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To cite this version:

HAL Id: hal-00807078
https://hal.archives-ouvertes.fr/hal-00807078
Submitted on 2 Apr 2013

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Perfect Conductor and Impedance Boundary Condition Corrections via a Finite Element Subproblem Method

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Abstract—A finite element subproblem method is developed to correct the inaccuracies proper to perfect conductor and impedance boundary condition models, in particular near edges and corners of conductors, for a large range of conductivities and frequencies. Local corrections, supported by fine local meshes, can be obtained from each model to a more accurate one, allowing efficient extensions of their domains of validity.

I. INTRODUCTION

Magnetodynamic finite element (FE) modeling of conductors can be tackled at various levels of precision, e.g. considering them via perfect conductor or impedance boundary conditions (BCs). Avoiding to mesh their interior allows to lighten the computational efforts, which is interesting for the preliminary stage of a design. Perfect conductor BCs are suitable for high conductivities or frequencies, i.e. for low skin depths [1]. For larger skin depths, impedance BCs (IBCs) lead to a better accuracy. Such conditions are nevertheless generally based on analytical solutions of ideal problems and are therefore only valid in practice far from any geometrical discontinuities, e.g., edges and corners. Local modifications of IBCs can be defined (e.g. [2]).

It is here proposed to perform successive FE refinements via a subproblem (SP) method (SPM) [3] to correct the models with approximate BCs. Accurate skin and proximity effects, i.e. distributions of fields and current densities, are to be obtained for accurate force and Joule loss density distributions as well as for accurate interactions with neighboring regions. Sequences of SP corrections are developed for the magnetic vector potential FE magnetodynamic formulation. Each model can be included in the SP sequence and corrected by the other models of higher accuracy, with the advantage of using a different mesh at each step. The developed technique will be illustrated and validated on application examples.

II. COUPLED SUBPROBLEMS

A. Sequence of subproblems

To allow a natural progression from simple to more elaborate models, a complete problem is split into a series of SPs that define a sequence of changes, with the complete solution being replaced by the sum of the SP solutions [3]. Each SP is defined in its particular domain, generally distinct from the complete one and usually overlapping those of the other SPs. At the discrete level, this aims to decrease the problem complexity and to allow distinct meshes with suitable refinements and possible domain overlapping. The successive sources are obtained by means of Galerkin projections between the meshes. These have to be properly discretized to assure the conformity of all the sequenced FE weak formulations.

B. Canonical form of magnetodynamic subproblems

A canonical magnetodynamic problem $p$, to be solved at step $p$ of the SPM, is defined in a domain $\Omega_p$, with boundary $\partial\Omega_p = \Gamma_p = \Gamma_{h,p} \cup \Gamma_{s,p}$. The eddy current conducting part of $\Omega_p$ is denoted $\Omega_{c,p}$ and the non-conducting one $\Omega_{c,p}^c$, with $\Omega_p = \Omega_{c,p} \cup \Omega_{c,p}^c$. Stranded conductors belong to $\Omega_{s} \subset \Omega_{c,p}^c$. Magnetic field $h_p$ and electric current density $j_p$ are related to magnetic flux density $b_p$ and electric field $e_p$, respectively, through the material relations

$$ h_p = \mu_p^{-1} b_p + h_{s,p}, \quad j_p = \sigma_p e_p + j_{s,p}, \quad (1a) $$

where $\mu_p$ is the magnetic permeability, $\sigma_p$ is the electric conductivity, and $h_{s,p}$ and $j_{s,p}$ are volume sources (VSs) defined by

$$ h_{s,p} = (\mu_p^{-1} - \mu_q^{-1}) b_q, \quad j_{s,p} = (\sigma_p - \sigma_q) e_q, \quad (2a-b) $$

for changes from $\mu_q$ and $\sigma_q$ for SP $q$ to $\mu_p$ and $\sigma_p$ for SP $p$ in some regions [3]. Also, BCs have to be defined for surface sources (SSs), possibly expressed from previous solutions, i.e. $n \times h_p|_{\Gamma_{h,p}} = j_{f,p} \quad n \cdot b_p|_{\Gamma_p} = j_{f,p} \quad n \times e_p|_{\Gamma_{s,p}} = k_{j,p}, \quad (3a-b-c)$

with $n$ the unit normal exterior to $\Omega$. Some paired portions of $\Gamma_p$ can define double layers, with the thin region in between exterior to $\Omega$; in particular, this will be the case with the perfect conductor and impedance BCs. They are denoted $\Gamma_p^+$ and $\Gamma_p^-$ and are geometrically defined as a single surface $\gamma_p$ with interface conditions (ICs), fixing the discontinuities $\{ n \times h_p \gamma_p = [j_{f,p}], \quad n \cdot h_p \gamma_p = [j_{f,p}], \quad n \times e_p \gamma_p = [k_{j,p}],\quad \} \quad (4a-b-c)$

With the magnetic vector potential $a_p$ and electric scalar potential $\psi_p$ defined via $h_p = \text{curl} a_p$ and $e_p = -\partial \psi_p \text{grad} \psi_p = -\partial_i a_p - u_p$, and the resulting BC and IC

$$ n \times a_p|_{\Gamma_{h,p}} = a_{f,p}, \quad n \times a_p|_{\Gamma_{s,p}} = [a_{f,p}]_{\gamma_p}, \quad (5a-b) $$

the $a_p$ weak formulation of the magnetodynamic problem is obtained from the weak form of the Ampère equation, i.e. [1]

$$ (\mu_p^{-1} \text{curl} a_p, \text{curl} a')_{\Omega_p} + (h_{s,p}, \text{curl} a')_{\Omega_p} - (j_{s,p}, a')_{\Omega_p} + (\sigma_p \partial_i a_p, a')_{\Omega_{c,p}} + (\sigma_p u_p a', a')_{\Omega_{s,p}} + n h_{p,a} a' \gamma_p + \langle n \times h_p a' \gamma_p, a' \rangle_{\gamma_p} = 0, \forall a' \in F_p^0(\Omega_p), \quad (6) $$
where \( F_p^{-1}(\Omega_p) \) is a curl-conform function space defined on \( \Omega_p \), gauged in \( \Omega_{c,p}^{-} \), and containing the basis functions for \( a_p \) as well as for the test function \( a' \) (at the discrete level, this space is defined by edge FEIs) (test function \( v' \) has to be used as well); the gauge is based on the tree-co-tree technique; \((\cdot,\cdot)_Q \) and \( <\cdot,\cdot>_T \) denote a volume integral in \( \Omega \) and a surface integral on \( \Gamma \), respectively, of the product of their field arguments.

### III. CONDUCTOR MODELING – VARIOUS APPROXIMATIONS

**A. Perfect conductor boundary condition (PCBC)**

A SP \( (p=pc) \) is defined in \( \Omega_p \) by considering some conductors \( \Omega_{c,p,i} \) (\( i \) is the conductor index) as being perfect, i.e. of infinite conductivity \( (\sigma_p \rightarrow \infty) \) \cite{1}. Its solution is thus independent of the conductivity and can serve as a reference solution for any conductivity further considered. This results in a zero skin depth and surface currents. The interior of \( \Omega_{c,p,i} \), with zero fields inside, can thus be extracted from the studied domain \( \Omega_p \) in (6) and treated via BC (3b) fixing zero traces of \( b_p \) and \( e_p \) on their boundaries \( \Gamma_{c,p,i} = \partial \Omega_{c,p,i} \), i.e.

\[
\begin{align*}
\text{n} \cdot \mathbf{b}_{\Gamma_{c,p,i}} &= 0, \quad \text{n} \times \mathbf{e}_{\Gamma_{c,p,i}} &= 0 \quad \text{or} \quad \text{n} \times a_p|_{\Gamma_{c,p,i}} = \text{n} \times \text{grad} \, u_p|_{\Gamma_{c,p,i}} \quad (7a-b-c)
\end{align*}
\]

with \( u_p \) any surface scalar potential. A non-zero trace \( \text{n} \times h_{\Gamma_{c,p,i}} \) will be part of the solution, thus giving a discontinuity \( [\text{n} \times h_p]|_{\Gamma_{c,p,i}} = \text{n} \times \mathbf{h}_{\Gamma_{c,p,i}} \).

**B. Impedance boundary condition (IBC)**

Some conductors can be extracted from \( \Omega_p \) by using IBCs on their boundaries \( \Gamma_{c,p,i} \), which defines SP \( p=ibc \). The BCs to define relate the tangential traces of \( h_p \) and \( e_p \) via \cite{2}

\[
\begin{align*}
\text{n} \times h_{\Gamma_{c,p,i}} &= Z_{c,p,i}^{-1} \text{n} \times (\text{n} \times e_p)|_{\Gamma_{c,p,i}} , \quad (8)
\end{align*}
\]

with \( Z_{c,p,i} \) the surface impedance for conductor \( \Omega_{c,p,i} \), i.e.

\[
Z_{c,p,i} = (\sigma_p \delta_p)^{-1} \left( 1 + j \right), \quad \text{with} \quad \delta_p = \sqrt{2/\left( \omega \sigma_p \mu_p \right)} ,
\]

with \( \omega \) the angular frequency \( (\omega = 2\pi f, \text{with} \, f \, \text{the frequency}) \) and \( j \) the imaginary unit \( (j = \sqrt{-1}) \). \( \mu_p \) is the magnetic permeability. BC (8) is thus to be expressed in (6) in term of the primal unknowns with \( \text{n} \times (\text{n} \times e_p)|_{\Gamma_{c,p,i}} = (\text{n} \times (\text{grad} \, u_p)) \times \text{n} \) of (4a).

**C. Modified impedance boundary condition (MIBC)**

A modified IBC can be defined in the vicinity of corners in 2-D or edges in 3-D, as given by a reference problem \cite{2}. If \( \Gamma_{c,p,i} \) has one corner singularity located at the origin \( X=0 \), of angle \( \beta \) in \( \Omega_{c,p,i} \), then the scaling \( X=X/\delta_p \) gives a “profile” term \( V_\beta \) that is independent of \( \delta_p \) and satisfies a reference problem described in details in \cite{2}, with a reference corner of the same opening \( \beta \) as in \( \Omega_{c,p,i} \). The surface impedance close to the corner can be then approximated by

\[
Z_{c,p,i}^{-1} \approx P_{\beta=0} Z_{c,p,i} \, V_\beta \left( \text{grad} \, u_p \right) \text{of} \left( \text{grad} \, u_p \right)
\]

**D. Corrections of BCs up to volume conductors**

Any SP \( p \) can be defined as a correction of a previous SP \( q \), without involving the already considered sources (e.g. inductors, previous VSSs and SSSs). For a change from a BC approximation to another, SSSs have to be defined. E.g., from a PCBC-SP \( q \) to an IBC-SP \( p \), \( \text{n} \times h_{q}\Gamma_{c,p} \) is to be subtracted from the right hand side of (8) for SP \( p \). It is thus involved in a surface integral term in (6), to be weakly expressed from (6) for SP \( q \), i.e. generally limited to

\[
< \text{n} \times h_q | a' >_{\Gamma_{c,p}} = - (a_q^{-1} \text{curl} \, a_q \cdot \text{curl} \, a')_{\Omega_q} \quad , \quad (11)
\]

A change from a PCBC-SP to an accurate volume model has been developed in \cite{1} and is here included in an extended correction procedure, with intermediary IBC-SP or MIBC-SP corrections. A change from an IBC-SP \( q \) to a volume SP \( p \) can involve both SSSs and VSSs. Considering a zero solution in \( \Omega_{c,p,i} \) and thus carrying all the fields in the double layer of its boundary, trace discontinuities of both \( h_q \) and \( e_q (a_q) \) occur: their opposite values then define SSS for SP \( p \) in (4a), weakly expressed as (11), and (4c) (and (5b)), strongly expressed in function space \( F_p^{-1} (\Omega) \). Also, a zero solution \( q \) in \( \Omega_{c,p,i} \) allows the VSSs (2a-b) to be zero. VSSs have thus to be only considered for further local changes of \( \mu_q \) and \( \sigma_q \), e.g. for nonlinear magnetic materials or temperature dependent conductivities.

### IV. REFERENCES