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Frobenius structure for rank one $p-$adic differential equations.

Andrea Pulita

Abstract. We generalize to all rank one $p-$adic differential equations over $\mathcal{R}$ the theorem 2.3.1 of [**Ch-Ch**] which provides the existence of a Frobenius structure of order h for soluble rank one operators of the form $\frac{d}{dx} + g(x)$, $g(x) \in x^{-2}K[x^{-1}]$. It follows a generalization of a theorem of Matsuda which asserts that the Robba's exponential $\exp(\sum_{i=0}^{m} \pi_{m-i} x^{p^i}/p^i)$ has a Frobenius structure. Namely our theorem works in the case $p = 2$.
In the appendix we describe the variation of the radius of convergence of a differential module by pull-back by a Kummer ramification.

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1. Notations

Let K be a complete field with respect to an ultra-metric absolute value $|\cdot|$. Let $\mathcal{O}_K = \{x \in K \mid |x| \leq 1\}$ be the ring of integers of K, and let $D(0, 1^-) = \{x \in K \mid |x| < 1\}$ be its maximal ideal. Let k be its residue field which will be supposed to be a perfect field of characteristic $p > 0$. Let $\overline{K}$ be an algebraic closure of K.

Let E (resp. E_ρ) be the completion of $K(x)$ for the Gauss norm (resp. the norm $|.|_\rho$ defined by $|\sum_i a_i x^i|_\rho := \sup_i |a_i|\rho^i$).

Let $\mathcal{E}$ be the Amice's ring. The elements of $\mathcal{E}$ are bounded series $f = \sum_{i\in\mathbb{Z}} a_i x^i$, $a_i \in K$, $|a_i| \to 0$ for $i \to -\infty$, for which there exists a constant $M(f) \in \mathbb{R}$ such that $|a_i| \leq M(f)$ for all $i \in \mathbb{Z}$. The topology of $\mathcal{E}$ is defined by the Gauss norm and $\mathcal{E}$ is a complete ring. We have a canonical isometric embedding $K(x) \subset E \subset \mathcal{E}$.

Let $I \subseteq \mathbb{R}_{\geq 0}$ be an interval. Let $\mathcal{C}(I) := \{x \in K \mid |x| \in I\}$. Let $\mathcal{A}(I)$ be the ring of analytic functions over $\mathcal{C}(I)$, the elements of $\mathcal{A}(I)$ are power series $f = \sum_{i\in\mathbb{Z}} a_i x^i$, $a_i \in K$, such that $\lim_{i\to\pm\infty} |a_i|\rho^i = 0$, for all $\rho \in I$. $\mathcal{A}(I)$ is complete for the topology defined by the family of absolute values $\{|.|_\rho\}_{\rho\in I}$, where

$$|f|_\rho := \sup_{i\in\mathbb{Z}} |a_i|\rho^i.$$

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Let $\mathcal{R}$ be the Robba's Ring. The elements f of $\mathcal{R}$ are germs of convergent analytic functions at the edge of $D(0,1^-)$, namely

$$\mathcal{R} = \cup_{0<\varepsilon<1}\mathcal{A}(]1-\varepsilon,1[). \tag{1.0.1}$$

In other words $\mathcal{R}$ is the inductive limit of the sequence $\mathcal{A}(r_1,1) \subset \mathcal{A}(r_2,1)$, $0 < r_1 < r_2 < 1$ and it is equipped with the limit topology.

All the rings E_ρ, $\mathcal{E}$, $\mathcal{A}(I)$, $\mathcal{R}$ are differential rings with respect to the continue derivation $\frac{d}{dx}$.

1.1. Berkovich spaces and $p-$adic differential equations. Let $\mathscr{M}(\mathcal{A}(I))$ be the analytic space (in the sense of Berkovich [**Ber**] 1.2) attached to the affinoïd algebra $\mathcal{A}(I)$ ([**Ber**] 2.1.1). The Berkovich's point defined by the norm $|.|_\rho$ can be (and will be) identified with the Dwork's generic point t_ρ of radius ρ ([**Ch-Ro**], 9.1.2). Following this identification let M be the differential module defined by $\frac{d}{dx}+G(x)$, $G(x)\in M_n(\mathcal{A}(I))$, then the radius of convergence of M at the point $|.|_\rho$ is defined as

$$Ray(M,|.|_\rho) := \inf\left(\liminf_s\left(\frac{|G_s(x)|_\rho}{|s!|}\right)^{-1/s},\rho\right) \tag{1.1.1}$$

where $G_s\in M_n(\mathcal{A}(I))$ is defined by the recursion formula

$$G_{s+1} = \frac{d}{dx}(G_s) + G_s\cdot G \ , \qquad G_1 := G. \tag{1.1.2}$$

The function $\rho\mapsto Ray(M,|.|_\rho)$ is continuous and there exists a partition $I=\cup_{j\in\mathbb{Z}}I_j$, $\sup I_j = \inf I_{j+1}$, such that $Ray(M,|.|_\rho) = \alpha_j\rho^{\beta_j}$, for all $\rho\in I_j$.

For simplicity we will write $Ray(M,\rho)$ instead of $Ray(M,|.|_\rho)$.

1.1.1. More generally let $|.|_t \in \mathscr{M}(\mathcal{A}(I))$ be a bounded multiplicative semi-norm ([**Ber**] 1.2). We define the *radius of the generic disk of center* $|.|_t$ as

$$\rho(|.|_t) := \inf(|x-a|_t \mid a\in\overline{K}) \tag{1.1.3}$$

and we put

$$Ray(M,|.|_t) := \inf\left(\liminf_s\left(\frac{|G_s(x)|_t}{|s!|}\right)^{-1/s},\rho(|.|_t)\right). \tag{1.1.4}$$

Remark 1.1. Observe that the function $\rho:\mathscr{M}(\mathcal{A}(I))\to[0,\sup I]$, $|.|_t\mapsto\rho(|.|_t)$ is semi-continuous in the sense that the set $\rho^{-1}([0,r[)$ is open in $\mathscr{M}(\mathcal{A}(I))$ [1], but ρ is not continuous. Indeed for all $a\in K$ such that $|a|\in I$ we define $|.|_a$ as the semi-norm given by $f\mapsto|f(a)|$. It is clear that $\rho(|.|_a)=0$. Now let $I:=[0,1]$ and let $|.|_1$ be the semi-norm attached to the unit disk $|f|_1 := \sup_{|a|\leq 1, a\in K}(|f(a)|)$.[2]Let $K=\overline{K}$. We choose a sequence $\{a_n\}_n$, $a_n\in K$, such that $\bar{a}_i\neq\bar{a}_j\in k$, $\forall i\neq j$, then we have $\lim_n|.|_{a_n} = |.|_1$ in $\mathscr{M}(\mathcal{A}(I))$, but $\rho(|.|_1)=1$. This results from the fact that every function $f\in\mathcal{A}(I)$ has only a finite number of zeros in $\mathcal{C}([0,1])$.

However let $\gamma: I\to\mathscr{M}(\mathcal{A}(I))$ be a continue section of ρ. Then the function $r\mapsto Ray(M,\gamma(r))$ is a continuous function ([**Ch-Dw**]).

[1]Observe that the open set $\rho^{-1}([0,r])$ is not an affinoïd in the sense of [**Ber**] 2.2.1.

[2]This norm is the Gauss norm.

DEFINITION 1.2 ([**Astx**]). A differential module M *over* E (resp. over $\mathcal{E}$) is called *soluble* if $Ray(M,1)=1$. A differential module M *over* $\mathcal{A}(I)$ is called *soluble at* $\rho \in I$ if $Ray(M,\rho)=\rho$. A differential module M *over* $\mathcal{R}$ is called *soluble* if

$$\lim_{\rho\to 1^-} Ray(M,\rho)=1. \tag{1.1.5}$$

1.1.2. The radius of convergence at $|.|_{c,r} \in \mathscr{M}(\mathcal{A}(I))$ of a differential module M can be viewed as the smallest radius of convergence of the solutions of M at some "incarnation" $t_{c,r}$ of $|.|_{c,r}$ (cf. 5.1.1). Observe that a function of $\mathcal{A}(I)$ has no poles and no zeros in the generic disk $D_\Omega(t_{c,r},r^-)$. Hence all points of $D_\Omega(t_{c,r},r^-)$ are non singular for all differential modules.

1.2. Frobenius structure. Let A be one of the rings E_ρ, $\mathcal{E}$, $\mathcal{A}(I)$ or $\mathcal{R}$ and let A^p be one of the rings E_{ρ^p}, $\mathcal{E}$, $\mathcal{A}(I^p)$ or $\mathcal{R}$ rispectively. Let $^\sigma : K\to K$ be an automorphism of K such that $|a^\sigma - a^p|<1$, for all $a\in\mathcal{O}_K$. For all functions $f(x)=\sum_i a_ix^i \in A^p$ we set $f^\sigma(x) := \sum_i a_i^\sigma x^i \in A$. We define a Frobenius map $\varphi : A^p \to A$ by

$$\varphi(f(x)) := f^\sigma(x^p).$$

This morphism defines an functor, called φ^* (cf. [**Astx**]), from the category of A^p-differential modules into the category of $A-$differential modules. Let M be the A^p-differential module, defined by $\frac{d}{dx}+G(x)$, $G(x)\in M_n(A^p)$. The Frobenius functor sends M into the module $\varphi^*(M)$ defined by $\frac{d}{dx}+px^{p-1}G^\sigma(x^p)$. In the appendix (cf. Corollary 5.7) we show that

$$Ray(\varphi^*(M),|.|_{c,r}) = \min\bigl(Ray(M,|.|_{c^p,r'})^{1/p}, |p|^{-1}\sup(|c|,r)^{1-p}Ray(M,|.|_{c^p,r'})\bigr)$$

where $r'=\max(r^p,|p||c|^{p-1}r)$ (cf. equation 5.2.3).

DEFINITION 1.3 (Frobenius structure). Let A be one of the rings E, $\mathcal{E}$ or $\mathcal{R}$. Let M be the differential module defined by $\frac{d}{dx}+G(x)$, $G(x)\in M_n(A)$ over A. We will say that M has a Frobenius structure of order h over A if there exists an $A-$isomorphism $M \xrightarrow{\sim} \varphi_h^*(M)$, where $\varphi_h^*(M)$ is the differential module defined by $\frac{d}{dx}+p^hx^{p^h-1}G^{\sigma^h}(x^{p^h})$. In other words there exists an invertible matrix $H(x)\in GL_n(A)$ such that

$$p^hx^{p^h-1}G^{\sigma^h}(x^{p^h}) = H(x)G(x)H^{-1}(x)+H(x)'H^{-1}(x).$$

THEOREM 1.4 ([**Ro**] 5.3). *Let $L:=\frac{d}{dx}+g(x)$, $g(x)\in K(x)$ be a soluble differential operator. By the Mittag-Leffler decomposition we may write $g(x)=g^+(x)+\sum_{-n\le i\le -1}a_ix^i$, $a_i\in K$, where $g^+(x)\in K(x)$ has no poles in $D(0,1^-)$. Then L is isomorphic over the ring $\mathcal{A}([0,1])[1/x]$ to the operator $\frac{d}{dx}+\sum_{-n\le i\le -1}a_ix^i$.*

THEOREM 1.5 ([**Ch-Ch**] 2.3.1). *Let k be perfect. Let $L=\frac{d}{dx}+\sum_{-n\le i\le -1}a_ix^i$, $a_i\in K$, be a* soluble *first order differential operator such that $a_{-1}\in\mathbb{Z}_{(p)}$. Then L has a Frobenius structure over $\mathcal{R}$. In other words there exist some $h>0$ and an invertible function $f(x)\in\mathcal{R}^\times$ for which the following equality holds*

$$\frac{f(x)'}{f(x)} = (\sum_{-n\le i\le -1} a_ix^i) - (p^hx^{p^h-1}\sum_{-n\le i\le -1} a_i^{\sigma^h}x^{ip^h})$$

2. Robba's exponentials

In this section $z = x^{-1}$. Let $\{\xi_m\}_{m\geq 0}$ be a sequence of (primitive) $p^{m+1}-$roots of 1 such that $\xi_m^{p^j} = \xi_{m-j}$, $j \geq 0$ and such that ξ_0 is a non trivial $p-$th root of 1. Let $\pi_m := \xi_m - 1$.

THEOREM 2.1. *For all $m \geq 0$ the function*

$$E_m(z) := \exp(\pi_m z + \pi_{m-1}\frac{z^p}{p} + \cdots + \pi_0 \frac{z^{p^m}}{p^m}) \in \overline{K}[[z]] \tag{2.0.1}$$

has radius of convergence equal to 1.

Proof: Let $E(z) = \exp(z + \frac{z^p}{p} + \frac{z^{p^2}}{p^2} + \cdots)$ be the Artin-Hasse exponential. If $(\lambda_0, \lambda_1, \ldots) \in W(\mathcal{O}_K)$, then by a straightforward computation ([**Bou**] exercice 58-b) one shows that

$$\prod_{i\geq 0} E(\lambda_i z^{p^i}) = \exp(\phi_0 z + \phi_1 \frac{z^p}{p} + \phi_2 \frac{z^{p^2}}{p^2} \cdots)$$

where $\phi_k = \lambda_0^{p^k} + p\lambda_1^{p^{k-1}} + \cdots + p^k \lambda_k$ are the phantom components of $(\lambda_0, \lambda_1, \ldots)$. Since $|\lambda_i| \leq 1$, hence this infinite product defines a *bounded* analytic function on $D(0, 1^-)$. If $(\lambda_0, \lambda_1, \ldots) = (\xi_m, 0, \ldots) - (1, 0, \ldots)$, then $\phi_i = \pi_{m-i}$. The fact that the radius of convergence of $E_m(z)$ is exactly 1 will be a consequence of the fact that the operator

$$\frac{d}{dx} + E_m(z^{-1})'/E_m(z^{-1}) = \frac{d}{dx} - (\pi_m z^{-2} + \pi_{m-1} z^{-p-1} + \cdots + \pi_0 z^{-p^m - 1})$$

is soluble[3] and its radius of convergence, for ρ close to 0, is ρ^{p^m+1} (cf. Corollary 3.1). Then, by the log $-$concavity property of the radius of convergence, we have that this operator has radius of covegence equal to ρ^{p^m+1}, for all $\rho < 1$. □

REMARK 2.2. Observe that for $|z| < 1$ close to 1, we have $|\pi_m z + \cdots + \pi_0 \frac{z^{p^m}}{p^m}| > |\pi_0|$. On the other hand, the analytic function $\exp(y)$ converges for $|y| < |\pi_0|$, so the convergent composition of $\pi_m z + \cdots + \pi_0 \frac{z^{p^m}}{p^m}$ and $\exp(y)$ does not exist. The precedent theorem asserts that the *formal composition*, after resommation, has radius of convergence equal to 1.

REMARK 2.3. Observe that in the formal case (cf. [**Man**]) a logarithmic derivative of a formal Laurent series has always an $x-$adic valuation ≥ -1. But in the $p-$adic case the Robba-Matsuda's exponentials give an example of logaritmic derivatives of negative $x-$adic valuation. Then the definitions of $p-$adic irregularity and formal ($x-$adic) irregularity must be different (cf. [**Ro**]).

THEOREM 2.4 ([**Ma**]). *Let $p \neq 2$. Then the exponential $E_m^{\sigma}(z^p)/E_m(z)$ is overconvergent. In other words, if p is different from 2, the differential operator*

$$\frac{d}{dx} + E'_m(x^{-1})/E_m(x^{-1}) = \frac{d}{dx} - (\pi_m x^{-2} + \pi_{m-1} x^{-p-1} + \cdots + \pi_0 x^{-p^m - 1})$$

has a Frobenius structure of order 1.

[3]The solubility of this operator is due to the fact that the convergent function $E_m(z^{-1})$ is a solution of this operator at infinity and $E_m(z^{-1})$ converges in the set $\{x \in K \mid |x| > 1\}$.

3. Formal slopes and $p-$adic slopes

LEMMA 3.1 (Young, cf.[**Astx**]). *Let $L := \sum_{s=0}^{r} g_s(x)(\frac{d}{dx})^s$ be a differential operator such that $g_r(x) = 1$, $g_s \in E_\rho$, $s = 0, \dots, r-1$. Let $\rho \in I$, then $R(M, \rho) < |\pi_0|\rho$ if and only if $|g_s|_\rho > \rho^{s-r}$ for some $s < r$, and in this case we have:*

$$R(M, |.|_\rho) = |\pi_0| \min_{0 \le s < r} (|g_s|_\rho^{-1/r-s})$$

Let M be a *soluble* $p-$adic differential module over $\mathcal{R}$. Then there exist $0 < \varepsilon < 1$ and a rational number $\beta \ge 0$ such that $Ray(M, \rho) = \rho^{\beta+1}$ for all $\rho \in]1-\varepsilon, 1[$ (cf. [**Astx**]). If M is defined in some basis by the operator $\frac{d}{dx} + G(x)$, with $G(x) \in M_n(\mathcal{A}([0, 1[)[1/x])$, then it is easy to show that there exist $0 < \delta < 1$ and a rational number $\alpha \ge 0$ such that $Ray(M, \rho) = \rho^{\alpha+1}$ for $\rho \in]0, \delta[$. By $\log-$concavity we have $\alpha \ge \beta$.

DEFINITION 3.2. The number β is called the $p-$adic *slope* of M. We set $pt(M) := \beta$. If M is defined by the operator $\frac{d}{dx} + G(x)$, $G(x) \in \mathcal{A}([0, 1[)[1/x]$ we set $pt_F(M) := \alpha$ and we will call $pt_F(M)$ the *formal slope.* We have $pt(M) \le pt_F(M)$.

REMARK 3.3. The precedent definition is justified by the fact that if M is defined by a linear differential operator $L := \sum_{s=0}^{r} g_s(x)(\frac{d}{dx})^s$, $g_r(x) = 1$, such that $g_s(x) \in \mathcal{A}([0, 1[)[1/x] \subset \mathcal{R}$, then $pt_F(M)$ is the usual formal slope defined by

$$pt_F(L) = \max\left(0, \max_s\left(\frac{s-r-v(g_s)}{r-s}\right)\right) \tag{3.0.2}$$

where $v(g_s)$ is the $x-$adic valuation of $g_s(x)$. The formal slope is the largest slope of the Formal Newton polygon of L.[4] This follows from lemma 3.1 and some continuity and convexity arguments. Indeed, as M is soluble, observe that, for ρ close to 0, by continuity and $\log-$concavity we have only two cases: $Ray(M, \rho) = \rho$ or $Ray(M, \rho) < |\pi_0|\rho$.

LEMMA 3.4. *Let $L = \frac{d}{dx} + \sum_{i \ge -d} a_i x^i$, $a_{-d} \ne 0$, $d \ge 1$ be a soluble rank one differential operator with $\sum_{i \ge -d} a_i x^i \in \mathcal{R}$. Then we have that $Ray(L, \rho) = |\pi_0||a_{-d}|^{-1}\rho^d$.*

4. Frobenius structure over $\mathcal{R}$

4.1. Reduction to a $K[x, x^{-1}]-$lattice. Let $L := \frac{d}{dx} + g(x)$, $g(x) = \sum_i a_i x^i \in \mathcal{R}$, be a rank one soluble differential operator. In this section we show that L is isomorphic over $\mathcal{R}$ to the operator $\tilde{L} = \frac{d}{dx} + \sum_{-d \le i \le -1} a_i x^i$ for a suitable $d \ge 1$ (cf. Theorem 4.7). Moreover, if $K = \overline{K}$, then d can be choosed equal to $pt(L) + 1$.

LEMMA 4.1. *The operator $L := \frac{d}{dx} + g(x)$, $g(x) = \sum_i a_i x^i \in \mathcal{R}$ is isomorphic over $\mathcal{R}$ to the operator $\tilde{L} := \frac{d}{dx} + \sum_{-d \le i \le \infty} a_i x^i$ for a suitable $d \ge 1$.*

Proof: Let $d \ge 1$ be an integer such that $\sup(|\frac{a_i}{i+1}|\rho^{i+1}) < |\pi_0|$, for all $i < -d$. Such an integer exists because $g(x) \in \mathcal{R}$. Then the series $f(x) := \exp(-\sum_{i<-d} \frac{a_i}{i+1} x^{i+1})$ lies in $\mathcal{R}$ and then L is isomorphic over $\mathcal{R}$ to the operator $\tilde{L} = \frac{d}{dx} + g(x) + f(x)'/f(x)$. $\Box$

[4] We recall that the formal Newton polygon is the convex hull of the set formed by the points of the form $(s, v(a_s)-s)$ and the two additional points $(-\infty, 0)$, $(0, +\infty)$. In particular we observe that in our case the last point is $(r, -r)$.

Lemma 4.2. *Let $L := \frac{d}{dx} + g(x)$, $g(x) = \sum_{-d\leq i\leq\infty} a_i x^i$, $d \geq 2$ be a soluble differential operator over $\mathcal{R}$. Then $|a_{-d}| \leq |\pi_0|$.*

Proof: Let $g_s(x) = g_{s-1}(x)' + g_s(x)g(x)$ be as in the equation 1.1.2. An explicit computation shows that $g_s(x) = a_{-d}^s x^{-sd} + \{\text{terms of degree } \geq -sd+1\}$. So the equation 1.1.1 shows that $Ray(L,\rho) \leq \frac{|\pi_0|\rho^d}{|a_{-d}|}$. The solubility implies $1 \leq |\pi_0|/|a_{-d}|$.$\square$

Lemma 4.3 ([**Ch-Ro**] 11.2.4). *Let $L = \frac{d}{dx} + a_{-1}x^{-1}$ be a differential equation. Then L is soluble if and only if $a_{-1} \in \mathbb{Z}_p$.*

Lemma 4.4 ([**Ch-Ro**] 18.4.4). *Let $L = \frac{d}{dx} + a_{-1}x^{-1}$ be a differential equation. Then L has a Frobenius structure if and only if $a_{-1} \in \mathbb{Z}_{(p)}$.*

Lemma 4.5. *Let $L = \frac{d}{dx} + a_{-1}x^{-1}$. Let*

$$\alpha(a_{-1}) := \limsup_n (|a_{-1}(a_{-1}-1)(a_{-1}-2)\cdots(a_{-1}-n+1)|^{\frac{1}{n}}).$$

Then for all $\rho > 0$ we have $Ray(L,\rho) = \frac{|\pi_0|}{\alpha(a_{-1})}\cdot\rho$.

Proof: A computation shows that $g_n(x) = \alpha_n(a_{-1})x^{-n}$, where $\alpha_n(a_{-1}) := a_{-1}(a_{-1}-1)\cdots(a_{-1}-n+1)$. So $Ray(L,\rho) = \liminf_n(|\alpha_n(a_{-1})|^{-1/n})|\pi_0|\rho$. $\square$

Lemma 4.6. *Let $L = \frac{d}{dx} + g(x)$, $g(x) = \sum_{i\geq -1} a_i x^i \in \mathcal{R}$ be a soluble differential equation. Then $a_{-1} \in \mathbb{Z}_p$. Moreover there exists an analytic function $f(x) \in \mathcal{A}([0,1[)$ such that $f'(x)/f(x) = \sum_{i\geq 0} a_i x^i$. In other words L is isomorphic to $\frac{d}{dx} + a_{-1}x^{-1}$.*

Proof: We proceed by absurd. L is the tensor product of $L_1 := \frac{d}{dx} + a_{-1}x^{-1}$ and $L_2 := \frac{d}{dx} + \sum_{i\geq 0} a_i x^i$. As L_2 has a convergent solution in 0, we have $Ray(L_2,\rho) = \rho$, for ρ sufficiently close to 0. On the other hand, by the lemma 4.5 we have $Ray(L_1,\rho) = \frac{|\pi_0|}{\alpha(a_{-1})}\cdot\rho$ for all $\rho > 0$. The radius of the tensor product of two operators with different radius is the minimum of the radius.[5] So, if L_1 is not soluble, then $Ray(L_1,1) = Ray(L_2,1) = R < 1$. Then $Ray(L_1,\rho) = R$ for all $\rho > 0$. But in this case, by convexity, $Ray(L_2,\rho) > R$ for $\rho < 1$. Then if $\rho < 1$ we would get $Ray(L,\rho) = \min(Ray(L_1,\rho), Ray(L_2,\rho)) = R$. Since L is soluble, hence by continuity of $Ray(L,\rho)$ we get a contraddiction.

We have shown that $Ray(L_1,\rho) = Ray(L_2,\rho) = Ray(L,\rho) = 1$, for all $0 < \rho \leq 1$. Now by transfert theorem ([**Ch-Ro**] 9.3.2) there exists a convergent solution $f(x)$ of L_2 in the disk $D(0,1^-)$. In particular $f \in \mathcal{R}$ and $f'(x)/f(x) = \sum_{i\geq 0} a_i x^i$.$\square$

Theorem 4.7. *Let $L := \frac{d}{dx} + g(x)$, $g(x) = \sum_i a_i x^i \in \mathcal{R}$ be a rank one soluble differential operator. Then $a_{-1} \in \mathbb{Z}_p$ and there exists a $d \geq \max(1, pt(M)+1)$ such that L is isomorphic, over $\mathcal{R}$, to $\frac{d}{dx} + \sum_{-d_1\leq i\leq -1} a_i x^i$, for all $d_1 \geq d$. Moreover*

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[5] Observe that this fact depends on the definition given in the equation 1.1.1. For example $(x-1)(x-1)^{-1} = 1$, so in this case the radius of 1 is not equal to the minimum of the other two radius. This depends on the definition of "radius". In general let $L_1 = \frac{d}{dx} + g(x)$, $L_2 = \frac{d}{dx} + f(x)$ be two differential operators. Suppose that $g, f \in \mathcal{A}(I)$. Let $s_i(x)$ be a power series solution of L_i at $a \in \mathcal{C}(I)$. Let $r = \inf(|a-b|, b \in \overline{K} - \mathcal{C}_{\overline{K}}(I))$. Then for all $\rho \leq r$ we have the equality

$$\min(Ray(s_1\cdot s_2),\rho) = \min(\min(Ray(s_1),\rho)\ ,\ \min(Ray(s_2),\rho)).$$

for all $-pt(M)-1 \leq i \leq -2$, *there exist* $b_i \in \overline{K}$ *such that* L *is isomorphic, over* $\mathcal{R}_{\overline{K}}$, *to* $\frac{d}{dx} + \sum_{-pt(M)-1 \leq i \leq -2} b_i x^i + a_{-1}x^{-1}$, *and* $|b_{-pt(M)-1}| = |\pi_0|$.

Proof: By the lemma 4.1 we can suppose that $L = \frac{d}{dx} + \sum_{-d \leq i \leq \infty} a_i x^i$, with $d \geq 1$. By the lemma 4.6 we can suppose that $d \geq 2$. The theorem will be proved applying the lemma 4.6. To apply this lemma we need that the operator $\frac{d}{dx} + \sum_{i \geq -1} a_i x^i$ is soluble. The solubility is invariant by extension of the field of constants and then we can actually suppose that $K = \overline{K}$.

Let $d-1 = n \cdot p^m$, $(n,p) = 1$ and let $(\frac{a_{-d}}{n \cdot \pi_0})^{1/p^m}$ be a p^m-th root of $\frac{a_{-d}}{n \cdot \pi_0}$. Let us introduce the following analytic function:

$$f_d(x) := E_m\left((\frac{a_{-d}}{n \cdot \pi_0})^{1/p^m} x^{-n} \right) = \exp(\pi_m (\frac{a_{-d}}{n \cdot \pi_0})^{1/p^m} x^{-n} + \cdots + a_{-d}\frac{x^{-(d-1)}}{d-1}).$$

By the lemma 4.2 we have $|a_{-d}| \leq |\pi_0|$. Then, as $|n| = 1$, f_d is an analytic function in $\mathcal{C}(]1,\infty[)$. The logarithmic derivative $f_d'/f_d = -a_{-d}x^{-d} + \cdots + (-n) \cdot \pi_m(\frac{a_{-d}}{n \cdot \pi_0})^{1/p^m} x^{-n-1}$ defines a differential operator

$$L_d := \frac{d}{dx} + f_d'/f_d$$

which is soluble because its solution at infinity is $f_d(x)$.

We proceed now by induction on $d \geq 2$. If $|a_{-d}| < |\pi_0|$, then f_d is an element of $\mathcal{R}$ and then the operator $\frac{d}{dx} + g(x)$ is isomorphic to $\frac{d}{dx} + g(x) + f_d'/f_d$. Now the $x-$adic valuation of the function $g(x) + f_d'/f_d$ is strictly larger than $-d$.

Otherwise, if $|a_{-d}| = |\pi_0|$, then the operator L_d is soluble and not trivial by the transfert theorem at infinity ([**Ch-Ro**] 9.3.2).[6] So the tensor product operator $L \otimes L_d$, defined by $\frac{d}{dx} + g(x) + f_d'/f_d$, is still soluble, and $g(x) + f_d'/f_d$ is of degree $\geq -d+1$ and we can proceed by induction on d. Observe that f_d'/f_d is a polynomial in $x^{-2}\overline{K}[x^{-1}]$. Iterating, we get that there exist functions $f_d, \cdots, f_2 \in \mathcal{A}_{\overline{K}}(]1,\infty[)$ which are analytic in $\mathcal{C}(]1,\infty[)$ and are such that

$$g(x) = f_d'/f_d + \cdots + f_2'/f_2 + (\sum_{i \geq -1} a_i x^i).$$

Then $\frac{d}{dx} + \sum_{i \geq -1} a_i x^i$ is soluble. Observe that, as in the proof of lemma 4.6, if $|a_{-d}| = |\pi_0|$ we have $Ray(L,\rho) = \min_i Ray(L_i,\rho) = Ray(L_d,\rho)$, then

$$Ray(L,\rho) = Ray(\frac{d}{dx} + f_d'/f_d, \rho) = |\pi_0||a_{-d}|^{-1}\rho^d = \rho^d, \qquad \forall\, 0 < \rho < 1.$$

And in this case $d = pt(M)+1$. Moreover if $d > pt(M)+1$, then it must be $|a_{-d}| < |\pi_0|$ and then $f_d \in \mathcal{R}_{\overline{K}}$. Iterating this process we can show that, over $\overline{K}$, we can obtain $d = \max(1, pt(M)+1)$.□

Remark 4.8. Observe that in the proof of the precedent lemma we show that, over $\overline{K}$, L is isomorphic to the operator

$$\frac{d}{dx} + f_d'/f_d + \cdots + f_2'/f_2 + a_{-1}x^{-1}. \tag{4.1.1}$$

In other words L is the tensor product of the operators $L_d := \frac{d}{dx} + f_d'/f_d, \ldots, L_2 := \frac{d}{dx} + f_2'/f_2$, and $L_{-1} := \frac{d}{dx} + a_{-1}x^{-1}$, where $f_i(x)$ are functions obtained from a

[6] Indeed if the operator L_d is trivial over $\mathcal{R}$ then $Ray(L_d,\rho) = \rho$ for all $\rho > 1-\varepsilon$, $\exists\, \varepsilon$ and by the transfert theorem the solution at infinity converges in the disk $\{|x| > 1-\varepsilon\}$.

Robba's exponential by a substitution of the variable. Moreover L_i is trivial or $Ray(L_i,\rho)=\rho^i$ for all $\rho<1$. Over $\overline{K}$ we can choose $d=pt(M)+1$. This is a kind of canonical form for rank one soluble differential equation over $\mathcal{R}_{\overline{K}}$.

4.2. Frobenius structure over $\mathcal{R}$.

THEOREM 4.9. *Let k be a perfect field of characteristic $p>0$. Let $L=\frac{d}{dx}+g(x)$, $g(x)=\sum_i a_i x^i\in\mathcal{R}$ be a soluble differential equation. Then L has a Frobenius structure if and only if $a_{-1}\in\mathbb{Z}_{(p)}$.*

Proof: The operator L is isomorphic to the operator $\frac{d}{dx}+\sum_{-d\leq i\leq -1}a_i x^i$. We can now apply the theorem of Christol-Chiarellotto 1.5. □

COROLLARY 4.10. *The Robba-Matsuda operator $\frac{\mathrm{d}}{\mathrm{dx}}+\frac{E'_m(x^{-1})}{E_m(x^{-1})}$ (cf. equation 2.0.1) has a Frobenius structure for all $m\geq 0$.*

Proof: We must show that the formal series $E_m^{\sigma^h}(x^{-p^h})/E_m(x^{-1})$ defines a function of $\mathcal{R}$ for some $h\geq 1$. Since the convergence does not change by extension of the field of constants, hence we can suppose $K=\overline{K}$. We can actually apply the theorem 4.9. □

REMARK 4.11. The corollary 4.10 works for all $p>0$. On the other hand, the proof of Matsuda (cf. Theorem 2.4) is a very strong computation and is stronger than our result because it shows that, if $p\neq 2$, the operator $\frac{d}{dx}+E_m(x^{-1})'/E_m(x^{-1})$ has a Frobenius structure *of order* 1.

5. Appendix: Variation of Radius of convergence by ramifications.

In this section we precise some known (but not pubblished) facts about the variation of the radius of convergence of the pull-back of a module by a covering of the form $x\mapsto x^n$. We study the ramification $\phi_n^*: f(x)\mapsto f(x^n)$ instead of φ, because the application $f(x)\mapsto f^{\sigma}(x)$ (cf. 1.2) defines an auto-equivalence of the category of differential modules which preserves the radius of convergence.

5.1. Scalar extension. Let I be a (non empty) interval. Let L/K be an extension of valued fields. Let $\mathcal{A}_L(I)$ denote the ring of analytic functions over I, with coefficients in L. Then we have the following diagram

$$\begin{array}{ccc} \mathscr{M}(\mathcal{A}_K(I)) & \overset{\psi_L}{\twoheadleftarrow} & \mathscr{M}(\mathcal{A}_L(I)) \\ \cup & & \cup \\ \mathcal{C}_K(I) & \subseteq & \mathcal{C}_L(I) \end{array}$$

where the vertical inclusions are the canonical inclusions $a\mapsto|.|_a$ (cf. remark 1.1) and the map ψ_L is the functorial morphism of analytic spaces corresponding to the inclusion $\mathcal{A}_K(I)\subseteq\mathcal{A}_L(I)$. This diagram is commutative in the sense that the inclusion $\mathcal{C}_K(I)\subseteq\mathcal{C}_L(I)$ is a section of the map ψ_L.

5.1.1. By [**Ch-Ro**] 9.1 there exists a field Ω such that $\psi_\Omega(\mathcal{C}_\Omega(I))=\mathscr{M}(\mathcal{A}_K(I))$. In this sense all points $|.|_t$ of $\mathscr{M}(\mathcal{A}_K(I))$ have an "incarnation" in a true point of $\mathcal{C}_\Omega(I)$. If $t\in\mathcal{C}_\Omega(I)$ is an incarnation of the seminorm $|.|_t\in\mathscr{M}(\mathcal{A}_K(I))$ (i.e. $\psi_\Omega(t)=|.|_t$) we will say that $D_\Omega(t,\rho(|.|_t)^-)$ is a generic disk for $|.|_t$ (cf. 1.1.1).

5.2. Ramifications and image of a point of Berkovich. Let $\phi_n^* : \mathcal{A}(I^n) \to \mathcal{A}(I)$ be the morphism $\sum_i a_i x^i \mapsto \sum_i a_i x^{ni}$. Let $\phi_n : \mathscr{M}(\mathcal{A}(I)) \to \mathscr{M}(\mathcal{A}(I^n))$ be the corresponding morphism of analytic spaces. In this section we compute the image of a point of $\mathscr{M}(\mathcal{A}(I))$ by the ramification ϕ_n.

5.2.1. By a result of Berkovich ([**Ber**] 1.4.4) we know that every point of $\mathscr{M}(\mathcal{A}(I))$ is a limit of points of the form $|.|_{c,r}$, $c \in K$

$$|f(x)|_{c,r} := \sup_{x\in D(c,r^-)} |f(x)|$$

In other words if $|.|_t$ is a point of $\mathscr{M}(\mathcal{A}(I))$, then $|.|_t$ is the seminorm attached to some disk, or $|.|_t$ is the seminorm attached to a totally ordered[7] sequence of disks. If K is spherically complete then all points are of the form $|.|_{c,r}$ and, since K is contained in some spherically complete and algebraically closed field K', hence we can suppose that all points of $\mathscr{M}(\mathcal{A}(I))$ and $\mathscr{M}(\mathcal{A}(I^n))$ are of the form $|.|_{c,r}$ for a suitable $c \in K'$. For simplicity we will suppose that $K = K'$.

Remark 5.1. Let $t_{c,r}$ be an incarnation of $|.|_{c,r}$. It is clear that $\rho(|.|_{c,r}) = r$ (cf. 1.1), and that $|t_{c,r}|_\Omega = \max(|c|, r)$.

5.2.2. Rational fractions are dense in $\mathcal{A}(I)$, hence to compute the seminorm $|.|_{c',r'} := \phi_n(|.|_{c,r})$ it is enough to know the value $|x-a|_{c',r'}$ for all $a \in K$. On the other hand $\phi_n(|.|_{c,r}) = |.|_{c,r} \circ \phi_n^*$, then $|x-a|_{c',r'} = |\phi_n^*(x-a)|_{c,r} = |x^n - a|_{c,r}$. We write $x^n - a = [(x-c)+c]^n - a = [\sum_{i=1}^n \binom{n}{i} c^{n-i}(x-c)^i] + (c^n - a)$. Then we have that

$$|x^n - a|_{c,r} = \sup_{1\le i\le n} (|\tbinom{n}{i}||c|^{n-i} r^i, |c^n - a|) \tag{5.2.1}$$

On the other hand $|x-a|_{c',r'} = \sup(r', |c'-a|)$. Therefore $c' = c^n$ and

$$r' = \sup_{1\le i\le n} (|\tbinom{n}{i}||c|^{n-i} r^i) = |c|^n \sup_{1\le i\le n} (|\tbinom{n}{i}|(r/|c|)^i). \tag{5.2.2}$$

We can compute r' in some particular case:

$$r' = \begin{cases} \min(r^p, |p||c|^{p-1} r) & \text{if} \quad n = p \\ \min(r^n, |c|^{n-1} r) & \text{if} \quad (n,p) = 1. \end{cases} \tag{5.2.3}$$

This process can be applied to compute the image of $|.|_{c,r}$ under the action of an arbitrary polynomial map instead of ϕ_n. To recover the value of r' it is sufficient to look at the Taylor's development of this polinomial at c.

Corollary 5.2 (Deformation of the Generic Disk). *Let $D_\Omega(t_{c,r}, r^-)$ be a generic disk for $|.|_{c,r}$ (cf. 5.1.1). Then $t_{c,r}^n$ is an incarnation of $|.|_{c^n,r'} := \phi_n(|.|_{c,r})$ and $\phi_n(D_\Omega(t_{c,r}, r^-)) \subseteq D_\Omega(t_{c,r}^n, r'^-)$. More precisely let $y \in D_\Omega(t_{c,r}, r^-)$. If $(n,p) = 1$, then*

$$|y^n - t_{c,r}^n| = |t_{c,r}|^{n-1} \cdot |y - t_{c,r}|. \tag{5.2.4}$$

If $n = p$, then

$$|y^p - t_{c,r}^p| \le \max(|t_{c,r}|^{p-1}|p||y - t_{c,r}|, |y - t_{c,r}|^p) \tag{5.2.5}$$

and the equality holds if $|t_{c,r}|^{p-1}|p||y - t_{c,r}| \neq |y - t_{c,r}|^p$.

[7]Ordered by inclusion.

Proof: In all cases we have that $|y-t_{c,r}|<|t_{c,r}|$ (cf. 5.1). If $(n,p)=1$ we have $|y^n-t_{c,r}^n|=|(y-t_{c,r}+t_{c,r})^n-t_{c,r}|=|t_{c,r}|^n|\sum_{i=1}^n\binom{n}{i}((y-t_{c,r})/t_{c,r})^i|$. Observe that $|\binom{n}{1}|=1$. If $n=p$, observe that $|\binom{p}{i}|=|p|$ for all $i=1,\dots,p-1$, then the same computation gives that $|y^n-t_{c,r}^n|\leq|t_{c,r}|^{p-1}|y-t_{c,r}|\cdot\sup(|p|,|(y-t_{c,r})/t_{c,r}|^{p-1})$.□

5.3. Variation of the radius of convergence by ramification. Let M be an $\mathcal{A}(I^p)-$differential module. In this section we compute the radius of convergence at $|.|_{c,r}$ of the pull-back $\mathcal{A}(I^p)-$differential module $\phi_n^*(M)$. Observe that the radius of $\varphi^*(M)$ and $\phi_p^*(M)$ are equal (cf. 5).

5.3.1. *Frobenius:* Let $s(x)=\sum_{i=0}^\infty a_i(x-t_{c,r}^p)$ be a convergent analytic function at $t_{c,r}^p$. Let $R=\liminf_i|a_i|^{-1/i}$ be the radius of convergence of $s(x)$ at $t_{c,r}^p$. Let $\phi_p^*(s)(y):=s(y^p)=\sum_i a_i(y^p-t_{c,r}^p)^i$ be its pull-back and R' the radius of $\phi_p^*(s)(y)$ at $t_{c,r}$. By composition we have (cf. corollary 5.2)

$$R'\geq\min(|p|^{-1}|t_{c,r}|^{1-p}R,R^{1/p}) \tag{5.3.1}$$

LEMMA 5.3. *Let $x_l\in\mathbb{R}$ be a sequence. Then for all $m\in\mathbb{Z}$ we have:*

$$\liminf_l(x_l)=\min_{0\leq\varepsilon\leq m-1}\big(\liminf_{l\in\varepsilon+m\mathbb{Z}}(x_l)\big)\leq\liminf_{l\in m\mathbb{Z}}(x_l).$$

LEMMA 5.4. *Let $i,n,l\in\mathbb{N}$, $i,n\geq1$, $i\leq l\leq n\cdot i$. Let*

$$B(i,l,n):=\sum_{\substack{j_1+\cdots+j_i=l\\1\leq j_k\leq n}}\binom{n}{j_1}\cdots\binom{n}{j_i}.$$

Then $B(i,i,n)=n^i$ and $B(i,in,n)=1$.

LEMMA 5.5. *Let $m\in\mathbb{N}$, $m\geq1$. Let $\{c_l\}_l$ be a sequence in some ultrametric ring. Let $R:=\liminf_l|c_l|^{-1/l}$. If $R>0$, then we have*

$$\liminf_{l\in m\mathbb{Z}}|c_{\frac{l}{m}}|^{-\frac{1}{l}}=R^{\frac{1}{m}} \tag{5.3.2}$$

Proof: This equation is equivalent to the equation $\liminf_l|c_l|^{-\frac{1}{ml}}=R^{\frac{1}{m}}$. □

THEOREM 5.6. *We have $R'=\min(R|p|^{-1}|t_{c,r}|^{1-p},R^{1/p})$.*

Proof: We write $y^p-t_{c,r}^p=(y-t_{c,r}+t_{c,r})^p-t_{c,r}^p$. We have $s(y^p)=\sum_i a_i\cdot(\sum_{j=1}^p\binom{p}{j}t_{c,r}^{p-j}y^j)^i$. After a resommation we get

$$s(y^p)=\sum_{l=0}^\infty\Big[\sum_{i=\lceil\frac{l}{p}\rceil}^{l}a_i\cdot B(i,l,p)\cdot t_{c,r}^{ip-l}\Big](y-t_{c,r})^l$$

where $\lceil l/p\rceil:=\min(i\in\mathbb{N}\mid i\geq l/p)$. Since the term $\sum_{i=\lceil\frac{l}{p}\rceil}^l a_i\cdot B(i,l,p)\cdot t_{c,r}^{ip-l}$ is a polynomial in $t_{c,r}$ with coefficients in K, hence its valuation is given by $\sup_{i=\lceil\frac{l}{p}\rceil,\dots,l}\big(|a_i|\cdot|B(i,l,p)|\cdot|t_{c,r}|^{ip-l}\big)$. We have then

$$R'=\liminf_l\Big(\sup_{i=\lceil\frac{l}{p}\rceil,\dots,l}\big(|a_i|\cdot|B(i,l,p)|\cdot|t_{c,r}|^{ip-l}\big)\Big)^{-1/l}.$$

Appliyng the lemma 5.3 we have

$$R'\leq\liminf_{l\in p\mathbb{Z}}\Big(\sup_{l/p\leq i\leq l}(|a_i|\cdot|B(i,l,p)|\cdot|t_{c,r}|^{ip-l})\Big)^{-1/l}. \tag{5.3.3}$$

By the lemma 5.4 we have $B(l/p,l,p)=1$ and $B(l,l,p)=p^l$ and then clearly $\sup_{l/p\leq i\leq l}\big(|a_i|\cdot|B(i,l,p)|\cdot|t_{c,r}|^{ip-l}\big)\geq\sup(|a_{l/p}|,|a_l|\cdot|p|^l\cdot|t_{c,r}|^{l(p-1)})$. This fact and the lemma 5.5 show that

$$R'\leq\liminf_{l\in p\mathbb{Z}}\Big(\sup(|a_{l/p}|,|a_l|\cdot|p|^l\cdot|t_{c,r}|^{l(p-1)})\Big)^{-\frac{1}{l}}=\inf(R^{1/p},R|p|^{-1}|t_{c,r}|^{1-p}).\ \Box$$

Recalling that $|t_{c,r}|=\sup(|c|,r)$ (cf. remark 5.1), we can state the following

COROLLARY 5.7. *Let A be one of the rings E_ρ, $\mathcal{A}(I)$, $\mathcal{E}$ or $\mathcal{R}$. Let A^p be the ring E_ρ^p, $\mathcal{A}(I^p)$, $\mathcal{E}$ or $\mathcal{R}$.respectively. Let M be a differential module over A^p, let $\phi_p^*(M)$ be its pull-back over A by the morphism $f(x)\to f(x^p):A^p\to A$. Then we have*

$$Ray(\phi_p^*(M),|.|_{c,r})=\min\Big(Ray(M,|.|_{c^p,r'})^{1/p},|p|^{-1}\sup(|c|,r)^{1-p}Ray(M,|.|_{c^p,r'})\Big)$$

where $r'=\max(r^p,|p||c|^{p-1}r)$ (cf. equation 5.2.3).

5.3.2. *Ramification prime to p.* Let $(n,p)=1$. Following the same method of the precedent discussion we get that $R'=|t_{c,r}|^{1-n}R$, and the following

THEOREM 5.8. *Let A be one of the rings E_ρ, $\mathcal{A}(I)$, $\mathcal{E}$ or $\mathcal{R}$. Let A^n be one of the rings E_ρ^n, $\mathcal{A}(I^n)$, $\mathcal{E}$ or $\mathcal{R}$. Let M be a differential module over A^n, let $\phi_n^*(M)$ be its pull-back over A. Then we have*

$$Ray(\phi_n^*(M),|.|_{c,r})=\sup(|c|,r)^{1-n}Ray(M,|.|_{c^n,r'})$$

where $r'=\max(r^n,|c|^{n-1}r)$ (cf. equation 5.2.3).

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