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Some Poisson-Lie sigma models

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Abstract
We calculate the Poisson-Lie sigma model for every 4-dimensional Manin triples (function of its structure constant) and we give the 6-dimensional models for the Manin triples

\((\text{sl}(2, \mathbb{C}) \oplus \text{sl}(2, \mathbb{C})^\ast, \text{sl}(2, \mathbb{C}), \text{sl}(2, \mathbb{C})^\ast)\),

\((\text{sl}(2, \mathbb{C}) \oplus \text{sl}(2, \mathbb{C})^\ast, \text{sl}(2, \mathbb{C})^\ast, \text{sl}(2, \mathbb{C}), \text{su}(2, \mathbb{C}))\) and

\((\text{sl}(2, \mathbb{C}), \text{su}(2, \mathbb{C}), \text{sb}(2, \mathbb{C}))\)
1 Introduction

A Manin triples \((D, g, \tilde{g})\) is a bialgebra \((g, \tilde{g})\) which don’t intersect each others and a direct sum of this bialgebra \(D = g \oplus \tilde{g}\). If the corresponding Lie groups have a Poisson structure, they are called Poisson-Lie groups. A Poisson-Lie sigma models is an action (3.13) calculated by a Poisson vector field matrix. [3] have deduced the extremal field which minimize the action of this models, which gives the motion equation (3.19). We calculate here the action and the equations of motion for some 6-dimensionals Manin triples and we give a general formula for each 4-dimensional Manin triples. The 6-dimensional Manin triples are \((\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})^*), (\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C})^*), \mathfrak{sl}(2, \mathbb{C}), (\mathfrak{sl}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C})^*), (\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}))\) and \((\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}))\).
2 Some Manin triples

The Drinfeld double $D$ is defined as a Lie group such that its Lie algebra $D$ equipped by a symmetric ad-invariant nondegenerate bilinear form $\langle ., . \rangle$ can be decomposed into a pair of maximally isotropic subalgebras $g, \tilde{g}$ such that $D$ as a vector space is the direct sum of $g$ and $\tilde{g}$. Any such decomposition written as an ordered set $(D, g, \tilde{g})$ is called a Manin triples $(D, g, \tilde{g}), (D, \tilde{g}, g)$.

One can see that the dimensions of the subalgebras are equal and that bases $\{T_i\}, \{\tilde{T}^i\}$ in the subalgebras can be chosen so that
\[
\langle T_i, T_j \rangle = 0, \quad \langle T_i, \tilde{T}^j \rangle = \langle \tilde{T}^i, T_j \rangle = \delta_i^j, \quad \langle \tilde{T}^i, \tilde{T}^j \rangle = 0
\]  
(2.1)

This canonical form of the bracket is invariant with respect to the transformations
\[
T_i' = T_k A_i^k, \quad \tilde{T}^i' = (A^{-1})^j_k \tilde{T}^j
\]  
(2.2)

Due to the ad-invariance of $\langle ., . \rangle$ the algebraic structure of $D$ is
\[
[T_i, T_j] = c_{ij}^k T_k, \quad [\tilde{T}^i, \tilde{T}^j] = f^{ij}k \tilde{T}^k
\]

\[
[T_i, \tilde{T}^j] = -c_{ik}f^{jk} \tilde{T}^k
\]

There are just four types of nonisomorphic four-dimensional Manin triples.

*Abelian Manin triples* :
\[
[T_i, T_j] = 0, \quad [\tilde{T}^i, \tilde{T}^j] = 0, \quad [T_i, \tilde{T}^j] = 0, \quad i, j = 1, 2
\]  
(2.3)

*Semi-Abelian Manin triples (only non trivial brackets are displayed)* :
\[
[\tilde{T}^1, \tilde{T}^2] = \tilde{T}^2, \quad [T_2, \tilde{T}^1] = T_2, \quad [T_2, \tilde{T}^2] = -T_1
\]  
(2.4)

*Type A non-Abelian Manin triples ($\beta \neq 0$)* :
\[
[T_1, T_2] = T_2, \quad [\tilde{T}^1, \tilde{T}^2] = \beta \tilde{T}^2
\]
\[
[T_1, \tilde{T}^2] = -\tilde{T}^2, \quad [T_2, \tilde{T}^1] = \beta T_2 \quad , [T_2, \tilde{T}^2] = -\beta T_1 + \tilde{T}^1
\]

*Type B non-Abelian Manin triples* :
\[
[T_1, T_2] = T_2, \quad [\tilde{T}^1, \tilde{T}^2] = \tilde{T}^1
\]
\[
[T_1, \tilde{T}^1] = T_2, \quad [T_1, \tilde{T}^2] = -T_1 - \tilde{T}^2 \quad , [T_2, \tilde{T}^2] = \tilde{T}^1
\]

Now we focus some six dimensional Manin triples. We recall that the commutation relations of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of the Lie group $SL(2, \mathbb{C})$ :
\[
[T_1, T_2] = 2T_2, \quad [T_1, T_3] = -2T_3, \quad [T_2, T_3] = T_1
\]  
(2.5)

The dual Lie algebra $\mathfrak{sl}(2, \mathbb{C})^*$ of the Lie algebra $\mathfrak{sl}(2, C)$ has the commutation relations :
\[
[\tilde{T}^1, \tilde{T}^2] = \frac{1}{4} \tilde{T}^2, \quad [\tilde{T}^1, \tilde{T}^3] = \frac{1}{4} \tilde{T}^3, \quad [\tilde{T}^2, \tilde{T}^3] = 0
\]  
(2.6)
There is a scalar product on $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*)$ such that (see [2]):
\[(T_i, \tilde{T}^j) = \delta^j_i\] (2.7)
Finally, we have that $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})^*)$ with this scalar product is a Manin triple. We note that $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C}))$ with this scalar product is also a Manin triples.

The Iwasawa decomposition allows us to decompose:
\[\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2, \mathbb{C}) \oplus \mathfrak{sb}(2, \mathbb{C})\] (2.8)
where $\mathfrak{su}(2, \mathbb{C})$ is the Lie algebra of the Lie group $SU(2)$ with commutation relations:
\[[T_1, T_2] = T_3, \quad [T_2, T_3] = T_1, \quad [T_3, T_1] = T_2\] (2.9)
$\mathfrak{sb}(2, \mathbb{C})$ is the Lie algebra of the Borel subgroup $SB(2, \mathbb{C})$ with commutation relations:
\[[\tilde{T}^1, \tilde{T}^2] = \tilde{T}^3, \quad [\tilde{T}^1, \tilde{T}^3] = \tilde{T}^2, \quad [\tilde{T}^2, \tilde{T}^3] = 0\] (2.10)
Here we can see in comparing (2.10) and (2.6) that $\mathfrak{sb}(2, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C})^*$.

The Iwasawa decomposition (2.8) allows us to identify $\mathfrak{sb}(2, \mathbb{C}) \simeq \mathfrak{su}(2, \mathbb{C})^*$. We define a scalar product on $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*)$ such that $(x, y) = \text{Im}(\text{Tr}(x|y))$. With this scalar product we have (see [2]):
\[(T_i, \tilde{T}^j) = \delta^j_i\] (2.11)
Finally we have that $(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}))$ with this scalar product is a Manin triple. We note that $(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}))$ with this scalar product is also a Manin triples.

3 Poisson-Lie sigma models

Given a Lie group $M$ and a Poisson structure on it. We define the action of this model (see [1]) as:
\[S_1 = \int_{\Sigma} \left( dgg^{-1}, A > -\frac{1}{2} < A, (r - Ad_g r Ad_g) A > \right)\] (3.12)
where $g \in G, A = A^i_\alpha d\xi^\alpha X_i$ and $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a classical $r$ matrix with $\mathfrak{g}$ as the Lie algebra of $G$ and $\{X_i\}$ as a basis of $\mathfrak{g}$. Note that the above action can be applied for simple or nonsemisimple Lie group $G$ with ad-invariant symmetric bilinear nondegenerate form $< X_i, X_j > = G_{ij}$ on the Lie algebra $\mathfrak{g}$. When the metric $G_{ij}$ of Lie algebra is denegerate then the above action is not good. Here we use the following action instead of the above one:
\[S_2 = \int_{\Sigma} (dX_i \wedge A_i - \frac{1}{2} P^{ij} A_i \wedge A_j)\] (3.13)
where $x$ are Lie group parameters with parametrization (e.g.)
\[\forall g \in G, g = e^{X_1 T_1} e^{X_2 T_2} ...\] (3.14)
where \( P_{ij} \) is the Poisson structure on the Lie group which for coboundary Poisson Lie groups it is obtained from
\[
(\mathcal{P}(g))_X = b(g)a(g)^{-1}
\]  
(3.15)

We can obtain \( a(g)^{-1} \) and \( b(g) \) in computing :
\[
(Ad_{g^{-1}})_X = \begin{pmatrix} a(g)^T & b(g)^T \\ 0 & d(g)^T \end{pmatrix} \]  
(3.16)
\[
(Ad_g)_X = \begin{pmatrix} a(g)^{-T} & -a(g)^{-T}b(g)^Td(g)^{-T} \\ 0 & d(g)^{-T} \end{pmatrix} \]  
(3.17)

where \( T \) denotes the transpose matrix.

The extremal fields \((X, A)\) which minimize the action (3.13) have to satisfy the equation written locally (see [3]) as :
\[
\begin{align*}
\text{d}X_i + P_{ij}(X)A_j &= 0 \\
\text{d}A_k + \frac{1}{2} \mathcal{P}_{ij}^k(X)A_i \wedge A_j &= 0
\end{align*}
\]  
(3.18) \hspace{1cm} (3.19)

where \( \mathcal{P}_{ij}^k = \partial_k \mathcal{P}_{ij} |_{X_k=0} \).

4 Poisson-Lie sigma model of any 4-dimensional Manin triple

We first calculate the matrix of the adjoint actions function of structure constant :
\[
\begin{align*}
ad_{T_1} &= \begin{pmatrix} 0 & c_{12}^1 & 0 & -f_{12}^1 \\ 0 & c_{12}^2 & f_{12}^1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c_{12}^1 & -c_{12}^2 \end{pmatrix} \\
ad_{T_2} &= \begin{pmatrix} -c_{12}^1 & 0 & 0 & -f_{12}^1 \\ -c_{12}^2 & 0 & f_{12}^1 & 0 \\ 0 & 0 & c_{12}^1 & c_{12}^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{align*}
\]

To obtain the matrix \( \mathcal{P} \), we calculate the adjoint action matrix of a general element \( g = \prod_{i=1}^2 e^{\alpha_i T_i} \) by the formula :
\[
(Ad_{\prod_{i=1}^2 e^{X_i T_i}})_X = \prod_{i=1}^2 e^{X_i(ad_{T_i})_X} \]  
(4.20)

Similarly, we have :
\[
(Ad_{\prod_{i=1}^2 e^{X_i T_i}^{-1}})_X = \prod_{i=1}^2 e^{-X_{3-i}(ad_{T_{3-i}})^{-1}}_X \]  
(4.21)

We can deduce the matrix \( \mathcal{P}_{ij}^{\prime} \) :
\[
\mathcal{P}_{ij}^{\prime} = \begin{pmatrix} 0 & -\mathcal{P}_{21}^{\prime} \\ \mathcal{P}_{21}^{\prime} & 0 \end{pmatrix}
\]

5
where

\[ \mathcal{P}^{21} = \frac{c_{12}^1}{c_{12}^1}(-1 + e^{c_{12}^2X_1})f_{12}^1 + \frac{c_{12}^2}{c_{12}^1}e^{c_{12}^2X_1} - c_{12}^1X_2(-1 + e^{c_{12}^1X_2})f_{12}^2 \]  

(4.22)

Now, we can calculate the action (3.13) of the model

\[ S_2 = \int_{\Sigma} \sum_{i=1}^{2} dX_i \wedge A_i - \mathcal{P}^{21} A_2 \wedge A_1 \]  

(4.23)

and the equations of motion (3.19):

\[ dX_1 - \mathcal{P}^{21} A_2 = 0 \]
\[ dX_2 + \mathcal{P}^{21} A_1 = 0 \]
\[ dA_1 - \frac{c_{12}^1 c_{12}^2}{c_{12}^1} f_{12}^1 + \frac{c_{12}^2}{c_{12}^1} e^{c_{12}^2X_1}(-1 + e^{-c_{12}^1X_2})f_{12}^2 A_1 \wedge A_2 = 0 \]
\[ dA_2 + \frac{c_{12}^2 e^{c_{12}^2X_1} c_{12}^1}{c_{12}^1} f_{12}^2 A_2 \wedge A_1 = 0 \]  

(4.24)

5 Poisson-Lie sigma model of \((\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})^*)\)

We first calculate the matrix of the adjoint actions:

\[ adT_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

\[ adT_2 = \frac{1}{4} \begin{pmatrix}
0 & 0 & 0 & 0 & -1 & 0 \\
-8 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

\[ adT_3 = \frac{1}{4} \begin{pmatrix}
0 & -4 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \]
To obtain the matrix $P$, we calculate the adjoint action matrix of a general element $g = \prod_{i=1}^{3} e^{\alpha_i T_i}$ by the formula:

$$(Ad_{\prod_{i=1}^{3} e^{\alpha_i T_i}})\chi = \prod_{i=1}^{3} e^{X_i (ad T_i)} \chi$$

(5.25)

Similarly, we have:

$$(Ad_{\prod_{i=1}^{3} e^{-\alpha_i T_i}})\chi = \prod_{i=1}^{3} e^{-X_i (ad T_i)} \chi$$

(5.26)

We can deduce the matrix $P_{ij}$:

$$P_{ij} = \begin{pmatrix}
0 & -X_2 (1 + X_2 X_3) e^{2X_1} & -X_3 e^{-2X_1} \\
X_2 (1 + X_2 X_3) e^{2X_1} & 0 & X_3 e^{-2X_1} \\
X_2 (1 + X_2 X_3) e^{2X_1} & -X_3 e^{-2X_1} & 0
\end{pmatrix}$$

Now, we can calculate the action (3.13) of the model

$$S_2 = \int_\Sigma \sum_{i=1}^{3} dX_i \wedge A_i + \left(\frac{X_2}{4} (1 + X_2 X_3) e^{2X_1} A_1 \wedge A_2 + \frac{X_3}{4} e^{-2X_1} A_1 \wedge A_3 - \frac{X_2 X_3}{2} A_2 \wedge A_3\right)$$

(5.27)

and the equations of motion (3.19):

$$dX_1 - \left(\frac{X_2}{4} (1 + X_2 X_3) e^{2X_1} A_2 - \frac{X_3}{4} e^{-2X_1} A_3\right) = 0$$

$$dX_2 + \left(\frac{X_2}{4} (1 + X_2 X_3) e^{2X_1} A_1 + \frac{X_2 X_3}{2} A_3\right) = 0$$

$$dX_3 + \frac{X_3}{4} e^{-2X_1} A_1 - \frac{X_2 X_3}{2} A_2 = 0$$

$$dA_1 - \frac{X_2}{2} (1 + X_2 X_3) A_1 \wedge A_2 + \frac{X_3}{2} A_1 \wedge A_2 = 0$$

$$dA_2 - \frac{e^{2X_1}}{4} A_1 \wedge A_2 + \frac{X_3}{2} A_2 \wedge A_3 = 0$$

$$dA_3 - \frac{X_2^2}{4} e^{2X_1} A_1 \wedge A_2 - \frac{X_3}{4} A_1 \wedge A_3 + X_2 A_2 \wedge A_3 = 0$$

6 Poisson-Lie sigma model of $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C}))$

Now to obtain this Poisson Lie sigma model, we have to change $T_i \rightarrow \tilde{T}^i$ and $\tilde{T}^i \rightarrow T_i$ of the previous model. And we can calculate the matrix of the adjoint actions as we do previously. With this we can deduce the matrix $P^{ij}$ for this model:

$$P^{ij} = \begin{pmatrix}
0 & -2 e^{\frac{X_1}{4}} X_2 & 2 e^{\frac{X_1}{4}} X_3 \\
2 e^{\frac{X_1}{4}} X_2 & 0 & \frac{2}{4} e^{\frac{X_1}{4}} (4 + X_2 X_3) \\
-2 e^{\frac{X_1}{4}} X_3 - 2 + \frac{1}{2} e^{\frac{X_1}{4}} (4 + X_2 X_3) & 2 - \frac{1}{2} e^{\frac{X_1}{4}} (4 + X_2 X_3) & 0
\end{pmatrix}$$
Now, we can calculate the action \((3.13)\) of the model

\[
S_2 = \int \sum_{i=1}^{3} dX_i \wedge A_i + 2e^\frac{X_1}{2}X_2A_1 \wedge A_2 - 2e^\frac{X_3}{2}X_3A_1 \wedge A_3 + (-2 + \frac{1}{2}e^\frac{X_1}{2}(4 + X_2X_3))A_2 \wedge A_3
\]  

(6.28)

and the equations of motion \((3.19)\) :

\[
\begin{align*}
    dX_1 - 2e^\frac{X_1}{2}X_2A_2 + 2e^\frac{X_3}{2}X_3A_3 &= 0 \\
    dX_2 + 2e^\frac{X_1}{2}X_2A_1 + (2 - \frac{1}{2}e^\frac{X_1}{2}(4 + X_2X_3))A_3 &= 0 \\
    dX_3 - 2e^\frac{X_3}{2}X_3A_1 - (2 - \frac{1}{2}e^\frac{X_3}{2}(4 + X_2X_3))A_2 &= 0 \\
    dA_1 - \frac{X_2}{2}A_1 \wedge A_2 + \frac{X_3}{4}A_1 \wedge A_3 - \frac{1}{4}(4 + X_2X_3)A_2 \wedge A_3 &= 0 \\
    dA_2 - 2e^\frac{X_1}{2}A_1 \wedge A_2 - \frac{1}{2}e^\frac{X_1}{2}A_2 \wedge A_3 &= 0 \\
    dA_3 + 2e^\frac{X_3}{2}A_1 \wedge A_3 - e^\frac{X_2}{2}A_2 \wedge A_3 &= 0
\end{align*}
\]

7 Poisson-Lie sigma model of \((\mathfrak{sl}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}))\)

We can calculate the matrix of the adjoint actions as we do previously. With this we can deduce the matrix \(\mathcal{P}^{ij}\) for this model :

\[
\mathcal{P}^{ij} = \begin{pmatrix}
0 & -\cos X_1 \cos X_3 \sin X_2 + \sin X_1 \sin X_3 & -\cos X_3 \sin X_1 \sin X_2 - \cos X_1 \sin X_3 \\
\cos X_1 \cos X_3 \sin X_2 - \sin X_1 \sin X_3 & 0 & -1 - \cos X_2 \cos X_3 \\
\cos X_3 \sin X_1 \sin X_2 + \cos X_1 \sin X_3 & 1 - \cos X_2 \cos X_3 & 0
\end{pmatrix}
\]

Now, we can calculate the action \((3.13)\) of the model

\[
S_2 = \int \sum_{i=1}^{3} dX_i \wedge A_i - (-\cos X_1 \cos X_3 \sin X_2 + \sin X_1 \sin X_3)A_1 \wedge A_2 \\
-(-\cos X_3 \sin X_1 \sin X_2 - \cos X_1 \sin X_3)A_1 \wedge A_3 - (-1 + \cos X_2 \cos X_3)A_2 \wedge A_3
\]

(7.29)

and the equations of motion \((3.19)\) :

\[
\begin{align*}
    dX_1 + (-\cos X_1 \cos X_3 \sin X_2 + \sin X_1 \sin X_3)A_2 + (-\cos X_3 \sin X_1 \sin X_2 - \cos X_1 \sin X_3)A_3 &= 0 \\
    dX_2 + (\cos X_1 \cos X_3 \sin X_2 - \sin X_1 \sin X_3)A_1 + (-1 + \cos X_2 \cos X_3)A_3 &= 0 \\
    dX_3 + (\cos X_3 \sin X_1 \sin X_2 + \cos X_1 \sin X_3)A_1(1 - \cos X_2 \cos X_3)A_2 &= 0 \\
    dA_1 + \sin X_3 A_1 \wedge A_2 - \cos X_3 \sin X_2 A_1 \wedge A_3 &= 0 \\
    dA_2 - \cos X_1 \cos X_3 A_1 \wedge A_2 - \cos X_3 \sin X_1 A_1 \wedge A_3 &= 0 \\
    dA_3 + \sin X_1 A_1 \wedge A_2 - \cos X_1 A_1 \wedge A_3 &= 0
\end{align*}
\]
8 Poisson-Lie sigma model of \((\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}))\)

We can calculate the matrix of the adjoint actions as we do previously. With this we can deduce the matrix \(\mathcal{P}^{ij}\) for this model:

\[
\mathcal{P}^{ij} = \begin{pmatrix}
0 & -e^{X_1}X_3 & -e^{X_1}X_2 \\
-e^{X_1}X_2 & 0 & \frac{1}{2}(1 - e^{2X_1}(1 + X_2^2 + X_3^2)) \\
e^{X_1}X_2 & \frac{1}{2}(-1 + e^{2X_1}(1 + X_2^2 + X_3^2)) & 0
\end{pmatrix}
\]

Now, we can calculate the action (3.13) of the model

\[
S_2 = \int_{\Sigma} \sum_{i=1}^{3} dX_i \wedge A_i + e^{X_1}X_3A_1 \wedge A_2 + e^{X_1}X_2A_1 \wedge A_3 - \frac{1}{2}(1 - e^{2X_1}(1 + 2X_2^2 + 2X_3^2))A_2 \wedge A_3
\]

and the equations of motion (3.19):

\[
\begin{align*}
dX_1 - e^{X_1}X_3A_2 - e^{X_1}X_2A_3 &= 0 \\
dX_2 + e^{X_1}X_3A_1 + \frac{1}{2}(1 - e^{2X_1}(1 + X_2^2 + X_3^2))A_3 &= 0 \\
dX_3 + e^{X_1}X_2A_1 - \frac{1}{2}(1 - e^{2X_1}(1 + X_2^2 + X_3^2))A_2 &= 0 \\
dA_1 - X_3A_1 \wedge A_2 - X_2A_1 \wedge A_3 - (1 + X_2^2 + X_3^2)A_2 \wedge A_3 &= 0 \\
dA_2 - e^{X_1}A_1 \wedge A_3 &= 0 \\
dA_3 - e^{X_1}A_1 \wedge A_2 &= 0
\end{align*}
\]

9 Discussion

We gives here the Poisson-Lie sigma models of some Manin triples. Concerning the general formula (9.32), we have to say that this is no problem when \(c_{12}^2\) and \(c_{12}^{-2}\) is zero because

\[
\mathcal{P}^{21} = \frac{(-1 + e^{c_{12}^{-2}X_1})f^{12}}{c_{12}^2} + \frac{e^{c_{12}^{-2}X_1-c_{12}^{-1}X_2}(-1 + e^{c_{12}^{-1}X_2})f^{12}}{c_{12}^2}
\]

which can be approximate by

\[
\mathcal{P}^{21} = (X_1 + \frac{c_{12}^{-2}}{2}X_1^2 + ...)f^{12} + e^{c_{12}^{-2}X_1-c_{12}^{-1}X_2}(X_2 + \frac{c_{12}^{-1}}{2}X_2^2 + ...)f^{12}
\]

We tried to obtain the equivalent formula for \(n = 3\) but the calculus was too hard.
Références

[1] Hajizadeh S., Rezaei-Aghdam A., Poisson-Lie Sigma models over low dimensional real Poisson-Lie groups
