdot u\|_{H^{s-1}}\Big)\\ &\leq C_0\big(\|(h,u)\|_{W^{1,\infty}}\big)\Big(\|V\|_{H^s}^2+\|H\|_{H^s}\,\|Hc(h)\|_{H^s}\Big).\end{aligned}$$

To estimate the last term, we use that $H=h^n$, which yields $h\partial_i H = nH\partial_i h$, thus

$$\partial_i\Big(Hc(h)\Big)=c(h)\partial_i H+c'(h)H\partial_i h=c(h)\partial_i H+\frac{1}{n}c'(h)h\partial_i H.$$

Consequently, by (60), (62), we get

$$\|Hc(h)\|_{H^s}\leq C\|c(h)\nabla H\|_{H^{s-1}}+C\|c'(h)h\nabla H\|_{H^{s-1}}\leq C_0\big(\|(h,u)\|_{W^{1,\infty}}\big)\Big(\|H\|_{H^s}+\|h\|_{H^{s-1}}\Big),$$

and (65) follows.

Estimate for $\mathcal{D}$. The term $\mathcal{D}$ involves derivatives of u of order $\leq s+1$, and we shall use the energy dissipation in (63). We prove that

$$C_1(\|h\|_{L^\infty})\,\epsilon\,\mathcal{D}\leq\frac{1}{2}\,\epsilon\,\|\nabla V\|_{H^s}^2+\epsilon\,C_0(\|h\|_{W^{1,\infty}})\Big(\|V\|_{H^s}^2+\|\nabla h\|_{H^{s-1}}^2\Big).\tag{66}$$

We have, on the one hand,

$$\Big|\int_{\mathbb{R}^d}c'(h)\nabla h\cdot\nabla\partial^\alpha u\cdot\partial^\alpha u\Big|\leq C_0(\|h\|_{W^{1,\infty}})\|\nabla u\|_{H^s}\,\|u\|_{H^s}\leq C_0(\|h\|_{W^{1,\infty}})\|\nabla V\|_{H^s}\,\|V\|_{H^s}.$$

On the other hand, for the second term (which vanishes if $n=1$), after one integration by parts when $|\alpha|>0$, we get

$$\begin{aligned}n(n-1)\Big|\int_{\mathbb{R}^d}\partial^\alpha\big(h^{n-2}|\nabla h|^2\big)\,\partial^\alpha H\Big| &\leq C\|\nabla H\|_{H^s}\,\|h^{n-2}|\nabla h|^2\|_{H^{s-1}}\\ &\leq C_0(\|h\|_{W^{1,\infty}})\,\|\nabla H\|_{H^s}\|\nabla h\|_{H^{s-1}},\end{aligned}$$

and if $\alpha=0$, since $H=h^n$ and $s\geq 1$,

$$n(n-1)\Big|\int_{\mathbb{R}^d}h^{n-2}|\nabla h|^2 H\Big|=\frac{n-1}{n}\int_{\mathbb{R}^d}|\nabla H|^2\leq C\|H\|_{H^s}^2.$$

Consequently,

$$\epsilon\,\mathcal{D}\leq\epsilon\,C_0(\|h\|_{W^{1,\infty}})\,\|\nabla V\|_{H^s}\Big(\|V\|_{H^s}+\|\nabla h\|_{H^{s-1}}\Big)+\epsilon\,C\|V\|_{H^s}^2,$$

and (66) follows from the standard inequality, for a, b, $\theta>0$, $ab\leq\theta a^2+\frac{b^2}{4\theta}$.

Estimate for $\mathcal{R}$. We prove that

$$C_1(\|h\|_{L^\infty})\,\mathcal{R} \le \frac{1}{2}\,\epsilon\,\|\nabla V\|_{H^s}^2 + C_0(\|(h,u)\|_{W^{1,\infty}})\,\|V\|_{H^s}^2. \tag{67}$$

By using the first equation in (58) for h and an integration by parts, we find, as for the first term in $\mathcal{D}$,

$$\begin{aligned}
\mathcal{R}^\alpha \;&\le\; C_0(\|(h,u)\|_{W^{1,\infty}})\,\|V\|_{H^s}^2 + \epsilon\,\frac{n}{4}\int_{\mathbb{R}^d} c'(h)\Delta h|\partial^\alpha u|^2\\
&\le\; C_0(\|(h,u)\|_{W^{1,\infty}})\,\|V\|_{H^s}^2 - \epsilon\,\frac{n}{4}\int_{\mathbb{R}^d} c'(h)\big((\nabla h\cdot\nabla)\partial^\alpha u\big)\cdot\partial^\alpha u - \epsilon\,\frac{n}{4}\int_{\mathbb{R}^d} c''(g)|\nabla h|^2\,|\partial^\alpha u|^2\\
&\le\; C_0(\|(h,u)\|_{W^{1,\infty}})\,\Big(\|V\|_{H^s}^2 + \epsilon\,\|\nabla V\|_{H^s}\,\|V\|_{H^s}\Big).
\end{aligned}$$

Then, (66) follows as above from the inequality $ab \le \theta a^2 + \frac{b^2}{4\theta}$.

Summing (65), (66) and (67), inserting this into (64) and cancelling the terms $\epsilon\,\|\nabla V\|_{H^s}^2$, we infer

$$\begin{aligned}
\|V(t)\|_{H^s}^2 \le C_1\big(\|h(t)\|_{L^\infty}\big)\Big(&\|V(0)\|_{H^s}^2\\
&+\int_0^t C_0(\|(h,u)(\tau)\|_{W^{1,\infty}})\Big[\|V(\tau)\|_{H^s}^2 + \|h(\tau)\|_{H^{s-1}}^2 + \epsilon\|\nabla h(\tau)\|_{H^{s-1}}^2\Big]\,d\tau\Big).
\end{aligned} \tag{68}$$

To close the estimate, it remains to evaluate $\|h\|_{H^{s-1}}^2$ and $\epsilon\int_0^t \|\nabla h\|_{H^{s-1}}^2$. We use the standard H^{s-1} estimate for the convection diffusion equation (58) which yields, as for (63), for $|\alpha| \le s-1$,

$$\frac{d}{dt}\Big[\frac{1}{2}\int_{\mathbb{R}^d}|\partial^\alpha h|^2\Big] + \epsilon\int_{\mathbb{R}^d}|\partial^\alpha h|^2 \le C_0\big(\big\|(h,u)\big\|_{W^{1,\infty}}\big)\Big(\|h\|_{H^{s-1}}^2 + \|h\|_{H^{s-1}}\|u\|_{H^s}\Big).$$

Summing for $|\alpha| \le s-1$ and integrating in time, this yields

$$\begin{aligned}
\frac{1}{2}\,\|h(t)\|_{H^{s-1}}^2 &+ \epsilon\int_0^t \|\nabla h(\tau)\|_{H^{s-1}}^2\,d\tau\\
&\le \frac{1}{2}\,\|h(0)\|_{H^{s-1}}^2 + \int_0^t C_0\big(\big\|(h,u)(\tau)\big\|_{W^{1,\infty}}\big)\Big(\|V(\tau)\|_{H^s}^2 + \|h(\tau)\|_{H^{s-1}}^2\Big)\,d\tau.
\end{aligned} \tag{69}$$

Finally, we can combine (68) and (69), to get

$$\begin{aligned}
&\|V(t)\|_{H^s}^2 + \|h(t)\|_{H^{s-1}}^2\\
&\quad\le C_0\big(\big\|(h,u)\big\|_{L^\infty([0,t],W^{1,\infty})}\big)\Big(\|V(0)\|_{H^s}^2 + \|h(0)\|_{H^{s-1}}^2 + \int_0^t \|V(\tau)\|_{H^s}^2 + \|h(\tau)\|_{H^{s-1}}^2\,d\tau\Big).
\end{aligned} \tag{70}$$

Since H^{s-1} is embedded in $W^{1,\infty}$ for $s > 2 + d/2$, we easily get by classical continuation arguments and the Gronwall lemma that the solution of (58) is defined on an interval of time $[0,T)$ independent of ϵ. Finally, (70) provides a uniform bound for (h,H,u) in $H^{s-1}\times H^s\times H^s$, which allows to prove in a classical way that (h_ϵ,u_ϵ) converges towards a solution of (57). This ends the proof of the existence of solution.

To prove the uniqueness, it suffices to use the same method as above and perform an L^2 energy estimate on the system satisfied by $h_1 - h_2, u_1 - u_2, H_1 - H_2$. This is left to the reader.

3.2 WKB expansions

We now turn to the construction of WKB expansions up to arbitrary order. Let us first notice that in Theorem 4, if the initial datum (a_0, u_0) is in $H^\infty \times H^\infty$, then the solution (a, u) is in $\mathcal{C}^0([0,T], H^{s-1} \times H^s)$ for every $s > 2 + d/2$, with T independent of $s > 2 + d/2$. In other words, the existence time of the maximal solution in $H^\infty \times H^\infty$ is positive. This fact follows easily from (70) and the Gronwall inequality (since $H^{s-1} \subset W^{1,\infty}$).

Lemma 2 *Consider $\Psi_0^\varepsilon = a_0^\varepsilon e^{i\varphi_0^\varepsilon/\varepsilon}$ with $a_0^\varepsilon \in H^\infty$, $\varphi_0^\varepsilon \in H^\infty$ and that for some $m \in \mathbb{N}$, there exists an expansion*

$$a_0^\varepsilon = \sum_{k=0}^{m} \varepsilon^k a_0^k + \varepsilon^{m+1} \boldsymbol{a_0^\varepsilon}, \quad \varphi_0^\varepsilon = \sum_{k=0}^{m} \varepsilon^k \varphi_0^k + \varepsilon^{m+1} \boldsymbol{\varphi_0^\varepsilon} \tag{71}$$

with $a_0^0 \in \mathbb{R}$, a_0^k, $\varphi_0^k \in H^\infty$, satisfying, for every s,

$$\sup_{\varepsilon \in (0,1)} \Big(\|\boldsymbol{a_0^\varepsilon}\|_{H^s} + \|\boldsymbol{\varphi_0^\varepsilon}\|_{H^s} \Big) < +\infty. \tag{72}$$

Let us denote $0 < T^* \leq +\infty$ the existence time of the maximal smooth (i.e. $H^\infty \times H^\infty$) solution (a^0, φ^0) for (55) with the initial condition (a_0^0, φ_0^0). Then, there exists an approximate smooth solution of (1) on $[0, T^*)$ under the form $\Psi^a = a^\varepsilon e^{i\varphi^\varepsilon/\varepsilon}$, with $a^\varepsilon, \varphi^\varepsilon \in H^\infty$ and a^ε complex-valued, solving

$$\begin{cases} \dfrac{\partial \varphi^\varepsilon}{\partial t} + f(|a^\varepsilon|^2) + \dfrac{1}{2}|\nabla \varphi^\varepsilon|^2 = R_\varphi^m \\[2ex] \dfrac{\partial a^\varepsilon}{\partial t} + \big(\nabla \varphi^\varepsilon\big) \cdot \nabla a^\varepsilon + \dfrac{a^\varepsilon}{2} \Delta \varphi^\varepsilon - \dfrac{\varepsilon}{2} J \Delta a^\varepsilon = R_a^m, \end{cases} \tag{73}$$

with the initial condition $\big(a^\varepsilon, \varphi^\varepsilon\big)_{/t=0} = \big(a_0^\varepsilon, \varphi_0^\varepsilon\big)$, and where, for every s and $0 < T < T^*$,

$$\sup_{[0,T]} \Big(\big\|R_a^m\big\|_{H^s} + \big\|R_\varphi^m\big\|_{H^s} \Big) \leq C_{s,T} \varepsilon^{m+2}. \tag{74}$$

Finally, for $0 < T < T^*$, a^ε verifies (38): $a^\varepsilon - a^0 = \mathcal{O}(\varepsilon)$ in $L^\infty([0,T], W^{s,\infty})$.

Note that Ψ^a is indeed an approximate solution of (1) since

$$i\varepsilon \frac{\partial \Psi^a}{\partial t} + \frac{\varepsilon^2}{2} \Delta \Psi^a - \Psi^a f(|\Psi^a|^2) = \Big(- i\varepsilon R_a^m + a^\varepsilon R_\varphi^m \Big) \exp\big(i \frac{\varphi^\varepsilon}{\varepsilon}\big).$$

By using the notation of section 2, we have $R^\varepsilon = -i\varepsilon R_a^m + a^\varepsilon R_\varphi^m$, hence

$$\sup_{[0,T]} \big\| R^\varepsilon \big\|_{H^s} \leq C_s \varepsilon^{m+2}. \tag{75}$$

Proof.

As in [9], we look for expansions

$$a^\varepsilon = \sum_{k=0}^{m} \varepsilon^k a^k + \varepsilon^{m+1} a^{m+1}, \qquad \varphi^\varepsilon = \sum_{k=0}^{m} \varepsilon^k \varphi^k + \varepsilon^{m+1} \varphi^{m+1}.$$

This yields that (a^0, φ^0) solves the nonlinear system

$$\begin{cases} \dfrac{\partial \varphi^0}{\partial t} + f(|a^0|^2) + \dfrac{1}{2}\,|\nabla\varphi^0|^2 = 0 \\[2ex] \dfrac{\partial a^0}{\partial t} + \big(\nabla\varphi^0\big)\cdot\nabla a^0 + \dfrac{a^0}{2}\,\Delta\varphi^0 = 0, \end{cases} \tag{76}$$

which is just (9), and that for $1 \le k \le m$, (a^k, φ^k) solves the linear system

$$\begin{cases} \dfrac{\partial \varphi^k}{\partial t} + 2f'(|a^0|^2)(a^0, a^k) + \nabla\varphi^0\cdot\nabla\varphi^k = S^k_\varphi \\[2ex] \dfrac{\partial a^k}{\partial t} + \big(\nabla\varphi^0\big)\cdot\nabla a^k + \nabla a^0\cdot\nabla\varphi^k + \dfrac{a^0}{2}\,\Delta\varphi^k + \dfrac{a^k}{2}\,\Delta\varphi^0 = S^k_a, \end{cases} \tag{77}$$

where the source terms (S^k_φ, S^k_a) depend only on $(a^j, \varphi^j)_{0\le j\le k-1}$, and S^k_a is complex-valued.

We first solve (76) (that is (9)) with the initial condition $\varphi^0_{/t=0} = \varphi^0_0$, $a^0_{/t=0} = a^0_0$. By introducing $u^0 \equiv \nabla\varphi^0$ and by taking the gradient of the first equation of (76), we find

$$\begin{cases} \partial_t a^0 + u^0\cdot\nabla a^0 + \dfrac{a^0}{2}\,\nabla\cdot u^0 = 0 \\[2ex] \partial_t u^0 + u^0\cdot\nabla u^0 + \nabla\Big(f\big((a^0)^2\big)\Big) = 0, \end{cases} \tag{78}$$

which is the compressible Euler type equation considered in the previous section. By using Theorem 4, we get the existence of a smooth solution $(a^0, u^0) \in H^{s-1}\times H^s$ for every s on $[0, T^*)$ (with T^* independent of s), with a^0 real-valued. Finally, to get φ^0, it is natural to set

$$\varphi^0(t,x) = \varphi^0_0(x) - \int_0^t \Big(f\big((a^0)^2\big) + \frac{1}{2}\,|u^0|^2\Big)(\tau, x)\,d\tau,$$

and the same argument as in [2] yields $u^0 = \nabla\varphi^0$.

We now turn to the resolution of (77). We solve it with the initial condition $\big(\varphi^k, a^k\big)_{/t=0} = \big(\varphi^k_0, a^k_0\big)$. By introducing again $u^k \equiv \nabla\varphi^k$, we can take the gradient in the first line of (77) to get

$$\begin{cases} \partial_t a^k + u^0\cdot\nabla a^k + \dfrac{a^0}{2}\nabla\cdot u^k + u^k\cdot\nabla a^0 + \dfrac{a^k}{2}\nabla\cdot u^0 = S^k_a, \\[2ex] \partial_t u^k + u^0\cdot\nabla u^k + \nabla\big(a^0, f'((a^0)^2)a^k\big) + u^k\cdot\nabla u^0 = \nabla S^k_\varphi. \end{cases} \tag{79}$$

Again, since $f'\big((a^0)^2\big)$ can vanish, the symmetrization of this linear hyperbolic system requires some care. We thus set

$$F^k(t,x) \equiv \begin{cases} \sqrt{2}\left(f'((a^0)^2)\right)^{\frac{1}{2}} a^k & \text{if } n \text{ is odd} \\ \sqrt{2}\, a^0 \left(\frac{f'((a^0)^2)}{(a^0)^2}\right)^{\frac{1}{2}} a^k & \text{if } n \text{ is even.} \end{cases}$$

Note that in both cases, we have

$$F^k(t,x) = \sqrt{2}\, g(a^0) a^k$$

with g smooth. Indeed, as we have seen, we can write $f'(\rho) = \rho^{n-1} q(\rho)$ with q smooth and positive, and we have in both cases :

$$g(a^0) = (a^0)^{n-1}\left(q((a^0)^2)\right)^{\frac{1}{2}}. \tag{80}$$

This is the natural generalization of the change of unknown used in [2]. Then, thanks to the equation on a^0, we get for (F^k, u^k) the system

$$\begin{cases} \partial_t F^k + u^0 \cdot \nabla F^k + \dfrac{1}{\sqrt{2}}\, a^0 g(a^0) \nabla \cdot u^k \quad + \sqrt{2}\, g(a^0)\, u^k \cdot \nabla a^0 + \dfrac{F^k}{2}\Big(1 + \dfrac{a^0 g'(a^0)}{g(a^0)}\Big)\nabla \cdot u^0 = \sqrt{2} g(a^0) S_a^k \\ \partial_t u^k + u^0 \cdot \nabla u^k \quad + \dfrac{1}{\sqrt{2}} \nabla\big(a^0 g(a^0), F^k\big) + u^k \cdot \nabla u^0 = \nabla S_\varphi^k. \end{cases}$$

Note that the coefficient $\frac{a^0 g'(a^0)}{g(a^0)}$ is smooth even when a^0 vanishes since g is under the form (80). We have obtained a linear symmetric hyperbolic system with a zero order term and a source term $\mathcal{S}^k$ depending only on (a^j, φ^j) for $0 \le j < k$ under the form

$$\partial_t U^k + \sum_{j=1}^{d} A^j(t,x)\partial_j U^k + L(t,x) U^k = \mathcal{S}^k, \quad U^k = \begin{pmatrix} F^k \\ u^k \end{pmatrix},$$

where $A^j(t,x)$ are smooth, real and symmetric and the matrix L is smooth. By the classical theory, there exists, on $[0, T^*)$, a smooth solution (F^k, u^k) in $H^\infty \times H^\infty$ of this system. Once u^k is built, we get a^k by solving the transport equation for a^k which is given by the first line of (79). Finally, we deduce the phase φ^k by integrating in time the first line of (77). We obtain

$$\varphi^k(t,x) = \varphi_0^k(x) - \int_0^t \Big(2f'(|a^0|^2)(a^0, a^k) + \nabla \varphi^0 \cdot u^k - S_\varphi^k\Big)(\tau, x) d\tau.$$

Finally, we choose in a similar way (a^{m+1}, φ^{m+1}) that solve (77) with the initial condition $\big(a^{m+1}, \varphi^{m+1}\big)_{/t=0} = \big(\boldsymbol{a_0^\varepsilon}, \boldsymbol{\varphi_0^\varepsilon}\big)$. Because of the assumption (72), we find that they are also uniformly bounded in $H^{s-1} \times H^s$ with respect to ε. This concludes the proof of Lemma 2. $\square$

4 Nonlinear stability

In this section, we give the proof of Theorem 2. We shall actually prove directly a more precise version which states the existence of a WKB expansion to any order.

Theorem 5 *Consider $\Psi_0^\varepsilon = a_0^\varepsilon e^{i\varphi_0^\varepsilon/\varepsilon}$ with $a_0^\varepsilon \in H^\infty$, $\varphi_0^\varepsilon \in H^\infty$ and that for some $m \in \mathbb{N}$, there exists an expansion (71) as in Lemma 2. We assume $(\mathcal{A})$ and let $(a^\varepsilon, \varphi^\varepsilon)$ be the smooth approximate solution given by Lemma 2 which is smooth on $[0, T^*)$. Then,*

- *if $m = 0$, there exists $\varepsilon_0 > 0$ and $T \in (0, T^*)$ such that for every $\varepsilon \in (0, \varepsilon_0]$, the solution of (1) with initial data Ψ_0^ε remains smooth on $[0, T]$ and satisfies for every $s \in \mathbb{N}$, the estimate*

$$\left\| \Psi^\varepsilon \exp\big(-\frac{i}{\varepsilon}\varphi^\varepsilon\big) - a^\varepsilon \right\|_{L^\infty([0,T],H^s)} \leq C_s \varepsilon.$$

- *if $m \geq 1$, for every $T \in (0, T^*)$, there exists $\varepsilon_0(T) > 0$ such that for every $\varepsilon \in (0, \varepsilon_0(T)]$, the solution of (1) with initial data Ψ_0^ε remains smooth on $[0, T]$ and satisfies for every $s \in \mathbb{N}$, the estimate*

$$\left\| \Psi^\varepsilon \exp\big(-\frac{i}{\varepsilon}\varphi^\varepsilon\big) - a^\varepsilon \right\|_{L^\infty([0,T],H^s)} \leq C_{s,T} \varepsilon^{m+1}.$$

Note that Theorem 2 is actually the special case $m = 0$ in Theorem 5.

Proof of Theorem 5.

Let $s > d/2$. We take $(a^\varepsilon, \varphi^\varepsilon)$ the approximate solutions given by Lemma 2 and look for the solution of (1) under the form $\Psi^\varepsilon = (a^\varepsilon + w)e^{i\varphi^\varepsilon/\varepsilon}$. We get for w the equation (21) with F^ε given by (22) and the initial condition $w_{/t=0} = 0$. For $s > d/2$, and every $\varepsilon > 0$, this semilinear equation is locally well-posed in H^s: we get very easily that there exists for some $T^\varepsilon > 0$ a unique maximal solution $w \in \mathcal{C}([0, T^\varepsilon), H^s)$ of (21) (see [5] for example). We shall prove that T^ε is bounded from below by some $T > 0$ if $m = 0$, and that $T^\varepsilon \geq T$ for every $T \in (0, T^*)$ for ε sufficiently small if $m \geq 1$. Let us define

$$\tau^\varepsilon \equiv \sup\big\{\tau \in (0, T^\varepsilon),\ \forall t \in [0, \tau],\ 2N_s^\varepsilon\big(w(t)\big) \leq \varepsilon^{2m+4}\big\}.$$

Note that $\tau^\varepsilon > 0$ since $w(0) = 0$ and that by Sobolev embedding, we have, for $t \leq \tau^\varepsilon$,

$$\big\|w(t)\big\|_{L^\infty}^2 \leq K^2 \varepsilon^{-2} N_s^\varepsilon\big(w(t)\big) \leq K^2 \varepsilon^{2m+2} \leq K^2,$$

for some K independent of ε.

We will apply Theorem 3 with F^ε given by (22). To estimate F^ε, we use the following lemma:

Lemma 3 *Let $R > 0$, $s > d/2$ and w such that $\|w\|_{L^\infty} \leq R$, and F^ε given by (22). Then, for a constant C depending only on $\|a^\varepsilon(t)\|_{W^{s+2,\infty}}$ and R, we have*

$$\|F^\varepsilon\|_{H^s}^2 + \frac{1}{\varepsilon^2}\|\mathrm{Im} F^\varepsilon\|_{H^{s-1}}^2 \leq C\varepsilon^{2m+4} + C\varepsilon^{2m} N_s^\varepsilon(w) + C\left[\frac{N_s^\varepsilon(w)}{\varepsilon^4} + \left(\frac{N_s^\varepsilon(w)}{\varepsilon^4}\right)^2\right] N_s^\varepsilon(w).$$

We postpone the proof of Lemma 3 to the end of the section. We can first easily end the proof of Theorem 5. Notice first that, by definition of Ψ^a, we have

$$R_a = R_a^m + \frac{i\varepsilon}{2}\Delta a^\varepsilon = \mathcal{O}_{H^k}(\varepsilon^{m+1}) + \mathcal{O}_{H^k}(\varepsilon) = \mathcal{O}_{H^k}(\varepsilon),$$

for every k, uniformly for $0 \le t \le T$, hence

$$\frac{1}{\varepsilon}\big\|R_a(t)\big\|_{W^{s-1,\infty}} \le C.$$

Applying Theorem 3 and Lemma 3 with $R \equiv K$, we infer that for $0 \le t \le \tau^\varepsilon$,

$$\frac{d}{dt}N_s^\varepsilon\big(w(t)\big) \le C\varepsilon^{2m+4} + C\varepsilon^{2m}N_s^\varepsilon\big(w(t)\big),$$

which gives immediately, since $w_{/t=0} = 0$, that

$$N_s^\varepsilon\big(w(t)\big) \le C\varepsilon^{2m+4}\left(e^{C\varepsilon^{2m}t} - 1\right) \le \frac{1}{2}\,\varepsilon^{2m+4}$$

in the following cases:

- for $m = 0$, $0 \le t \le T$ with $0 < T < T^*$ sufficiently small independent of ε,
- for $m \ge 1$, $T \in (0, T^*)$ is arbitrary, $0 \le t \le T$ and $\varepsilon \le \varepsilon_0(T)$ with $\varepsilon_0(T)$ sufficiently small.

As a consequence, $\tau^\varepsilon \ge T$ as desired and

$$\big\|w\big\|_{L^\infty([0,T],H^s(\mathbb{R}^d))} \le C_{s,T}\varepsilon^{m+1}.$$

It remains to prove Lemma 3.

Proof of Lemma 3.

We recall that F^ε is given by

$$F^\varepsilon = R^\varepsilon + Q^\varepsilon(w) = R^\varepsilon + (a^\varepsilon + w)\Big(f(|a^\varepsilon + w|^2) - f(|a^\varepsilon|^2)\Big) - 2(w, a^\varepsilon)f'(|a^\varepsilon|^2)a^\varepsilon.$$

As a first try, we could use the rough estimate

$$Q^\varepsilon(w) = \mathcal{O}(|w|^2) \qquad \text{as} \;\; w \to 0,$$

which would lead to

$$\big\|Q^\varepsilon\big\|_{H^s}^2 + \frac{1}{\varepsilon^2}\big\|\operatorname{Im} Q^\varepsilon\big\|_{H^{s-1}}^2 \le \frac{C}{\varepsilon^2}\big\|w\big\|_{H^s}^4 \le \frac{C}{\varepsilon^6}N_s^\varepsilon(w)^2,$$

which does not allow to conclude in the proof of Theorem 5 for $m = 0$ and does not give the sharp result for the existence time if $m = 1$. To get the refined estimate of Lemma 3, the idea is then to use a Taylor expansion for Q^ε w.r.t. w up to second order, and write

$$Q^\varepsilon(w) = |w|^2 f'(|a^\varepsilon|^2)a^\varepsilon + 2f'(|a^\varepsilon|^2)(w, a^\varepsilon)w + 2a^\varepsilon f''(|a^\varepsilon|^2)(w, a^\varepsilon)^2 + G^\varepsilon(x, w),$$

so that for fixed x, we have as $w \to 0$,

$$G^\varepsilon(x, w) = \mathcal{O}\big(|w|^3\big).$$

We turn now to estimate each term in F^ε.

Estimate for $R^\varepsilon = i\varepsilon R_a^m - R_\varphi^m a^\varepsilon$. Thanks to (75), we have

$$\left\|R^\varepsilon\right\|_{H^s}^2 \le C\varepsilon^{2m+4}.$$

Moreover, since R_φ^m is real-valued and since, from (38), $\operatorname{Im} a^\varepsilon = \mathcal{O}_{W^{s,\infty}}(\varepsilon)$, we also have

$$\frac{1}{\varepsilon^2}\left\|\operatorname{Im} R^\varepsilon\right\|_{H^{s-1}}^2 \le C\varepsilon^{2m+4}$$

thanks to (74). We have thus proven that

$$\left\|R^\varepsilon\right\|_{H^s}^2 + \frac{1}{\varepsilon^2}\left\|\operatorname{Im} R^\varepsilon\right\|_{H^{s-1}}^2 \le C\varepsilon^{2m+4}.$$

Estimate for $G^\varepsilon(x,w)$. The estimate relies on Lemma 5 in the appendix. Indeed, it is clear from the Taylor formula that G^ε may be written under the form

$$\big(\operatorname{Re} w\big)^2 h_{11}\big(x,w(x)\big) + \big(\operatorname{Re} w\big)\big(\operatorname{Im} w\big) h_{12}\big(x,w(x)\big) + \big(\operatorname{Im} w\big)^2 h_{22}\big(x,w(x)\big),$$

where h_{11}, h_{12}, $h_{22} : \mathbb{R}^d \times \mathbb{C} \to \mathbb{C}$ are of class $\mathcal{C}^\infty$ and $\forall x \in \mathbb{R}^d$, $h_{11}(x,0) = h_{12}(x,0) = h_{22}(x,0) = 0$. Moreover, h_{11}, h_{12} and h_{22} verify the hypothesis of Lemma 5 in the Appendix since $a^\varepsilon \in L^\infty([0,T], W^{s,\infty})$. As a consequence, if $\left\|w\right\|_{L^\infty} \le R$,

$$\left\|G^\varepsilon\right\|_{H^s} \le C\left\|w\right\|_{H^s}^3,$$

which implies

$$\left\|G^\varepsilon\big(x,w(x)\big)\right\|_{H^s}^2 + \frac{1}{\varepsilon^2}\left\|\operatorname{Im} G^\varepsilon\big(x,w(x)\big)\right\|_{H^{s-1}}^2 \le \frac{2}{\varepsilon^2}\left\|G^\varepsilon\big(x,w(x)\big)\right\|_{H^s}^2 \le \frac{C}{\varepsilon^8} N_s^\varepsilon(w)^3.$$

The estimate for the quadratic terms in $Q^\varepsilon(w)$ will rely crucially on the fact that a^ε is *real* to first order and that (w, a^ε) is estimated in H^{s-1} by $N_s^\varepsilon(w)$ and not just by $\varepsilon^{-2} N_s^\varepsilon(w)$.

Estimate for $F_1^\varepsilon \equiv |w|^2 f'(|a^\varepsilon|^2) a^\varepsilon$. We have

$$\left\|F_1^\varepsilon\right\|_{H^s}^2 \le \frac{C}{\varepsilon^4} N_s^\varepsilon(w)^2,$$

and in view of (38), Im $a^\varepsilon = \mathcal{O}_{W^{s,\infty}}(\varepsilon)$, thus

$$\frac{1}{\varepsilon^2}\left\|\operatorname{Im} F_1^\varepsilon\right\|_{H^{s-1}}^2 \le C\left\||w|^2\right\|_{H^{s-1}}^2 \le \frac{C}{\varepsilon^4} N_s^\varepsilon(w)^2.$$

Estimate for $F_2^\varepsilon \equiv 2f'(|a^\varepsilon|^2)(w, a^\varepsilon)w$. We begin with the rough estimate

$$\left\|F_2^\varepsilon\right\|_{H^s}^2 \le \frac{C}{\varepsilon^4} N_s^\varepsilon(w)^2.$$

Moreover, one has

$$\|f'(|a^\varepsilon|^2)(w,a^\varepsilon)\|^2_{H^{s-1}} \le CN_s^\varepsilon(w). \tag{81}$$

Indeed, let $\mu \in \mathbb{N}^d$ with $|\mu| \le s-1$. Then,

$$\partial^\mu\big(f'(|a^\varepsilon|^2)(w,a^\varepsilon)\big) = \sum_{\alpha+\beta+\lambda=\mu} * \,\partial^\lambda\big[f'(|a^\varepsilon|^2)\big]\big(\partial^\alpha w, \partial^\beta a^\varepsilon\big),$$

where $*$ is a coefficient depending only on α, β and λ. Since $|\mu| \le s-1$, the terms $(\partial^\alpha w, \partial^\beta a^\varepsilon)$ are bounded in L^2 by $\Sigma(w)^{\frac12} + \varepsilon\|w\|_{H^{s-2}}$ as soon as $|\alpha| \le s-2$. The term in the sum with $|\alpha| = s-1$ (hence $\mu = \alpha$ and $\beta = \lambda = 0$) is $f'(|a^\varepsilon|^2)\big(\partial^\mu w, a^\varepsilon\big)$ is bounded in L^2 by $N^\varepsilon(\partial^\mu w)$. Hence, (81) follows.

As a consequence, by (60) and Sobolev embedding, we obtain

$$\|f'(|a^\varepsilon|^2)(w,a^\varepsilon)w\|_{H^{s-1}} \le C_s\|w\|_{L^\infty}\Big(\|f'(|a^\varepsilon|^2)(w,a^\varepsilon)\|_{H^{s-1}} + \|w\|_{H^{s-1}}\Big) \le \frac{C}{\varepsilon^2}N_s^\varepsilon(w).$$

Consequently,

$$\|F_2^\varepsilon\|^2_{H^s} + \frac{1}{\varepsilon^2}\|\mathrm{Im}\,F_2^\varepsilon\|^2_{H^{s-1}} \le \frac{C}{\varepsilon^4}N_s^\varepsilon(w)^2.$$

Estimate for $F_3^\varepsilon \equiv 2a^\varepsilon f''(|a^\varepsilon|^2)(w,a^\varepsilon)^2$. We find as for F_1^ε

$$\|F_3^\varepsilon\|^2_{H^s} \le \frac{C}{\varepsilon^4}N_s^\varepsilon(w)^2,$$

and once again in view of (38),

$$\frac{1}{\varepsilon^2}\|\mathrm{Im}\,F_3^\varepsilon\|^2_{H^{s-1}} \le C\|w\|^4_{H^{s-1}} \le \frac{C}{\varepsilon^4}N_s^\varepsilon(w)^2.$$

We conclude the proof of Lemma 3 summing these estimates. □

5 Geometric optics in a half-space

In this section, we consider the Gross-Pitaevskii equation in a half-space in dimension $d \le 3$

$$GP(\Psi^\varepsilon) \equiv i\varepsilon\partial_t\Psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\Psi^\varepsilon - \Psi^\varepsilon(|\Psi^\varepsilon|^2 - 1) = 0, \quad x \in \mathbb{R}^d_+ \equiv \mathbb{R}^{d-1}\times(0,+\infty). \tag{82}$$

We consider the Neumann boundary condition (12) on the boundary and the condition (13) at infinity, that is

$$\frac{\partial\Psi^\varepsilon}{\partial n}_{/\partial\mathbb{R}^d_+} = \frac{\partial\Psi^\varepsilon}{\partial z}_{/z=0} = 0 \qquad \text{and} \qquad \exp\Big(\frac{i}{2\varepsilon}|u^\infty|^2\,t - \frac{i}{\varepsilon}u^\infty\cdot x\Big)\Psi^\varepsilon \to 1 \quad |x| \to +\infty$$

by using the notation $x = (y,z) \in \mathbb{R}^{d-1}\times(0,+\infty)$.

5.1 Construction of the WKB expansion

In this section, we shall consider a smooth solution (a,u), with a real-valued, of

$$\begin{cases} \partial_t a + u\cdot\nabla a + \dfrac{1}{2}\, a\,\nabla\cdot u = 0 \\ \\ \partial_t u + u\cdot\nabla u + \nabla(a^2) = 0, \end{cases} \tag{83}$$

with the boundary condition $u_d(t,y,0)=0$ and the condition at infinity

$$u(t,x)\to u^\infty, \qquad a(t,x)\to 1 \qquad \text{when } \ |x|\to+\infty.$$

Since we look for a real-valued, the resolution of this system is made in [14] (Theorem 2). Given $s\in\mathbb{N}^*$, if the initial datum a_0 is positive and $(a_0-1,u_0-u^\infty)\in H^s$, and under some compatibility conditions for (a_0,u_0) on the boundary $\partial\mathbb{R}^d_+$ of sufficiently high order on the initial data, there exists $T_0\in(0,+\infty)$ and a solution (a,u) on $[0,T_0]$ with $(a-1,u-u^\infty)\in\mathcal{C}^0([0,T_0],H^s)\cap\mathcal{C}^1([0,T_0],H^{s-1})$, such that

$$a(t,x)\geq\alpha>0, \quad \forall t\in[0,T_0],\, \forall x\in\overline{\mathbb{R}^d_+}. \tag{84}$$

for some $\alpha>0$. We also define the phase φ by

$$\varphi(t,x)\equiv\varphi_0(x)-\int_0^t\Big(\frac{1}{2}|u|^2+|a|^2-1\Big)(\tau,x)\;d\tau.$$

In view of the condition (13) at infinity, φ is not in H^s but $\varphi(t,.)-u^\infty\cdot x+\frac{t}{2}|u^\infty|^2\in H^s$. As we have seen and as in [2], $u=\nabla\varphi$.

The aim of this subsection is to prove the existence of WKB expansion (which involves boundary layers since the solution of (83) does not match the Neumann boundary condition (12)) up to arbitrary orders for (82), (12), (13) starting from a smooth (a,u) which verifies (84).

We define the set of boundary layer profiles $\mathcal{S}_{exp}$ as

$$\mathcal{S}_{exp}=\Big\{A(t,y,Z)\in H^\infty(\mathbb{R}_+\times\mathbb{R}^{d-1}\times\mathbb{R}_+), \quad \forall k,\,\alpha,\,l, \quad \exists\gamma>0, \quad |\partial_t^k\partial_y^\alpha\partial_Z^l A|\leq C_{k,\alpha,l}\exp(-\gamma Z)\Big\}.$$

Lemma 4 *Let $s\in\mathbb{N}$ and $m\in\mathbb{N}^*$ be fixed. Then, there exists a smooth function $\Psi^{a,m}=a^\varepsilon e^{i\frac{\varphi^\varepsilon}{\varepsilon}}$ on $[0,T_m]$ verifying the Neumann condition (12) and the condition (13) at infinity and such that $\Psi^{a,m}$ is an approximate solution of (82) on $[0,T_m]$:*

$$GP(\Psi^{a,m})=\varepsilon^m R^\varepsilon e^{i\frac{\varphi^\varepsilon}{\varepsilon}}, \tag{85}$$

where R^ε can be written under the form

$$R^\varepsilon=-a^\varepsilon\Big(R^{int,m}_\varphi(t,x)+R^{\flat,m}_\varphi(t,y,\frac{z}{\varepsilon})\Big)+i\Big(\varepsilon R^{int,m}_a(t,x)+R^{\flat,m}_a(t,y,\frac{z}{\varepsilon})\Big), \tag{86}$$

with $R_\varphi^{int,m}$, $R_a^{int,m}$ smooth and uniformly bounded in H^s and $R_a^{\flat,m}(t,y,Z)$, $R_\varphi^{\flat,m}(t,y,Z) \in \mathcal{S}_{exp}$. Moreover, a^ε is real-valued and a^ε, φ^ε have smooth expansions under the form

$$a^\varepsilon = a + \sum_{k=1}^{m-1} \varepsilon^k \Big(a^k(t,x) + A^k(t,y,\frac{z}{\varepsilon}) \Big) + \varepsilon^m A^m(t,y,\frac{z}{\varepsilon}), \tag{87}$$

$$\varphi^\varepsilon = \varphi + \sum_{k=1}^{m-1} \varepsilon^k \Big(\varphi^k(t,x) + \Phi^k(t,y,\frac{z}{\varepsilon}) \Big) + \varepsilon^m \Phi^m(t,y,\frac{z}{\varepsilon}). \tag{88}$$

The boundary layer profiles $A^k(t,y,Z)$, $\Phi^k(t,y,Z)$ belong to $\mathcal{S}_{exp}$ and are such that

$$\partial_Z A^1(t,y,0) = -\partial_z a(t,y,0), \qquad \partial_Z \Phi^1(t,y,0) = -\partial_z \varphi(t,y,0),$$

$$\partial_Z A^k(t,y,0) = -\partial_z a^{k-1}(t,y,0), \qquad \partial_Z \Phi^k(t,y,0) = -\partial_z \varphi^{k-1}(t,y,0) \qquad \forall 2 \le k \le m. \tag{89}$$

Proof.

Since $\Psi^{a,m} = a^\varepsilon \exp\left(i\frac{\varphi^\varepsilon}{\varepsilon}\right)$, we want to solve approximately

$$-a^\varepsilon \Big(\partial_t \varphi^\varepsilon + \frac{1}{2}|\nabla \varphi^\varepsilon|^2 + |a^\varepsilon|^2 - 1 \Big) + i\varepsilon \Big(\partial_t a^\varepsilon + \nabla \varphi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2}\, a^\varepsilon \Delta \varphi^\varepsilon \Big) + \frac{\varepsilon^2}{2} \Delta a^\varepsilon = 0. \tag{90}$$

Since, in this section, we are looking for a^ε real-valued, we can split the system (90) into

$$\begin{cases} \partial_t a^\varepsilon + \nabla \varphi^\varepsilon \cdot \nabla a^\varepsilon + \dfrac{1}{2}\, a^\varepsilon \Delta \varphi^\varepsilon = 0 \\[2ex] \partial_t \varphi^\varepsilon + \dfrac{1}{2}|\nabla \varphi^\varepsilon|^2 + (a^\varepsilon)^2 - 1 = \dfrac{\varepsilon^2}{2}\, \dfrac{\Delta a^\varepsilon}{a^\varepsilon} \end{cases} \quad \text{for } \ t \ge 0, \quad x \in \mathbb{R}^d_+. \tag{91}$$

Note that in this section, the division by a^ε in the right-hand side of the second equation of (91) is not a problem since $a^0 = a$ verifies (84) and hence does not vanish.

We thus plug the expansions (87), (88) in (91) and we cancel the powers of ε. To separate interior and boundary layer terms, we use the general theory of [11]. In particular, we use that for every smooth function f and $V \in \mathcal{S}_{exp}$, we have the expansion

$$f\Big(u(t,x) + V(t,y,z/\varepsilon)\Big) = f\Big(u(t,x)\Big) + f\Big(u(t,y,0) + V(t,y,z/\varepsilon)\Big) - f\Big(u(t,y,0)\Big) + \varepsilon \mathcal{R},$$

where $\mathcal{R} \in \mathcal{S}_{exp}$. This yields that the boundary layer part of $f\big(u(t,x) + V(t,y,z/\varepsilon)\big)$ is given by $f\big(u(t,y,0) + V(t,y,z/\varepsilon)\big) - f\big(u(t,y,0)\big)$. In the following, we use the notation $W_b = W(t,y,0)$ for every $W(t,x)$. At first, the ε^{-1} term in the equation only gives

$$a_b \partial_{ZZ} \Phi^1 = 0$$

and hence we have $\Phi^1 = 0$, since $a_b \ge \alpha > 0$ and $\Phi^1 \in \mathcal{S}_{exp}$. Note that this is coherent with the fact that $u_d(t,y,0) = \big(\partial_z \varphi\big)_b = 0$ so that we do not need a boundary layer to correct the boundary condition. The ε^0 term gives, as expected,

$$\begin{cases} \partial_t \varphi + \dfrac{1}{2}|\nabla \varphi|^2 + a^2 - 1 = 0 \\[2ex] \partial_t a + \nabla \varphi \cdot \nabla a + \dfrac{1}{2}\, a\, \Delta \varphi = 0 \end{cases} \quad \text{for } \ t \ge 0, \quad x \in \mathbb{R}^d_+ \tag{92}$$

for the interior part, and for the boundary layer terms, for $(t,y) \in \mathbb{R}^+ \times \mathbb{R}^{d-1}$,

$$a_b \partial_{ZZ}\Phi^2 = -(\partial_z\varphi)_b\, \partial_Z A^1 = 0 \qquad \text{for} \quad Z > 0, \tag{93}$$

since $(\partial_z\varphi)_b = u_d(t,y,0) = 0$. Consequently, we also find $\Phi^2 = 0$. Next, the order ε gives

$$\begin{cases} \partial_t a^1 + \nabla\varphi\cdot\nabla a^1 + \nabla\varphi^1\cdot\nabla a + \dfrac{1}{2}(a\Delta\varphi^1 + a^1\Delta\varphi) = 0 \\ \partial_t\varphi^1 + 2a\,a^1 + \nabla\varphi\cdot\nabla\varphi^1 = 0 \end{cases} \qquad \text{for} \quad t \geq 0, \quad x \in \mathbb{R}^d_+$$

in the interior and for the boundary layer terms

$$\begin{cases} \dfrac{1}{2}\partial_{ZZ}A^1 = A^1\Big(\partial_t\varphi + \dfrac{1}{2}|\nabla\varphi|^2 + a^2 - 1\Big)_b + 2a_b^2 A^1 = 2a_b^2 A^1 \\ a_b\partial_{ZZ}\Phi^3 = G^3 \end{cases} \qquad \text{for} \quad Z > 0, \tag{94}$$

where $G^3 \in \mathcal{S}_{exp}$ depends only on (a, A^1, a^1) and (φ, φ^1). Consequently, the boundary layer A^1 is given by

$$A^1 \equiv \frac{(\partial_z a)_b}{2a_b} e^{-2a_b Z}$$

in order to match (89). Finally, the ε^k, $k \geq 2$ terms give

$$\begin{cases} \partial_t\varphi^k + 2a\,a^k + \nabla\varphi\cdot\nabla\varphi^k = S^k_\varphi \\ \partial_t a^k + \nabla\varphi\cdot\nabla a^k + \nabla a\cdot\nabla\varphi^k + \dfrac{a}{2}\Delta\varphi^k + \dfrac{a^k}{2}\Delta\varphi = S^k_a \end{cases} \qquad \text{for} \quad t \geq 0, \quad x \in \mathbb{R}^d_+ \tag{95}$$

and

$$\begin{cases} \partial_{ZZ}A^k = 4a_b^2 A^k + F^k \\ \partial_{ZZ}\Phi^k = G^k \end{cases} \qquad \text{for} \quad Z > 0, \tag{96}$$

where S^k_φ and S^k_a depend only on (a,φ) and $(a^j,\varphi^j)_{1\leq j\leq k-1}$; $F^k \in \mathcal{S}_{exp}$ depends only on (a,φ), $(a^j,\varphi^j,A^j,\Phi^j)_{1\leq j\leq k-1}$ and Φ^k; and $G^k \in \mathcal{S}_{exp}$ depends on (a,φ), $(a^j,\varphi^j,A^j,\Phi^j)_{1\leq j\leq k-1}$. Therefore, if we want to solve by induction these equations, one has to determine first Φ^k, then (a^k,φ^k) and finally A^k.

To solve the cascade of equations by induction, we first determine (a^1,φ^1). As before, we notice that $(a^1, u^1 \equiv \nabla\varphi^1)$ solves a symmetrizable hyperbolic system (there is no problem with the vacuum since we are in the same situation as in [9]). Since the condition at infinity is already absorbed by (a,φ), one can look for (a^1,u^1) in H^s. Moreover, we solve the system in $\mathbb{R}^d_+$ with the boundary condition $u^1_d(t,y,0) = 0$ which is needed in order to match (89) since we have already found that $\Phi^2 = 0$. The existence of a smooth solution for this linear system with the boundary condition $u^1_d(t,y,0) = 0$ which is maximal dissipative and an initial condition satisfying suitable compatibility conditions can be obtained by the classical theory [17]. Then, one finds φ^1 by the formula

$$\varphi^1(t,x) = \varphi^1_0(x) - \int_0^t \big(2a\,a^1 + u\cdot u^1\big)(\tau,x)\, d\tau.$$

Furthermore, since $F^2 \in \mathcal{S}_{exp}$ and $a_b \geq \alpha > 0$, the first equation in (96) (with $k = 2$) has a unique solution $A^2 \in \mathcal{S}_{exp}$. We have therefore found $(a^1, A^1, \varphi^1, \Phi^1, A^2, \Phi^2)$.

We now proceed by induction. Assume that, for some $m \geq 2$, we have determined $(a^j, \varphi^j)_{1\leq j\leq m-1}$ and $(A^j, \Phi^j)_{1\leq j\leq m}$. Then, we wish to solve (95) and (96) with $k = m + 1$. Since G^{m+1} is already determined and $G^{m+1} \in \mathcal{S}_{exp}$, the differential equation $\partial_{ZZ}\Phi^{m+1} = G^{m+1}$ has a unique solution in $\mathcal{S}_{exp}$ and

$$\partial_Z \Phi^{m+1}(t, y, Z) = -\int_Z^{+\infty} \frac{G^{m+1}(t, y, \zeta)}{a_b(t, y)}\, d\zeta.$$

This determines the boundary condition for $u^{m+1} \equiv \nabla\varphi^{m+1}$. Indeed, to match (89) we shall need to impose

$$u_d^{m+1}(t, y, 0) = (\partial_z \varphi^{m+1})(t, y, 0) = -(\partial_Z \Phi^{m+1})(t, y, 0) = \int_0^{+\infty} \frac{G^{m-1}(t, y, \zeta)}{a_b(t, y)}\, d\zeta, \qquad (97)$$

which is non-zero in general. We then solve (96) in the following way: $(a^{m+1}, u^{m+1} \equiv \nabla\varphi^{m+1})$ still solves a linear symmetrizable hyperbolic system, with source terms S_φ^{m+1} and S_a^{m+1} already known, with the maximal dissipative boundary condition (97). It has then a smooth solution by the above mentionned theory. Then, we recover φ^{m+1} as usual by

$$\varphi^{m+1}(t, x) \equiv \varphi_0^{m+1}(x) + \int_0^t \big(S_\varphi^{m+1} - 2a\, a^{m+1} - u \cdot u^{m+1}\big)(\tau, x)\ d\tau.$$

Finally, the first equation in (96) (with $k = m + 1$) is a linear ODE for A^{m+1}, with source term $F^{m+1} \in \mathcal{S}_{exp}$ now determined, for which we can write down explicitly the unique exponentially decreasing solution satisfying $\partial_Z A^k(t, y, 0) = -\partial_z a^k(t, y, 0)$.

Consequently, we have constructed an approximate solution of (91) such that

$$\begin{cases} \partial_t a^\varepsilon + \nabla\varphi^\varepsilon \cdot \nabla a^\varepsilon + \dfrac{1}{2}\, a^\varepsilon \Delta\varphi^\varepsilon & = \varepsilon^m \big(R_a^{int,m}(t, x) + \varepsilon^{-1} R_a^{\flat,m}(t, y, z/\varepsilon)\big) \\[2ex] \partial_t \varphi^\varepsilon + \dfrac{1}{2}\, |\nabla\varphi^\varepsilon|^2 + \big(a^\varepsilon\big)^2 - 1 & = \dfrac{\varepsilon^2}{2} \dfrac{\Delta a^\varepsilon}{a^\varepsilon}(t, x) + \varepsilon^m \big(R_\varphi^{int,m}(t, x) + R_\varphi^{\flat,m}(t, y, z/\varepsilon)\big), \end{cases}$$

where $R_a^{int,m}(t, x)$, $R_\varphi^{int,m}(t, x)$ are smooth bounded functions and $R_a^{\flat,m}$, $R_\varphi^{\flat,m} \in \mathcal{S}_{exp}$. We can thus write the error R^ε in the GP equation as

$$R^\varepsilon(t, x) = \varepsilon^m \Big(- a^\varepsilon \big(R_\varphi^{int,m}(t, x) + R_\varphi^{\flat,m}(t, y, z/\varepsilon)\big) + i\big(\varepsilon R_a^{int,m}(t, x) + R_a^{\flat,m}(t, y, z/\varepsilon)\big)\Big).$$

This ends the proof of Lemma 4. $\square$

5.2 Validity of the WKB expansion

We shall now prove the stability of the WKB expansion built in Lemma 4.

Theorem 6 *Let $\Psi^{a,m} = a^\varepsilon e^{i\frac{\varphi^\varepsilon}{\varepsilon}}$ a WKB expansion defined on $[0, T_m]$ given by Lemma 4. Then for $d \leq 3$ and $m \geq 4$ there exists a unique smooth solution Ψ^ε also defined on $[0, T_m]$ of (82), (12), (13) such that $\Psi^\varepsilon_{/t=0} = \Psi^{a,m}_{/t=0}$. Moreover, we have the estimate*

$$\varepsilon \big\| \Psi^\varepsilon e^{\frac{-i\varphi^\varepsilon}{\varepsilon}} - a^\varepsilon \big\|_{H^1(\mathbb{R}^d_+)} + \varepsilon^3 \big\| \Psi^\varepsilon e^{-i\frac{\varphi^\varepsilon}{\varepsilon}} - a^\varepsilon \big\|_{H^3(\mathbb{R}^d_+)} \leq C_m \varepsilon^{m-\frac{1}{2}}, \quad \forall t \in [0, T_m]$$

and in particular

$$\left\|\Psi^\varepsilon e^{-i\frac{\varphi^\varepsilon}{\varepsilon}} - \left(a+\varepsilon A^1\right)\right\|_{W^{1,\infty}(\mathbb{R}^d_+)} \le C_m \max\{\varepsilon, \varepsilon^{m-\frac{7}{2}}\}. \tag{98}$$

Remark 3 For simplicity, we have restricted ourselves to dimension $d \le 3$. Note however that it is possible to get H^s estimates for every s. By contrast with Theorem 2, we emphasize that the initial condition in Theorem 6 is exactly the WKB approximate solution $\Psi^{a,m}$. In particular, this initial datum has to verify some compatibility condition on the boundary.

Proof.

As in the proof of Theorem 5, we set

$$\Psi^\varepsilon = \Psi^{a,m} + w\, e^{\frac{i\varphi^\varepsilon}{\varepsilon}}$$

and we study the equation for w *i.e.* (19). Note that we are now seeking for w which tends to zero at infinity since the boundary condition at infinity is already absorbed in the WKB expansion. Again the first step is to get estimates for the linear equation (21) in Ω with the Neumann boundary condition

$$\partial_z w(t,y,0) = 0. \tag{99}$$

As we can check in the proof of Lemma 1, in all the integration by parts that are performed, the boundary terms vanish due to the Neumann boundary condition or the fact that $u_d^\varepsilon(t,y,0) = 0$, and hence the proof of the L^2 stability will be almost the same as the one in the whole space. Nevertheless, we have to pay attention to the presence of boundary layer terms in the coefficients. At first, we note that since $\Phi^1 = 0$ and $\Phi^2 = 0$ in the WKB expansion, we still have that M (which is defined in Lemma 1) is independent of ε. Indeed, for the worse term which is $\nabla(\nabla\cdot u^\varepsilon)$, we have

$$\nabla(\nabla\cdot u^\varepsilon) = \partial_{ZZZ}\Phi^3 + \nabla\Delta\varphi + \mathcal{O}_{L^\infty}(\varepsilon).$$

Next, keeping the definitions of R_a and R_φ given in (17), (18) and by construction of the WKB expansion, we have

$$\|R_a\|_{L^\infty} \le C\varepsilon^m. \tag{100}$$

Nevertheless, again by construction of the WKB expansion, we only have

$$R_\varphi = R_\varphi^m + \frac{\varepsilon^2}{2}\frac{\Delta a^\varepsilon}{a^\varepsilon}$$

and due to the presence of boundary layers in a^ε, we can split R_φ into

$$R_\varphi = \varepsilon^2 R_\varphi^{int}(t,y,z) + \varepsilon R_\varphi^\flat(t,y,\frac{z}{\varepsilon}), \tag{101}$$

where R_φ^{int} is smooth and bounded whereas $R_\varphi^\flat \in \mathcal{S}_{exp}$ and we see that $\varepsilon\,\|R_\varphi^\flat\|_{L^\infty} = \mathcal{O}(\varepsilon)$, $\varepsilon\,\|\nabla R_\varphi^\flat\|_{L^\infty} = \mathcal{O}(1)$, hence the estimate (23) of Lemma 1 would be useless. Moreover, the fact that $R^\flat_\varphi$ belongs to $\mathcal{S}_{exp}$ does not seem to improve the estimates. The way to overcome this difficulty seems to incorporate this new singular term into the functional. Let us define the operator

$$\mathcal{S}^\varepsilon_+ w = -\frac{\varepsilon^2}{2}\Delta w + 2(w, a^\varepsilon)a^\varepsilon + \varepsilon R^\flat_\varphi w,$$

our weighted norm in this section will be

$$N^\varepsilon_+(w) = \int_\Omega \Big((\mathcal{S}^\varepsilon_+ w, w) + K\,\varepsilon^2\,|w|^2 \Big)\,dx = \frac{1}{2}\int_\Omega \Big(\varepsilon^2 |\nabla w|^2 + 4(w, a^\varepsilon)^2 + 2\varepsilon R^\flat_\varphi\,|w|^2 + 2K\,\varepsilon^2 |w|^2 \Big)\,dx.$$

Note that R_φ has no sign, nevertheless, $N^\varepsilon_+(w)$ can be bounded from below by a weighted H^1 norm if K is chosen sufficiently large. Indeed, since $R^\flat_\varphi$ belongs to $\mathcal{S}_{exp}$ we can write

$$2\varepsilon\Big|\int_\Omega R^\flat_\varphi\,|w|^2\,dx\Big| \leq C\varepsilon \int_\Omega e^{\frac{-\gamma z}{\varepsilon}}|w|^2\,dx$$

and then use the one-dimensional Sobolev inequality

$$|w(t,y,z)|^2 \leq C\Big(\int_{\mathbb{R}_+} |w(t,y,\zeta)|^2\,d\zeta\Big)^{\frac{1}{2}}\,\Big(\int_{\mathbb{R}_+} |\partial_z w(t,y,\zeta)|^2\,d\zeta\Big)^{\frac{1}{2}}$$

to get

$$\varepsilon\int_\Omega e^{-\frac{\gamma z}{\varepsilon}}|w|^2 \leq C\varepsilon \|w\|_{L^2}\,\|\nabla w\|_{L^2}\int_{\mathbb{R}_+} e^{-\frac{\gamma z}{\varepsilon}}\,dz \leq C\varepsilon^2\,\|w\|_{L^2}\,\|\nabla w\|_{L^2}. \tag{102}$$

In particular, we have proven that

$$2\varepsilon\Big|\int_\Omega R^\flat_\varphi\,|w|^2\,dx\Big| \leq C\varepsilon^2\,\|w\|_{L^2}\,\|\nabla w\|_{L^2}. \tag{103}$$

This yields thanks to the Young inequality

$$2\varepsilon\Big|\int_\Omega R^\flat_\varphi\,|w|^2\,dx\Big| \leq \frac{1}{2}\,\varepsilon^2\|\nabla w\|^2_{L^2} + C\varepsilon^2\|w\|^2_{L^2} \tag{104}$$

where C is independent of ε. Consequently, if K is chosen such that $2K > C$, we get

$$N^\varepsilon_+(w) \geq C_0\Big(\varepsilon^2\|w\|^2_{H^1} + \int_\Omega (w, a^\varepsilon)^2\,dx\Big), \quad C_0 > 0.$$

Note that in this section, we have

$$a^\varepsilon = a + \mathcal{O}(\varepsilon)$$

with $a \geq \alpha$, this finally yields that $N^\varepsilon_+(w)$ is equivalent to the weighted norm

$$N^\varepsilon_+(w) \sim \varepsilon^2\|w\|^2_{H^1} + \|\mathrm{Re}\ w\|^2_{L^2}. \tag{105}$$

The first step in the proof of Theorem 6 is to prove the equivalent of Lemma 1. We shall prove the estimate

$$\begin{aligned} \frac{d}{dt}N^\varepsilon_+\big(w(t)\big) \quad &\leq \quad CN^\varepsilon_+\big(w(t)\big) \\ &+\|F^\varepsilon\|^2_{L^2} + \int_\Omega \frac{4}{\varepsilon}(w, a^\varepsilon)(ia^\varepsilon, F^\varepsilon) - \int_\Omega (i\varepsilon\Delta w, F^\varepsilon) - \int_\Omega (iF^\varepsilon, R^\flat_\varphi w), \end{aligned} \tag{106}$$

where C is independent of ε.

Proof of (106)**.**

The proof follows the same lines as the proof of Lemma 1. At first, since $\mathcal{S}_+^\varepsilon$ is self adjoint, we have

$$\frac{d}{dt}\int_\Omega \big(\mathcal{S}_+^\varepsilon w, w\big)\,dx = \int_\Omega \Big(2\big(\mathcal{S}_+^\varepsilon w, \partial_t w\big) + 4(w, a^\varepsilon)(w, \partial_t a^\varepsilon) + 2\varepsilon\,\partial_t R_\varphi^\flat\,|w|^2\Big)\,dx.$$

Since $\partial_t R_\varphi^\flat \in \mathcal{S}_{exp}$, we can still use (102) to get

$$2\varepsilon \int_\Omega \partial_t R_\varphi^\flat\,|w|^2 \le CN_+^\varepsilon(w).$$

Next, as in the proof of Lemma 1, we use (21) to express $\partial_t w$ as

$$\partial_t w = -\frac{i}{\varepsilon}\mathcal{S}_+^\varepsilon w - \big(u^\varepsilon\cdot\nabla w + \frac{1}{2} w\,\nabla\cdot u^\varepsilon\big) - i\frac{\varepsilon^2 R_\varphi^{int}}{\varepsilon} w - \frac{iF^\varepsilon}{\varepsilon}$$

to get

$$2\int_\Omega \big(\partial_t w, \mathcal{S}_+^\varepsilon w\big)dx = 2\int_\Omega \Big(- \big(u^\varepsilon\cdot\nabla w + \frac{1}{2}\,w\,\nabla\cdot u^\varepsilon\big) - \frac{i\varepsilon^2 R_\varphi^{int}}{\varepsilon} w - i\frac{F^\varepsilon}{\varepsilon}, \mathcal{S}_+^\varepsilon w\Big)dx. \qquad (107)$$

Moreover, since R_φ^{int} and $R_\varphi^\flat$ are real, we have the cancellation

$$\int_\Omega (iR_\varphi^{int} w, R_\varphi^\flat w)\ dx = 0.$$

Therefore, the only terms in the right-hand side of (107) which are not present in (25) are $-\int_\Omega (iF^\varepsilon, R_\varphi^\flat w)$ and

$$\mathcal{I} = -2\int_\Omega \Big(u^\varepsilon\cdot\nabla w + \frac{1}{2}\,w\,\nabla\cdot u^\varepsilon, \varepsilon R_\varphi^\flat w\Big).$$

To estimate $\mathcal{I}$, we note that we have a bound on the second term by using again (102). It remains to estimate the first term. Integrating by parts and using that $u_d^\varepsilon(t,y,0) = 0$, we get

$$-2\int_\Omega \big(u^\varepsilon\cdot\nabla w, \varepsilon R_\varphi^\flat w\big) = \varepsilon\int_\Omega \nabla\cdot\big(R_\varphi^\flat u^\varepsilon\big)|w|^2 = \int_\Omega \nabla\cdot u^\varepsilon\,\varepsilon R_\varphi^\flat |w|^2 + \int_\Omega \varepsilon\,u^\varepsilon\cdot\nabla R_\varphi^\flat\,|w|^2.$$

Again, the first term can be bounded thanks to (102). For the second one, we first notice that since $u_d^\varepsilon(t,y,0) = 0$ and $R_\varphi^\flat \in \mathcal{S}_{exp}$, we have

$$\varepsilon\,\big|u^\varepsilon\cdot\nabla R_\varphi^\flat\big| \le C\varepsilon\Big(|\nabla_y R_\varphi^\flat| + |z\partial_z R_\varphi^\flat|\Big) \le C\varepsilon e^{-\frac{\gamma z}{\varepsilon}}.$$

This finally yields

$$\mathcal{I} \le CN_+^\varepsilon(w)$$

thanks to a new use of (102).

The end of the proof of (106) is then exactly the same as the proof of Lemma 1, since all the integration by parts do not create boundary terms either because of the Neumann boundary condition or because u_d^ε vanishes on the boundary.

Higher order estimates.

The estimates of higher order derivatives are more involved than in the whole space. There are two main reasons. The first one is that there is a new singular term $\varepsilon R_{\varphi}^{\flat} w$ which creates bad terms when we take the derivatives of the equation. The second reason is that to recover estimates on the normal derivatives, we need to use the equation which gives in particular that $\varepsilon^2 \partial_z^2$ behaves like $\varepsilon \partial_t$ and $\varepsilon \nabla$. This anisotropy in the weights does not seem to allow to construct high order functionals like $N_s^\varepsilon(w)$ which allows to get H^s estimates without additional loss of ε. Let us use the notation

$$\Lambda = (\Lambda_0, \cdots, \Lambda_d) = \big(\partial_t, \nabla_y, p(z)\partial_z\big)^t$$

where the weight $p(z)$ is given by $p(z) = z/(1+z)$. Note that we can apply Λ to the equation since Λw still satisfies the Neumann boundary condition. The use of Λ is classical in hyperbolic characteristic initial boundary value problems (see [17] for example) The weighted norm that we shall estimate is

$$Y_+^\varepsilon(w) \equiv N_+^\varepsilon(w) + N_+^\varepsilon(\varepsilon \Lambda w).$$

In dimension $d \leq 3$, this is sufficient to get the nonlinear stability. We shall see in the proof why the use of Λ_d is necessary.

We shall prove that

$$\frac{d}{dt} Y_+^\varepsilon(w) \leq C\Big(Y_+^\varepsilon(w) + X^\varepsilon(F^\varepsilon) + X^\varepsilon(\varepsilon \Lambda F^\varepsilon)\Big) \tag{108}$$

for some $C > 0$ independent of ε where we have set

$$X^\varepsilon(F) \equiv \|F\|_{H^1}^2 + \frac{\|F\|_{L^2}^2}{\varepsilon} + \frac{\|\mathrm{Im}\ F\|_{L^2}^2}{\varepsilon^2}.$$

Proof of (108)**.**

As a preliminary, we shall rewrite (106) in a more convenient form. We can use that $a^\varepsilon = a + \mathcal{O}(\varepsilon)$ with a real, perform an integration by parts and use (102) to get from (106) that

$$\frac{d}{dt} N_+^\varepsilon\big(w(t)\big) \quad \leq \quad C N_+^\varepsilon\big(w(t)\big) + X^\varepsilon(F^\varepsilon) \tag{109}$$

where

$$X^\varepsilon(F^\varepsilon) = \|F^\varepsilon\|_{H^1}^2 + \frac{\|F\|_{L^2}^2}{\varepsilon} + \frac{\|\mathrm{Im}\ F^\varepsilon\|_{L^2}^2}{\varepsilon^2}.$$

To prove (108), we start with the estimate of $N_+^\varepsilon(\varepsilon \partial_t w)$. When we apply $\varepsilon \partial_t$ to (21), we find

$$\Big(i\varepsilon\partial_t + \mathcal{L}^\varepsilon\Big)\varepsilon\partial_t w = R_\varphi\, \varepsilon \partial_t w + \varepsilon \partial_t F^\varepsilon + \mathcal{C} \tag{110}$$

where the commutator $\mathcal{C}$ can be splitted into

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 \tag{111}$$

with

$$\mathcal{C}_1 \equiv \varepsilon\big(\partial_t R_\varphi\big)w,$$
$$\mathcal{C}_2 \equiv 2\varepsilon\Big((\partial_t a^\varepsilon, w)a^\varepsilon + (a^\varepsilon, w)\partial_t a^\varepsilon\Big),$$
$$\mathcal{C}_3 \equiv -i\varepsilon^2\Big(\partial_t u^\varepsilon \cdot \nabla w + \frac{1}{2}\partial_t(\nabla \cdot u^\varepsilon)\, w\Big).$$

Consequently, we can apply (109) to (110) with the new source term $\varepsilon\partial_t F^\varepsilon + \mathcal{C}$ to get

$$\frac{d}{dt}N_+^\varepsilon\big(\varepsilon\partial_t w(t)\big) \quad \leq \quad CN_+^\varepsilon\big(\varepsilon\partial_t w(t)\big) + X^\varepsilon(\varepsilon\partial_t F^\varepsilon) + X^\varepsilon(\mathcal{C}). \tag{112}$$

Thus it remains to estimate $X^\varepsilon(\mathcal{C})$. Let us begin with $X^\varepsilon(\mathcal{C}_1)$. Thanks to the expansion (101), we easily get

$$\begin{aligned} X^\varepsilon(\mathcal{C}_1) \quad &\lesssim \quad N_+^\varepsilon(w) + \int_\Omega |\partial_t R_\varphi^\flat|^2\, \varepsilon^4|w|^2 + \varepsilon^4|\nabla w|^2) + \varepsilon^4|\nabla\partial_t R_\varphi^\flat|^2\, |w|^2 \\ &\lesssim \quad N_+^\varepsilon(w). \end{aligned} \tag{113}$$

Note that we could have a better estimate by using that $R_\varphi^\flat \in \mathcal{S}_{exp}$ and (102). Next, we turn to the estimate of $X^\varepsilon(\mathcal{C}_2)$. By using that $a^\varepsilon = a + \mathcal{O}(\varepsilon)$ with a real, we find

$$X^\varepsilon(\mathcal{C}_2) \lesssim N_+^\varepsilon(w) + \varepsilon\|\mathrm{Re}\ w\|_{L^2}^2 + \varepsilon^2\|\nabla w\|_{L^2}^2 \lesssim N^\varepsilon(w). \tag{114}$$

Note that the above estimate was sharp. This is for the estimate of this commutator $\mathcal{C}_2$ that we had to chose the weight ε in front of the time derivative. Finally, we estimate $X^\varepsilon(\mathcal{C}_3)$ using that $\partial_t u_d^\varepsilon$ vanishes on the boundary which implies that

$$|\partial_t u_d^\varepsilon| \lesssim p(z).$$

Thanks to this remark, we find

$$X^\varepsilon(\mathcal{C}_3) \lesssim N_+^\varepsilon(w) + \varepsilon^4\|\nabla\Lambda w\|_{L^2}^2 \lesssim Y_+^\varepsilon(w). \tag{115}$$

Note that this is for the control of this commutator that we are obliged to add the vector field $p(z)\partial_z$ in the definition of the functional space. Consequently, the combination of (112), (113), (114) and (115) gives

$$\frac{d}{dt}N_+^\varepsilon\big(\varepsilon\partial_t w(t)\big) \quad \lesssim \quad Y_+^\varepsilon\big(w(t)\big) + X^\varepsilon(\varepsilon\partial_t F^\varepsilon). \tag{116}$$

The estimate of $\varepsilon\nabla_y w$ follows exactly the same lines, and we also find

$$\frac{d}{dt}N_+^\varepsilon\big(\varepsilon\nabla_y w(t)\big) \quad \lesssim \quad Y_+^\varepsilon\big(w(t)\big) + X^\varepsilon(\varepsilon\nabla_y F^\varepsilon). \tag{117}$$

The estimate of $\varepsilon\Lambda_d w = \varepsilon p(z)\partial_z w$ requires some additional work since the vector field Λ_d does not commute with the Laplacian. By applying $\varepsilon\Lambda_d$ to (21), we get

$$\Big(i\varepsilon\partial_t + \mathcal{L}^\varepsilon\Big)\varepsilon\Lambda_d w = R_\varphi\varepsilon\Lambda_d w + \varepsilon\Lambda_d F^\varepsilon + \mathcal{C} + \mathcal{C}_4 \tag{118}$$

where $\mathcal{C}$ is defined as in (111) above with ∂_t replaced by Λ_d and $\mathcal{C}_4$ is given by

$$\mathcal{C}_4 \equiv -\frac{\varepsilon^3}{2}[\Lambda_d, \Delta]w = -\frac{\varepsilon^3}{2}\Big(2\big(\partial_z p\big)\,\partial_{zz}w + \big(\partial_{zz}p\big)\,\partial_z w\Big).$$

Next, we can apply (106) to get

$$\begin{aligned}\frac{d}{dt}N_+^\varepsilon\big(\varepsilon\Lambda_d w(t)\big) \quad &\lesssim \quad N_+^\varepsilon\big(\varepsilon\Lambda_d w(t)\big) + X^\varepsilon(\varepsilon\Lambda_d F^\varepsilon) + X^\varepsilon(\mathcal{C}) + \|\mathcal{C}_4\|_{H^1}^2 \\ &\quad + \frac{4}{\varepsilon}\int_\Omega (\varepsilon\Lambda_d w, a^\varepsilon)(ia^\varepsilon, \mathcal{C}_4) - \int_\Omega \big(i\mathcal{C}_4, R_\varphi^\flat\,\varepsilon\Lambda_d w\big).\end{aligned}$$

Since one can easily check that $X^\varepsilon(\mathcal{C})$ still satisfies the bounds (113), (114), (115), we obtain

$$\begin{aligned}\frac{d}{dt}N_+^\varepsilon\big(\varepsilon\Lambda_d w(t)\big) \quad &\lesssim \quad Y_+^\varepsilon(w) + X^\varepsilon(\varepsilon\Lambda_d F^\varepsilon) + \|\mathcal{C}_4\|_{H^1}^2 \\ &\quad + \frac{4}{\varepsilon}\int_\Omega (\varepsilon\Lambda_d w, a^\varepsilon)(ia^\varepsilon, \mathcal{C}_4) - \int_\Omega \big(i\mathcal{C}_4, R_\varphi^\flat\,\varepsilon\Lambda_d w\big).\end{aligned}$$

Next, we note that

$$\|\mathcal{C}_4\|_{H^1}^2 \lesssim \varepsilon^6\|w\|_{H^3}^2$$

and that

$$\begin{aligned}\frac{4}{\varepsilon}\Big|\int_\Omega (\varepsilon\Lambda_d w, a^\varepsilon)(ia^\varepsilon, \mathcal{C}_4)\Big| \lesssim \frac{4}{\varepsilon}\int_\Omega \varepsilon|\partial_z w|\,|p(z)\mathcal{C}_4| \quad &\lesssim \quad \varepsilon^2 N_+^\varepsilon(w)^{\frac{1}{2}}\Big(\|p\partial_{zz}w\|_{L^2} + \|\partial_z w\|_{L^2}\Big) \\ &\lesssim \quad N_+^\varepsilon(w)^{\frac{1}{2}}Y_+^\varepsilon(w)^{\frac{1}{2}}.\end{aligned}$$

In a similar way, we also get

$$\Big|\int_\Omega \big(i\mathcal{C}_4, R_\varphi^\flat\,\varepsilon\Lambda_d w\big)\Big| \lesssim \varepsilon\|\partial_z w\|_{L^2}\,\|p\,\mathcal{C}_4\|_{L^2} \lesssim Y_+^\varepsilon(w).$$

Consequently, we have proven that

$$\frac{d}{dt}N_+^\varepsilon\big(\varepsilon\Lambda_d w(t)\big) \lesssim Y_+^\varepsilon(w) + X^\varepsilon(\varepsilon\Lambda_d F^\varepsilon) + \varepsilon^6\|w\|_{H^3}^2. \tag{119}$$

To conclude, it remains to estimate $\varepsilon^6\|w\|_{H^3}^2$. As usual, this is done thanks to the equation (19) and the standard regularity result for elliptic equations. We rewrite (19) as the equation

$$\varepsilon^2\Delta w = G^\varepsilon, \quad \partial_z w(t, y, 0) = 0 \tag{120}$$

where the source term enjoys the estimates

$$\begin{aligned}&\|G^\varepsilon\|_{L^2}^2 \lesssim \varepsilon^2\|\Lambda w\|_{L^2}^2 + \|w\|_{L^2}^2 + \|F^\varepsilon\|_{L^2}^2, \\ &\|\nabla G^\varepsilon\|_{L^2}^2 \lesssim \varepsilon^2\|\nabla\Lambda w\|_{L^2}^2 + \|w\|_{H^1}^2 + \|\nabla F^\varepsilon\|_{L^2}^2.\end{aligned}$$

Consequently, we get from (120) by standard elliptic regularity that

$$\varepsilon^6\|w\|_{H^3}^2 \lesssim Y_+^\varepsilon(w) + \|F^\varepsilon\|_{H^1}^2. \tag{121}$$

By replacing this last estimate in (119), we finally obtain

$$\frac{d}{dt}N_+^\varepsilon\big(\varepsilon\Lambda_d w(t)\big) \lesssim Y_+^\varepsilon(w) + X^\varepsilon(\varepsilon\Lambda_d F^\varepsilon) + \|F^\varepsilon\|_{H^1}^2. \tag{122}$$

To conclude, it suffices to sum the estimates (109), (116), (117) and (122) to get (108).

The estimate (108) is sufficient to prove the nonlinear stability stated in Theorem 6 for $d \leq 3$. Nevertheless, it is possible to prove by induction that for every s,

$$\frac{d}{dt}\Big(\sum_{m\leq s} N_+^\varepsilon\big((\varepsilon\Lambda)^m w\big)\Big) \lesssim \sum_{m\leq s}\Big(X^\varepsilon\big((\varepsilon\Lambda)^m F^\varepsilon\big) + N_+^\varepsilon\big((\varepsilon\Lambda)^m w\big)\Big).$$

Nonlinear stability.

Thanks to (108) and Gronwall inequality, we get for $0 \leq T \leq T_m$,where T_m is the existence time of the approximate solution given by Lemma 4,

$$\sup_{[0,T]} Y_+^\varepsilon(w) \lesssim Y_+^\varepsilon(0) + Te^{\gamma T}\sup_{[0,T]}\Big(X^\varepsilon(F^\varepsilon) + X^\varepsilon(\varepsilon\Lambda F^\varepsilon)\Big)$$

for some $\gamma > 0$ independent of ε. Combining this last estimate with (121), we get

$$\sup_{[0,T]} Z_+^\varepsilon(w) \leq C_{T_m}\Big(Y_+^\varepsilon(0) + \sup_{[0,T]}\Big(X^\varepsilon(F^\varepsilon) + X^\varepsilon(\varepsilon\Lambda F^\varepsilon)\Big)\Big), \tag{123}$$

with

$$Z_+^\varepsilon(w) \equiv Y_+^\varepsilon(w) + \varepsilon^6\|w\|_{H^3}^2.$$

Thanks to this a priori estimate, one can easily prove by standard fixed point argument the existence of a unique solution of (19) with the neumann condition $\partial_z w_{|z=0} = 0$ on some interval of time $[0, T^\varepsilon] \subset [0, T_m]$ such that $Z_+^\varepsilon(w)$ remains finite.

By using that $w_{/t=0} = 0$ and the equation to compute the time derivative, we find

$$Y_+^\varepsilon(w)_{/t=0} = N_+^\varepsilon(\varepsilon\partial_t w)_{/t=0} \leq C_{T_m}\varepsilon^{2m}.$$

Moreover, using that $F^\varepsilon = \varepsilon^m R^\varepsilon + Q^\varepsilon$, we have thanks to (86) that

$$\sup_{[0,T_m]}\Big(X_+^\varepsilon(R^\varepsilon) + X_+^\varepsilon(\Lambda R^\varepsilon)\Big) \leq C_{T_m}\varepsilon^{2m-1}.$$

Inserting this into (123) yields, for $0 \leq t \leq T^\varepsilon$,

$$\sup_{[0,T]} Z_+^\varepsilon(w) \leq K_{T_m}\varepsilon^{2m-1} + C_{T_m}\sup_{[0,T]}\Big(X^\varepsilon(Q^\varepsilon) + X^\varepsilon(\varepsilon\Lambda Q^\varepsilon)\Big). \tag{124}$$

We can thus define $\tau^\varepsilon \in (0, T_m]$ as the maximal time such that the solution w of (19) satisfies $Z_+^\varepsilon(w(t)) \leq 2K_{T_m}\varepsilon^{2m-1}$ on $[0, \tau^\varepsilon]$. As in the proof of Theorem 5, we shall prove that for ε sufficiently small, we have $\tau^\varepsilon = T_m$. Here, the expression of $Q^\varepsilon(w)$ is given by

$$Q^\varepsilon(w) = a^\varepsilon|w|^2 + 2(w, a^\varepsilon)w + w|w|^2.$$

To conclude, we need to bound the right hand side of (124). To estimate the nonlinear term, we use that for $d \leq 3$, we have

$$\|w\|_{L^\infty}^2 \lesssim \|\nabla^2 w\| \, \|w\|_{H^1},$$

which gives

$$\|w\|_{L^\infty}^2 \lesssim \frac{Z_+^\varepsilon(w)}{\varepsilon^4} \lesssim \varepsilon^{2m-5} \quad \forall t \in [0, \tau^\varepsilon).$$

We shall take m such that $2m > 5$ in order to get $\|w\|_{L^\infty} \leq 1$ for $t \in [0, \tau^\varepsilon)$. This implies

$$\|Q^\varepsilon\|_{H^1}^2 \lesssim \left(\|w\|_{L^\infty}^2 + \|w\|_{L^\infty}^4\right) \|w\|_{H^1}^2 \lesssim \frac{Z_+^\varepsilon(w)^2}{\varepsilon^6}.$$

Next, since $H^1(\mathbb{R}^d) \subset L^4$ for $d \leq 3$, we also have

$$\frac{\|Q^\varepsilon\|_{L^2}^2}{\varepsilon^2} \lesssim \frac{\|w\|_{H^1}^4}{\varepsilon^2}(1 + \|w\|_{L^\infty}^2) \lesssim \frac{Z_+^\varepsilon(w)^2}{\varepsilon^6}.$$

Consequently, we have already proven that

$$X^\varepsilon(Q^\varepsilon) \lesssim \frac{Z_+^\varepsilon(w)^2}{\varepsilon^6}. \tag{125}$$

Next, we evaluate $X^\varepsilon(\varepsilon \Lambda Q^\varepsilon)$. At first, we write

$$\varepsilon^2 \|\Lambda Q^\varepsilon\|_{H^1}^2 \lesssim \varepsilon^2 \|\Lambda w\|_{H^1}^2 \left(\|w\|_{L^\infty}^2 + \|w\|_{L^\infty}^4\right) + \varepsilon^2 \|\Lambda w\|_{L^4}^2 \, \|\nabla w\|_{L^4}^2 \left(1 + \|w\|_{L^\infty}^2\right)$$

and by using for $d \leq 3$, the Sobolev embedding $H^1 \subset L^4$ and the Gagliardo-Nirenberg inequality

$$\|\nabla f\|_{L^4}^2 \lesssim \|f\|_{H^1}^{\frac{1}{2}} \, \|\nabla^2 f\|_{L^2}^{\frac{3}{2}},$$

we get for $0 \leq t \leq \tau^\varepsilon$:

$$\varepsilon^2 \|\Lambda Q^\varepsilon\|_{H^1}^2 \lesssim \frac{Z_+^\varepsilon(w)^2}{\varepsilon^4} + \varepsilon^2 \|\nabla w\|_{H^1}^2 \|w\|_{H^1}^{\frac{1}{2}} \, \|\nabla^2 w\|_{L^2}^{\frac{3}{2}} \lesssim \frac{Z_+^\varepsilon(w)^2}{\varepsilon^6}.$$

Finally, by similar arguments, we also have

$$\frac{\|\varepsilon \Lambda Q^\varepsilon\|_{L^2}^2}{\varepsilon^2} \lesssim \|\Lambda w\|_{L^4}^{\frac{1}{2}} \, \|w\|_{L^4}^{\frac{1}{2}} \lesssim \|\Lambda w\|_{H^1}^2 \, \|w\|_{H^1}^2 \lesssim \frac{Z_+^\varepsilon(w)^2}{\varepsilon^6}.$$

We have thus proven that

$$X^\varepsilon(\varepsilon \Lambda Q^\varepsilon) \lesssim \frac{Z_+^\varepsilon(w)^2}{\varepsilon^6}. \tag{126}$$

Consequently, inserting (125), (126) into (124), we get

$$\sup_{[0,\tau^\varepsilon]} Z_+^\varepsilon(w) \leq K_{T_m} \varepsilon^{2m-1} + C_{T_m} \sup_{[0,\tau^\varepsilon]} \frac{Z_+^\varepsilon(w)^2}{\varepsilon^6} \leq K_{T_m} \varepsilon^{2m-1} + 2K_{T_m} C_{T_m} \varepsilon^{2m-7} \sup_{[0,\tau^\varepsilon]} Z_+^\varepsilon(w).$$

By choosing $m \geq 4$, this allows to get for ε sufficiently small that $\tau^\varepsilon = T_m$ and that

$$\sup_{[0,T_m]} Z_+^\varepsilon(w) \leq C \varepsilon^{2m-1}.$$

Finally, the estimate (98) follows by Sobolev embedding. This ends the proof of Theorem 6. $\square$

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A A Lemma about composition in Sobolev spaces

During the proof of Lemma 3, we have used a result about composition in Sobolev spaces. This result is very standard when h does not depend on x (see, for instance, [18]).

Lemma 5 *Let $R > 0$, $s \in \mathbb{N}$ and $h = h(x, w) \in \mathcal{C}^{s+1}(\mathbb{R}^d \times \mathbb{R}^2, \mathbb{R})$, satisfying $h(x, 0) = 0$ for all $x \in \mathbb{R}^d$. Assume moreover*

$$A \equiv \sup \left\{ \left\| \partial_x^\alpha \partial_w^\beta h \right\|_{L^\infty(\mathbb{R}^d \times B_R)}, \ \alpha \in \mathbb{N}^d, \ \beta \in \mathbb{N}^2, \ |\alpha| \leq s, \ |\alpha| + |\beta| \leq s + 1 \right\} < +\infty.$$

Then, there exists C, depending only on A, s and R, such that, for any $w \in H^s(\mathbb{R}^d)$ satisfying $|w|_{L^\infty(\mathbb{R}^d)} \leq R$, we have $h\big(x, w(x)\big) \in H^s(\mathbb{R}^d)$ and

$$\left\| h\big(x, w(x)\big) \right\|_{H^s} \leq C \left\| w \right\|_{H^s}.$$

Proof. The proof is by induction on $s \in \mathbb{N}$ and relies on the Gagliardo-Nirenberg inequality. If $s = 0$, it suffices to notice that since $h(x, 0) = 0$, then for $w \in B_R$,

$$|h(x, w)| \leq A|w|.$$

Assume then the result for $s - 1 \in \mathbb{N}$. Let $\mu \in \mathbb{N}^d$ with $|\mu| = s$. One has easily

$$\partial^\mu \big(h(x, w(x))\big) = \sum * \big(\partial_x^\alpha \partial_w^{\beta+\gamma} h\big)\big(x, w(x)\big) \big(\partial^\beta w_1\big)^p \big(\partial^\gamma w_2\big)^q,$$

where $\alpha \in \mathbb{N}^d$, $\alpha \leq \mu$, β, $\gamma \in \mathbb{N}^2$, p, $q \in \mathbb{N}^*$ depend on β and γ, $|\alpha| + p|\beta| + q|\gamma| = s$, and $*$ is a coefficient depending only on μ, α, β and γ. Furthermore, since $w \in H^s \cap L^\infty$, the Gagliardo-Nirenberg inequality yields, for $1 \leq k \leq s$,

$$\left\| w \right\|_{W^{k, \frac{2s}{k}}} \leq C_{k,s} \left\| w \right\|_{H^s}^{\frac{k}{s}} \left\| w \right\|_{L^\infty}^{1 - \frac{k}{s}}.$$

As a consequence, by interpolation, if $w \in H^s \cap L^\infty$ and $\left\| w \right\|_{L^\infty} \leq R$, then for $\gamma \in \mathbb{N}^d$, $|\gamma| \leq s$, and $2 \leq p \leq \frac{2s}{|\gamma|}$,

$$\left\| \partial^\gamma w \right\|_{L^p} \leq C_{s,p,R} \left\| w \right\|_{H^s}^{\frac{2}{p}}.$$

Therefore, in view of $|\alpha| + p|\beta| + q|\gamma| = s$, by Hölder inequality, we can estimate the terms in $\partial^\mu \big(h(x, w(x))\big)$ for which $\alpha \neq \mu$ (thus $|\alpha| < s$) as

$$\left\| \big(\partial_x^\alpha \partial_w^{\beta+\gamma} h\big)\big(x, w(x)\big) \big(\partial^\beta w_1\big)^p \big(\partial^\gamma w_2\big)^q \right\|_{L^2} \leq A \left\| \partial^\beta w_1 \right\|_{L^{2\frac{s-|\alpha|}{|\beta|}}}^p \left\| \partial^\gamma w_2 \right\|_{L^{2\frac{s-|\alpha|}{|\gamma|}}}^q \leq C_{s,p,R} A \left\| w \right\|_{H^s}.$$

For the term for which $\alpha = \mu$, we note that since $h(x, 0) = 0$ for $x \in \mathbb{R}^d$, then $(\partial_x^\alpha h)(x, 0) = 0$ for any $x \in \mathbb{R}^d$, so that if $w \in B_R \subset \mathbb{R}^2$,

$$\left| (\partial_x^\alpha h)(x, w) \right| \leq A|w|,$$

which implies

$$\left\|(\partial_x^\alpha h)\big(x, w(x)\big)\right\|_{L^2} \leq A\|w\|_{L^2} \leq A\|w\|_{H^s}.$$

Combining these two estimates gives

$$\left\|\partial^\mu\big(h(x, w(x))\big)\right\|_{L^2} \leq C_{s,p,R} A\|w\|_{H^s}$$

and the proof of the Lemma is complete. □

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