Perfect Conductor and Impedance Boundary Condition Corrections via a Finite Element Subproblem Method

Patrick Dular, Victor Péron, Ronan Perrussel, Laurent Krähenbühl, Christophe Geuzaine

Perfect Conductor and Impedance Boundary Condition Corrections via a Finite Element Subproblem Method

Patrick Dular[1,2], Victor Péron[3], Ronan Perrussel[4], Laurent Krähenbühl[5], Christophe Geuzaine[1]

[1] University of Liège, Dept. of Electrical Engineering and Computer Science, ACE, B-4000 Liège, Belgium

[2] F.R.S.-FNRS, Fonds de la Recherche Scientifique, Belgium

[3] Université de Pau et des Pays de l'Adour, MAGIQUE3D (CNRS UMR5142-INRIA Bordeaux-Sud-Ouest), F-64013 Pau, France

[4] Université de Toulouse, LAPLACE (CNRS UMR5213), F-31400 Toulouse, France

[5] Université de Lyon, Ampère (CNRS UMR5005), École Centrale de Lyon, F-69134 Écully Cedex, France

Abstract— A finite element subproblem method is developed to correct the inaccuracies proper to perfect conductor and impedance boundary condition models, in particular near edges and corners of conductors, for a large range of conductivities and frequencies. Local corrections, supported by fine local meshes, can be obtained from each model to a more accurate one, allowing efficient extensions of their domains of validity.

I. INTRODUCTION

Magnetodynamic finite element (FE) modeling of conductors can be tackled at various levels of precision, e.g. considering them via perfect conductor or impedance boundary conditions (BCs). Avoiding to mesh their interior allows to lighten the computational efforts, which is interesting for the preliminary stage of a design. Perfect conductor BCs are suitable for high conductivities or frequencies, i.e. for low skin depths [1]. For larger skin depths, impedance BCs (IBCs) lead to a better accuracy. Such conditions are nevertheless generally based on analytical solutions of ideal problems and are therefore only valid in practice far from any geometrical discontinuities, e.g., edges and corners. Local modifications of IBCs can be defined (e.g. [2]).

It is here proposed to perform successive FE refinements via a subproblem (SP) method (SPM) [3] to correct the models with approximate BCs. Accurate skin and proximity effects, i.e. distributions of fields and current densities, are to be obtained for accurate force and Joule loss density distributions as well as for accurate interactions with neighboring regions. Sequences of SP corrections are developed for the magnetic vector potential FE magnetodynamic formulation. Each model can be included in the SP sequence and corrected by the other models of higher accuracy, with the advantage of using a different mesh at each step. The developed technique will be illustrated and validated on application examples.

II. COUPLED SUBPROBLEMS

A. Sequence of subproblems

To allow a natural progression from simple to more elaborate models, a complete problem is split into a series of SPs that define a sequence of changes, with the complete solution being replaced by the sum of the SP solutions [3]. Each SP is defined in its particular domain, generally distinct from the complete one and usually overlapping those of the other SPs. At the discrete level, this aims to decrease the problem complexity and to allow distinct meshes with suitable refinements and possible domain overlapping. The successive sources are obtained by means of Galerkin projections between the meshes. These have to be properly discretized to assure the conformity of all the sequenced FE weak formulations.

B. Canonical form of magnetodynamic subproblems

A canonical magnetodynamic problem p, to be solved at step p of the SPM, is defined in a domain Ω_p, with boundary $\partial\Omega_p = \Gamma_p = \Gamma_{h,p} \cup \Gamma_{b,p}$. The eddy current conducting part of Ω_p is denoted $\Omega_{c,p}$ and the non-conducting one $\Omega_{c,p}{}^C$, with $\Omega_p = \Omega_{c,p} \cup \Omega_{c,p}{}^C$. Stranded inductors belong to $\Omega_s \subset \Omega_{c,p}{}^C$. Magnetic field $\boldsymbol{h}_p$ and electric current density $\boldsymbol{j}_p$ are related to magnetic flux density $\boldsymbol{b}_p$ and electric field $\boldsymbol{e}_p$, respectively, through the material relations

$$\boldsymbol{h}_p = \mu_p{}^{-1}\, \boldsymbol{b}_p + \boldsymbol{h}_{s,\mathrm{p}}\,, \quad \boldsymbol{j}_p = \sigma_p\, \boldsymbol{e}_p + \boldsymbol{j}_{s,\mathrm{p}}\,, \tag{1a-b}$$

where μ_p is the magnetic permeability, σ_p is the electric conductivity, and $\boldsymbol{h}_{\mathrm{s,p}}$ and $\boldsymbol{j}_{\mathrm{s,p}}$ are volume sources (VSs) defined by

$$\boldsymbol{h}_{s,p} = (\mu_p{}^{-1} - \mu_q{}^{-1})\, \boldsymbol{b}_q\,, \quad \boldsymbol{j}_{s,p} = (\sigma_p - \sigma_q)\, \boldsymbol{e}_q\,, \tag{2a-b}$$

for changes from μ_q and σ_q for SP q to μ_p and σ_p for SP p in some regions [3]. Also, BCs have to be defined for surface sources (SSs), possibly expressed from previous solutions, i.e.

$$\boldsymbol{n} \times \boldsymbol{h}_p|_{\Gamma_{h,p}} = \boldsymbol{j}_{f,p}\,, \quad \boldsymbol{n} \cdot \boldsymbol{b}_p|_{\Gamma_{b,p}} = \boldsymbol{f}_{f,p}\,, \quad \boldsymbol{n} \times \boldsymbol{e}_p|_{\Gamma_{e,p} \subset \Gamma_{b,p}} = \boldsymbol{k}_{f,p}\,, \tag{3a-b-c}$$

with $\boldsymbol{n}$ the unit normal exterior to Ω. Some paired portions of Γ_p can define double layers, with the thin region in between exterior to Ω_p; in particular, this will be the case with the perfect conductor and impedance BCs. They are denoted $\gamma_p{}^+$ and $\gamma_p{}^-$ and are geometrically defined as a single surface γ_p with interface conditions (ICs), fixing the discontinuities ($[\,\cdot\,]_{\gamma_p} = \cdot\,|_{\gamma_p{}^+} - \cdot\,|_{\gamma_p{}^-}$)

$$[\boldsymbol{n} \times \boldsymbol{h}_p]_{\gamma_p} = [\boldsymbol{j}_{f,p}]_{\gamma_p}, \; [\boldsymbol{n} \cdot \boldsymbol{b}_p]_{\gamma_p} = [\boldsymbol{f}_{f,p}]_{\gamma_p}, \; [\boldsymbol{n} \times \boldsymbol{e}_p]_{\gamma_p} = [\boldsymbol{k}_{f,p}]_{\gamma_p}. \tag{4a-b-c}$$

With the magnetic vector potential $\boldsymbol{a}_p$ and electric scalar potential v_p defined via $\boldsymbol{b}_p = \mathrm{curl}\, \boldsymbol{a}_p$ and $\boldsymbol{e}_p = -\partial_t \boldsymbol{a}_p - \mathrm{grad}\, v_p = -\partial_t \boldsymbol{a}_p - \boldsymbol{u}_p$, and the resulting BC and IC

$$\boldsymbol{n} \times \boldsymbol{a}_p|_{\Gamma_{b,p}} = \boldsymbol{a}_{f,p}\,, \quad [\boldsymbol{n} \times \boldsymbol{a}_p]_{\gamma_p} = [\boldsymbol{a}_{f,p}]_{\gamma_p}, \tag{5a-b}$$

the $\boldsymbol{a}_p$ weak formulation of the magnetodynamic problem is obtained from the weak form of the Ampère equation, i.e. [1]

$$\begin{aligned} &(\mu_p^{-1}\, \mathrm{curl}\, \boldsymbol{a}_p, \mathrm{curl}\, \boldsymbol{a}')_{\Omega_p} + (\boldsymbol{h}_{s,p}, \mathrm{curl}\, \boldsymbol{a}')_{\Omega_p} - (\boldsymbol{j}_{s,p}, \boldsymbol{a}')_{\Omega_p} \\ &+ (\sigma_p\, \partial_t\, \boldsymbol{a}_p, \boldsymbol{a}')_{\Omega_{c,p}} + (\sigma_p\, \boldsymbol{u}_p, \boldsymbol{a}')_{\Omega_{c,p}} + < \boldsymbol{n} \times \boldsymbol{h}_p, \boldsymbol{a}' >_{\Gamma_{h,p} \setminus \gamma_p} \\ &+ < [\boldsymbol{n} \times \boldsymbol{h}_p]_{\gamma_p}, \boldsymbol{a}' >_{\gamma_p} = 0\,, \forall\, \boldsymbol{a}' \in F_p^1(\Omega_p), \end{aligned} \tag{6}$$

where $F_p^1(\Omega_p)$ is a curl-conform function space defined on Ω_p, gauged in $\Omega_{c,p}^C$, and containing the basis functions for $\boldsymbol{a}_p$ as well as for the test function $\boldsymbol{a}'$ (at the discrete level, this space is defined by edge FEs) (test function v' has to be used as well); the gauge is based on the tree-co-tree technique); $(\cdot,\cdot)_\Omega$ and $<\cdot,\cdot>_\Gamma$ denote a volume integral in Ω and a surface integral on Γ, respectively, of the product of their field arguments.

III. Conductor Modeling – Various Approximations

A. Perfect conductor boundary condition (PCBC)

A SP ($p \equiv pc$) is defined in Ω_p by considering some conductors $\Omega_{c,p,i}$ (i is the conductor index) as being perfect, i.e. of infinite conductivity ($\sigma_p \rightarrow \infty$) [1]. Its solution is thus independent of the conductivity and can serve as a reference solution for any conductivity further considered. This results in a zero skin depth and surface currents. The interior of $\Omega_{c,p,i}$, with zero fields inside, can thus be extracted from the studied domain Ω_p in (6) and treated via BC (3b) fixing zero traces of $\boldsymbol{b}_p$ and $\boldsymbol{e}_p$ on their boundaries $\Gamma_{c,p,i} = \partial\Omega_{c,p,i}$, i.e.

$$\boldsymbol{n}\cdot\boldsymbol{b}_p|_{\Gamma_{c,p,i}} = 0,\ \boldsymbol{n}\times\boldsymbol{e}_p|_{\Gamma_{c,p,i}} = 0 \text{ or } \boldsymbol{n}\times\boldsymbol{a}_p|_{\Gamma_{c,p,i}} = \boldsymbol{n}\times\operatorname{grad} u_p|_{\Gamma_{c,p,i}}, \quad \text{(7a-b-c)}$$

with u_p any surface scalar potential. A non-zero trace $\boldsymbol{n}\times\boldsymbol{h}_p|_{\partial\Omega_{c,i}}$ will be part of the solution, thus giving a discontinuity $[\boldsymbol{n}\times\boldsymbol{h}_p]_{\Gamma_{c,p,i}} = \boldsymbol{n}\times\boldsymbol{h}_p|_{\Gamma_{c,p,i}}$.

B. Impedance boundary condition (IBC)

Some conductors can be extracted from Ω_p by using IBCs on their boundaries $\Gamma_{c,p,i}$, which defines SP $p \equiv ibc$. The BCs to define relate the tangential traces of $\boldsymbol{h}_p$ and $\boldsymbol{e}_p$ via [2]

$$\boldsymbol{n}\times\boldsymbol{h}_p|_{\Gamma_{c,p,i}} = Z_{c,p,i}^{-1}\ \boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{e}_p)|_{\Gamma_{c,p,i}}\,, \quad (8)$$

with $Z_{c,p,i}$ the surface impedance for conductor $\Omega_{c,p,i}$, i.e.

$$Z_{c,p,i} = (\sigma_p\delta_p)^{-1}(1+j)\,,\ \text{with}\ \delta_p = \sqrt{2/(\omega\sigma_p\mu_p)}\,, \quad \text{(9a-b)}$$

with ω the angular frequency ($\omega = 2\pi f$, with f the frequency) and j the imaginary unit ($\partial_t \equiv j\omega$ in the frequency domain). BC (8) is thus to be expressed in (6) in term of the primal unknowns with $\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{e}_p)|_{\Gamma_{c,p,i}} = (\boldsymbol{n}\times(\partial_t\boldsymbol{a}_p + \boldsymbol{u}_p))\times\boldsymbol{n}|_{\Gamma_{c,p,i}}$.

C. Modified impedance boundary condition (MIBC)

A modified IBC can be defined in the vicinity of corners in 2-D or edges in 3-D, as given by a reference problem [2]. If $\Gamma_{c,p,i}$ has one corner singularity located at the origin $X = 0$, of angle β in $\Omega_{c,p,i}$, then the scaling $X = x/\delta_p$ gives a "profile" term V_α that is independent of δ_p and satisfies a reference problem described in details in [2], with a reference corner of the same opening β as in $\Omega_{c,p,i}$. The surface impedance close to the corner can be then approximated by

$$Z_{c,p,i}' \underset{\rho\to 0}{\approx} Z_{c,p,i}\, V_\alpha(\cdot/\delta_p)\big/(\partial_n V_\alpha)(\cdot/\delta_p)\,. \quad (10)$$

D. Corrections of BCs up to volume conductors

Any SP p can be defined as a correction of a previous SP q, without involving the already considered sources (e.g. inductors, previous VSs and SSs). For a change from a BC approximation to another, SSs have to be defined. E.g., from a PCBC-SP q to an IBC-SP p, SS $\boldsymbol{n}\times\boldsymbol{h}_q|_{\Gamma_{c,p,i}}$ is to be subtracted from the right hand side of (8) for SP p. It is thus involved in a surface integral term in (6), to be weakly expressed from (6) for SP q, i.e. generally limited to

$$<\boldsymbol{n}\times\boldsymbol{h}_q, \boldsymbol{a}'>_{\Gamma_{c,p,i}} = -(\mu_q^{-1}\operatorname{curl}\boldsymbol{a}_q, \operatorname{curl}\boldsymbol{a}')_{\Omega_q}\,. \quad (11)$$

A change from a PCBC-SP to an accurate volume model has been developed in [1] and is here included in an extended correction procedure, with intermediary IBC-SP or MIBC-SP corrections. A change from an IBC-SP q to a volume SP p can involve both SSs and VSs. Considering a zero solution in $\Omega_{c,p,i}$, and thus carrying all the fields in the double layer of its boundary, trace discontinuities of both $\boldsymbol{h}_q$ and $\boldsymbol{e}_q$ ($\boldsymbol{a}_q$) occur: their opposite values then define SSs for SP p in (4a), weakly expressed as (11), and (4c) (and (5b)), strongly expressed in function space $F_p^1(\Omega)$. Also, a zero solution q in $\Omega_{c,p,i}$ allows the VSs (2a-b) to be zero. VSs have thus to be only considered for further local changes of μ_p and σ_p, e.g. for nonlinear magnetic materials or temperature dependent conductivities.

Fig. 1 shows some solutions and their corrections for PCBC, IBC and MIBC models, with clear increasing accuracy. Fig. 2 shows the corresponding surface impedances before correction.

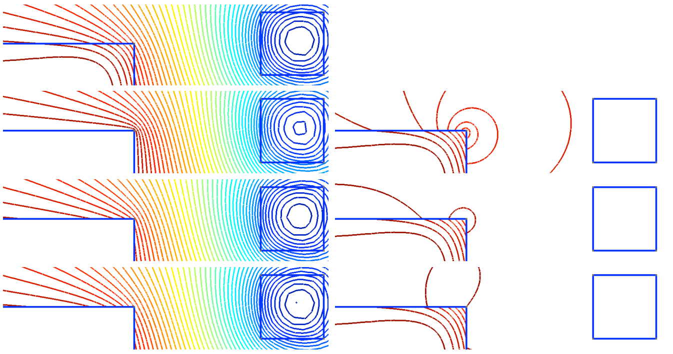

Fig. 1. Field lines near a conductor corner, *from top to bottom:* complete solution, *left:* PCBC, IBC, MIBC models, *right:* associated correction of each model; same scale for each figure pointing out the decrease of the required correction.

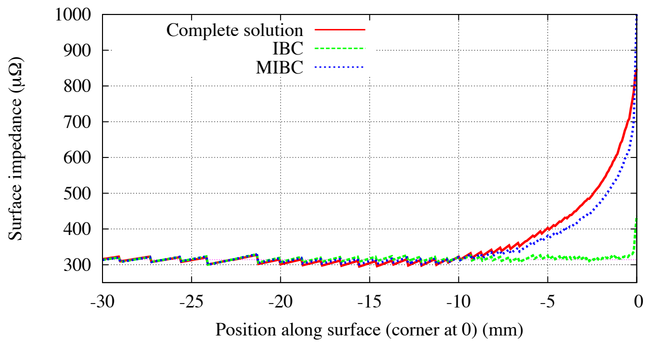

Fig. 2. Post-computed surface impedance along horizontal boundary (skin depth 4.5 mm).

IV. References

[1] P. Dular, R.V. Sabariego and L. Krähenbühl, "Subdomain Perturbation Finite-Element Method for Skin and Proximity Effects", *IEEE Trans. Magn.*, vol. 44, no. 6, pp. 738-741, 2008.

[2] F. Buret, M. Dauge, P. Dular, L. Krähenbühl, V. Péron, R. Perrussel, C. Poignard and D. Voyer, "Eddy Currents and Corner Singularities", *IEEE Trans. Magn.*, vol. 48, no. 2, pp. 679-682, 2012.

[3] P. Dular, R. V. Sabariego, C. Geuzaine, M. V. Ferreira da Luz, P. Kuo-Peng and L. Krähenbühl, "Finite Element Magnetic Models via a Coupling of Subproblems of Lower Dimensions," *IEEE Trans. Magn.*, vol. 46, no. 8, pp. 2827-2830, 2010.