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► **To cite this version:**

Pierre Bayard, Marie-Amélie Lawn, Julien Roth. Spinorial representation of surfaces in four-dimensional Space Forms. *Annals of Global Analysis and Geometry*, Springer Verlag, 2013, 44 (4), pp.433-453. hal-00807496

HAL Id: hal-00807496

<https://hal.archives-ouvertes.fr/hal-00807496>

Submitted on 3 Apr 2013

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Spinorial Representation of Surfaces into 4-dimensional Space Forms

Pierre Bayard*, Marie-Amélie Lawn† and Julien Roth‡

Abstract

In this paper we give a geometrically invariant spinorial representation of surfaces in four-dimensional space forms. In the Euclidean space, we obtain a representation formula which generalizes the Weierstrass representation formula of minimal surfaces. We also obtain as particular cases the spinorial characterizations of surfaces in \mathbb{R}^3 and in S^3 given by T. Friedrich and by B. Morel.

keywords: Dirac Operator, Isometric Immersions, Weierstrass Representation.

2000 Mathematics Subject Classification: 53C27, 53C40.

1 Introduction

The Weierstrass representation describes a conformal minimal immersion of a Riemann surface M into the three-dimensional Euclidean space \mathbb{R}^3 . Precisely, the immersion is expressed using two holomorphic functions $f, g : M \rightarrow \mathbb{C}$ by the following integral formula

$$(x_1, x_2, x_3) = \Re e \left(\int f(1 - g^2) dz, \int i f(1 + g^2) dz, \int 2f g dz \right) : M \rightarrow \mathbb{R}^3.$$

On the other hand, the spinor bundle ΣM over M is a two-dimensional complex vector bundle splitting into

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M = \Lambda^0 M \oplus \Lambda^{0,1} M.$$

Hence, a pair of holomorphic functions (g, f) can be considered as a spinor field $\varphi = (g, f dz)$. Moreover, the Cauchy-Riemann equations satisfied by f and g are equivalent to the Dirac equation

$$D\varphi = 0.$$

This representation is still valid for arbitrary surfaces. In the general case, the functions f and g are not holomorphic and the Dirac equation becomes

$$D\varphi = H\varphi,$$

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where H is the mean curvature of the immersion. This fact is well-known and has been studied in the last years by many authors (see [8, 9, 15, 16]).

In [5], T. Friedrich gave a geometrically invariant spinorial representation of surfaces in \mathbb{R}^3 . This approach was generalized to surfaces of other three-dimensional spaces [13, 14] and also in the pseudo-Riemannian case [10, 11].

The aim of the present paper is to extend this approach to the case of codimension 2 and then provide a geometrically invariant representation of surfaces in the 4-dimensional space form $\mathbb{M}^4(c)$ of sectional curvature c by spinors solutions of a Dirac equation.

2 Preliminaries

2.1 The fundamental theorem of surfaces in $\mathbb{M}^4(c)$

Let (M^2, g) be an oriented surface isometrically immersed into the four-dimensional space form $\mathbb{M}^4(c)$. Let us denote by E its normal bundle and by $B : TM \times TM \rightarrow E$ its second fundamental form defined by

$$B(X, Y) = \bar{\nabla}_X Y - \nabla_X Y,$$

where ∇ and $\bar{\nabla}$ are the Levi-Civita connections of M and $\mathbb{M}^4(c)$ respectively. For $\xi \in \Gamma(E)$, the shape operator associated to ξ is defined by

$$S_\xi(X) = -(\bar{\nabla}_X \xi)^T,$$

for all $X \in \Gamma(TM)$, where the upper index T means that we take the component of the vector tangent to M . Then, the following equations hold:

1. $K = \langle B(e_1, e_1), B(e_2, e_2) \rangle - |B(e_1, e_2)|^2 + c$, (Gauss equation)
2. $K_N = -\langle (S_{e_3} \circ S_{e_4} - S_{e_4} \circ S_{e_3})(e_1), e_2 \rangle$, (Ricci equation)
3. $(\nabla'_X B)(Y, Z) - (\nabla'_Y B)(X, Z) = 0$, (Codazzi equation)

where K and K_N are the curvatures of (M, g) and E , (e_1, e_2) and (e_3, e_4) are orthonormal and positively oriented bases of TM and E respectively, and where ∇' is the natural connection induced on the bundle $T^*M^{\otimes 2} \otimes E$. Reciprocally, there is the following theorem:

Theorem (Tenenblat [17]). *Let (M^2, g) be a Riemannian surface and E a vector bundle of rank 2 on M , equipped with a metric $\langle \cdot, \cdot \rangle$ and a compatible connection. We suppose that M and E are oriented. Let $B : TM \times TM \rightarrow E$ be a bilinear map satisfying the Gauss, Ricci and Codazzi equations above, where, if $\xi \in E$, the shape operator $S_\xi : TM \rightarrow TM$ is the symmetric operator such that*

$$g(S_\xi(X), Y) = \langle B(X, Y), \xi \rangle$$

for all $X, Y \in TM$. Then, there exists a local isometric immersion $V \subset M \rightarrow \mathbb{M}^4(c)$ so that E is identified with the normal bundle of M into $\mathbb{M}^4(c)$ and with B as second fundamental form.

2.2 Twisted spinor bundle

Let (M^2, g) be an oriented Riemannian surface, with a given spin structure, and E an oriented and spin vector bundle of rank 2 on M . We consider the spinor bundle Σ over M twisted by E and defined by

$$\Sigma = \Sigma M \otimes \Sigma E.$$

We endow Σ with the spinorial connection ∇ defined by

$$\nabla = \nabla^{\Sigma M} \otimes \text{Id}_{\Sigma E} + \text{Id}_{\Sigma M} \otimes \nabla^{\Sigma E}.$$

We also define the Clifford product \cdot by

$$\begin{cases} X \cdot \varphi = (X \cdot_M \alpha) \otimes \bar{\sigma} & \text{if } X \in \Gamma(TM) \\ X \cdot \varphi = \alpha \otimes (X \cdot_E \sigma) & \text{if } X \in \Gamma(E) \end{cases}$$

for all $\varphi = \alpha \otimes \sigma \in \Sigma M \otimes \Sigma E$, where \cdot_M and \cdot_E denote the Clifford products on ΣM and on ΣE respectively and where $\bar{\sigma} = \sigma^+ - \sigma^-$. We finally define the Dirac operator D on $\Gamma(\Sigma)$ by

$$D\varphi = e_1 \cdot \nabla_{e_1} \varphi + e_2 \cdot \nabla_{e_2} \varphi,$$

where (e_1, e_2) is an orthonormal basis of TM .

We note that Σ is also naturally equipped with a hermitian scalar product $\langle \cdot, \cdot \rangle$ which is compatible to the connection ∇ , since so are ΣM and ΣE , and thus also with a compatible real scalar product $\Re \langle \cdot, \cdot \rangle$. We also note that the Clifford product \cdot of vectors belonging to $TM \oplus E$ is antihermitian with respect to this hermitian product. Finally, we stress that the four subbundles $\Sigma^{\pm\pm} := \Sigma^\pm M \otimes \Sigma^\pm E$ are orthogonal with respect to the hermitian product. Throughout the paper we will assume that the hermitian product is \mathbb{C} -linear w.r.t. the first entry, and \mathbb{C} -antilinear w.r.t. the second entry.

2.3 Spin geometry of surfaces in $\mathbb{M}^4(c)$

It is a well-known fact (see [1, 6, 7]) that there is an identification between the spinor bundle $\Sigma \mathbb{M}^4(c)|_M$ of $\mathbb{M}^4(c)$ over M , and the spinor bundle of M twisted by the normal bundle $\Sigma := \Sigma M \otimes \Sigma E$. Moreover, we have the spinorial Gauss formula: for any $\varphi \in \Gamma(\Sigma)$ and any $X \in TM$,

$$\tilde{\nabla}_X \varphi = \nabla_X \varphi + \frac{1}{2} \sum_{j=1,2} e_j \cdot B(X, e_j) \cdot \varphi,$$

where $\tilde{\nabla}$ is the spinorial connection of $\Sigma \mathbb{M}^4(c)$ and ∇ is the spinorial connection of Σ defined by

$$\nabla = \nabla^{\Sigma M} \otimes \text{Id}_{\Sigma E} + \text{Id}_{\Sigma M} \otimes \nabla^{\Sigma E}.$$

Here \cdot is the Clifford product on $\mathbb{M}^4(c)$. Therefore, if φ is a Killing spinor of $\mathbb{M}^4(c)$, that is satisfying

$$\tilde{\nabla}_X \varphi = \lambda X \cdot \varphi,$$

where the Killing constant λ is 0 for the Euclidean space, $\pm\frac{1}{2}$ for the sphere and $\pm\frac{i}{2}$ for the hyperbolic space, that is, $4\lambda^2 = c$, then its restriction over M satisfies

$$\nabla_X \varphi = -\frac{1}{2} \sum_{j=1,2} e_j \cdot B(X, e_j) \cdot \varphi + \lambda X \cdot \varphi. \quad (1)$$

Taking the trace in (1), we obtain the following Dirac equation

$$D\varphi = \vec{H} \cdot \varphi - 2\lambda\varphi, \quad (2)$$

where we have again $D\varphi = \sum_{j=1}^2 e_j \cdot \nabla_{e_j} \varphi$ and $\vec{H} = \frac{1}{2} \sum_{j=1}^2 B(e_j, e_j)$ is the mean curvature vector of M in $\mathbb{M}^4(c)$.

Let us consider $\omega_4 = -e_1 \cdot e_2 \cdot e_3 \cdot e_4$. We recall that $\omega_4^2 = 1$ and ω_4 has two eigenspaces for eigenvalues 1 and -1 of same dimension. We denote by Σ^+ and Σ^- these subbundles. They decompose as follows:

$$\begin{cases} \Sigma^+ = (\Sigma^+ M \otimes \Sigma^+ E) \oplus (\Sigma^- M \otimes \Sigma^- E) \\ \Sigma^- = (\Sigma^+ M \otimes \Sigma^- E) \oplus (\Sigma^- M \otimes \Sigma^+ E), \end{cases}$$

where $\Sigma^\pm M$ and $\Sigma^\pm E$ are the spaces of half-spinors for M and E respectively. In the sequel, for $\varphi \in \Sigma$, we will use the following convention:

$$\varphi = \varphi^{++} + \varphi^{--} + \varphi^{+-} + \varphi^{-+},$$

with

$$\begin{cases} \varphi^{++} \in \Sigma^{++} := \Sigma^+ M \otimes \Sigma^+ E, \\ \varphi^{--} \in \Sigma^{--} := \Sigma^- M \otimes \Sigma^- E, \\ \varphi^{+-} \in \Sigma^{+-} := \Sigma^+ M \otimes \Sigma^- E, \\ \varphi^{-+} \in \Sigma^{-+} := \Sigma^- M \otimes \Sigma^+ E. \end{cases}$$

Finally, we set

$$\varphi^+ = \varphi^{++} + \varphi^{--} \quad \text{and} \quad \varphi^- = \varphi^{+-} + \varphi^{-+}.$$

If φ is a Killing spinor of $\mathbb{M}^4(c)$, an easy computation yields

$$X|\varphi^+|^2 = 2\Re\langle \lambda X \cdot \varphi^-, \varphi^+ \rangle \quad \text{and} \quad X|\varphi^-|^2 = 2\Re\langle \lambda X \cdot \varphi^+, \varphi^- \rangle.$$

3 Main result

Theorem 1. *Let (M^2, g) be an oriented Riemannian surface, with a given spin structure, and E an oriented and spin vector bundle of rank 2 on M . Let $\Sigma = \Sigma M \otimes \Sigma E$ be the twisted spinor bundle and D its Dirac operator. We assume that λ is a constant belonging to $\mathbb{R} \cup i\mathbb{R}$ and \vec{H} is a section of E . Then the three following statements are equivalent:*

1. *There exists a spinor $\varphi \in \Gamma(\Sigma)$ solution of the Dirac equation*

$$D\varphi = \vec{H} \cdot \varphi - 2\lambda\varphi \quad (3)$$

such that φ^+ and φ^- do not vanish and satisfy

$$X|\varphi^+|^2 = 2\Re\langle \lambda X \cdot \varphi^-, \varphi^+ \rangle \quad \text{and} \quad X|\varphi^-|^2 = 2\Re\langle \lambda X \cdot \varphi^+, \varphi^- \rangle. \quad (4)$$

2. There exists a spinor $\varphi \in \Gamma(\Sigma)$ solution of

$$\nabla_X \varphi = -\frac{1}{2} \sum_{j=1,2} e_j \cdot B(X, e_j) \cdot \varphi + \lambda X \cdot \varphi,$$

where $B : TM \times TM \rightarrow E$ is bilinear and $\frac{1}{2} \text{tr}(B) = \vec{H}$ and such that φ^+ and φ^- do not vanish.

3. There exists a local isometric immersion of (M, g) into $\mathbb{M}^4(c)$ with normal bundle E , second fundamental form B and mean curvature \vec{H} .

The form B and the spinor field φ are linked by (6).

In order to prove Theorem 1 we consider the following equivalent technical

Proposition 3.1. *Let M , E and Σ as in Theorem 1 and assume that there exists a spinor $\varphi \in \Gamma(\Sigma)$ solution of*

$$D\varphi = \vec{H} \cdot \varphi - 2\lambda\varphi \quad (5)$$

with φ^+ and φ^- nowhere vanishing spinors satisfying (4). Then the symmetric bilinear map

$$B : TM \times TM \rightarrow E$$

defined by

$$\begin{aligned} \langle B(X, Y), \xi \rangle &= \frac{1}{2|\varphi^+|^2} \Re \langle X \cdot \nabla_Y \varphi^+ + Y \cdot \nabla_X \varphi^+ + 2\lambda \langle X, Y \rangle \varphi^-, \xi \cdot \varphi^+ \rangle \\ &+ \frac{1}{2|\varphi^-|^2} \Re \langle X \cdot \nabla_Y \varphi^- + Y \cdot \nabla_X \varphi^- + 2\lambda \langle X, Y \rangle \varphi^+, \xi \cdot \varphi^- \rangle \end{aligned} \quad (6)$$

for all $X, Y \in \Gamma(TM)$ and all $\xi \in \Gamma(E)$ satisfies the Gauss, Codazzi and Ricci equations and is such that

$$\vec{H} = \frac{1}{2} \text{tr} B.$$

Remark 1. *If $\lambda = 0$, and if $\varphi \in \Gamma(\Sigma)$ is a solution of*

$$D\varphi = \vec{H} \cdot \varphi \quad \text{with} \quad |\varphi^+| = |\varphi^-| = 1,$$

formula (6) simplifies to

$$\begin{aligned} \langle B(X, Y), \xi \rangle &= \frac{1}{2} \Re \langle X \cdot \nabla_Y \varphi + Y \cdot \nabla_X \varphi, \xi \cdot \varphi \rangle \\ &= \Re \langle X \cdot \nabla_Y \varphi, \xi \cdot \varphi \rangle, \end{aligned} \quad (7)$$

since this last expression is in fact symmetric in X and Y .

To prove proposition 3.1 we first state the following lemma.

Lemma 3.2. *Assume that φ is a solution of the Dirac equation (5) with φ^+ and φ^- non-vanishing spinors satisfying (4). Then, for all $X \in \Gamma(TM)$,*

$$\nabla_X \varphi = \eta(X) \cdot \varphi + \lambda X \cdot \varphi, \quad (8)$$

with

$$\eta(X) = -\frac{1}{2} \sum_{j=1}^2 e_j \cdot B(e_j, X), \quad (9)$$

where the bilinear map B is defined by (6).

The proof of this lemma will be given in Section 4.

Proof of Proposition 3.1: The equations of Gauss, Codazzi and Ricci appear to be the integrability conditions of (8). Indeed computing the spinorial curvature \mathcal{R} for φ , we first observe that (9) implies

$$X \cdot \eta(Y) - \eta(Y) \cdot X = B(X, Y) = Y \cdot \eta(X) - \eta(X) \cdot Y$$

for all $X, Y \in TM$. Then, a direct computation yields

$$\begin{aligned} \mathcal{R}(X, Y)\varphi &= d^\nabla \eta(X, Y) \cdot \varphi + (\eta(Y) \cdot \eta(X) - \eta(X) \cdot \eta(Y)) \cdot \varphi \quad (10) \\ &\quad + \lambda^2(Y \cdot X - X \cdot Y) \cdot \varphi, \end{aligned}$$

where

$$d^\nabla \eta(X, Y) = \nabla_X(\eta(Y)) - \nabla_Y(\eta(X)) - \eta([X, Y]).$$

Here we also denote by ∇ the natural connection on $Cl(TM \oplus E) \simeq Cl(M) \hat{\otimes} Cl(E)$, the graded tensor product of the Clifford algebras $Cl(M)$ and $Cl(E)$.

Lemma 3.3. *We have:*

1. *The left-hand side of (10) satisfies*

$$\mathcal{R}(e_1, e_2)\varphi = -\frac{1}{2}K e_1 \cdot e_2 \cdot \varphi - \frac{1}{2}K_N e_3 \cdot e_4 \cdot \varphi.$$

2. *The first term of the right-hand side of (10) satisfies*

$$d^\nabla \eta(X, Y) = -\frac{1}{2} \sum_{j=1}^2 e_j \cdot \left((\nabla'_X B)(Y, e_j) - (\nabla'_Y B)(X, e_j) \right)$$

where ∇' stands for the natural connection on $T^*M \otimes T^*M \otimes E$.

3. *The second term of the right-hand side of (10) satisfies*

$$\begin{aligned} \eta(e_2) \cdot \eta(e_1) - \eta(e_1) \cdot \eta(e_2) &= \frac{1}{2} (|B(e_1, e_2)|^2 - \langle B(e_1, e_1), B(e_2, e_2) \rangle) e_1 \cdot e_2 \\ &\quad + \frac{1}{2} \langle (S_{e_3} \circ S_{e_4} - S_{e_4} \circ S_{e_3})(e_1), e_2 \rangle e_3 \cdot e_4. \end{aligned}$$

Proof: First, we compute $\mathcal{R}(e_1, e_2)\varphi$. We recall that $\Sigma = \Sigma M \otimes \Sigma E$ and suppose that $\varphi = \alpha \otimes \sigma$ with $\alpha \in \Sigma M$ and $\sigma \in \Sigma E$. Thus,

$$\mathcal{R}(e_1, e_2)\varphi = \mathcal{R}^M(e_1, e_2)\alpha \otimes \sigma + \alpha \otimes \mathcal{R}^E(e_1, e_2)\sigma,$$

where \mathcal{R}^M and \mathcal{R}^E are the spinorial curvatures on M and E respectively. Moreover, by the Ricci identity on M , we have

$$\mathcal{R}^M(e_1, e_2)\alpha = -\frac{1}{2}K e_1 \cdot e_2 \cdot \alpha,$$

where K is the Gauss curvature of (M, g) . Similarly, we have

$$\mathcal{R}^E(e_1, e_2)\sigma = -\frac{1}{2}K_N e_3 \cdot e_4 \cdot \sigma,$$

where K_N is the curvature of the connection on E . These last two relations give the first point of the lemma.

For the second point of the lemma, we choose e_j so that at $p \in M$, $\nabla e_j|_p = 0$. Then, we have

$$\begin{aligned}
d^\nabla \eta(X, Y) &= \nabla_X(\eta(Y)) - \nabla_Y(\eta(X)) - \eta([X, Y]) \\
&= \sum_{j=1,2} -\frac{1}{2} \nabla_X(e_j \cdot B(Y, e_j)) + \frac{1}{2} \nabla_Y(e_j \cdot B(X, e_j)) + \frac{1}{2} e_j \cdot B([X, Y], e_j) \\
&= \sum_{j=1,2} -\frac{1}{2} e_j \cdot \nabla_X^E(B(Y, e_j)) + \frac{1}{2} e_j \cdot \nabla_Y^E(B(X, e_j)) + \frac{1}{2} e_j \cdot B(\nabla_X Y, e_j) \\
&\quad - \frac{1}{2} e_j \cdot B(\nabla_Y X, e_j) \\
&= -\frac{1}{2} \sum_{j=1,2} e_j \cdot \left((\nabla'_X B)(Y, e_j) - (\nabla'_Y B)(X, e_j) \right)
\end{aligned}$$

since $[X, Y] = \nabla_X Y - \nabla_Y X$ and $(\nabla'_X B)(Y, e_j) = \nabla_X^E(B(Y, e_j)) - B(\nabla_X Y, e_j)$. Here ∇^E stands for the given connection on E .

We finally prove the third assertion of the lemma. In order to simplify the notation, we set $B(e_i, e_j) = B_{ij}$. We have

$$\begin{aligned}
\eta(e_2) \cdot \eta(e_1) - \eta(e_1) \cdot \eta(e_2) &= -\frac{1}{4} \sum_{j,k=1}^2 e_j \cdot B_{1j} \cdot e_k \cdot B_{2k} + \frac{1}{4} \sum_{j,k=1}^2 e_j \cdot B_{2j} \cdot e_k \cdot B_{1k} \\
&= \frac{1}{4} \left[-e_1 \cdot B_{11} \cdot e_1 \cdot B_{21} - e_1 \cdot B_{11} \cdot e_2 \cdot B_{22} - e_2 \cdot B_{12} \cdot e_1 \cdot B_{21} - e_2 \cdot B_{12} \cdot e_2 \cdot B_{22} \right. \\
&\quad \left. + e_1 \cdot B_{21} \cdot e_1 \cdot B_{11} + e_1 \cdot B_{21} \cdot e_2 \cdot B_{12} + e_2 \cdot B_{22} \cdot e_1 \cdot B_{11} + e_2 \cdot B_{22} \cdot e_2 \cdot B_{12} \right] \\
&= \frac{1}{2} \left[|B_{12}|^2 - \langle B_{11}, B_{22} \rangle \right] e_1 \cdot e_2 + \frac{1}{4} \left[-B_{11} \cdot B_{21} + B_{21} \cdot B_{11} - B_{12} \cdot B_{22} + B_{22} \cdot B_{12} \right].
\end{aligned}$$

Now, if we write $B_{ij} = B_{ij}^3 e_3 + B_{ij}^4 e_4$, we have

$$-B_{11} \cdot B_{21} + B_{21} \cdot B_{11} = 2(-B_{11}^3 B_{21}^4 + B_{21}^3 B_{11}^4) e_3 \cdot e_4$$

and

$$-B_{12} \cdot B_{22} + B_{22} \cdot B_{12} = 2(-B_{12}^3 B_{22}^4 + B_{12}^4 B_{22}^3) e_3 \cdot e_4.$$

Moreover

$$-B_{11}^3 B_{21}^4 + B_{21}^3 B_{11}^4 - B_{12}^3 B_{22}^4 + B_{12}^4 B_{22}^3 = \langle (S_{e_3} \circ S_{e_4} - S_{e_4} \circ S_{e_3})(e_1), e_2 \rangle$$

since for $j \in \{1, 2\}$ and $k \in \{3, 4\}$, we have $S_{e_k} e_j = B_{j1}^k e_1 + B_{j2}^k e_2$. The formula follows. \square

Now, we give this final lemma

Lemma 3.4. *If T is an element of $Cl(M) \hat{\otimes} Cl(E)$ of order 2, that is of*

$$\Lambda^2 M \otimes 1 \oplus TM \otimes E \oplus 1 \otimes \Lambda^2 E,$$

so that

$$T \cdot \varphi = 0,$$

where φ is a spinor field of Σ such that φ^+ and φ^- do not vanish, then $T = 0$.

Proof: We have

$$\mathcal{Cl}_2 \hat{\otimes} \mathcal{Cl}_2 \simeq \mathcal{Cl}_4 \simeq \mathbb{H}(2),$$

where $\mathbb{H}(2)$ is the set of 2×2 matrices with quaternionic coefficients. The spinor bundle Σ and the Clifford product come from the representation

$$\mathbb{H}(2) \longrightarrow \text{End}_{\mathbb{H}}(\mathbb{H} \oplus \mathbb{H}).$$

The first factor of $\mathbb{H} \oplus \mathbb{H}$ correspond to Σ^+ and the second to Σ^- . Moreover, elements of order 2 of \mathcal{Cl}_4 are matrices

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix},$$

where p, q are purely imaginary quaternions. Hence $T \cdot \varphi = 0$ is equivalent to

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} \alpha \\ \sigma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with α, σ non zero quaternions. Thus $p = q = 0$, and so T vanishes identically. \square

We deduce from (10) and Lemma 3.4 and comparing terms, that

$$\begin{cases} K = \langle B(e_1, e_1), B(e_2, e_2) \rangle - |B(e_1, e_2)|^2 + 4\lambda^2, \\ K_N = -\langle (S_{e_3} \circ S_{e_4} - S_{e_4} \circ S_{e_3})(e_1), e_2 \rangle, \\ (\nabla'_X B)(Y, e_j) - (\nabla'_Y B)(X, e_j) = 0, \quad \forall j = 1, 2, \end{cases}$$

which are respectively the Gauss, Ricci and Codazzi equations. \square

From Proposition 3.1 and the fundamental theorem of submanifolds, we deduce that a spinor field solution of (5) such that (4) holds defines a local isometric immersion of M into $\mathbb{M}^4(c)$ with normal bundle E and second fundamental form B . This implies the equivalence between assertions 1 and 3 in Theorem 1. The equivalence between assertions 1 and 2 is given by Lemma 3.2 and will be proven in the next section.

Remark 2. *If in Theorem 1 we assume moreover that the manifold is simply connected, the spinor field solution of (5) defines a global isometric immersion of M into $\mathbb{M}^4(c)$.*

4 Proof of Lemma 3.2

In order to prove Lemma 3.2, we need some preliminary results. First, we remark that

$$\begin{cases} D\varphi^{--} = \vec{H} \cdot \varphi^{++} - 2\lambda\varphi^{+-}, \\ D\varphi^{++} = \vec{H} \cdot \varphi^{--} - 2\lambda\varphi^{-+}, \\ D\varphi^{+-} = \vec{H} \cdot \varphi^{-+} - 2\lambda\varphi^{--}, \\ D\varphi^{-+} = \vec{H} \cdot \varphi^{+-} - 2\lambda\varphi^{++}. \end{cases}$$

We fix a point $p \in M$, and consider e_3 a unit vector in E_p so that $\vec{H} = |\vec{H}|e_3$ at p . We complete e_3 by e_4 to get a positively oriented and orthonormal frame of E_p . We first assume that $\varphi^{--}, \varphi^{++}, \varphi^{+-}$ and φ^{-+} do not vanish at p . We see easily that

$$\left\{ e_1 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|} \right\}$$

is an orthonormal frame of Σ^{++} for the real scalar product $\Re \langle \cdot, \cdot \rangle$. Indeed, we have

$$\begin{aligned} \Re \langle e_1 \cdot e_3 \cdot \varphi^{--}, e_2 \cdot e_3 \cdot \varphi^{--} \rangle &= \Re \langle \varphi^{--}, e_3 \cdot e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \rangle \\ &= \Re (i|\varphi^{--}|^2) = 0. \end{aligned}$$

Analogously,

$$\begin{aligned} &\left\{ e_1 \cdot e_3 \cdot \frac{\varphi^{++}}{|\varphi^{++}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{++}}{|\varphi^{++}|} \right\}, \\ &\left\{ e_1 \cdot e_3 \cdot \frac{\varphi^{-+}}{|\varphi^{-+}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{-+}}{|\varphi^{-+}|} \right\}, \\ &\left\{ e_1 \cdot e_3 \cdot \frac{\varphi^{+-}}{|\varphi^{+-}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{+-}}{|\varphi^{+-}|} \right\} \end{aligned}$$

are orthonormal frames of Σ^{--}, Σ^{+-} and Σ^{-+} respectively. We define the following bilinear forms

$$\begin{aligned} F_{++}(X, Y) &= \Re \langle \nabla_X \varphi^{++}, Y \cdot e_3 \cdot \varphi^{--} \rangle, \\ F_{--}(X, Y) &= \Re \langle \nabla_X \varphi^{--}, Y \cdot e_3 \cdot \varphi^{++} \rangle, \\ F_{+-}(X, Y) &= \Re \langle \nabla_X \varphi^{+-}, Y \cdot e_3 \cdot \varphi^{-+} \rangle, \\ F_{-+}(X, Y) &= \Re \langle \nabla_X \varphi^{-+}, Y \cdot e_3 \cdot \varphi^{+-} \rangle, \end{aligned}$$

and

$$\begin{aligned} B_{++}(X, Y) &= -\Re \langle \lambda X \cdot \varphi^{-+}, Y \cdot e_3 \cdot \varphi^{--} \rangle, \\ B_{--}(X, Y) &= -\Re \langle \lambda X \cdot \varphi^{+-}, Y \cdot e_3 \cdot \varphi^{++} \rangle, \\ B_{+-}(X, Y) &= -\Re \langle \lambda X \cdot \varphi^{--}, Y \cdot e_3 \cdot \varphi^{-+} \rangle, \\ B_{-+}(X, Y) &= -\Re \langle \lambda X \cdot \varphi^{++}, Y \cdot e_3 \cdot \varphi^{+-} \rangle. \end{aligned}$$

We have this first lemma:

Lemma 4.1. *We have*

1. $\text{tr}(F_{++}) = -|\vec{H}||\varphi^{--}|^2 + 2\Re \langle \lambda \varphi^{-+}, e_3 \cdot \varphi^{--} \rangle,$
2. $\text{tr}(F_{--}) = -|\vec{H}||\varphi^{++}|^2 + 2\Re \langle \lambda \varphi^{+-}, e_3 \cdot \varphi^{++} \rangle,$
3. $\text{tr}(F_{+-}) = -|\vec{H}||\varphi^{-+}|^2 + 2\Re \langle \lambda \varphi^{--}, e_3 \cdot \varphi^{-+} \rangle,$
4. $\text{tr}(F_{-+}) = -|\vec{H}||\varphi^{+-}|^2 + 2\Re \langle \lambda \varphi^{++}, e_3 \cdot \varphi^{+-} \rangle.$

This second lemma gives the defect of symmetry:

Lemma 4.2. *We have*

1. $F_{++}(e_1, e_2) = F_{++}(e_2, e_1) - 2\Re \langle \lambda \varphi^{-+}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \rangle,$
2. $F_{--}(e_1, e_2) = F_{--}(e_2, e_1) - 2\Re \langle \lambda \varphi^{+-}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{++} \rangle,$
3. $F_{+-}(e_1, e_2) = F_{+-}(e_2, e_1) - 2\Re \langle \lambda \varphi^{--}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{-+} \rangle,$
4. $F_{-+}(e_1, e_2) = F_{-+}(e_2, e_1) - 2\Re \langle \lambda \varphi^{++}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{+-} \rangle.$

For sake of brevity, we only prove Lemma 4.2. The proof of Lemma 4.1 is very similar.

Proof: We have

$$\begin{aligned}
F_{++}(e_1, e_2) &= \Re \langle \nabla_{e_1} \varphi^{++}, e_2 \cdot e_3 \cdot \varphi^{--} \rangle \\
&= \Re \langle e_1 \cdot \nabla_{e_1} \varphi^{++}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \rangle \\
&= \Re \left\langle \vec{H} \cdot \varphi^{--} - 2\lambda \varphi^{-+} - e_2 \cdot \nabla_{e_2} \varphi^{++}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \right\rangle.
\end{aligned}$$

The first term is

$$\begin{aligned}
\Re \left\langle \vec{H} \cdot \varphi^{--}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \right\rangle &= -\Re \left\langle \varphi^{--}, i\vec{H} \cdot e_3 \cdot \varphi^{--} \right\rangle \\
&= -\Re \left(i|\vec{H}| |\varphi^{--}|^2 \right) = 0.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
F_{++}(e_1, e_2) + 2\Re \langle \lambda \varphi^{-+}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \rangle &= -\Re \langle e_2 \cdot \nabla_{e_2} \varphi^{++}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \rangle \\
&= \Re \langle \nabla_{e_2} \varphi^{++}, e_2 \cdot e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \rangle \\
&= \Re \langle \nabla_{e_2} \varphi^{++}, e_1 \cdot e_3 \cdot \varphi^{--} \rangle \\
&= F_{++}(e_2, e_1).
\end{aligned}$$

The proof is similar for the three other forms. \square

By analogous computations, we also get the following lemmas:

Lemma 4.3. *We have*

1. $tr(B_{++}) = -2\Re \langle \lambda \varphi^{-+}, e_3 \cdot \varphi^{--} \rangle,$
2. $tr(B_{--}) = -2\Re \langle \lambda \varphi^{+-}, e_3 \cdot \varphi^{++} \rangle,$
3. $tr(B_{+-}) = -2\Re \langle \lambda \varphi^{--}, e_3 \cdot \varphi^{-+} \rangle,$
4. $tr(B_{-+}) = -2\Re \langle \lambda \varphi^{++}, e_3 \cdot \varphi^{+-} \rangle.$

Lemma 4.4. *We have*

1. $B_{++}(e_1, e_2) = B_{++}(e_2, e_1) + 2\Re \langle \lambda \varphi^{-+}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \rangle,$
2. $B_{--}(e_1, e_2) = B_{--}(e_2, e_1) + 2\Re \langle \lambda \varphi^{+-}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{++} \rangle,$
3. $B_{+-}(e_1, e_2) = B_{+-}(e_2, e_1) + 2\Re \langle \lambda \varphi^{--}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{-+} \rangle,$
4. $B_{-+}(e_1, e_2) = B_{-+}(e_2, e_1) + 2\Re \langle \lambda \varphi^{++}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{+-} \rangle.$

Now, we set

$$\begin{cases} A_{++} := F_{++} + B_{++}, \\ A_{--} := F_{--} + B_{--}, \\ A_{+-} := F_{+-} + B_{+-}, \\ A_{-+} := F_{-+} + B_{-+}, \end{cases}$$

and

$$F_+ = \frac{A_{++}}{|\varphi^{--}|^2} - \frac{A_{--}}{|\varphi^{++}|^2} \quad \text{and} \quad F_- = \frac{A_{+-}}{|\varphi^{-+}|^2} - \frac{A_{-+}}{|\varphi^{+-}|^2}.$$

From the last four lemmas we deduce immediately that F_+ and F_- are symmetric and trace-free. Moreover, by a direct computation using the conditions (4) on the norms of φ^+ and φ^- , we get the following lemma:

Lemma 4.5. *The symmetric operators F^+ and F^- of TM associated to the bilinear forms F_+ and F_- , defined by*

$$F^+(X) = F_+(X, e_1)e_1 + F_+(X, e_2)e_2 \quad \text{and} \quad F^-(X) = F_-(X, e_1)e_1 + F_-(X, e_2)e_2$$

for all $X \in TM$, satisfy

1. $\Re \langle F^+(X) \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle = 0,$
2. $\Re \langle F^-(X) \cdot e_3 \cdot \varphi^{-+}, \varphi^{+-} \rangle = 0.$

Proof. Since

$$A_{++}(X, Y) = \Re \langle \nabla_X \varphi^{++} - \lambda X \cdot \varphi^{-+}, Y \cdot e_3 \cdot \varphi^{--} \rangle,$$

and since $(e_1 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|})$ is an orthonormal frame of Σ^{++} , we have

$$\begin{aligned} & \Re \langle \nabla_X \varphi^{++} - \lambda X \cdot \varphi^{-+}, \varphi^{++} \rangle \\ = & \frac{A_{++}}{|\varphi^{--}|^2}(X, e_1) \Re \langle e_1 \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle + \frac{A_{++}}{|\varphi^{--}|^2}(X, e_2) \Re \langle e_2 \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle. \end{aligned}$$

Similarly,

$$\begin{aligned} & \Re \langle \nabla_X \varphi^{--} - \lambda X \cdot \varphi^{+-}, \varphi^{--} \rangle \\ = & \frac{A_{--}}{|\varphi^{++}|^2}(X, e_1) \Re \langle e_1 \cdot e_3 \cdot \varphi^{++}, \varphi^{--} \rangle + \frac{A_{--}}{|\varphi^{++}|^2}(X, e_2) \Re \langle e_2 \cdot e_3 \cdot \varphi^{++}, \varphi^{--} \rangle \\ = & -\frac{A_{--}}{|\varphi^{++}|^2}(X, e_1) \Re \langle e_1 \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle - \frac{A_{--}}{|\varphi^{++}|^2}(X, e_2) \Re \langle e_2 \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle. \end{aligned}$$

These two formulas imply that

$$\Re \langle F^+(X) \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle = \Re \langle \nabla_X \varphi^+ - \lambda X \cdot \varphi^-, \varphi^+ \rangle;$$

by the first condition in (4), this last expression is zero. \square

Hence, the operators F^+ and F^- are of rank at most ≤ 1 . Since they are symmetric and trace-free, they vanish identically.

Using again that $(e_1 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|})$ is an orthonormal frame of Σ^{++} , we have

$$\nabla_X \varphi^{++} = F_{++}(X, e_1) e_1 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|} + F_{++}(X, e_2) e_2 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|}.$$

Since $F_{++} = A_{++} - B_{++}$ and denoting by A^{++} and B^{++} the symmetric operators of TM associated to A_{++} and B_{++} and defined by

$$A^{++}(X) = A_{++}(X, e_1) e_1 + A_{++}(X, e_2) e_2, \quad B^{++}(X) = B_{++}(X, e_1) e_1 + B_{++}(X, e_2) e_2,$$

we get

$$\nabla_X \varphi^{++} = \frac{1}{|\varphi^{--}|^2} [A^{++}(X) \cdot e_3 \cdot \varphi^{--} - B^{++}(X) \cdot e_3 \cdot \varphi^{--}]. \quad (11)$$

Similarly, if A^{--} and B^{--} denote the symmetric operators of TM associated to A_{--} and B_{--} , we have

$$\nabla_X \varphi^{--} = \frac{1}{|\varphi^{++}|^2} [A^{--}(X) \cdot e_3 \cdot \varphi^{++} - B^{--}(X) \cdot e_3 \cdot \varphi^{++}]. \quad (12)$$

Moreover, we easily get

$$B^{++}(X) \cdot e_3 \cdot \varphi^{--} = -|\varphi^{--}|^2 \lambda X \cdot \varphi^{-+} \quad \text{and} \quad B^{--}(X) \cdot e_3 \cdot \varphi^{++} = -|\varphi^{++}|^2 \lambda X \cdot \varphi^{+-}.$$

Thus

$$\begin{aligned} \nabla_X \varphi^+ &= \frac{1}{|\varphi^{--}|^2} A^{++}(X) \cdot e_3 \cdot \varphi^{--} + \lambda X \cdot \varphi^{-+} \\ &\quad + \frac{1}{|\varphi^{++}|^2} A^{--}(X) \cdot e_3 \cdot \varphi^{++} + \lambda X \cdot \varphi^{+-}. \end{aligned}$$

Setting $A^+ = A^{++} + A^{--}$ we get from the definition of A^{++} and A^{--} and from $F^+ = 0$ that $\frac{A^+}{|\varphi^+|^2} = \frac{A^{++}}{|\varphi^{++}|^2}$. Bearing in mind that $|\varphi^+|^2 = |\varphi^{++}|^2 + |\varphi^{--}|^2$, we get finally

$$\frac{A^+}{|\varphi^+|^2} = \frac{A^{++}}{|\varphi^{--}|^2} = \frac{A^{--}}{|\varphi^{++}|^2}. \quad (13)$$

Thus

$$\nabla_X \varphi^+ = \frac{1}{|\varphi^+|^2} A^+(X) \cdot e_3 \cdot \varphi^+ + \lambda X \cdot \varphi^-. \quad (14)$$

Similarly, denoting by A^{+-} and A^{-+} the symmetric operators of TM associated to A_{+-} and A_{-+} , setting $A^- = A^{+-} + A^{-+}$ and using $F^- = 0$ we get

$$\begin{aligned} \nabla_X \varphi^- &= \frac{1}{|\varphi^{+-}|^2} A^{-+}(X) \cdot e_3 \cdot \varphi^{+-} + \lambda X \cdot \varphi^{++} \\ &\quad + \frac{1}{|\varphi^{-+}|^2} A^{+-}(X) \cdot e_3 \cdot \varphi^{-+} + \lambda X \cdot \varphi^{--} \\ &= \frac{1}{|\varphi^-|^2} A^-(X) \cdot e_3 \cdot \varphi^- + \lambda X \cdot \varphi^+. \end{aligned} \quad (15)$$

We now observe that formulas (14) and (15) also hold if φ^{++} or φ^{--} , (resp. φ^{+-} or φ^{-+}) vanishes at p : indeed, assuming for instance that $\varphi^{++}(p) = 0$,

and thus that $\varphi^{--}(p) \neq 0$ since $\varphi^+(p) \neq 0$, equation (11) holds, and, from the first condition in (4),

$$\Re \langle \nabla_X \varphi^{--} - \lambda X \cdot \varphi^{+-}, \varphi^{--} \rangle = 0.$$

Since $\left(\frac{\varphi^{--}}{|\varphi^{--}|}, i \frac{\varphi^{--}}{|\varphi^{--}|} \right)$ is an orthonormal basis of Σ^{--} , we deduce that

$$\nabla_X \varphi^{--} - \lambda X \cdot \varphi^{+-} = i\alpha(X) \frac{\varphi^{--}}{|\varphi^{--}|}$$

for some real 1-form α . Since $D\varphi^{--} + 2\lambda\varphi^{+-} = 0$ ($\varphi^{++} = 0$ at p), this implies that

$$(\alpha(e_1)e_1 + \alpha(e_2)e_2) \cdot \frac{\varphi^{--}}{|\varphi^{--}|} = 0,$$

and thus that $\alpha = 0$. We thus get $\nabla_X \varphi^{--} = \lambda X \cdot \varphi^{+-}$ instead of (12), which, together with (11), easily implies (14).

Now, we set

$$\eta^+(X) = \left(\frac{1}{|\varphi^+|^2} A^+(X) \cdot e_3 \right)^+ \quad \text{and} \quad \eta^-(X) = \left(\frac{1}{|\varphi^-|^2} A^-(X) \cdot e_3 \right)^-$$

where, if σ belongs to $\mathcal{C}l^0(TM \oplus E)$, we denote by $\sigma^+ := \frac{1+\omega_4}{2} \cdot \sigma$ and by $\sigma^- := \frac{1-\omega_4}{2} \cdot \sigma$ the parts of σ acting on Σ^+ and on Σ^- only, i.e., such that

$$\sigma^+ \cdot \varphi = \sigma \cdot \varphi^+ \in \Sigma^+ \quad \text{and} \quad \sigma^- \cdot \varphi = \sigma \cdot \varphi^- \in \Sigma^-.$$

Setting $\eta = \eta^+ + \eta^-$ we thus get

$$\nabla_X \varphi = \eta(X) \cdot \varphi + \lambda X \cdot \varphi,$$

as claimed in Lemma 3.2.

Explicitely, setting $A_+(X, Y) := \langle A^+(X), Y \rangle$ and $A_-(X, Y) := \langle A^-(X), Y \rangle$, the form η is given by

$$\begin{aligned} \eta(X) &= \frac{1}{2|\varphi^+|^2} [A_+(X, e_1)(e_1 \cdot e_3 - e_2 \cdot e_4) + A_+(X, e_2)(e_2 \cdot e_3 + e_1 \cdot e_4)] \\ &\quad + \frac{1}{2|\varphi^-|^2} [A_-(X, e_1)(e_1 \cdot e_3 + e_2 \cdot e_4) + A_-(X, e_2)(e_2 \cdot e_3 - e_1 \cdot e_4)] \end{aligned}$$

with

$$A_+(X, Y) = \Re \langle \nabla_X \varphi^+ - \lambda X \cdot \varphi^-, Y \cdot e_3 \cdot \varphi^+ \rangle$$

and

$$A_-(X, Y) = \Re \langle \nabla_X \varphi^- - \lambda X \cdot \varphi^+, Y \cdot e_3 \cdot \varphi^- \rangle.$$

By direct computations we get that

$$B(X, Y) := X \cdot \eta(Y) - \eta(Y) \cdot X$$

is a vector belonging to E which is such that

$$\begin{aligned} \langle B(X, Y), \xi \rangle &= \frac{1}{|\varphi^+|^2} \Re \langle X \cdot \nabla_Y \varphi^+ - \lambda X \cdot Y \cdot \varphi^-, \xi \cdot \varphi^+ \rangle \\ &\quad + \frac{1}{|\varphi^-|^2} \Re \langle X \cdot \nabla_Y \varphi^- - \lambda X \cdot Y \cdot \varphi^+, \xi \cdot \varphi^- \rangle \end{aligned}$$

for all $\xi \in E$. This last expression appears to be symmetric in X, Y (the proof is analogous to the proof of the symmetry of $A_{++} = F_{++} + B_{++}$ above). Computing

$$\langle B(X, Y), \xi \rangle = \frac{1}{2} (\langle B(X, Y), \xi \rangle + \langle B(Y, X), \xi \rangle)$$

we finally obtain that B is given by formula (6).

Since $B(e_j, X) = e_j \cdot \eta(X) - \eta(X) \cdot e_j$, we obtain

$$\sum_{j=1,2} e_j \cdot B(e_j, X) = -2\eta(X) - \sum_{j=1,2} e_j \cdot \eta(X) \cdot e_j.$$

Writing $\eta(X)$ in the form $\sum_k e_k \cdot n_k$ for some vectors n_k belonging to E , we easily get that $\sum_j e_j \cdot \eta(X) \cdot e_j = 0$. Thus

$$\eta(X) = -\frac{1}{2} \sum_{j=1,2} e_j \cdot B(e_j, X).$$

The last claim in Lemma 3.2 is proved. \square

5 Weierstrass representation of surfaces in \mathbb{R}^4

We are interested here in isometric immersions in euclidean space \mathbb{R}^4 (thus $c = \lambda = 0$); we obtain that the immersions are given by a formula which generalizes the representation formula given by T. Friedrich in [5]. Such a formula was also found in [4] using a different method involving twistor theory.

We consider the scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ defined on Σ^+ by

$$\begin{aligned} \langle\langle \cdot, \cdot \rangle\rangle : \Sigma^+ \times \Sigma^+ &\rightarrow \mathbb{H} \\ (\varphi^+, \psi^+) &\mapsto \overline{[\psi^+]}[\varphi^+], \end{aligned}$$

where $[\varphi^+]$ and $[\psi^+] \in \mathbb{H}$ represent the spinors φ^+ and ψ^+ in some frame, and where, if $q = q_1 I + q_2 I + q_3 J + q_4 K$ belongs to \mathbb{H} ,

$$\bar{q} = q_1 1 - q_2 I - q_3 J - q_4 K.$$

We also define the product $\langle\langle \cdot, \cdot \rangle\rangle$ on Σ^- by an analogous formula:

$$\begin{aligned} \langle\langle \cdot, \cdot \rangle\rangle : \Sigma^- \times \Sigma^- &\rightarrow \mathbb{H} \\ (\varphi^-, \psi^-) &\mapsto \overline{[\psi^-]}[\varphi^-]. \end{aligned}$$

The following properties hold: for all $\varphi, \psi \in \Sigma$ and all $X \in TM \oplus E$,

$$\langle\langle \varphi^+, \psi^+ \rangle\rangle = \overline{\langle\langle \psi^+, \varphi^+ \rangle\rangle}, \quad \langle\langle \varphi^-, \psi^- \rangle\rangle = \overline{\langle\langle \psi^-, \varphi^- \rangle\rangle} \quad (16)$$

and

$$\langle\langle X \cdot \varphi^+, \psi^- \rangle\rangle = -\langle\langle \varphi^+, X \cdot \psi^- \rangle\rangle. \quad (17)$$

Assume that we have a spinor φ solution of the Dirac equation $D\varphi = \vec{H} \cdot \varphi$ so that $|\varphi^+| = |\varphi^-| = 1$, and define the \mathbb{H} -valued 1-form ξ by

$$\xi(X) = \langle\langle X \cdot \varphi^-, \varphi^+ \rangle\rangle \in \mathbb{H}.$$

Proposition 5.1. *The form $\xi \in \Omega^1(M, \mathbb{H})$ is closed.*

Proof: By a straightforward computation, we get

$$d\xi(e_1, e_2) = \langle \langle e_2 \cdot \nabla_{e_1} \varphi^-, \varphi^+ \rangle \rangle - \langle \langle e_1 \cdot \nabla_{e_2} \varphi^-, \varphi^+ \rangle \rangle + \langle \langle e_2 \cdot \varphi^-, \nabla_{e_1} \varphi^+ \rangle \rangle - \langle \langle e_1 \cdot \varphi^-, \nabla_{e_2} \varphi^+ \rangle \rangle.$$

First observe that

$$\begin{aligned} \langle \langle e_2 \cdot \nabla_{e_1} \varphi^-, \varphi^+ \rangle \rangle - \langle \langle e_1 \cdot \nabla_{e_2} \varphi^-, \varphi^+ \rangle \rangle &= -\langle \langle e_1 \cdot \nabla_{e_1} \varphi^-, e_1 \cdot e_2 \cdot \varphi^+ \rangle \rangle + \langle \langle e_2 \cdot \nabla_{e_2} \varphi^-, e_2 \cdot e_1 \cdot \varphi^+ \rangle \rangle \\ &= -\langle \langle D\varphi^-, e_1 \cdot e_2 \cdot \varphi^+ \rangle \rangle \end{aligned}$$

and similarly that

$$\begin{aligned} \langle \langle e_2 \cdot \varphi^-, \nabla_{e_1} \varphi^+ \rangle \rangle - \langle \langle e_1 \cdot \varphi^-, \nabla_{e_2} \varphi^+ \rangle \rangle &= \langle \langle e_1 \cdot e_2 \cdot \varphi^-, e_1 \cdot \nabla_{e_1} \varphi^+ \rangle \rangle - \langle \langle e_2 \cdot e_1 \cdot \varphi^-, e_2 \cdot \nabla_{e_2} \varphi^+ \rangle \rangle \\ &= \langle \langle e_1 \cdot e_2 \cdot \varphi^-, D\varphi^+ \rangle \rangle. \end{aligned}$$

Thus

$$d\xi(e_1, e_2) = \langle \langle e_1 \cdot e_2 \cdot D\varphi^-, \varphi^+ \rangle \rangle + \langle \langle e_1 \cdot e_2 \cdot \varphi^-, D\varphi^+ \rangle \rangle.$$

Since $D\varphi = \vec{H} \cdot \varphi$, then $D\varphi^+ = \vec{H} \cdot \varphi^+$ and $D\varphi^- = \vec{H} \cdot \varphi^-$, which implies

$$d\xi(e_1, e_2) = \langle \langle (e_1 \cdot e_2 \cdot \vec{H} - \vec{H} \cdot e_1 \cdot e_2) \cdot \varphi^-, \varphi^+ \rangle \rangle = 0.$$

□

Assuming that M is simply connected, there exists a function $F : M \rightarrow \mathbb{H}$ so that $dF = \xi$. We now identify \mathbb{H} to \mathbb{R}^4 in the natural way.

Theorem 2. 1. *The map $F = (F_1, F_2, F_3, F_4) : M \rightarrow \mathbb{R}^4$ is an isometry.*

2. *The map*

$$\begin{aligned} \Phi_E : E &\rightarrow M \times \mathbb{R}^4 \\ X \in E_m &\mapsto (F(m), \xi_1(X), \xi_2(X), \xi_3(X), \xi_4(X)) \end{aligned}$$

is an isometry between E and the normal bundle $N(F(M))$ of $F(M)$ into \mathbb{R}^4 , preserving connections and second fundamental forms.

Proof. Note first that the euclidean norm of $\xi \in \mathbb{R}^4 \simeq \mathbb{H}$ is

$$|\xi|^2 = \langle \xi, \xi \rangle = \bar{\xi} \xi \in \mathbb{R},$$

and more generally that the real scalar product $\langle \xi, \xi' \rangle$ of $\xi, \xi' \in \mathbb{R}^4 \simeq \mathbb{H}$ is the component of 1 in $\langle \langle \xi, \xi' \rangle \rangle = \bar{\xi}' \xi \in \mathbb{H}$. We first compute, for all X, Y belonging to $E \cup TM$,

$$\begin{aligned} \overline{\xi(Y)} \xi(X) &= \overline{\langle \langle Y \cdot \varphi^-, \varphi^+ \rangle \rangle} \langle \langle X \cdot \varphi^-, \varphi^+ \rangle \rangle = \overline{([\varphi^+][Y \cdot \varphi^-])} ([\varphi^+][X \cdot \varphi^-]) \\ &= \overline{[Y \cdot \varphi^-]} [X \cdot \varphi^-] \end{aligned}$$

since $[\varphi^+][\overline{[\varphi^+]}] = 1$ ($|\varphi^+| = 1$). Here and below the brackets $[\cdot]$ stand for the components ($\in \mathbb{H}$) of the spinor fields in some local frame. Thus

$$\overline{\xi(Y)} \xi(X) = \langle \langle X \cdot \varphi^-, Y \cdot \varphi^- \rangle \rangle, \quad (18)$$

which in particular implies (considering the components of I of these quaternions)

$$\langle \xi(X), \xi(Y) \rangle = \Re \langle X \cdot \varphi^-, Y \cdot \varphi^- \rangle. \quad (19)$$

This last identity easily gives

$$\langle \xi(X), \xi(Y) \rangle = 0 \quad \text{and} \quad |\xi(Z)|^2 = |Z|^2 \quad (20)$$

for all $X \in TM$, $Y \in E$ and $Z \in E \cup TM$. Thus $F = \int \xi$ is an isometry, and ξ maps isometrically the bundle E into the normal bundle of $F(M)$ in \mathbb{R}^4 .

We now prove that ξ preserves the normal connection and the second fundamental form: let $X \in TM$ and $Y \in \Gamma(E) \cup \Gamma(TM)$; then $\xi(Y)$ is a vector field normal or tangent to $F(M)$. Considering $\xi(Y)$ as a map $M \rightarrow \mathbb{R}^4 \simeq \mathbb{H}$, we have

$$\begin{aligned} d(\xi(Y))(X) &= d\langle Y \cdot \varphi^-, \varphi^+ \rangle(X) \\ &= \langle \nabla_X Y \cdot \varphi^-, \varphi^+ \rangle + \langle Y \cdot \nabla_X \varphi^-, \varphi^+ \rangle + \langle Y \cdot \varphi^-, \nabla_X \varphi^+ \rangle \end{aligned} \quad (21)$$

where the connection $\nabla_X Y$ denotes the connection on E (if $Y \in \Gamma(E)$) or the Levi-Civita connection on TM (if $Y \in \Gamma(TM)$). We will need the following formulas:

Lemma 5.2. *We have*

$$\begin{aligned} \langle \langle \nabla_X Y \cdot \varphi^-, \varphi^+ \rangle, \xi(\nu) \rangle &= \Re \langle \nabla_X Y \cdot \varphi^-, \nu \cdot \varphi^- \rangle \\ &= \Re \langle \nabla_X Y \cdot \varphi^+, \nu \cdot \varphi^+ \rangle, \end{aligned}$$

$$\langle \langle Y \cdot \nabla_X \varphi^-, \varphi^+ \rangle, \xi(\nu) \rangle = \Re \langle Y \cdot \nabla_X \varphi^-, \nu \cdot \varphi^- \rangle$$

and

$$\langle \langle Y \cdot \varphi^-, \nabla_X \varphi^+ \rangle, \xi(\nu) \rangle = \Re \langle Y \cdot \nabla_X \varphi^+, \nu \cdot \varphi^+ \rangle.$$

In the expressions above, $\langle \cdot, \cdot \rangle$ defined on \mathbb{H} for the left-hand side and $\Re \langle \cdot, \cdot \rangle$ defined on Σ for the right-hand side of each identity, stand for the natural real scalar products.

Proof. The first identity is a consequence of (19) and the second identity may be obtained by a very similar computation observing that, by (16)-(17),

$$\langle \langle \nabla_X Y \cdot \varphi^-, \varphi^+ \rangle, \xi(\nu) \rangle = \langle \langle \nabla_X Y \cdot \varphi^+, \varphi^- \rangle, \langle \nu \cdot \varphi^+, \varphi^- \rangle \rangle.$$

The last two identities may be obtained by very similar computations. \square

From (21) and the lemma, we readily get the formula

$$\langle d(\xi(Y))(X), \xi(\nu) \rangle = \frac{1}{2} \Re \langle \nabla_X Y \cdot \varphi, \nu \cdot \varphi \rangle + \Re \langle Y \cdot \nabla_X \varphi, \nu \cdot \varphi \rangle. \quad (22)$$

We first suppose that $X, Y \in \Gamma(TM)$. The first term in the right-hand side of the equation above vanishes in that case since $\nabla_X Y \in \Gamma(TM)$, $\nu \in \Gamma(E)$. Recalling (7), we get then

$$\langle d(\xi(Y))(X), \xi(\nu) \rangle = \Re \langle Y \cdot \nabla_X \varphi, \nu \cdot \varphi \rangle = \langle B(X, Y), \nu \rangle = \langle \xi(B(X, Y)), \xi(\nu) \rangle.$$

Hence the component of $d(\xi(Y))(X)$ normal to $F(M)$ is given by

$$(d(\xi(Y))(X))^N = \xi(B(X, Y)). \quad (23)$$

We now suppose that $X \in \Gamma(TM)$ and $Y \in \Gamma(E)$. We first observe that the second term in the right-hand side of equation (22) vanishes. Indeed, if (e_3, e_4) stands for an orthonormal basis of E , for all $i, j \in \{3, 4\}$ we have

$$\Re \langle e_i \cdot \nabla_X \varphi, e_j \cdot \varphi \rangle = -\Re \langle \nabla_X \varphi, e_i \cdot e_j \cdot \varphi \rangle = -\Re \langle \eta(X) \cdot \varphi, e_i \cdot e_j \cdot \varphi \rangle,$$

which is a sum of terms of the form $\Re \langle e \cdot \varphi, e' \cdot \varphi \rangle$ with e and e' belonging to TM and E respectively; these terms are therefore all equal to zero. Thus, (22) simplifies to

$$\langle d(\xi(Y))(X), \xi(\nu) \rangle = \frac{1}{2} \Re \langle \nabla_X Y \cdot \varphi, \nu \cdot \varphi \rangle = \langle \xi(\nabla_X Y), \xi(\nu) \rangle.$$

Hence

$$(d(\xi(Y))(X))^N = \xi(\nabla_X Y). \quad (24)$$

Equations (23) and (24) mean that $\Phi_E = \xi$ preserves the second fundamental form and the normal connection respectively. \square

Remark 3. *The immersion $F : M \rightarrow \mathbb{R}^4$ given by the fundamental theorem is thus*

$$F = \int \xi = \left(\int \xi_1, \int \xi_2, \int \xi_3, \int \xi_4 \right).$$

This formula generalizes the classical Weierstrass representation: let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the \mathbb{C} -linear forms defined by

$$\alpha_k(X) = \xi_k(X) - i\xi_k(JX),$$

for $k = 1, 2, 3, 4$, where J is the natural complex structure of M . Let z be a conformal parameter of M , and let $\psi_1, \psi_2, \psi_3, \psi_4 : M \rightarrow \mathbb{C}$ be such that

$$\alpha_1 = \psi_1 dz, \quad \alpha_2 = \psi_2 dz, \quad \alpha_3 = \psi_3 dz, \quad \alpha_4 = \psi_4 dz.$$

By an easy computation using $D\varphi = \vec{H} \cdot \varphi$, we see that $\alpha_1, \alpha_2, \alpha_3$ and α_4 are holomorphic forms if and only if M is a minimal surface ($\vec{H} = \vec{0}$). Then if M is minimal,

$$\begin{aligned} F &= \operatorname{Re} \left(\int \alpha_1, \int \alpha_2, \int \alpha_3, \int \alpha_4 \right) \\ &= \operatorname{Re} \left(\int \psi_1 dz, \int \psi_2 dz, \int \psi_3 dz, \int \psi_4 dz \right) \end{aligned}$$

where $\psi_1, \psi_2, \psi_3, \psi_4$ are holomorphic functions. This is the Weierstrass representation of minimal surfaces.

Remark 4. *Theorem 2 also gives a spinorial proof of the fundamental theorem. We may integrate the Gauss, Ricci and Codazzi equations in two steps:*

1- first solving

$$\nabla_X \varphi = \eta(X) \cdot \varphi, \quad (25)$$

where

$$\eta(X) = -\frac{1}{2} \sum_{j=1,2} e_j \cdot B(e_j, X)$$

(there is a unique solution in $\Gamma(\Sigma)$, up to the natural right-action of $Spin(4)$);

2- then solving

$$dF = \xi$$

where $\xi(X) = \langle \langle X \cdot \varphi^-, \varphi^+ \rangle \rangle$ (the solution is unique, up to translations).

Indeed, equation (25) is solvable, since its conditions of integrability are exactly the Gauss, Ricci and Codazzi equations; see the proof of Theorem 1. Moreover, the multiplication of φ on the right by a constant belonging to $Spin(4)$ in the first step, and the addition to F of a constant belonging to \mathbb{R}^4 in the second step, correspond to a rigid motion in \mathbb{R}^4 .

6 Surfaces in \mathbb{R}^3 and S^3 .

The aim of this section is to obtain as particular cases the spinor characterizations of T. Friedrich [5] and B. Morel [13] of surfaces in \mathbb{R}^3 and S^3 . Assume that $M^2 \subset \mathcal{H}^3 \subset \mathbb{R}^4$, where \mathcal{H}^3 is a hyperplane, or a sphere of \mathbb{R}^4 . Let N be a unit vector field such that

$$T\mathcal{H} = TM \oplus_{\perp} \mathbb{R}N.$$

The intrinsic spinors of M identify with the spinors of \mathcal{H} restricted to M , which in turn identify with the positive spinors of \mathbb{R}^4 restricted to M :

Proposition 6.1. *There is an identification*

$$\begin{aligned} \Sigma M &\xrightarrow{\sim} \Sigma_M^+ \\ \psi &\mapsto \psi^* \end{aligned}$$

such that

$$(\nabla\psi)^* = \nabla(\psi^*)$$

and such that the Clifford actions are linked by

$$(X \cdot_M \psi)^* = N \cdot X \cdot \psi^*$$

for all $X \in TM$ and all $\psi \in \Sigma M$.

Using this identification, the intrinsic Dirac operator on M defined by

$$D_M\psi := e_1 \cdot_M \nabla_{e_1}\psi + e_2 \cdot_M \nabla_{e_2}\psi$$

is linked to D by

$$(D_M\psi)^* = N \cdot D\psi^*.$$

If $\varphi \in \Gamma(\Sigma)$ is a solution of

$$D\varphi = \vec{H} \cdot \varphi \quad \text{and} \quad |\varphi^+| = |\varphi^-| = 1$$

then $\varphi^+ \in \Sigma^+$ may be considered as belonging to ΣM ; it satisfies

$$D_M\varphi^+ = N \cdot D\varphi^+ = N \cdot \vec{H} \cdot \varphi^+. \quad (26)$$

We examine separately the case of a surface in a hyperplane, and in a 3-dimensional sphere:

1. If \mathcal{H} is a hyperplane, then \vec{H} is of the form HN , and (26) reads

$$D_M\varphi^+ = -H\varphi^+. \quad (27)$$

This is the equation considered by T. Friedrich in [5].

2. If $\mathcal{H} = S^3$, then \vec{H} is of the form $HN - \nu$, where ν is the outer unit normal of S^3 , and (26) reads

$$D_M\varphi^+ = -H\varphi^+ - i\overline{\varphi^+}. \quad (28)$$

This equation is obtained by B. Morel in [13].

Conversely, we now suppose that ψ is an intrinsic spinor field on M solution of (27) or (28). The aim is to construct a spinor field φ in dimension 4 which induces an immersion in a hyperplane, or in a 3-sphere. Define $E = M \times \mathbb{R}^2$, with its natural metric $\langle \cdot, \cdot \rangle$ and its trivial connection ∇' , and consider $\nu, N \in \Gamma(E)$ such that

$$|\nu| = |N| = 1, \quad \langle \nu, N \rangle = 0 \quad \text{and} \quad \nabla'\nu = \nabla'N = 0.$$

We first consider the case of an hyperplane:

Proposition 6.2. *Let $\psi \in \Gamma(\Sigma M)$ be a solution of*

$$D_M\psi = -H\psi$$

of constant length $|\psi| = 1$. There exists $\varphi \in \Gamma(\Sigma)$ solution of

$$D\varphi = \vec{H} \cdot \varphi \quad \text{and} \quad |\varphi^+| = |\varphi^-| = 1, \quad (29)$$

with $\vec{H} = HN$, such that

$$\varphi^+ = \psi$$

and the normal vector field

$$\xi(\nu) = \langle \langle \nu \cdot \varphi^-, \varphi^+ \rangle \rangle$$

has a fixed direction in \mathbb{H} . In particular, the immersion given by φ belongs to the hyperplane $\xi(\nu)^\perp$ of \mathbb{H} . The spinor field φ is unique, up to the natural right-action of S^3 on φ^- .

Proof: define $\varphi = (\varphi^+, \varphi^-)$ by

$$\varphi^+ = \psi, \quad \varphi^- = -\nu \cdot \psi.$$

We compute:

$$D\varphi^- = \nu \cdot D\varphi^+ = \nu \cdot \vec{H} \cdot \varphi^+ = \vec{H} \cdot \varphi^-,$$

$$\xi(\nu) = \langle \langle \nu \cdot \varphi^-, \varphi^+ \rangle \rangle = 1,$$

and, for all $X \in TM$,

$$\xi(X) = \langle \langle X \cdot \varphi^-, \varphi^+ \rangle \rangle = -\langle \langle X \cdot \nu \cdot \psi, \psi \rangle \rangle = \langle \langle \psi, X \cdot \nu \cdot \psi \rangle \rangle = \overline{\langle \langle X \cdot \nu \cdot \psi, \psi \rangle \rangle} = -\overline{\xi(X)},$$

that is $\xi(X) \in \Im m(\mathbb{H})$, the hyperplane of pure imaginary quaternions. Thus $F = \int \xi$ also belongs to the hyperplane $\Im m(\mathbb{H})$. Uniqueness is straightforward. \square

We now consider the case of the 3-sphere:

Proposition 6.3. *Let $\psi \in \Gamma(\Sigma M)$ be a solution of*

$$D_M \psi = -H\psi - i\bar{\psi}$$

of constant length $|\psi| = 1$. There exists $\varphi \in \Gamma(\Sigma)$ solution of

$$D\varphi = \vec{H} \cdot \varphi \quad \text{and} \quad |\varphi^+| = |\varphi^-| = 1, \quad (30)$$

with $\vec{H} = HN - \nu$, such that

$$\varphi^+ = \psi$$

and the immersion F defined by φ is given by the unit normal vector field $\xi(\nu)$:

$$F = \xi(\nu) = \langle \langle \nu \cdot \varphi^-, \varphi^+ \rangle \rangle.$$

In particular $F(M)$ belongs to the sphere $S^3 \subset \mathbb{H}$. The spinor field φ is unique, up to the natural right-action of S^3 on φ^- .

Proof: The system

$$\begin{cases} F = \langle \langle \nu \cdot \varphi^-, \varphi^+ \rangle \rangle \\ dF(X) = \langle \langle X \cdot \varphi^-, \varphi^+ \rangle \rangle \end{cases}$$

is equivalent to

$$\varphi^- = -\nu \cdot \varphi^+ \cdot F$$

where $F : M \rightarrow \mathbb{H}$ solves the equation

$$dF(X) = \beta(X)F \quad (31)$$

in \mathbb{H} , with

$$\beta(X) = -\langle \langle X \cdot \nu \cdot \varphi^+, \varphi^+ \rangle \rangle.$$

By a direct computation, the compatibility equation

$$d\beta(X, Y) = \beta(X)\beta(Y) - \beta(Y)\beta(X)$$

of (31) is satisfied, and equation (31) is solvable. Uniqueness is straightforward. \square

Remark 5. *Let M be a minimal surface in S^3 and N be such that*

$$TM \oplus_{\perp} \mathbb{R}N = TS^3.$$

For any $x \in S^3$, denote by $\vec{x} = \overrightarrow{0x}$ the position vector of x . At $x \in M$, $\vec{H} = -\vec{x}$. Thus, $M \subset S^3$ is represented by a solution $\varphi \in \Gamma(\Sigma)$ of

$$D\varphi = -\vec{x} \cdot \varphi.$$

The spinor field

$$\tilde{\varphi} := (\varphi^+, N \cdot \varphi^+)$$

defines a surface of constant mean curvature $H = -1$ in $\Im m(\mathbb{H}) \simeq \mathbb{R}^3$. This is a classical transformation, described by B. Lawson in [12], and by T. Friedrich using spinors in dimension 3 in [5].

Remark 6. In [13] B. Morel also obtained a spinor characterization of an isometric immersion of a surface in three-dimensional hyperbolic space \mathbb{H}^3 . Very similarly to the case of a surface in \mathbb{R}^3 or in S^3 described above, such a characterization may be naturally obtained from a spinor characterization of a spacelike surface in four-dimensional Minkowski space; see [3].

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