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# Dynamics of elastic bodies connected by a thin soft viscoelastic layer

C. Licht\*, A. Léger†, S. Orankitjaroen ‡, and A. Ould Khaoua §

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**Abstract** A dynamic study was performed on a structure consisting of two three-dimensional linearly elastic bodies connected by a thin soft nonlinear Kelvin-Voigt viscoelastic adhesive layer. The adhesive is assumed to be viscoelastic of Kelvin-Voigt generalized type, which makes it possible to deal with a relatively wide range of physical behavior by choosing suitable dissipation potentials. In the static and purely elastic case, convergence results when geometrical and mechanical parameters tend to zero have already been obtained using variational convergence methods. To obtain convergence results in the dynamic case, the main tool, as in the quasistatic case, is a nonlinear version of Trotter’s theory of approximation of semigroups acting on variable Hilbert spaces. The limit problem involves a mechanical constraint imposed along the surface to which the layer shrinks. The meaning of this limit with respect to the relative behavior of the parameters is discussed. The problem applies in particular to wave phenomena in bonded domains.

**Keywords:** Approximation of semigroups, Bonding problem, Kelvin-Voigt viscoelasticity, Trotter’s theory, Variational convergence, Wave propagation.

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# 1 Introduction

The aim of this study is to perform a mathematical analysis of the so-called bonding problem in the case of two massive linearly elastic solids connected by a thin soft nonlinear Kelvin-Voigt viscoelastic adhesive layer. The two solids are the adherents, and the thin layer is the adhesive consisting of glue or a weld, for example. In the static case, models of this kind, which are classically referred to as the junction problem, describe how this bonded structure behaves when the thickness of the adhesive is smaller and smaller. The key point addressed here in order to obtain a simplified but sufficiently accurate model is determining the conditions under which the thickness of the adhesive, which is very small in the physical problem, can be “approximated” by zero, the model itself and its accuracy being given by a convergence result. From this point of view, these bonding problems are very similar to the problem of justifying models for plates or shells. In the dynamic case, bonding problems mostly deal with vibrations or wave phenomena concerning in particular the transmission of acoustic waves through the thin layer. The field of applications is very wide, ranging from seismology to nondestructive testing. The results obtained can usually be applied to detect the damage or delamination of the thin layer.

The first models for thin adhesive layers were developed in the fifties by physicists (see for example [1]), and in the first rheological models in the field of seismology, the adhesive layer was replaced by an areal distribution of springs. Although these models were widely used, their range of applicability was not established, which was not an easy task in dynamics, nor was the physical behavior of the layer, even when the values of the parameters were improved by making comparisons with experimental data. Although the use of finite element calculations recently made it possible to take complex behavior of the adhesive into account [2], it was still difficult to interpret the adhesive as an interface constraint, and the thinness of the mesh required in the adhesive made the problem increasingly unwieldy, so that dynamic studies on large time intervals were practically impossible, or led to more and more ill-conditioned numerical problems as the adhesive became increasingly thin.

Models for bonding problems supported by mathematical justifications are always based nowadays on asymptotic analysis. Up to very recently, only the static case has been dealt with in this way. The basic tools used for this purpose were the same as those used to justify structural models. In the linearly elastic case, asymptotic expansions were inserted into the equilibrium equations, which led to families of problems depending on the thinness. Based on studies on these families, convergence results were obtained by making the thinness parameter tend to zero. This approach has yielded a large number of results relating to the theory of structures, homogenization and bonding problems. But the mathematical foundations of the analysis have gradually changed. Even when the results had already been obtained, as in the proof of the equilibrium equations for thin linearly elastic structures, the use of asymptotic expansions has gradually been replaced by that of variational convergence methods [3]. These methods consist basically in establishing the convergence of a sequence  $(\mathcal{F}_n)$  of energy functionals towards a functional  $\mathcal{F}_\infty$ , in such a way that the minima and the minimizers of the  $\mathcal{F}_n$  are also converging towards those of  $\mathcal{F}_\infty$ . As a basic tool, this convergence requires the proof that some lower and upper bounds of sequences of bounded energy do coincide. Now the use of variational convergence has yielded new results for bonding problems, first in the framework of general nonlinear elasticity with superlinear growth of the energy density [4], which was previously only obtained formally using asymptotic expansions, and then for the case of linear growth of the energy density, which includes some models of cracks or plasticity [5].

But since variational convergence is closely related to minimization problems, it can give convergence results only in the case of equilibrium problems, and other theoretical tools are required for analyzing quasistatic and dynamic problems. These tools have been given by Trotter’s theory of convergence of semigroups of operators acting on variable Hilbert spaces [6]. In short, Trotter’s theorem states that if the equilibrium problems converge, then the corresponding evolution problems will also converge, which seemed to be particularly relevant in the case of the dynamic analysis of bonding problems since the static case has already been dealt with using variational convergence methods. This approach was first presented in [7] in the case of a linearly elastic adhesive. The main qualitative result was not only that the thin layer can be replaced by a mechanical constraint, but also that this constraint is the same as that obtained for the limit of stationary bonding problems. Since we were strongly motivated by physical considerations as regards the behavior of the adhesive, it was proposed to deal with the dissipative case,

focusing in a first step for the sake of clarity on the case of thin soft viscoelastic layer of the nonlinear Kelvin-Voigt type. The more general case of a generalized standard material [8], [9] will be treated in a forthcoming study.

This paper consists of the following main sections:

- The elastodynamic problem is stated in section 2. The geometry of the domain, the behavior of its various constituents, and the set of parameters of interest are presented. The dynamic equations are then given in the classical local and weak forms.
- In section 3, the problem is rewritten in the form of a nonlinear evolution equation posed in a parametrized Hilbert space of possible states with finite energy. Since this equation is governed by a maximal monotone operator, existence and uniqueness follow.
- The next section deals with the asymptotic analysis, which is the main part of this study. It is performed in several steps, starting with some assumptions about the parameters, and arriving at the convergence in the sense of Trotter of the solutions to the sequence of nonlinear evolution equations.
- Lastly, the limit problem is given. In particular, the mechanical constraint, which can be used instead of the thin adhesive layer, is given explicitly, and discussed in terms of the relative asymptotic behavior of the geometrical and mechanical parameters.

## 2 Setting the problem

As usual, we make no difference between  $\mathbb{R}^3$  and the physical Euclidean space, the orthonormal basis of which is denoted by  $\{e^1, e^2, e^3\}$ , and for all  $\xi = (\xi_1, \xi_2, \xi_3)$  in  $\mathbb{R}^3$ ,  $\widehat{\xi}$  stands for  $(\xi_1, \xi_2)$ . We will study the dynamic response of a structure consisting of two adherents connected by a thin adhesive layer, which is subjected to a given load. More specifically, the reference configuration of the structure is a bounded connected open subset  $\Omega$  of  $\mathbb{R}^3$  with a Lipschitz-continuous boundary  $\partial\Omega$ . Its intersection  $S$  with  $\{x_3 = 0\}$  is assumed to have a positive two-dimensional Hausdorff measure  $\mathcal{H}^2(S)$ , and it is also assumed that there exists  $\varepsilon_0 > 0$  such that  $B_{\varepsilon_0} := \{x \in \Omega; |x_3| < \varepsilon_0\}$  is equal to  $S \times (-\varepsilon_0, \varepsilon_0)$ . Let  $\varepsilon < \varepsilon_0$ , then the adhesive occupies the layer  $B_\varepsilon$  while each of the two adherents occupies  $\Omega_\varepsilon^\pm := \{x \in \Omega; \pm x_3 > \varepsilon\}$ , and let  $\Omega_\varepsilon = \Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$ . Adherents and adhesive are assumed to be perfectly stuck together along  $S_\varepsilon = S_\varepsilon^+ \cup S_\varepsilon^-$ ,  $S_\varepsilon^\pm = \{x \in \Omega; x_3 = \pm\varepsilon\}$ . The structure is clamped on a part  $\Gamma_0$  of  $\partial\Omega$ , with  $\mathcal{H}^2(\Gamma_0) > 0$ , and is subjected to body forces in  $\Omega$  and surface forces on  $\Gamma_1 = \partial\Omega \setminus \Gamma_0$  having densities  $f$  and  $g$ , respectively, during the time interval  $[0, T]$ ; let  $\Gamma_0^\pm = \Gamma_0 \cap \{\pm x_3 > 0\}$ . The adherents are modeled as linearly elastic materials with a strain energy density  $W$  such that

$$\begin{cases} W(x, e) = \frac{1}{2}a(x)e \cdot e & \text{a.e. } x \in \Omega, \forall e \in S^3 \\ a \in L^\infty(\Omega; \text{Lin}(S^3)); \quad \exists \alpha, \beta > 0 \text{ s.t. } \alpha|e|^2 \leq a(x)e \cdot e \leq \beta|e|^2 & \forall e \in S^3 \end{cases} \quad (2.1)$$

where  $S^3$  is the space of  $(3 \times 3)$  symmetric matrices with the usual inner product and norm denoted by  $\cdot$  and  $|\cdot|$  (as for  $\mathbb{R}^3$ ), and  $\text{Lin}(S^3)$  denotes the space of linear mappings from  $S^3$  into  $S^3$ . The adhesive is assumed to be homogeneous, isotropic and ‘‘viscoelastic of Kelvin-Voigt generalized type’’. Its strain energy density reads as:

$$W_{\lambda\mu}(e) = \frac{\lambda}{2}(\text{tr } e)^2 + \mu|e|^2, \quad \text{tr } e = e_{11} + e_{22} + e_{33}, \quad \forall e \in S^3, \quad (2.2)$$

while its dissipation potential is denoted by  $b\mathcal{D}$ , where  $\lambda, \mu, b$  are positive real numbers and  $\mathcal{D}$  is a convex function satisfying

$$\exists p \in [1, 2], \exists \alpha', \beta' > 0; \quad \alpha'|e|^p \leq \mathcal{D}(e) \leq \beta'(1 + |e|^p) \quad \forall e \in S^3. \quad (2.3)$$

Let  $\rho > 0$  and  $\bar{\rho}_M > \bar{\rho}_m > 0$ . If  $\bar{\rho}$  is a measurable function such that  $\bar{\rho}_m \leq \bar{\rho}(x) \leq \bar{\rho}_M$  a.e.  $x$  in  $\Omega$ , the density  $\gamma$  of the structure is

$$\gamma(x) = \begin{cases} \bar{\rho}(x) & \text{a.e. } x \text{ in } \Omega_\varepsilon \\ \rho & \text{a.e. } x \text{ in } B_\varepsilon. \end{cases} \quad (2.4)$$

Hence, the problem of determining the dynamic evolution of the structure involves a quintuplet  $s := (\varepsilon, \lambda, \mu, b, \rho)$  of data and the equations satisfied by the fields of displacement  $u_s$  and stress  $\sigma_s$  are:

$$(P_s) \quad \begin{cases} \gamma \frac{\partial^2 u_s}{\partial t^2} = \operatorname{div} \sigma_s + f & \text{in } \Omega \times (0, T] \\ \sigma_s = ae(u_s) & \text{in } \Omega_\varepsilon \times (0, T] \\ \sigma_s \in \lambda \operatorname{tr}(e(u_s))I + 2\mu e(u_s) + b\partial\mathcal{D}\left(e\left(\frac{\partial u_s}{\partial t}\right)\right) & \text{in } B_\varepsilon \times (0, T] \\ u_s = 0 & \text{on } \Gamma_0 \times (0, T] \\ \sigma_s n = g & \text{on } \Gamma_1 \times (0, T] \\ \left(u_s(\cdot, 0), \frac{\partial u_s}{\partial t}(\cdot, 0)\right) = (u_s^0, v_s^0) := U_s^o & \text{in } \Omega \end{cases}$$

where  $t$  naturally denotes the time and  $U_s^o := (u_s^0, v_s^0)$  is the initial state,  $I$  is the identity matrix of  $S^3$ ,  $e(u)$  is the linearized strain tensor associated with the vector field  $u$  (the symmetric part of  $\nabla u$ , the gradient of  $u$ ) and, from now on,  $\partial J(v)$  will systematically denote the subdifferential at  $v$  of any lower semicontinuous convex function  $J$ , while  $DJ(v)$  denotes the differential at  $v$  of any Fréchet-differentiable function  $J$ . A “formally equivalent” formulation of  $(P_s)$  will clearly be

$$(P_s) \quad \begin{cases} \text{Find } u_s \text{ sufficiently smooth in } \Omega \times [0, T] \text{ such that } u_s = 0 \text{ on } \Gamma_0 \times (0, T], \\ \left(u_s(\cdot, 0), \frac{\partial u_s}{\partial t}(\cdot, 0)\right) = U_s^o \text{ and there exists } \xi \text{ in } \partial\mathcal{D}\left(e\left(\frac{\partial u_s}{\partial t}\right)\right) \text{ satisfying:} \\ \int_\Omega \gamma \frac{\partial^2 u_s}{\partial t^2} \cdot v \, dx + \int_{\Omega_\varepsilon} ae(u_s) \cdot e(v) \, dx + \int_{B_\varepsilon} DW_{\lambda\mu}(e(u_s)) \cdot e(v) \, dx + b \int_{B_\varepsilon} \xi \cdot e(v) \, dx \\ = \int_\Omega f \cdot v \, dx + \int_{\Gamma_1} g \cdot v \, d\mathcal{H}^2 \\ \text{for all } v \text{ sufficiently smooth in } \Omega \text{ and vanishing on } \Gamma_0. \end{cases}$$

We will use this formulation which results directly from the principle of virtual power to show in the next section that  $(P_s)$  has a unique solution in a suitable sense and, in section 4, to study the asymptotic behavior of  $u_s$  when  $s$ , *which is regarded as a parameter*, tends to its natural limit. In what follows,  $C$  denotes various constants which can differ from one line to another.

### 3 Existence and uniqueness

Assuming

$$(f, g) \in BV(0, T; L^2(\Omega; \mathbb{R}^3)) \times BV^{(2)}(0, T; L^2(\Gamma_1; \mathbb{R}^3)) \quad (\text{H1})$$

where, for any Banach space  $X$ ,  $BV(0, T; X)$  is the subspace of  $L^1(0, T; X)$  consisting of all the elements whose time derivative in the sense of distributions is a bounded  $X$ -valued measure on  $(0, T)$ , and  $BV^{(2)}(0, T; X)$  is the subspace of  $BV(0, T; X)$  consisting of all elements whose time derivative in the sense of distributions belongs to  $BV(0, T; X)$ .

We seek  $u_s$  having the form

$$u_s = u_s^e + u_s^r, \quad (3.1)$$

where  $u_s^e$  is the unique solution to

$$u_s^e(t) \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3); \quad \varphi_s(u_s^e(t), v) = L(t)(v) \quad \forall v \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \quad \forall t \in [0, T], \quad (3.2)$$

where

$$\begin{aligned} \varphi_s(v, v') &:= \int_{\Omega_\varepsilon} ae(v) \cdot e(v') \, dx + \int_{B_\varepsilon} DW_{\lambda\mu}(e(v)) \cdot e(v') \, dx \quad \forall v, v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3), \\ \Phi_s(v) &:= \varphi_s(v, v), \end{aligned} \quad (3.3)$$

$$L(t)(v) := \int_{\Gamma_1} g(x, t) \cdot v(x) d\mathcal{H}^2 \quad \forall v \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \quad \forall t \in [0, T], \quad (3.4)$$

and where  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  is the closed subspace of  $H^1(\Omega; \mathbb{R}^3)$  consisting of the elements with vanishing traces on  $\Gamma_0$ . Note that this notation  $W_g^{1,q}(G; \mathbb{R}^n)$  will be systematically used for any  $G \subset \mathbb{R}^n$ ,  $g \subset \partial G$  and Sobolev space  $W^{1,q}(G; \mathbb{R}^n)$ ,  $1 \leq q \leq \infty$ . Since  $g \mapsto u_s^e$  is linear continuous from  $L^2(\Gamma_1; \mathbb{R}^3)$  into  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$ , we have

$$u_s^e \in BV^{(2)}(0, T; H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)). \quad (3.5)$$

The remaining part  $u_s^r$  of  $u_s$  will therefore satisfy an evolution equation governed by a maximal monotone operator  $A_s$  defined in a Hilbert space  $H_s$  of possible states with finite total mechanical (kinetic + strain) energy. The space of velocities,  $L^2(\Omega; \mathbb{R}^3)$ , is equipped with the following inner product  $k_s$  and the square of norm  $K_s$  associated with the true kinetic energy:

$$k_s(v, v') := \int_{\Omega} \gamma(x) v(x) \cdot v'(x) dx, \quad K_s(v) := k_s(v, v), \quad \forall v, v' \in L^2(\Omega; \mathbb{R}^3) \quad (3.6)$$

while the space of displacements,  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$ , is equipped with the inner product  $\varphi_s$  defined in (3.3), which is equivalent to the usual one by Korn inequality. Hence

$$H_s := H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \quad (3.7)$$

where, for all  $U = (u, v)$  and  $U' = (u', v')$  in  $H_s$ , the inner product and norm are

$$(U, U')_s := \varphi_s(u, u') + k_s(v, v'), \quad |U|_s^2 := (U, U)_s, \quad (3.8)$$

while  $A_s$  is defined by

$$\left\{ \begin{array}{l} D(A_s) = \left\{ U = (u, v) \in H_s; \begin{array}{l} \text{i) } v \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \\ \text{ii) } \exists (w, \xi) \in L^2(\Omega; \mathbb{R}^3) \times \partial\mathcal{D}(e(v)) \text{ with} \\ k_s(w, v') + \varphi_s(u, v') + b \int_{B_\varepsilon} \xi \cdot e(v') dx = 0 \quad \forall v' \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \end{array} \right\} \\ A_s U = (-v, 0) + \{ (0, -w); w \text{ satisfies ii) of definition of } D(A_s) \}. \end{array} \right. \quad (3.9)$$

**Proposition 3.1.**

The operator  $A_s$  is a maximal monotone operator and, for all  $\psi = (\psi^1, \psi^2)$  in  $H_s$ ,

$$\left\{ \begin{array}{l} \bar{U}_s = (\bar{u}_s, \bar{v}_s) \text{ s.t.} \\ \bar{U}_s + A_s \bar{U}_s \ni \psi \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} J_s(\bar{v}_s) \leq J_s(v) \quad \forall v \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3), \\ J_s(v) := \frac{1}{2} K_s(v) - k_s(\psi^2, v) + \frac{1}{2} \Phi_s(v) + \varphi_s(\psi^1, v) + b \int_{B_\varepsilon} \mathcal{D}(e(v)) dx \\ \bar{u}_s = \bar{v}_s + \psi^1. \end{array} \right.$$

*Proof.* Let  $U = (u, v)$  and  $U' = (u', v')$  in  $D(A_s)$ ,  $-(v, w)$  in  $A_s U$  and  $-(v', w')$  in  $A_s U'$ , then the definition of  $D(A_s)$  shows that there exists  $(\xi, \xi')$  in  $\partial\mathcal{D}(e(v)) \times \partial\mathcal{D}(e(v'))$  such that

$$\varphi_s(-v + v', u - u') + k_s(-w + w', v - v') = b \int_{B_\varepsilon} (\xi' - \xi) \cdot e(v' - v) dx \geq 0,$$

and hence the monotonicity of  $A_s$  stems from that of  $\partial\mathcal{D}$ .

If  $\bar{U}_s + A_s \bar{U}_s \ni \psi$ , the very definition of  $A_s$  means that  $\bar{u}_s - \bar{v}_s = \psi^1$  and that there exists  $\xi$  in  $\partial\mathcal{D}(e(\bar{v}_s))$  such that

$$k_s(\bar{v}_s - \psi^2, v) + \varphi_s(\bar{v}_s + \psi^1, v) + b \int_{B_\varepsilon} \xi \cdot e(v) dx = 0 \quad \forall v \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \quad (3.10)$$

that is to say,  $\bar{v}_s$  is the unique minimizer on  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  of the strictly convex, continuous and coercive function  $J_s$ . Conversely, if  $\bar{v}_s$  is the minimizer of  $J_s$  then there exists  $\xi$  in  $\partial\mathcal{D}(e(\bar{v}_s))$  satisfying (3.10) so that  $\bar{v}_s - \psi^2$  satisfies the point ii) in the definition of  $D(A_s)$ . Thus  $\bar{U}_s := (\bar{u}_s := \bar{v}_s + \psi^1, \bar{v}_s)$  belongs to  $D(A_s)$  and  $\bar{U}_s + A_s \bar{U}_s \ni \psi$ .  $\square$

Then, taking into account (H1), (3.1), (3.2), (3.5), (3.9), it can be checked straightforwardly that  $(P_s)$  is “formally equivalent” to

$$\begin{cases} \frac{dU_s^r}{dt} + A_s U_s^r \ni F_s \\ U_s^r(0) = U_s^o - (u_s^e(0), 0) \end{cases} \quad (3.11)$$

where

$$F_s = \left( -\frac{du_s^e}{dt}, f/\gamma \right). \quad (3.12)$$

A result of [11] therefore yields

**Theorem 3.1.**

If  $(f, g)$  satisfies (H1) and  $U_s^o \in (u_s^e(0), 0) + D(A_s)$ , then (3.11) has a unique solution such that  $U_s^r$  belongs to  $W^{1,\infty}(0, T; H_s)$  and the first line of (3.11) is satisfied almost everywhere in  $[0, T]$ . Hence, there exists a unique  $u_s$  in  $W^{1,\infty}(0, T; H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)) \cap W^{2,\infty}(0, T; L^2(\Omega; \mathbb{R}^3))$  which does satisfy

$$\left\{ \begin{array}{l} \exists \xi \in \partial \mathcal{D} \left( e \left( \frac{du_s}{dt} \right) \right) \text{ such that} \\ \int_{\Omega} \gamma \frac{d^2 u_s}{dt^2} v \, dx + \int_{\Omega_\varepsilon} a e(u_s) \cdot e(v) \, dx + \int_{B_\varepsilon} DW_{\lambda\mu}(e(u_s)) \cdot e(v) \, dx + b \int_{B_\varepsilon} \xi \cdot e(v) \, dx \\ \quad = \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_1} g \cdot v \, d\mathcal{H}^2, \quad \forall v \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \text{ a.e. } t \in (0, T] \\ u_s(0) = u_s^o, \quad \frac{du_s}{dt}(0) = v_s^o. \end{array} \right. \quad (3.13)$$

We set

$$U_s^e = (u_s^e, 0), \quad U_s = U_s^r + U_s^e. \quad (3.14)$$

## 4 Asymptotic behavior

We will now present a simplified but accurate enough model for the initial physical situation by determining the asymptotic behavior when the quintuplet  $s$  of geometrical and mechanical *data* is regarded as a quintuplet of *parameters* taking values in a countable subset of  $[0, +\infty]^5$  with a unique cluster point  $\bar{s}$ . Moreover, *taking into account* the low thickness and stiffness of the layer and the fact that its density may be low, we assume:

$$\left\{ \begin{array}{l} \text{i)} \quad \bar{s} \in \{0\} \times [0, +\infty)^2 \times [0, +\infty) \times [0, +\infty) \\ \text{ii)} \quad \exists (\bar{\lambda}, \bar{\mu}) \in [0, +\infty)^2 \text{ s.t. } (\lambda/2\varepsilon, \mu/2\varepsilon) \rightarrow (\bar{\lambda}, \bar{\mu}) \\ \text{iii)} \quad \lim_{s \rightarrow \bar{s}} b\varepsilon = 0, \exists \bar{b} \in [0, +\infty) \text{ s.t. } b/(2\varepsilon)^{p-1} \rightarrow \bar{b} \\ \text{iv)} \quad \bar{\mu} \in (0, +\infty] \text{ if } \min\{\mathcal{H}^2(\Gamma_0^\pm)\} = 0 \\ \text{v)} \quad \overline{\lim}_{s \rightarrow \bar{s}} \varepsilon^2/\mu < +\infty \\ \text{vi)} \quad \exists r \in [0, 1) \text{ s.t. } \overline{\lim}_{s \rightarrow \bar{s}} \varepsilon^r/\rho \leq C. \end{array} \right. \quad (\text{H2})$$

Note that  $\lambda, \mu, \rho$  may remain bounded. Assumptions (H2), iv)-vi) say that the stiffness and density are not “too low”, and in addition (H2), v)-vi) are appropriate from the mathematical point of view for stating some convergence results in standard functional spaces.

### 4.1 A candidate for the limit behavior

From a previous study [10] on the *quasistatic* evolution of a thin dissipative layer, it is easy to guess what the limit behavior may be, and we therefore introduce the following concepts. We will bring out three cases indexed by  $I$ :  $I = 1$  if  $(\bar{\lambda}, \bar{\mu}) \in [0, +\infty)^2$ ;  $I = 2$  if  $(\bar{\lambda}, \bar{\mu}) \in \{+\infty\} \times [0, +\infty)$ ;  $I = 3$  if  $\bar{\mu} = +\infty$ . Let

$${}^1\mathbf{H}^1 = H_{\Gamma_0}^1(\Omega \setminus S; \mathbb{R}^3), \quad {}^2\mathbf{H}^1 = \{u \in H_{\Gamma_0}^1(\Omega \setminus S; \mathbb{R}^3); [u]_3 = 0\}, \quad {}^3\mathbf{H}^1 = H_{\Gamma_0}^1(\Omega; \mathbb{R}^3), \quad (4.1)$$

where, since any element in  $H_{\Gamma_0}^1(\Omega \setminus S; \mathbb{R}^3)$  has restrictions  $u^\pm$  to  $\Omega^\pm$  in  $H^1(\Omega^\pm; \mathbb{R}^3)$ , we denote the difference between the traces on  $S$  of  $u^+$  and  $u^-$  by  $[u]$  which belongs to  $L^2(S; \mathbb{R}^3)$ . Let us introduce the following bilinear form and the associated quadratic form which, from (H2), is continuous and coercive on  ${}^1\mathbf{H}^1$ :

$${}^1\varphi(u, v) := \int_{\Omega \setminus S} ae(u) \cdot e(v) dx + \int_S D\overline{W}_{\overline{\lambda\mu}}([u]) \cdot [v] d\hat{x}, \quad {}^1\Phi(u) := {}^1\varphi(u, u) \quad (4.2)$$

where for all  $u$  in  $H_{\Gamma_0}^1(\Omega \setminus S; \mathbb{R}^3)$  we still keep  $e(u)$  to denote the symmetric part of the gradient of  $u$  in the sense of the distributions of  $\mathcal{D}'(\Omega \setminus S)$  and  $\overline{W}_{\overline{\lambda\mu}}$  is the quadratic form on  $L^2(S; \mathbb{R}^3)$  defined by

$$\begin{cases} \bullet \text{ I} = 1 & \overline{W}_{\overline{\lambda\mu}}(u) = W_{\overline{\lambda\mu}}(u \otimes_S e^3), \\ \bullet \text{ I} = 2 & \overline{W}_{\overline{\lambda\mu}}(u) = W_{\overline{\lambda\mu}}(\hat{u}, 0) \otimes_S e^3, \\ \bullet \text{ I} = 3 & \overline{W}_{\overline{\lambda\mu}}(u) = 0, \end{cases} \quad (4.3)$$

where  $(\xi \otimes_S \zeta)_{ij} = \frac{1}{2}(\xi_i \zeta_j + \xi_j \zeta_i) \forall i, j \in \{1, 2, 3\}$ ,  $\forall \xi, \zeta \in \mathbb{R}^3$ . For all I in  $\{1, 2, 3\}$ , there obviously exists a unique  ${}^1u^e$  in  $BV^{(2)}(0, T; {}^1\mathbf{H}^1)$  such that

$${}^1\varphi({}^1u^e(t), v) = L(t)(v) \quad \forall v \in {}^1\mathbf{H}^1, \quad \forall t \in [0, T]. \quad (4.4)$$

As in the previous section, the expected limit of  $u_s$  will be the sum of  ${}^1u^e$  and some  ${}^1u^r$  solution to an evolution equation set in the following framework.

The space of velocities,  $L^2(\Omega; \mathbb{R}^3)$ , is equipped with the following inner product  $k$  and square of norm  $K$  equivalent to the usual ones and associated with the “limit” kinetic energy:

$$k(u, v) := \int_{\Omega} \overline{\rho}(x) u(x) \cdot v(x) dx, \quad K(u) := k(u, u) \quad \forall u, v \in L^2(\Omega; \mathbb{R}^3) \quad (4.5)$$

while, the space of displacement,  ${}^1\mathbf{H}^1$ , is equipped with the inner product  ${}^1\varphi$  so that the Hilbert space of possible states with finite mechanical energy is

$${}^1\mathbf{H} = {}^1\mathbf{H}^1 \times L^2(\Omega; \mathbb{R}^3) \quad (4.6)$$

where, for all  $U = (u, v)$  and  $U' = (u', v')$  in  ${}^1\mathbf{H}$ , the inner product and norm are

$$((U, U'))_{\mathbf{I}} := {}^1\varphi(u, u') + k(v, v') \quad \|U\|_{\mathbf{I}}^2 := (U, U)_{\mathbf{I}}. \quad (4.7)$$

Denoting the limit dissipative function in  $L^2(S; \mathbb{R}^3)$  by

$$\overline{\mathcal{D}}(q) = \begin{cases} \overline{b}\mathcal{D}^{\infty, p}(q \otimes_S e^3) & \text{if } \overline{b} < +\infty \\ I_{\{0\}}(q) & \text{if } \overline{b} = \infty, \end{cases}$$

where  $I_C$  is the indicator function of any convex set  $C$  and

$$\mathcal{D}^{\infty, p}(e') = \overline{\lim}_{t \rightarrow \infty} \mathcal{D}(te')/t^p,$$

where it is assumed that:

$$\exists \delta > 0 \text{ and } \theta \in (0, p); \quad |\mathcal{D}(e) - \mathcal{D}^{\infty, p}(e)| \leq \delta(1 + |e|^\theta) \quad \forall e \in S^3, \quad (\text{H3})$$

we can define the evolution operator  ${}^1A$  by:

$$\left\{ \begin{array}{l} D({}^1A) = \left\{ U = (u, v) \in {}^1\mathbf{H}; \begin{array}{l} \text{i) } v \in {}^1\mathbf{H}^1 \text{ and } [v] = 0 \text{ if } \overline{b} = \infty, \\ \text{ii) } \exists (w, \xi) \in L^2(\Omega; \mathbb{R}^3) \times \partial\overline{\mathcal{D}}([v]) \text{ s.t.} \\ \quad k(w, v') + {}^1\varphi(u, v') + \int_S \xi \cdot [v'] d\hat{x} = 0 \quad \forall v' \text{ satisfying i) } \end{array} \right\} \\ {}^1AU = (-v, 0) + \{ (0, -w); w \text{ satisfying ii) } \}. \end{array} \right. \quad (4.8)$$



Arguing as in the case of  $A_s$ , it can easily be checked that  ${}^1A$  is maximal monotone in  ${}^1\mathbf{H}$  and especially, that for all  $\psi = (\psi^1, \psi^2)$  in  ${}^1\mathbf{H}$ :

$$\begin{cases} {}^1\bar{U} = ({}^1\bar{u}, {}^1\bar{v}) \\ {}^1\bar{U} + {}^1A{}^1\bar{U} \ni \psi \end{cases} \Leftrightarrow \begin{cases} {}^1J({}^1\bar{v}) \leq {}^1J(v) := \frac{1}{2}K(v) - k(\psi^2, v) + \frac{1}{2}{}^1\Phi(v) + {}^1\varphi(\psi^1, v) + \int_S \bar{D}([v]) d\hat{x} \quad \forall v \in {}^1\mathbf{H}^1 \\ {}^1\bar{u} = {}^1\bar{v} + \psi^1 \end{cases} \quad (4.9)$$

We are now in a position to introduce an evolution equation in  ${}^1\mathbf{H}$  which will describe the asymptotic behavior of  $u_s$ :

$$\begin{cases} \frac{d{}^1U^r}{dt} + {}^1A{}^1U^r \ni {}^1F \\ {}^1U^r(0) = {}^1U^{r_0} \end{cases} \quad (4.10)$$

where  ${}^1U_0^r$  will be specified later on and

$${}^1F = \left( -\frac{d{}^1u^e}{dt}, f/\bar{\rho} \right). \quad (4.11)$$

As with (3.11) (3.12), a classical result of [11] gives

**Proposition 4.1.**

*If  ${}^1U^{r_0}$  belongs to  $D({}^1A)$  and if  $(f, g)$  satisfies (H1), then (4.10) has a unique solution such that  $U^r$  belongs to  $W^{1,\infty}(0, T; {}^1\mathbf{H}) \cap W^{2,\infty}(0, T; L^2(\Omega; \mathbb{R}^3))$  and the first line of (4.10) holds almost everywhere.*

We set:

$${}^1U^e = ({}^1u^e, 0), \quad {}^1U = {}^1U^e + {}^1U^r. \quad (4.12)$$

## 4.2 Convergence

To prove the convergence of  $u_s$  toward  ${}^1u = {}^1u^e + {}^1u^r$ , we will use the framework of a nonlinear version of Trotter's theory of approximation of semigroups acting on variable spaces (see the Appendix of [12]) because  $u_s^r$  and  ${}^1u^r$  do not live in the same space. First, we introduce  ${}^1P_s$ , which is linear continuous from  ${}^1\mathbf{H}$  to  $H_s$ , in order to "compare" the elements in  ${}^1\mathbf{H}$  and  $H_s$ . For this purpose, we take a smoothing operator, which is also linear continuous from  ${}^1\mathbf{H}^1$  to  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  and defined by

$$R_\varepsilon u(x) = \begin{cases} u^s(x) + \text{Min}\{|x_3|/\varepsilon, 1\}u^a(x) & \forall x \text{ in } B_\varepsilon \\ u(x) & \forall x \text{ in } \Omega_\varepsilon \end{cases} \quad (4.13)$$

where  $u^s(x) = \frac{1}{2}(u(\hat{x}, x_3) + u(\hat{x}, -x_3))$ ,  $u^a(x) = \frac{1}{2}(u(\hat{x}, x_3) - u(\hat{x}, -x_3))$ . If

$$\begin{aligned} {}^1P_s^1 u &:= R_\varepsilon u, & {}^2P_s^1 u &:= R_\varepsilon(\hat{u}, 0) + (0, u_3), & {}^3P_s^1 u &:= u \quad \forall u \in {}^1\mathbf{H}^1, \\ {}^1P_s^2 v &:= v \quad \forall v \in L^2(\Omega; \mathbb{R}^3) \quad \forall I \in \{1, 2, 3\}, \end{aligned} \quad (4.14)$$

then  ${}^1P_s = ({}^1P_s^1, {}^1P_s^2)$  has the fundamental properties:

**Proposition 4.2.** i) *There exists a strictly positive constant  $C$  such that  $|{}^1P_s U|_s \leq C\|U\|_I, \forall U \in {}^1\mathbf{H}$ .*

ii) *When  $s$  tends to  $\bar{s}$ ,  ${}^1P_s$  satisfies  $\lim_{s \rightarrow \bar{s}} |{}^1P_s U|_s = \|U\|_I$ .*

*Proof.* Let  $U = (u, v)$  be arbitrary in  ${}^1\mathbf{H}$ . The boundedness of  $\rho$  obviously means that  $K_s({}^1P_s^2 v) \leq CK(v)$  and  $\lim_{s \rightarrow \bar{s}} K_s({}^1P_s^2 v) = K(v)$ . It still remains to deal with  ${}^1P_s^1$ , where

$$\Phi_s({}^1P_s^1 u) = 2 \left( \int_{\Omega_\varepsilon} W(e(u)) dx + \int_{B_\varepsilon} W_{\lambda\mu}(e({}^1P_s^1 u)) dx \right).$$

In fact, recalling that  $B_\varepsilon = S \times (-\varepsilon, \varepsilon)$ ,  $\forall \varepsilon \in (0, \varepsilon_0)$ , we have the following estimate:

**Lemma 4.1.**

$$\int_{B_\varepsilon} \left| e(R_\varepsilon u) - \frac{[u] \otimes_S e^3}{2\varepsilon} \right|^q dx \leq C \int_{B_\varepsilon \setminus S} |\nabla u|^q dx \quad \forall u \in W_{\Gamma_0}^{1,q}(\Omega \setminus S; \mathbb{R}^3), \quad \forall q \in [1, +\infty).$$

*Proof.* Obviously

$$\begin{aligned} e(R_\varepsilon u) &= e(u^s) + \text{Min}\{|x_3|/\varepsilon, 1\}e(u^a) + \frac{\text{sgn } x_3}{\varepsilon} (u^a \otimes_S e^3), \\ \int_{B_\varepsilon \setminus S} |e(u^s) + \text{Min}\{|x_3|/\varepsilon, 1\}e(u^a)|^q dx &\leq C \int_{B_\varepsilon \setminus S} |\nabla u|^q dx, \\ \left| \text{sgn } x_3 (u^a(x) \otimes_S e^3) - \frac{[u](\hat{x})}{2} \otimes_S e^3 \right| &\leq \left| u^a(x) - \frac{[u](\hat{x})}{2} \right|. \end{aligned}$$

Since  $\frac{[u]}{2}$  is the trace on  $S$  of  $\pm(u^a)^\pm = \text{sgn } x_3 u^a$ , simply integrating with respect to  $x_3$  therefore gives the result required.  $\square$

Hence, when  $I = 1$ , the convexity of  $W_{\lambda\mu}$  and Lemma 4.1 yield

$$\begin{aligned} \Phi_s({}^I P_s^1 u) &\leq 2 \int_{\Omega \setminus S} W(e(u)) dx + 4 \left[ \int_{B_\varepsilon} W_{\lambda\mu} \left( \frac{[u] \otimes_S e^3}{2\varepsilon} \right) dx + \int_{B_\varepsilon} W_{\lambda\mu} \left( e(R_\varepsilon u) - \frac{[u] \otimes_S e^3}{2\varepsilon} \right) dx \right] \\ &\leq 2 \int_{\Omega \setminus S} W(e(u)) dx + C \left( \int_S \overline{W}_{\lambda\mu}([u]) d\hat{x} + \int_{B_\varepsilon \setminus S} |\nabla u|^2 dx \right) \\ &\leq C^I \Phi(u) \end{aligned}$$

which proves i). To establish ii), it suffices to note that

$$\begin{aligned} &\left| \int_{B_\varepsilon} W_{\lambda\mu}(e(R_\varepsilon u)) dx - \int_{B_\varepsilon} W_{\lambda\mu} \left( \frac{[u] \otimes_S e^3}{2\varepsilon} \right) dx \right| \\ &\leq C \text{Max}\{\lambda, \mu\} \left| e(R_\varepsilon u) - \frac{[u] \otimes_S e^3}{2\varepsilon} \right|_{L^2(B_\varepsilon; S^3)} \left( \left| \frac{[u] \otimes_S e^3}{2\varepsilon} \right|_{L^2(B_\varepsilon; S^3)} + \left| e(R_\varepsilon u) - \frac{[u] \otimes_S e^3}{2\varepsilon} \right|_{L^2(B_\varepsilon; S^3)} \right) \\ &\leq C |\nabla u|_{L^2(B_\varepsilon \setminus S; S^3)} \left( {}^I \Phi(u) \right)^{1/2}. \end{aligned}$$

When  $I = 3$ , i) and ii) stem immediately from the boundedness of  $\lambda, \mu$ , so that in the intermediate case  $I = 2$ , it suffices to combine the previous arguments.  $\square$

Next, we will say that:

$$U_s \text{ in } H_s \text{ converges in the sense of Trotter toward } U \text{ in } {}^I H \text{ if } \lim_{s \rightarrow \bar{s}} |{}^I P_s U - U_s|_s = 0. \quad (4.15)$$

Even if this notion is the ‘‘right one’’ from the mechanical point of view, it is useful to relate this convergence to some classical ones as stated in the following Proposition 4.3. First, let us recall that for all sets  $G$  contained in  $\Omega$ ,  $1_G$  denotes the characteristic function of  $G$  and that

$$\begin{aligned} LD(\Omega) &:= \{ v \in L^1(\Omega; \mathbb{R}^3); e(v) \in L^1(\Omega; S^3) \}, \\ BD(\Omega) &:= \{ v \in L^1(\Omega; \mathbb{R}^3); e(v) \text{ is a bounded measure on } \Omega \} \end{aligned}$$

are Banach spaces. We shall also say that:

$$\begin{cases} (u_s) \text{ } \tau\text{-converges toward } u \text{ if } (u_s) \text{ converges toward } u \\ \text{ in } L^2(\Omega; \mathbb{R}^3) \text{ when } \overline{\lim}_{s \rightarrow \bar{s}} (\varepsilon^2/\mu) = 0, \\ \text{ or in } L^q(\Omega; \mathbb{R}^3) \forall q < 2 \text{ when } \overline{\lim}_{s \rightarrow \bar{s}} (\varepsilon^2/\mu) \in (0, +\infty). \end{cases} \quad (4.16)$$

Then we have the following properties of Trotter convergence.

**Proposition 4.3.** For all  $U = (u, v)$  in  ${}^1\mathbf{H}$ , if  $U_s = (u_s, v_s)$  in  $H_s$  converges in the sense of Trotter toward  $U$ , then:

- i) for all positive  $\eta$ , the sequence  $(u_s)$  converges strongly in  $H_{\Gamma_0}^1(\Omega_\eta; \mathbb{R}^3)$  toward  $u$ ;
- ii)  $1_{\Omega_\varepsilon} e(u_s)$  converges strongly in  $L^2(\Omega \setminus S; S^3)$  toward  $e(u)$ ;
- iii)  $(u_s)$  converges weakly in  $BD(\Omega)$  toward  $u$  when  $\overline{\lim}_{s \rightarrow \bar{s}} (\varepsilon/\mu) < +\infty$ ;  $(u_s)$  converges strongly in  $LD(\Omega)$  toward  $u$  when  $\bar{\mu} = \infty$ ;  $(u_s)$  converges strongly in  $W^{1,q}(\Omega; \mathbb{R}^3)$  toward  $u$  when  $\bar{\mu} = \infty$  and if there exists  $q$  in  $[1, 2)$  such that  $\overline{\lim}_{s \rightarrow \bar{s}} (\varepsilon^{2-q}/\mu) < +\infty$ ;
- iv)  $1_{\Omega_\varepsilon} u_s$  converges strongly in  $L^2(\Omega; \mathbb{R}^3)$  toward  $u$ ;
- v) the traces on  $S_\varepsilon^\pm$  of  $(u_s)$ , regarded as elements of  $L^2(S; \mathbb{R}^3)$ , converge strongly in  $L^2(S; \mathbb{R}^3)$  toward the traces on  $S$  of  $u^\pm$ ;
- vi)  $(u_s)$  is bounded in  $L^2(\Omega; \mathbb{R}^3)$  and  $\tau$ -converges toward  $u$ ;
- vii)  $\lim_{s \rightarrow \bar{s}} \frac{\mu}{\varepsilon^2} |u_s - {}^1P_s^1 u|_{L^2(B_\varepsilon; \mathbb{R}^3)}^2 = 0$ ;
- viii)  $1_{\Omega_\varepsilon} v_s$  converges strongly in  $L^2(\Omega; \mathbb{R}^3)$  toward  $v$  and  $v_s$  converges strongly in  $L^{2/(1+r)}(\Omega; \mathbb{R}^3)$  toward  $u$ .

*Proof.* It is divided into two main steps.

**Step 1:** proof of points i) - iii).

The coercivity of  $W$  and  $W_{\lambda\mu}$  entails that

$$\lim_{s \rightarrow \bar{s}} \int_{\Omega_\varepsilon} |e(u_s - u)|^2 dx = 0 \quad (4.17)$$

$$\lim_{s \rightarrow \bar{s}} \mu \int_{B_\varepsilon} |e(u_s - {}^1P_s^1 u)|^2 dx = 0, \quad (4.18)$$

Point ii) is therefore obvious.

When  $\text{Min}\{\mathcal{H}^2(\Gamma_0^\pm)\} > 0$ , i) stems from Korn's inequality.

If one of  $\mathcal{H}^2(\Gamma_0^\pm)$  vanishes, say  $\mathcal{H}^2(\Gamma_0^-)$ , then taking (H2)-ii), (4.17), (4.18) and the Cauchy-Schwarz inequality into account yields that  $u_s - {}^1P_s^1 u$  converges strongly toward 0 in  $LD(\Omega)$  so that also converges strongly toward 0 in  $L^q(\Omega; \mathbb{R}^3) \forall q \in [1, 3/2]$  by the standard Sobolev-like embedding (see [13]).

As  ${}^1P_s^1 u$  obviously converges strongly toward  $u$  in  $L^2(\Omega; \mathbb{R}^3)$ , then  $u_s$  converges strongly in  $L^q(\Omega; \mathbb{R}^3)$  toward  $u$ . On the other hand, since (4.17) means that  $(u_s)$  converges strongly in  $L^2(\Omega_\eta^-; \mathbb{R}^3)/\mathcal{R}$  for all positive  $\eta$ , where  $\mathcal{R}$  is the (finite dimensional) set of rigid displacements, we deduce that  $u_s$  converges strongly in  $L^2(\Omega_\eta; \mathbb{R}^3)$  for all positive  $\eta$  which consequently complete the proof of point i).

Next, point iii) results from (4.17), (4.18) and Hölder inequality, and Lemma 4.1 when  $I < 3$  or  ${}^3P_s^1 u = u$ .

**Step 2:** proof of points iv) - viii).

To establish the other convergences we take into account the special geometry of  $B_{\varepsilon_0} : B_{\varepsilon_0} = S \times (-\varepsilon_0, \varepsilon_0)$  by splitting  $u_s$  into two parts  $\check{u}_s$  and  $\tilde{u}_s$ . Let  $\xi$  in  $C_c^\infty(\mathbb{R})$  be such that  $\xi(t) = 1$  if  $|t| \leq \varepsilon_0/3$ ,  $0 \leq \xi(t) \leq 1$  if  $\varepsilon_0/3 < |t| < 2\varepsilon_0/3$ ,  $\xi(t) = 0$  if  $|t| > 2\varepsilon_0/3$  and  $\check{u}_s$  defined by

$$\check{u}_s(x) = \begin{cases} \xi(x_3)u_s(x) & \text{if } x \in B_{\varepsilon_0} \\ 0 & \text{if } x \in \Omega_{\varepsilon_0}. \end{cases}$$

$\check{u}_s$  clearly belongs to  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  and the restriction of  $\check{u}_s$  to  $B_{\varepsilon_0}$  belongs to  $H_{S_{\varepsilon_0}}^1(B_{\varepsilon_0}; \mathbb{R}^3)$ . Then,  $\tilde{u}_s := u_s - \check{u}_s$  belongs to  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  and has a support included in  $\Omega_{\varepsilon_0/3}$ . Moreover

$$|e(\tilde{u}_s)|_{L^2(\Omega; S^3)} \leq |e(u_s)|_{L^2(\Omega_{\varepsilon_0/3}; S^3)} + C|u_s|_{L^2(B_{2\varepsilon_0/3} \setminus B_{\varepsilon_0/3}; \mathbb{R}^3)} \leq C$$

by duly taking (4.17) into account, as well as the previous convergence in  $H_{\Gamma_0}^1(\Omega_{\varepsilon_0/3}; \mathbb{R}^3)$ . Hence, arguing as in the case of  $(u_s)$ , we deduce that  $(\tilde{u}_s)$  is strongly relatively compact in  $L^2(\Omega; \mathbb{R}^3)$ .

To prove that  $(\tilde{u}_s)$  also  $\tau$ -converges, we introduce a kind of translation operator  $T_\varepsilon$  for  $\varepsilon \in (0, \varepsilon_0)$  which is linear continuous on  $H_{S_{\varepsilon_0}^\pm}^1(B_{\varepsilon_0}^\pm; \mathbb{R}^3)$  and defined by:

$$(T_\varepsilon w)(x) = \begin{cases} w(\hat{x}, x_3 + \text{sgn}(x_3)\varepsilon) & \text{if } |x_3| \leq \varepsilon_0 - \varepsilon \\ 0 & \text{if } |x_3| > \varepsilon_0 - \varepsilon. \end{cases}$$

Then, equation (4.14) and the weak convergence of  $(u_s)$  in  $H^1(\Omega_{\varepsilon_0/3}; \mathbb{R}^3)$  imply

$$\int_{B_{\varepsilon_0}^\pm} |e(T_\varepsilon \tilde{u}_s)|^2 dx = \int_{B_{\varepsilon_0}^\pm \setminus B_\varepsilon^\pm} |e(\tilde{u}_s)|^2 dx \leq C \left( \int_{B_{2\varepsilon_0/3} \setminus B_\varepsilon} |e(u_s)|^2 dx + \int_{B_{2\varepsilon_0/3} \setminus B_{\varepsilon_0/3}} |u_s|^2 dx \right) \leq C,$$

so that  $(T_\varepsilon \tilde{u}_s)$  is strongly relatively compact in  $L^2(B_{\varepsilon_0}; \mathbb{R}^3)$ . In addition, since  $\int_{B_{\varepsilon_0}} 1_{B_{\varepsilon_0} \setminus B_\varepsilon} |\tilde{u}_s|^2 dx = \int_{B_{\varepsilon_0}} |T_\varepsilon \tilde{u}_s|^2 dx$  and

$$\forall \varphi \in C_c^\infty(B_{\varepsilon_0}) \quad \lim_{s \rightarrow \bar{s}} \int_{B_{\varepsilon_0}} 1_{B_{\varepsilon_0} \setminus B_\varepsilon} \tilde{u}_s \varphi dx = \lim_{s \rightarrow \bar{s}} \int_{B_{\varepsilon_0}} T_\varepsilon \tilde{u}_s \varphi dx, \quad (4.19)$$

we deduce that  $1_{B_{\varepsilon_0} \setminus B_\varepsilon} \tilde{u}_s$  is relatively compact in  $L^2(B_{\varepsilon_0}; \mathbb{R}^3)$ .

Since  $(\tilde{u}_s)$  is strongly relatively compact in  $L^2(\Omega; \mathbb{R}^3)$ , point i) above and equation (4.19) on the one hand imply point iv), and on the other hand also imply that  $(T_\varepsilon \tilde{u}_s)^\pm$  converges weakly in  $H^1(B_{\varepsilon_0/3}^\pm; \mathbb{R}^3)$  toward  $u^\pm$ . In addition, since the traces on  $S_\varepsilon^\pm$  of  $u_s$  are the traces on  $S$  of  $(T_\varepsilon \tilde{u}_s)^\pm$ , we deduce that, considered as elements of  $L^2(S; \mathbb{R}^3)$ , they converge strongly in  $L^2(S; \mathbb{R}^3)$  toward the traces on  $S$  of  $u^\pm$ , which establishes point v).

We can improve the convergence result of  $\tilde{u}_s$  and consequently obtain the  $\tau$ -convergence result given at point vi) by duly accounting for (H2)-iv). For all  $w$  in  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  we classically have

$$\begin{aligned} \frac{1}{2} \int_{B_\varepsilon} |w|^2 dx &\leq \varepsilon \int_{S_\varepsilon} |w|^2 d\hat{x} + \varepsilon^2 \int_{B_\varepsilon} |\nabla w|^2 dx \\ &\leq \varepsilon \int_{S_\varepsilon} |w|^2 d\hat{x} + \varepsilon^2 \int_{\Omega} |\nabla w|^2 dx \\ &\leq \varepsilon \int_{S_\varepsilon} |w|^2 dx + C\varepsilon^2 \int_{\Omega} |e(w)|^2 dx \quad (\text{by Korn inequality in } H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)) \\ &= \varepsilon \int_{S_\varepsilon} |w|^2 d\hat{x} + C \frac{\varepsilon^2}{\mu} \cdot \mu \int_{B_\varepsilon} |e(w)|^2 dx + C\varepsilon^2 \int_{\Omega_\varepsilon} |e(w)|^2 dx. \end{aligned} \quad (4.20)$$

Hence (4.17), (4.18), Lemma 4.1, the previously established convergence of traces and Hölder inequality mean that  $\tilde{u}_s$   $\tau$ -converges toward 0 and consequently, that  $u_s$   $\tau$ -converges toward  $u$ . Then taking inequality (4.20) with  $w = {}^1P_s^1 u - u_s$  gives point vii).

Lastly, point viii) stems from  $\lim_{s \rightarrow \bar{s}} \int_{\Omega_\varepsilon} |v_s - v|^2 dx = 0$ ,  $\lim_{s \rightarrow \bar{s}} \int \rho |v_s - v|^2 dx = 0$  and assumption (H2)-v).  $\square$

We will now conclude by using a suitable nonlinear version (see Appendix of [12]) of Trotter's theory of approximation of semigroups of linear operators acting on variable spaces [6]:

**Theorem 4.1.** *Let  $H_n, H$  be Hilbert spaces and let  $A_n: H_n \mapsto 2^{H_n}$ ,  $A: H \mapsto 2^H$  be maximal monotone multivalued operators. Let  $P_n: (H, | \cdot |) \mapsto (H_n, | \cdot |_n)$  such that  $P_n \in \mathcal{L}(H, H_n)$  and*

i)  $|P_n x|_n \leq C|x| \forall x \in H$ , where  $C$  is a constant independent of  $n$ ,

ii)  $|P_n x|_n \rightarrow |x| \forall x \in H$ .

Let  $f_n \in L^1(0, T; H_n)$  and  $f \in L^1(0, T; H)$ ,  $u_n^0 \in \overline{D(A_n)}$  and  $u^0 \in \overline{D(A)}$ . Let  $u_n$  and  $u$  be the weak solutions to the equations:

$$\begin{cases} \frac{du_n}{dt} + A_n u_n \ni f_n, \\ u_n(0) = u_n^0 \end{cases}, \quad \begin{cases} \frac{du}{dt} + Au \ni f, \\ u(0) = u^0. \end{cases}$$

If  $|P_n u^0 - u_n^0|_n \rightarrow 0$ ,  $\int_0^T |P_n f(t) - f_n(t)|_n dt \rightarrow 0$ ,  $|(I + A_n)^{-1} P_n z - P_n (I + A)^{-1} z|_n \rightarrow 0$  when  $n \rightarrow \infty$ ,  $\forall z \in H$ , then  $|P_n u(t) - u_n(t)|_n \rightarrow 0$  when  $n \rightarrow \infty$ , uniformly on  $[0, T]$ .

Thus, to prove the convergence in the sense of Trotter of  $U_s$  toward  ${}^1U$  uniformly on  $[0, T]$ , it suffices to make a suitable additional assumption about the initial state and to establish the following two propositions:

**Proposition 4.4.**

$$\forall \psi \in {}^1H, \quad \lim_{s \rightarrow \bar{s}} |{}^1P_s (I + {}^1A)^{-1} \psi - (I + A_s)^{-1} {}^1P_s \psi|_s = 0.$$

**Proposition 4.5.**

$$\text{i) } \lim_{s \rightarrow \bar{s}} \int_0^T |{}^1P_s F(t) - F_s(t)|_s dt = 0$$

$$\text{ii) } \lim_{s \rightarrow \bar{s}} |{}^1P_s {}^1U^e(t) - U_s^e(t)|_s = 0 \text{ uniformly on } [0, T].$$

Actually, to establish Proposition 4.5, which takes into account the external loading  $(f, g)$ , we need an additional assumption:

$$\begin{aligned} \text{i) } & f \in BV(0, T; L^{2/(1-r)}(B_{\varepsilon_0}; \mathbb{R}^3)) \text{ where } r \text{ was defined in (H2)-vi).} \\ \text{ii) } & \text{supp}(g) \cap \overline{B_{\varepsilon_0}} = \emptyset \quad \forall t \in [0, T] \text{ and} \\ & \text{if } \text{Min}\{\mathcal{H}^2(\Gamma_0^\pm)\} = 0, \text{ say } \mathcal{H}^2(\Gamma_0^-) = 0, \text{ then } \text{supp } g \cap (\partial\Omega_{\varepsilon_0}^-) = \emptyset. \end{aligned} \tag{H4}$$

Assumption (H4)-i) says that if  $\rho$  tends to zero then  $f$  has to be a little more smooth than  $L^2(\Omega; \mathbb{R}^3)$ . Note that in most previous studies on static and quasistatic cases, the support of  $f$  is assumed to be located outside  $B_{\varepsilon_0}$ , so that (H4)-i) is satisfied. On the other hand, in practice, the body forces reduce to the weight where  $f = -C\bar{\gamma}e^3$  so that  $F_s^2 = {}^1F^2 = -Ce^3$  and  $\int_0^T K({}^1P_s^2 {}^1F^2(t) - F_s^2(t)) dt = 0$ . Assumption (H4)-ii) says that the support of  $g$  is outside  $\overline{B_{\varepsilon_0}}$  and that if the lower adherent is not clamped, there are no surface forces imposed on its boundary. This will mean that  $u_s^e$  converges toward  $u^e$ .

*proof of proposition 4.4:* The proof is obtained in four steps. The main idea is to take advantage of Proposition 3.1 and of (4.9) and to establish the variational convergence of  $\tilde{J}_s = \frac{1}{2}K_s - k_s(\psi^2, \cdot) + \frac{1}{2}\Phi_s + \varphi_s({}^1P^1\psi^1, \cdot) + b \int_{B_\varepsilon} \mathcal{D}(e(\cdot)) dx$  toward  ${}^1J$ .

**First step** (Compactness properties of any sequence such that  $\tilde{J}_s(w_s) \leq C$ ) :

**Lemma 4.2.** *Let  $(w_s)$  be a sequence such that  $\tilde{J}_s(w_s) \leq C$ , then there exists  $w$  in  ${}^1H^1$  and a non-relabelled subsequence such that:*

- i) for all positive  $\eta$ ,  $(w_s)$  converges weakly in  $H_{\Gamma_0}^1(\Omega_\eta; \mathbb{R}^3)$  toward  $w$ ,
- ii)  $(1_{\Omega_\varepsilon} e(w_s))$  converges weakly in  $L^2(\Omega \setminus S; S^3)$  toward  $e(w)$ ,
- iii)  $(1_{\Omega_\varepsilon} w_s)$  converges strongly in  $L^2(\Omega; \mathbb{R}^3)$  toward  $w$ ,
- iv) the traces on  $S_\varepsilon^\pm$  of  $w_s$ , which are taken to be elements of  $L^2(S; \mathbb{R}^3)$ , converge strongly in  $L^2(S; \mathbb{R}^3)$  toward the traces on  $S$  of  $w^\pm$ ,
- v)  $(w_s)$  is bounded in  $L^2(\Omega; \mathbb{R}^3)$  and  $\tau$ -converges toward  $w$ .

*Proof.* This proof is similar to that of Proposition 4.3, except that weak convergences take place due to the initial assumption:

$$K_s(w_s) + \Phi_s(w_s) + b|e(w_s)|_{L^p(B_\varepsilon; S^3)}^p \leq C \quad (4.21)$$

deduced from Proposition 4.2 and the coercivity of  $W$ ,  $W_{\lambda\mu}$ ,  $\mathcal{D}$ . The Korn inequality, (4.21) and a standard diagonalization argument show that there exists  $w$  in  $\cup_{\eta>0} H_{\Gamma_0}^1(\Omega_\eta; \mathbb{R}^3)$  such that a non-relabeled subsequence  $(w_s)$  converges weakly toward  $w$  in  $H^1(\Omega_\eta; S^3)$ ,  $\forall \eta > 0$ . In addition, (4.21) implies that there exist some  $e^*$  in  $L^2(\Omega; S^3)$  such that  $e^*$  is the weak limit in  $L^2(\Omega; S^3)$  of a non-relabeled subsequence (of the previous one) of  $(1_{\Omega_\varepsilon} e(w_s))$ . But the restriction to each  $\Omega_\eta$  of  $e^*$  is necessarily the restriction to each  $\Omega_\eta$  of  $e(w)$ , where  $e(w)$  denotes the symmetric part of the gradient in the sense of distributions of  $\mathcal{D}'(\Omega \setminus S)$  of  $w$ . Hence  $e(w)$  belongs to  $L^2(\Omega \setminus S; S^3)$ , as, in the same way,  $w$  appears to be the weak limit in  $L^2(\Omega_\varepsilon; \mathbb{R}^3)$  of  $1_{\Omega_\varepsilon} w_s$ , then  $w$  belongs to  $H_{\Gamma_0}^1(\Omega \setminus S; \mathbb{R}^3)$ . The rest of the proof is exactly as in Proposition 4.3, but it remains to be established that  $w$  belongs to  ${}^1\mathbf{H}^1$  when  $I > 1$ . If  $p$  is arbitrary in  $C_0^\infty(S)$  and  $\sigma^i = e^i \otimes_S e^3$ ,  $i \in \{1, 2, 3\}$ , then it suffices to let  $s$  tend to  $\bar{s}$  (with  $i = 3$  for  $I = 2$ ,  $i = 1, 2, 3$  for  $I = 3$ ) in the identity

$$\int_{B_\varepsilon} p(\hat{x}) \sigma^i \cdot e(w_s) dx = - \int_{B_\varepsilon} \operatorname{div}(p(\hat{x}) \sigma^i) \cdot w_s dx + \int_{S_\varepsilon^\pm} p(\hat{x}) (\sigma^i e^3) \cdot (\pm w_s) d\hat{x}, \quad (4.22)$$

by accounting duly for  $\varepsilon \int_{B_\varepsilon} |e(w_s)|^2 dx \leq \frac{\varepsilon}{\mu} C$ ,  $\varepsilon \int_{B_\varepsilon} |\operatorname{tr} e(w_s)|^2 dx \leq \frac{\varepsilon}{\lambda} C$ ,  $\int_{B_\varepsilon} |w_s|^2 dx \leq C$  and the convergence of the traces on  $S_\varepsilon^\pm$  of  $w_s$ .  $\square$

### Second step (Upper bound for $\tilde{J}_s(w_s)$ )

**Lemma 4.3.** *For all  $w$  in  ${}^1\mathbf{H}^1$ , there exists a sequence  $(w_s)$  in  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  which  $\tau$ -converges toward  $w$  such that each term of  $\tilde{J}_s(w_s)$  converges to the corresponding term of  ${}^1J(w)$ .*

*Proof.* As  ${}^1J$  is continuous on  ${}^1\mathbf{H}^1$ , it suffices to prove the result on a dense subset, namely  $W^{1,\infty}(\Omega \setminus S; \mathbb{R}^3) \cap {}^1\mathbf{H}^1$ , and to conclude by adding a diagonalization argument [3]. Let us prove that  $w_s = {}^1P_s^1 w$  works well. As obviously,

$$|{}^1P_s^1 w|_{L^2(B_\varepsilon; \mathbb{R}^3)} \leq C |w|_{L^2(B_\varepsilon; \mathbb{R}^3)}, \quad (4.23)$$

the convergence of the kinetic terms  $K_s({}^1P_s^1 w)$  and  $k_s({}^1P_s^2 \psi^2, w_s)$  stems from the boundedness of  $\rho$ . The convergence of  $\Phi_s({}^1P_s^1 w)$  was proved from Proposition 4.2 ii), while this same Proposition also means that

$$\begin{aligned} \lim_{s \rightarrow \bar{s}} \varphi_s({}^1P_s^1 \psi^1, {}^1P_s^1 w) &= \lim_{s \rightarrow \bar{s}} \left( \frac{1}{2} \left[ \Phi_s({}^1P_s^1(\psi^1 + w)) - \Phi_s({}^1P_s^1 \psi^1) - \Phi_s({}^1P_s^1 w) \right] \right) \\ &= \frac{1}{2} \left[ {}^1\Phi(\psi^1 + w) - {}^1\Phi(\psi^1) - {}^1\Phi(w) \right] = {}^1\varphi(\psi^1, w). \end{aligned}$$

If  $\bar{b} = \infty$ , then  $[w] = 0$ , and consequently,  ${}^1P^1 w = w$ , and  $b \int_{B_\varepsilon} \mathcal{D}(e({}^1P^1 w)) dx$  is less than  $Cb\varepsilon$ , which tends to zero from (H2)-iii). In the other case, the convexity and growth of order  $p$  of  $\mathcal{D}$ , the Hölder inequality and Lemma 4.1 give

$$\begin{aligned} & \lim_{s \rightarrow \bar{s}} \left| b \int_{B_\varepsilon} \mathcal{D}(e({}^1P_s^1 w)) dx - b \int_{B_\varepsilon} \mathcal{D}\left(\frac{[w] \otimes_S e^3}{2\varepsilon}\right) dx \right| \\ & \leq \lim_{s \rightarrow \bar{s}} b \left| e({}^1P_s^1 w) - \frac{[w] \otimes_S e^3}{2\varepsilon} \right|_{L^p(B_\varepsilon; S^3)} \left( \varepsilon + \left| e({}^1P_s^1 w) - \frac{[w] \otimes_S e^3}{2\varepsilon} \right|_{L^p(B_\varepsilon; S^3)}^{p-1} + \left| \frac{w \otimes_S e^3}{2\varepsilon} \right|_{L^p(B_\varepsilon; S^3)}^{p-1} \right) = 0 \end{aligned}$$

and we conclude with assumption (H3) and the Hölder inequality.  $\square$

**Third step** (Lower bound for  $\tilde{J}_s(w_s)$ ):

**Lemma 4.4.** *For all  $w$  in  $L^2(\Omega; \mathbb{R}^3)$  and for all sequences  $(w_s)$  which  $\tau$ -converge toward  $w$ , each term of  ${}^1J(w)$  is less than or equal to the  $\varliminf_{s \rightarrow \bar{s}}$  of the corresponding term of  $\tilde{J}_s(w_s)$ .*

*Proof.* Of course, we can restrict ourselves to the case where all the  $\varliminf_{s \rightarrow \bar{s}}$  are finite, so that  $(w_s)$  has the compactness properties of Lemma 4.2, especially iii) and v), which suffices to prove the desired inequality for the kinetic terms (we recall that  $\rho$  is bounded). With the terms  $\Phi_s(w_s)$  and  $\varphi_s({}^1P_s^1 \psi^1, w_s)$ , we take only the two cases  $I = 1$  and  $I = 3$ , and the other case can be handled by combining some appropriate arguments from those used in these two basic cases. Point ii) of Lemma 4.2 gives

$$\varliminf_{s \rightarrow \bar{s}} \int_{\Omega_\varepsilon} ae(w_s) \cdot e(w_s) dx \geq \int_{\Omega} ae(w) \cdot e(w) dx. \quad (4.24)$$

For all positive  $h$ , there exists  $w_h$  such that  $w_h^\pm \in C^\infty(\overline{\Omega^\pm}; \mathbb{R}^3) \cap H_{\Gamma_0^\pm}^1(\Omega^\pm; \mathbb{R}^3)$  and

$$|e(w) - e(w_h)|_{L^2(\Omega; S^3)} < h$$

while the subdifferential inequality yields:

$$\int_{B_\varepsilon} W_{\lambda\mu}(e(w_s)) dx \geq \int_{B_\varepsilon} W_{\lambda\mu}(e(R_\varepsilon w_h)) dx + \int_{B_\varepsilon} DW_{\lambda\mu}(e(R_\varepsilon w_h)) \cdot e(R_\varepsilon w_h - w_s) dx.$$

Since

$$\int_{B_\varepsilon} W_{\lambda\mu}(e(R_\varepsilon w_h)) dx \leq C, \quad \int_{B_\varepsilon} W_{\lambda\mu}(e(w_s)) dx \leq C, \quad (4.25)$$

Lemma 4.1 implies that

$$\lim_{s \rightarrow \bar{s}} \left( \int_{B_\varepsilon} DW_{\lambda\mu}(e(R_\varepsilon w_h)) \cdot e(R_\varepsilon w_h - w_s) dx - \int_{B_\varepsilon} DW_{\lambda\mu} \left( \frac{[w_h] \otimes_S e^3}{2\varepsilon} \right) \cdot e(R_\varepsilon w_h - w_s) dx \right) = 0$$

thus

$$\begin{aligned} \lim_{s \rightarrow \bar{s}} \int_{B_\varepsilon} DW_{\lambda\mu}(e(R_\varepsilon w_h)) \cdot e(R_\varepsilon w_h - w_s) dx &= \lim_{s \rightarrow \bar{s}} \int_{B_\varepsilon} DW_{\lambda\mu} \left( \frac{[w_h] \otimes_S e^3}{2\varepsilon} \right) \cdot e(R_\varepsilon w_h - w_s) dx \\ &= \lim_{s \rightarrow \bar{s}} \int_{B_\varepsilon} D\overline{W}_{\overline{\lambda\mu}}([w_h]) \cdot e(R_\varepsilon w_h - w_s) dx \\ &= \lim_{s \rightarrow \bar{s}} \left\{ \int_{B_\varepsilon} -\operatorname{div} D\overline{W}_{\overline{\lambda\mu}}([w_h]) \cdot (R_\varepsilon w_h - w_s) dx + \int_{S_\varepsilon^\pm} D\overline{W}_{\overline{\lambda\mu}}([w_h]) e^3 \cdot (\pm(w_h - w_s)) d\hat{x} \right\} \\ &= \int_S D\overline{W}_{\overline{\lambda\mu}}([w_h]) e^3 \cdot [w_h - w] d\hat{x} + o(h) \end{aligned}$$

by taking Lemma 4.2 and (4.23) duly into account. Hence the desired result is obtained when  $I = 1$  by using Lemma 4.3 and letting  $h$  tend to zero. When  $I = 3$ , due to (4.24), it suffices to establish  $[w] = 0$ , which stems from Lemma 4.2. Next, Lemma 4.3 gives

$$\begin{aligned} \lim_{s \rightarrow \bar{s}} \varphi_s({}^1P_s^1 \psi^1, w_s) &= \lim_{s \rightarrow \bar{s}} \varphi_s({}^1P_s^1 \psi^1, {}^1P_s^1 w) + \lim_{s \rightarrow \bar{s}} \varphi_s({}^1P_s^1 \psi^1, w_s - {}^1P_s^1 w) \\ &= {}^1\varphi(\psi^1, w) + \lim_{s \rightarrow \bar{s}} \varphi_s({}^1P_s^1 \psi^1, w_s - {}^1P_s^1 w) \\ &= {}^1\varphi(\psi^1, w) \end{aligned}$$

by arguing as previously for  $\lim_{s \rightarrow \bar{s}} \varphi_s(R_\varepsilon w_h, w_s - R_\varepsilon w_h)$  in the case  $I = 1$ . When  $I = 3$ , we have

$$\begin{aligned} \lim_{s \rightarrow \bar{s}} \varphi_s({}^3P_s^1 \psi^1, w_s) &= \lim_{s \rightarrow \bar{s}} \int_{\Omega} e(\psi^1) \cdot 1_{\Omega_\varepsilon} e(w_s) dx + \int_{B_\varepsilon} DW_{\lambda\mu}(e(\psi^1)) \cdot e(w_s) dx \\ &= \int_{\Omega} e(\psi^1) \cdot e(w) dx \end{aligned}$$

by taking Lemma 4.2 and (4.25) duly into account because

$$\int_{B_\varepsilon} DW_{\lambda\mu}(e(\psi^1)) \cdot e(w_s) dx \leq 2 \left\{ \int_{B_\varepsilon} W_{\lambda\mu}(e(\psi^1)) \right\}^{1/2} \cdot \left\{ \int_{B_\varepsilon} W_{\lambda\mu}(e(w_s)) \right\}^{1/2}$$

Lastly, when  $\bar{b}$  is finite, the boundedness of  $b|e(w_s)|_{L^p(B_\varepsilon; \mathbb{S}^3)}^p$ , (H3) and the Hölder inequality give

$$\lim_{s \rightarrow \bar{s}} \left| b \int_{B_\varepsilon} \mathcal{D}(e(w_s)) dx - b \int_{B_\varepsilon} \mathcal{D}^{\infty, p}(e(w_s)) dx \right| = 0$$

and we conclude by using the same argument as for  $\int_{B_\varepsilon} W_{\lambda\mu}(e(w_s)) dx$ . If  $\bar{b} = \infty$ , it suffices to prove  $[w] = 0$ , which can be done based on (4.22).  $\square$

**Fourth step** (Convergence in the sense of Trotter)

By combining Lemmas 4.2, 4.3 and 4.4, we establish classically (see [3], [14]) that the unique minimizer  $\bar{v}_s$  of  $\tilde{J}_s$   $\tau$ -converges toward the unique minimizer  ${}^I\bar{v}$  of  ${}^I J$  and

$${}^I J({}^I\bar{v}) = \lim_{s \rightarrow \bar{s}} J_s(\bar{v}_s), \quad (4.26)$$

Actually, we have more:

$$\lim_{s \rightarrow \bar{s}} \Phi_s({}^I P_s^1 {}^I\bar{v} - \bar{v}_s) = 0. \quad (4.27)$$

Indeed, Proposition 4.2 says  ${}^I\Phi({}^I\bar{v}) = \lim_{s \rightarrow \bar{s}} \Phi_s({}^I P_s^1 {}^I\bar{v})$ , while the following three points can be deduced from Lemma 4.3 and 4.4 :

i)  ${}^I\Phi(\bar{v}) \leq \liminf_{s \rightarrow \bar{s}} \Phi_s(\bar{v}_s)$ ,

ii)

$$\begin{aligned} \overline{\lim}_{s \rightarrow \bar{s}} \frac{1}{2} \Phi_s(\bar{v}_s) &= \overline{\lim}_{s \rightarrow \bar{s}} \left( \tilde{J}_s(\bar{v}_s) - \frac{1}{2} K_s(\bar{v}_s) - k_s({}^I P_s^2 \psi^2, \bar{v}_s) - \varphi_s({}^I P_s^1 \psi^1, \bar{v}_s) - b \int_{B_\varepsilon} \mathcal{D}(e(\bar{v}_s)) dx \right) \\ &\leq \lim_{s \rightarrow \bar{s}} \tilde{J}_s(\bar{v}_s) - \lim_{s \rightarrow \bar{s}} \frac{1}{2} K_s(\bar{v}_s) - \lim_{s \rightarrow \bar{s}} k_s({}^I P_s^2 \psi^2, \bar{v}_s) - \lim_{s \rightarrow \bar{s}} \varphi_s({}^I P_s^1 \psi^1, \bar{v}_s) - \lim_{s \rightarrow \bar{s}} b \int_{B_\varepsilon} \mathcal{D}(e(\bar{v}_s)) dx \\ &\leq {}^I J({}^I\bar{v}) - \frac{1}{2} K({}^I\bar{v}) - k(\psi^2, {}^I\bar{v}) - {}^I\varphi(\psi^1, {}^I\bar{v}) - \bar{b} \int_S \overline{\mathcal{D}}([{}^I\bar{v}]) d\hat{x} \\ &= \frac{1}{2} {}^I\Phi({}^I\bar{v}) \end{aligned}$$

iii)  $\lim_{s \rightarrow \bar{s}} \varphi_s({}^I P_s^1 {}^I\bar{v}, {}^I\bar{v}) = {}^I\Phi({}^I\bar{v})$ .

Therefore

$$\begin{aligned} \lim_{s \rightarrow \bar{s}} \Phi_s({}^I P_s^1 {}^I\bar{v} - \bar{v}_s) &= \lim_{s \rightarrow \bar{s}} \left[ \Phi_s({}^I P_s^1 {}^I\bar{v}) - 2\varphi_s({}^I P_s^1 {}^I\bar{v}, \bar{v}_s) + \Phi_s(\bar{v}_s) \right] \\ &= {}^I\Phi({}^I\bar{v}) - 2{}^I\Phi({}^I\bar{v}) + {}^I\Phi({}^I\bar{v}) = 0. \end{aligned}$$

Since  $\bar{u}_s = \bar{v}_s - {}^I P_s^1 \psi^1$ ,  ${}^I\bar{u} = {}^I\bar{v} - \psi^1$ , we have

$$\lim_{s \rightarrow \bar{s}} \Phi_s({}^I P_s^1 {}^I\bar{u} - \bar{u}_s) = 0.$$

Proceeding as previously we obtain

$$K({}^I\bar{v}) = \lim_{s \rightarrow \bar{s}} K_s(\bar{v}_s), \quad K({}^I\bar{v}) = \lim_{s \rightarrow \bar{s}} K_s({}^I P_s^2 {}^I\bar{v}), \quad K({}^I\bar{v}) = \lim_{s \rightarrow \bar{s}} k_s({}^I P_s^2 {}^I\bar{v}, \bar{v}_s)$$

which gives

$$\lim_{s \rightarrow \bar{s}} K_s({}^I P_s^2 {}^I\bar{v} - \bar{v}_s) = 0$$

and the proof of Proposition 4.4 is complete.



□

*Proof of Proposition 4.5:*

Assumption (H4)-ii) means that  $L(t)$  is either a sum of two linear continuous forms on  $H_{\Gamma_0^\pm}^1(\Omega_{\varepsilon_0}^\pm; \mathbb{R}^3)$  or a linear continuous form on  $H_{\Gamma_0^+}^1(\Omega_{\varepsilon_0}^+; \mathbb{R}^3)$ , so that Korn's inequality in  $H_{\Gamma_0^\pm}^1(\Omega_{\varepsilon_0}^\pm; \mathbb{R}^3)$  yields

$$\begin{aligned}\Phi_s(u_s^e) &= L(t)(u_s^e) \leq C|g(t)|_{L^2(\Gamma_1; \mathbb{R}^3)}|e(u_s^e)|_{L^2(\Omega_{\varepsilon_0}; S^3)} \\ &\leq C|g(t)|_{L^2(\Gamma_1; \mathbb{R}^3)}\Phi_s(e(u_s^e))^{1/2} \quad \forall t \in [0, T]\end{aligned}$$

hence  $g \mapsto u_s^e$  is a linear mapping with

$$\Phi_s(u_s^e) \leq C|g(t)|_{L^2(\Gamma_1; \mathbb{R}^3)} \quad \forall t \in [0, T]. \quad (4.28)$$

On the other hand,  $K_s(F_s^2) = \int_{\Omega_\varepsilon} \frac{1}{\bar{\rho}}|f|^2 dx + \frac{1}{\bar{\rho}} \int_{B_\varepsilon} |f|^2 dx$ . Hence, based on assumptions (H1), (H4),  $|{}^I P_s F_s(t) - {}^I F(t)|_s \leq C$ . Since  ${}^I u^e$  and  $u_s^e$  are minimizers of  $1/2\Phi_s - L(t)(\cdot)$  and  $\frac{1}{2}{}^I \Phi - L(t)(\cdot)$ , the arguments in Proposition 4.4 and (H1) give

$$\lim_{s \rightarrow \bar{s}} |{}^I P_s {}^I U^e(t) - U_s^e(t)|_s = \lim_{s \rightarrow \bar{s}} \left| {}^I P_s \frac{d{}^I U^e}{dt}(t) - \frac{dU_s^e}{dt}(t) \right|_s = 0 \quad \text{a.e. } t \text{ in } (0, T).$$

But (H1) shows that  $t \in [0, T] \mapsto |{}^I P_s {}^I U^e(t) - U_s^e(t)|_s$  is equicontinuous, so that the second part of Proposition 4.5 is established. In addition, (H4) gives

$$\lim_{s \rightarrow \bar{s}} K_s({}^I P_s^2(f/\bar{\rho}) - f/\gamma) = \lim_{s \rightarrow \bar{s}} \int_{B_\varepsilon} |f|^2 \left( \frac{1}{\bar{\rho}} - \frac{1}{\rho} \right) dx = 0 \quad \text{a.e. } t \in [0, T]$$

and the first part of Proposition 4.5 results from the Lebesgue dominated convergence theorem. □

Now we have to make an additional assumption about the initial states to be able to state our convergence result:

$$\exists {}^I U^o \in {}^I U^e(0) + D({}^I A); \quad U_s^o \in U_s^e(0) + D(A_s) \text{ and } \lim_{s \rightarrow \bar{s}} |{}^I P_s {}^I U^o - U_s^o|_s = 0. \quad (H5)$$

The first condition is a compatibility condition between the initial state and the initial loading conditions; and the second is a convergence condition which, because of Proposition 4.4, is satisfied by

$$U_s^o = U_s^e(0) + (I + A_s)^{-1} {}^I P_s (I + {}^I A)^{-1} ({}^I U^o - {}^I U^e(0)).$$

Hence, from the nonlinear Trotter theorem 4.1, we deduce that the solution to (3.11) converges uniformly on  $[0, T]$  in the sense of Trotter toward the solution to (4.10) with  ${}^I U^{ro} = {}^I U^o - {}^I U^e(0)$ .

Based on all these propositions, the convergence can be expressed more explicitly as follows

**Theorem 4.2.** *The solution to*

$$\begin{cases} \frac{dU_s}{dt} + A_s(U_s - U_s^e) \ni (0, f/\gamma) \\ U_s(0) = U_s^o \end{cases} \quad (4.29)$$

*converges toward the solution to*

$$\begin{cases} \frac{d{}^I U}{dt} + {}^I A({}^I U - {}^I U^e) \ni (0, f/\bar{\rho}) \\ {}^I U(0) = {}^I U^o \end{cases} \quad (4.30)$$

*in the sense  $\lim_{s \rightarrow \bar{s}} |{}^I P_s {}^I U(t) - U_s(t)|_s = 0$  uniformly on  $[0, T]$ , where in addition  $\lim_{s \rightarrow \bar{s}} |U_s(t)|_s = \|{}^I U(t)\|_I$  uniformly on  $[0, T]$ .*

## 5 Concluding result

A more explicit way of writing (4.30) is:

- if  $\bar{b} < +\infty$ ,  $\exists \xi \in \partial \bar{\mathcal{D}}\left(\left[\frac{d^{\text{I}}u}{dt}\right]\right)$  such that

$$\begin{aligned} \int_{\Omega} \bar{\rho} \frac{d^2 \text{I}u}{dt^2} \varphi \, dx + \int_{\Omega \setminus S} a e(\text{I}u) \cdot e(\varphi) \, dx + \int_S D\bar{W}_{\bar{\lambda}\bar{\mu}}(\text{I}u) \cdot [\varphi] \, d\hat{x} + \int_S \xi \cdot ([\varphi] \otimes_S e^3) \, d\hat{x} \\ = \int_{\Omega} f \cdot \varphi \, dx + \int_{\Gamma_1} g \cdot \varphi \, d\mathcal{H}^2 \quad \forall \varphi \in \text{I}\mathbf{H}^1. \end{aligned}$$

- if  $\bar{b} = \infty$ ,  $\left[\frac{d^{\text{I}}u}{dt}\right] = 0$  and

$$\begin{aligned} \int_{\Omega} \bar{\rho} \frac{d^2 \text{I}u}{dt^2} \varphi \, dx + \int_{\Omega \setminus S} a e(\text{I}u) \cdot e(\varphi) \, dx + \int_S D\bar{W}_{\bar{\lambda}\bar{\mu}}(\text{I}u) \cdot [\varphi] \, d\hat{x} \\ = \int_{\Omega} f \cdot \varphi \, dx + \int_{\Gamma_1} g \cdot \varphi \, d\mathcal{H}^2 \quad \forall \varphi \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3). \end{aligned}$$

Hence, the limit behavior describes the dynamic response to the real loads  $(f, g)$  of a structure consisting of two linearly elastic adherents occupying  $\Omega^{\pm}$ , which are linked along  $S$  by a dissipative mechanical constraint which can be written as follows:

$$\text{I}\sigma e^3 \in D\bar{W}_{\bar{\lambda}\bar{\mu}}(\text{I}u) + \partial \bar{\mathcal{D}}\left(\left[\frac{d^{\text{I}}u}{dt}\right]\right) \quad (5.1)$$

where  $\text{I}\sigma e^3$  is the stress vector along  $S$ . It is nothing but the relation obtained in [10] in the quasistatic case. Based on the present study, (5.1) appears to be the actual constitutive equation given by the thin adhesive because it is obtained whatever the pattern of the evolution of the assembly. This constitutive equation is of the same form as that for the layer (nonlinear viscoelastic of the Kelvin-Voigt type), which may degenerate when the values of one of the coefficients  $\bar{\lambda}$ ,  $\bar{\mu}$ ,  $\bar{b}$  is in  $\{0, +\infty\}$ .

- $\bar{b} = 0$

		$\bar{\mu}$		
		0	finite and positive	$+\infty$
$\bar{\lambda}$	0	$\sigma e^3 = 0$	$\sigma_T = \bar{\mu}[u]_T$ $\sigma_N = 2\bar{\mu}[u]_N$	$[u] = 0$
	finite and positive	$\sigma_T = 0$ $\sigma_N = \bar{\lambda}[u]_N$	$\sigma_T = \bar{\mu}[u]_T$ $\sigma_N = (\bar{\lambda} + 2\bar{\mu})[u]_N$	$[u] = 0$
	$+\infty$	$\sigma_T = 0$ $[u]_N = 0$	$\sigma_T = \bar{\mu}[u]_T$ $[u]_N = 0$	$[u] = 0$

with  $[u]_N = [u]_3$ ,  $[u]_T = [u] - [u]_N e^3$ ,  $\sigma_N = \sigma e^3 \cdot e^3$ ,  $\sigma_T = \sigma e^3 - \sigma_N e^3$ . These are the standard elastic constraints which occur when the layer is isotropically and linearly elastic [7].

- $\bar{b} \in (0, +\infty)$ : In the previous left upper  $2 \times 2$  block, we have to add  $\xi_N$  and  $\xi_T$ , where  $\xi$  is some element of  $\partial \bar{\mathcal{D}}\left(\left[\frac{d^{\text{I}}u}{dt}\right]\right)$ . The other boxes on the right are not changed whereas we have to add  $\xi_T$  to the left bottom boxes. Thus, as seen in [10], the case  $(\bar{\lambda}, \bar{\mu}) = (+\infty, 0)$  corresponds to Norton Hoff ( $1 < p \leq 2$ ) or Tresca friction ( $p = 1$ ) with bilateral contact.
- $\bar{b} = \infty$ : The jump in the displacement along  $S$  is always equal to its initial value (which is of course zero if  $\bar{\mu} = \infty$  and has a vanishing third component if  $\bar{\lambda} = \infty$ ). Whatever the values of  $\bar{\lambda}$ ,  $\bar{\mu}$  may be, the relative motion along  $S$  is frozen!

Since, in practice, the geometrical and mechanical data obviously “do not go to some limits”, the simplified but accurate enough model proposed for the behavior of the structure is that obtained with  $\text{I} = 1$  by replacing  $\bar{\lambda}$ ,  $\bar{\mu}$ ,  $\bar{b}$  by the real values  $\lambda/2\varepsilon$ ,  $\mu/2\varepsilon$  and  $b/(2\varepsilon)^{p-1}$ !

To illustrate our intention, keeping nevertheless the discussion within reasonable limits, we considered the case of a layer with an isotropic Hooke-like strain energy and a potential of dissipation involving only one viscosity coefficient which, in the purely elastic case, yield a decoupling of the tangential and normal effects in the limit constraint. It would be straightforward to deal with the general case where the density of the strain energy and that of the potential of dissipation are a quadratic convex function  $W$  and a convex function  $\mathcal{D}$  with growth of order  $p$ , respectively. The limit constraint will then involves terms stemming from  $\overline{W}([u] \otimes_S e^3)$  and  $\overline{\mathcal{D}}\left(\left[\frac{du}{dt}\right] \otimes_S e^3\right)$ , where  $\overline{W}$  and  $\overline{\mathcal{D}}$  describe the asymptotic behavior of the functions  $2\varepsilon W(\cdot/2\varepsilon)$  and  $2\varepsilon \mathcal{D}^{\infty,p}(\cdot/2\varepsilon)$ .

We recalled in the introduction that more general behavior of the adhesive including models of cracks and plasticity have been taken into account in the case of equilibrium problems, but this remains to be tackled in the dynamic case. Our method can be applied to the case of a generalized standard material with a coercive quadratic energy density. This will be the subject of a subsequent paper.

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