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Pierre Del Moral[1] and Julian Tugaut[2]

[1]INRIA Bordeaux
[2]Université Jean Monnet, Institut Camille Jordan, 23, rue du docteur Paul Michelon, CS 82301, 42023 Saint-Étienne Cedex 2, France.

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Abstract

We are interested in nonlinear diffusions in which the own law intervenes in the drift. This kind of diffusions corresponds to the hydrodynamical limit of some particle system. One also talks about propagation of chaos. It is well-known, for McKean-Vlasov diffusions, that such a propagation of chaos holds on finite-time interval. However, it has been proven that the lack of convexity of the external force implies that there is no uniform propagation of chaos if the diffusion coefficient is small enough. We here aim to establish a uniform propagation of chaos even if the external force is not convex, with a diffusion coefficient sufficiently large. The idea consists in combining the propagation of chaos on a finite-time interval with a functional inequality, already used by Bolley, Gentil and Guillin, see [BGG12a, BGG12b]. Here, we also deal with a case in which the system at time $t = 0$ is not chaotic and we show under easily checked assumptions that the system becomes chaotic as the number of particles goes to infinity together with the time. This yields the first result of this type for mean field particle diffusion models as far as we know.

Introduction

We are interested in some nonlinear processes in $\mathbb{R}^d$ defined by an equation in which the own law of the process intervenes in the drift. An example of such diffusion is the McKean-Vlasov one:

$$X_t = X_0 + \sigma B_t - \int_0^t \nabla V\left(X_s\right) ds - \int_0^t \Big(\nabla F * \mathcal{L}\left(X_s\right)\Big)(X_s) ds\,, \tag{I}$$

where V and F respectively are called the confinement and the interaction potentials and $\{B_t\,;\, t \geq 0\}$ is a d-dimensional Wiener process. The notation $*$

is used for denoting the convolution.

The infinitesimal generator of Diffusion (I) therefore is

$$A\mu := \frac{\sigma^2}{2}\Delta\mu - \nabla.\left\{\left[\nabla V + \nabla F * \mu\right]\mu\right\}\,.$$

The associated semi-group is denoted by $(P_t)_{t\geq 0}$. In other words, one has $\mu_t = \mu_0 P_t$ where $\mu_t := \mathcal{L}(X_t)$. We notice that X_t, μ_t, P_t and A depend on σ. We do not write it for simplifying the reading.

This equation is nonlinear in the sense of McKean, see [McK67, McK66].

It is well-known, see [McK67], that the law $\mathcal{L}(X_t)$ is absolutely continuous with respect to the Lebesgue measure for all $t > 0$, provided some regularity hypotheses on V and F. Moreover, its density, which is denoted by u_t, satisfies the so-called granular media equation,

$$\frac{\partial}{\partial t}u_t = \nabla.\left\{\frac{\sigma^2}{2}\nabla u_t + \left(\nabla V + \nabla F * u_t\right)u_t\right\}\,.$$

The setting of this work is restricted to the McKean-Vlasov case. However, we could apply to more general hypotheses. Let us notice that we do not assume any global convex properties on the confinement potential nor on the interaction one. Under easily checked assumptions, Diffusion (I) corresponds to the hydrodynamical limit of the following particle system

$$X_t^i = X_0^i + \sigma B_t^i - \int_0^t \left[\nabla V\left(X_s^i\right) + \sum_{j=1}^N \frac{1}{N}\nabla F\left(X_s^i - X_s^j\right)\right]ds\,, \tag{II}$$

$\left\{B_t^i\,;\,t\geq 0\right\}$ being N independent d-dimensional Wiener processes. We also assume that $\left\{X_0^i\,;\,i\in\mathbb{N}^*\right\}$ is a family of independent random variables, identically distributed with common law $\mathcal{L}(X_0)$ (and independent from the Brownian motions). The particles therefore are exchangeable. We notice that $X_t^1,\cdots,X_t^N$ depend on N and on σ. We do not write it for simplifying the reading. We here focus on the first diffusion. By $\mu_t^{1,N}$, we denote the law at time t of the diffusion X^1.

One says, in this work, that simple propagation of chaos holds on interval $[0;T]$ with $T>0$ if we have the limit

$$\lim_{N\to+\infty}\sup_{0\leq t\leq T}\mathbb{W}_2\left(\mu_t^{1,N}\,;\,\mu_0 P_t\right) = \lim_{N\to+\infty}\sup_{0\leq t\leq T}\mathbb{W}_2\left(\mu_t^{1,N}\,;\,\mu_t\right) = 0\,,$$

$\mathbb{W}_2$ standing for the Wasserstein distance . This means that X^1 is a good approximation of Diffusion (I) as N goes to infinity.

A consequence of the uniform propagation of chaos for the nonlinear diffusion is the uniqueness of the invariant probability μ^σ and the weak convergence toward this measure. However, without global convex properties, it is proven in [HT10, Tug13c, Tug12] that there is non-uniqueness of the invariant probabilities under simple assumptions, provided that the diffusion coefficient σ is sufficiently small.

But, as pointed out in [Tug13c], if σ is large enough, we have a unique invariant probability. The question thus is: does uniform propagation of chaos hold if σ is sufficiently large? Also, can we reciprocally use the convergence toward the unique invariant probability to obtain this uniform propagation of chaos?

We positively answer to the two questions by using the simple propagation of chaos and a so-called WJ-inequality already used in [BGG12b].

The analysis of interaction jump particle models clearly differs from the more traditional coupling analysis of the McKean-Vlasov diffusion models developed in the present article. The common feature is to enter the stability properties of the limiting nonlinear semigroup into the estimation of the propagation of chaos properties of the finite particle systems, to deduce L_p-mean error estimates of order $1/\sqrt{N}^{\delta}$, for any $0 < \delta < 1$ (cf. for instance theorem 2.11 in [DMM00], at the level of the empirical processes). In our context using these techniques, we obtain a variance and a W_2^2-estimate of order $1/N^{\delta}$, for some $0 \leq \delta < 1$. We underline that in the context of Feynman-Kac particle models, the order $1/N$ can be obtained under stronger mixing conditions, using backward semigroup techniques. Thus, we conjecture that this decay rate is also met in our context.

The other subject of the paper is the creation of chaos. We show that under suitable assumptions, there is creation of chaos then propagation of this chaos for a mean-field system of particles without assuming that the initial random variables are independent. In other words, the particles become independent as the time t goes to infinity if the number of particles is large.

The existence problem of a solution to (I) is not investigated here. However, we take assumptions which ensure that there exists a unique strong solution $(X_t)_{t\geq 0}$. The method consists in applying a fixed-point theorem, see [BRTV98, HIP08].

In a first section, we give the assumptions of the paper and its main results. The second section is devoted to the framework of the WJ-inequality and we establish some functional inequalities based on the work in [BGG12a, BGG12b]. In Section 3, we provide some results on the simple propagation of chaos. In Section 4, we prove the main results about the uniform propagation of chaos when the coefficient diffusion is sufficiently large. Section 5 is devoted to the proofs of the results about the creation of chaos.

1 Hypotheses and main results

We now present the exact assumptions of the paper on the potentials V and F and on the initial measure of probability, μ_0. First, we give the hypotheses on the confining potential V.

Assumption (A-1): *V is a $\mathcal{C}^2$-continuous function.*

Assumption (A-2): *For all $\lambda > 0$, there exists $R_\lambda > 0$ such that $\nabla^2 V(x) > \lambda$,* *for any* $||x|| \geq R_\lambda$.

We can observe that under assumptions (A-1) and (A-2), there exist a convex potential V_0 and $\theta \in \mathbb{R}$ such that $V(x) = V_0(x) - \frac{\theta}{2}||x||^2$.

Assumption (A-3) *The gradient* ∇V *is slowly increasing: there exist* $m \in \mathbb{N}^*$ *and* $C > 0$ *such that* $||\nabla V(x)|| \leq C\left(1 + ||x||^{2m-1}\right)$*, for all* $x \in \mathbb{R}$.

This assumption together with the same kind of assumptions on F ensure us that there is a global solution if some moments of μ_0 are finite.

Let us present now the assumptions on the interaction potential F:

Assumption (A-4): *There exists a function* G *from* $\mathbb{R}_+$ *to* $\mathbb{R}$ *such that* $F(x) = G(||x||)$.

Assumption (A-5): G *is an even polynomial function such that* $\deg(G) =: 2n \geq 2$ *and* $G(0) = 0$.

This hypothesis is used for simplifying the study of the invariant probabilities. Indeed, see [HT10, Tug13c, Tug12], the research of an invariant probability is equivalent to a fixed-point problem in infinite dimension. Nevertheless, under Assumption (A-5), it reduces to a fixed-point problem in finite dimension.

Assumption (A-6): *And,* $\lim_{r\to+\infty} G''(r) = +\infty$.

Immediately, we deduce the existence of an even polynomial and convex function G_0 such that $F(x) = G_0(||x||) - \frac{\alpha}{2}||x||^2$, α being a real constant.

Assumption (A-7) *there exist a strictly convex function* Θ *such that* $\Theta(y) > \Theta(0) = 0$ *for all* $y \in \mathbb{R}^d$ *and* $p \in \mathbb{N}$ *such that the following limit holds for any* $y \in \mathbb{R}^d$*:* $\lim_{r\to+\infty} \frac{V(ry)}{r^{2p}} = \Theta(y)$.

Assumption (A-8) *the following inequality holds:* $p > 2n$.

We also need hypotheses on the initial measure μ_0:

Assumption (A-9) *the* $8q^2$*th moment of the measure* μ_0 *is finite with* $q := \max\{m, n\}$.

Assumption (A-10) *the measure* μ_0 *admits a* $\mathcal{C}^\infty$*-continuous density* u_0 *with respect to the Lebesgue measure. And, the entropy* $-\int_{\mathbb{R}^d} u_0(x)\log(u_0(x))dx$ *is finite.*

The last two hypotheses concern the law μ_0. Hypothesis (A-9) is required to prove the existence of a solution to the nonlinear stochastic differential equation (I), see [HIP08, BRTV98, CGM08]. And, Hypothesis (A-10) is necessary to apply the result in [AGS08] which characterizes the dissipation of the Wasserstein distance. This hypothesis was also assumed in order to get the weak convergence of the law of X_t as t goes to infinity, see [Tug13a].

One says that the set of Assumptions (A) is satisfied if Hypotheses (A-1)–(A-10) are assumed.

Under Assumptions (A-1)–(A-10), Equation (I) admits a unique strong solution. Indeed, the assumptions of Theorem 2.13 in [HIP08] are satisfied: ∇V and ∇F are locally Lipschitz, G' is odd, ∇F grows polynomially, ∇V is continuously differentiable and there exists a compact $\mathcal{K}$ such that $\nabla^2 V$ is uniformly positive on $\mathcal{K}^c$. Moreover, we have the following inequality for a positive M_0:

$$\max_{1\leq j\leq 8q^2}\sup_{t\in\mathbb{R}_+}\mathbb{E}\left[||X_t||^j\right]\leq M_0\,. \tag{1.1}$$

In the following, we use the long-time convergence of the measure μ_t toward an invariant probability μ and the rate of convergence. We need a complementary hypothesis:

Assumption (B) *Diffusion* (I) *admits a unique invariant probability μ. Moreover, there exists $C_\sigma>0$ such that*

$$\mathbb{W}_2\left(\mu_t;\mu\right)\leq e^{-C_\sigma t}\mathbb{W}_2\left(\mu_0;\mu\right)$$

for any initial measure μ_0 which is absolutely continuous with respect to the Lebesgue measure and with finite entropy.

Under the Hypotheses (A)-(B), the probability measure μ_t converges exponentially for Wasserstein distance to the unique invariant probability μ as soon as the initial measure μ_0 is absolutely continuous with respect to the Lebesgue measure and with finite entropy.

Let us briefly justify why we can extend this inequality by starting from a Dirac measure: $\mu_0=\delta_{x_0}$ with $x_0\in\mathbb{R}$. To do so, we consider a sequence of probability measures with finite entropy $\left(\mu_0^{(n)}\right)_{n\geq1}$ which converges for the Wasserstein distance to μ_0. By μ_t (respectively $\mu_t^{(n)}$), we denote the law at time t of the McKean-Vlasov diffusion starting from the law μ_0 (respectively the law $\mu_0^{(n)}$). Then, we have :

$$\mathbb{W}_2\left(\mu_t;\mu\right)\leq\mathbb{W}_2\left(\mu_t;\mu_t^{(n)}\right)+\mathbb{W}_2\left(\mu_t^{(n)};\mu\right)\,.$$

By applying the inequality in Hypothesis (B) to $\mu_t^{(n)}$, we get

$$\mathbb{W}_2\left(\mu_t;\mu\right)\leq\mathbb{W}_2\left(\mu_t;\mu_t^{(n)}\right)+e^{-C_\sigma t}\mathbb{W}_2\left(\mu_0^{(n)};\mu\right)\,.$$

By making a coupling, one can easily show that the quantity $\mathbb{W}_2\left(\mu_t;\mu_t^{(n)}\right)$ converges to 0. Finally, since $\mathbb{W}_2\left(\mu_0^{(n)};\mu_0\right)$ goes to 0 as n tends to infinity, we obtain the formula

$$\mathbb{W}_2\left(\mu_t;\mu\right)\leq e^{-C_\sigma t}\mathbb{W}_2\left(\mu_0;\mu\right)\,.$$

Consequently, μ_t goes to μ as t goes to infinity.

We now give the main results of the paper.

Theorem A: *We assume that V, F and μ_0 satisfy the set of Hypotheses (A). Thus, there exists $\sigma^c>0$ such that $\sigma>\sigma^c$ implies Diffusion* (I) *admits a unique invariant probability μ^σ. Moreover, we have the following convergence with exponential decay if $\sigma>\sigma^c$:*

$$\mathbb{W}_2\left(\mu_t\,;\,\mu^\sigma\right)\leq\exp\left[-C(\sigma)t\right]\mathbb{W}_2\left(\mu_0\,;\,\mu^\sigma\right)\,,$$

$C(\sigma)$ being a positive constant such that $C(\sigma) \geq |\alpha| + |\theta|$.
Proposition B: *We assume that V, F and μ_0 satisfy the set of Hypotheses (A). Let $X_0^1, \cdots, X_0^N$ be N random variables with common law μ_0.* We do not assume these variables to be independent but they are exchangeable. *We consider the two following particle systems:*

$$X_t^i = X_0^i + \sigma B_t^i - \int_0^t \nabla V\left(X_s^i\right) ds - \int_0^t \nabla F * \eta_s^N \left(X_s^i\right) ds\,,$$

where $\eta_s^N := \left(\frac{1}{N}\sum_{j=1}^N \delta_{X_0^j}\right) P_s$ and

$$Z_t^i = X_0^i + \sigma B_t^i - \int_0^t \nabla V\left(Z_s^i\right) ds - \frac{1}{N}\sum_{j=1}^N \int_0^t \nabla F\left(Z_s^i - Z_s^j\right) ds\,,$$

$B^1, \cdots, B^N$ being N independent Brownian motions (and independent from the initial random variables). Then, for any $T > 0$, we have the following inequality:

$$\sup_{t\in[0;T]} \mathbb{E}\left\{\left|\left|X_t^i - Z_t^i\right|\right|^2\right\} \leq \frac{C(\mu_0)}{(\theta + 2\alpha)^2 N} \exp\left[2\left(\theta + 2\alpha\right) T\right]\,,$$

where $C(\mu_0)$ is a positive function of $\int_{\mathbb{R}^d} ||x||^{8q^2} \mu_0(dx)$.
Theorem C: *We assume that V, F and μ_0 satisfy the set of Hypotheses (A). Also, we assume that the initial random variables are independent. If $\sigma > \sigma_c$ (where σ_c is defined in Theorem A) and if $-\alpha > \theta$, we have the uniform propagation of chaos. In other words, we have the limit*

$$\lim_{N\to+\infty} \sup_{t\geq 0} \mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right) = 0\,. \tag{1.2}$$

Moreover, we can compute the rate of convergence by dealing with $\psi(t)$, where ψ is defined by

$$\sup_{t\in[0;T]} \mathbb{E}\left\{\left|\left|X_t^i - Z_t^i\right|\right|^2\right\} \leq \left(\frac{\exp\left[\psi(T)\right]}{\sqrt{N}}\right)^2\,.$$

In the previous inequality, we are dealing with the notations X_t^i and Z_t^i of Proposition (B).
First case: *The quantity $\frac{C(\sigma)t}{\psi(t)}$ goes to $\lambda \in \mathbb{R}_+^* \bigcup \{+\infty\}$ as t goes to infinity. Thus, for all $0 < \delta < 1$, we have:*

$$\lim_{N\to+\infty} N^{\frac{1}{2(1+1/\lambda)}-\delta} \sup_{t\geq 0} \mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right) = 0\,. \tag{1.3}$$

Second case: *The quantity $\frac{C(\sigma)t}{\psi(t)}$ goes to 0 as t goes to infinity. Thus, for all $\delta \in]0;1[$, we have:*

$$\limsup_{N\to+\infty} \exp\left\{C(\sigma)\psi^{-1}\left[\frac{1}{2}(1-\delta)\log(N)\right]\right\} \sup_{t\geq 0} \mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right) < \infty\,. \tag{1.4}$$

Let us point out that the assumption $-\alpha > \theta$ is purely technical. We also point out that it has been used in previous work (like in [CMV03]). However, this was used jointly with the assumption that the center of mass is fixed. And, we do not know any case in which this last hypothesis is satisfied except if both V, F and μ_0 are symmetric, which is a very strong restriction.

Let us give a corollary of Theorem C.

Corollary D: *Let us assume that V, F and μ_0 satisfy the set of Assumptions (A) and that* $\max\{\alpha\,;\,\theta\} \leq 0$. *Let $X_0^1, \cdots, X_0^N$ be N random variables with common law μ_0.* We do not assume these variables to be independent but they are exchangeable. *For any $\sigma > 0$, we have the following uniform propagation of chaos result:*

$$\lim_{N\to\infty} N^{1-\delta} \sup_{t\geq 0} \mathbb{W}_2^2\left(\mu_t\,;\,\mu_t^{1,N}\right) = 0$$

for any $0 < \delta < 1$.

We now give the main results concerning the creation of chaos, when $X_0^1 = \cdots = X_0^N = x_0 \in \mathbb{R}$.

Theorem E: *Let f_1 and f_2 be two Lipschitz-continuous functions. Under the sets of Assumptions (A) and (B), for all $\epsilon > 0$, for all $T > 0$, there exist $t_0(\epsilon)$ and $N_0(\epsilon)$ such that*

$$\sup_{N\geq N_0(\epsilon)} \sup_{t\in[t_0(\epsilon);t_0(\epsilon)+T]} \left|\mathrm{Cov}\left[f_1\left(X_t^{1,N}\right)\,;\,f_2\left(X_t^{2,N}\right)\right]\right| \leq \epsilon\,.$$

We can remark that a small covariance implies a phenomenon of chaos. Consequently, we have creation of chaos after time $t_0(\epsilon)$. And, there is propagation of this chaos on an interval of length T.

Theorem F: *Let f_1 and f_2 be two Lipschitz-continuous functions. Under the sets of Assumptions (A) and (B), if moreover, V and F are convex then, for all $\epsilon > 0$, there exist $t_0(\epsilon)$ and $N_0(\epsilon)$ such that*

$$\sup_{N\geq N_0(\epsilon)} \sup_{t\geq t_0(\epsilon)} \left|\mathrm{Cov}\left[f_1\left(X_t^{1,N}\right)\,;\,f_2\left(X_t^{2,N}\right)\right]\right| \leq \epsilon\,.$$

Here, we have a uniform propagation of chaos after the creation of chaos. Let us remark that, in Theorem E and in Theorem F, we consider only two particles but we have the same result with any integer k.

We also have results about the empirical measure of the system. In case of chaos, this measure is close to a measure of the form $\nu^{\otimes N}$.

Theorem G: *Let f_1 and f_2 be two Lipschitz-continuous functions. Under the sets of Assumptions (A) and (B), for all $\epsilon > 0$, for all $T > 0$, there exist $t_1(\epsilon)$ and $N_1(\epsilon)$ such that*

$$\sup_{N\geq N_1(\epsilon)} \sup_{t\in[t_1(\epsilon);t_1(\epsilon)+T]} \left|\mathrm{Cov}\left[\eta_t^N(f_1)\,;\,\eta_t^N(f_2)\right]\right| \leq \epsilon$$

with $\eta_t^N(f) := \frac{1}{N}\sum_{i=1}^N f_i\left(X_t^{i,N}\right)$. If, moreover, both V and F are convex, we *have*

$$\sup_{N\geq N_1(\epsilon)}\sup_{t\geq t_1(\epsilon)}\left|\mathrm{Cov}\left[\eta_t^N(f_1)\,;\,\eta_t^N(f_2)\right]\right|\leq\epsilon$$

We conjecture that, by using the same technics, one should be able to obtain creation of chaos for more general mean-field models providing that the hydrodynamical limit is stable in long-time.

2 Functional inequality

Let us give the framework (definitions and basic propositions) of the current work. For any probability measures on $\mathbb{R}^d$, μ and ν, the Wasserstein distance between μ and ν is

$$\mathbb{W}_2(\mu\,;\,\nu):=\sqrt{\inf\ \mathbb{E}\left\{||X-Y||^2\right\}},$$

where the infimum is taken over the random variables X and Y with law μ and ν respectively. The Wasserstein distance can be characterized in the following way, thanks to Brenier's theorem, see [Bre91].

Let μ and ν be two probability measures which admit a finite second moment on $\mathbb{R}^d$. If μ is absolutely continuous with respect to the Lebesgue measure, there exists a convex function τ from $\mathbb{R}^d$ to $\mathbb{R}$ such that the following equality occurs for every bounded test function g:

$$\int_{\mathbb{R}^d}g(x)\nu(dx)=\int_{\mathbb{R}^d}g\left(\nabla\tau(x)\right)\mu(dx)\,.$$

Then, we write

$$\nu=\nabla\tau\#\mu\,,$$

and we have the following equality

$$\mathbb{W}_2(\mu\,;\,\nu)=\sqrt{\int_{\mathbb{R}^d}||x-\nabla\tau(x)||^2\,\mu(dx)}\,.$$

The key-idea of the paper is a so-called $WJ_{V,F}$-inequality. Let us present the expression that we denote by $J_{V,F}(\nu\mid\mu)$ if μ is absolutely continuous with respect to the Lebesgue measure:

$$\begin{aligned}J_{V,F}(\nu\mid\mu)&:=\frac{\sigma^2}{2}\int_{\mathbb{R}^d}\Big(\Delta\tau(x)+\Delta\tau^*\left(\nabla\tau(x)\right)-2d\Big)\mu(dx)\\&+\int_{\mathbb{R}^d}\left\langle\nabla V\left(\nabla\tau(x)\right)-\nabla V(x)\,;\,\nabla\tau(x)-x\right\rangle\mu(dx)\\&+\frac{1}{2}\iint_{\mathbb{R}^{2d}}\left\langle\nabla F\left(Z(x,y)\right)-\nabla F(x-y)\,;\,Z(x,y)-(x-y)\right\rangle\mu(dx)\mu(dy)\,,\end{aligned}\tag{2.1}$$

with $Z(x,y):=\nabla\tau(x)-\nabla\tau(y)$ and where τ^* denotes the Legendre transform of τ. Here, we have $\nu=\nabla\tau\#\mu$. We now present the transportation inequality, already used in [BGG12a, BGG12b], on which the article is based.

Definition 2.1. *Let μ be a probability measure on $\mathbb{R}^d$ absolutely continuous with respect to the Lebesgue measure and $C > 0$. We say that μ satisfies a $WJ_{V,F}(C)$-inequality if the inequality*

$$C\mathbb{W}_2^2\left(\nu\,;\,\mu\right) \leq J_{V,F}\left(\nu\,|\,\mu\right) \tag{2.2}$$

holds for any probability measure ν on $\mathbb{R}^d$.

In the following, we aim to establish a $WJ_{V,F}$-inequality for an invariant probability μ^σ of Diffusion (I). However, it is well known that μ^σ is absolutely continuous with respect to the Lebesgue measure. Consequently, we can apply Brenier's theorem. So, the $WJ_{V,F}$-inequality consists in obtaining an inequality on the convex function τ from $\mathbb{R}^d$ to $\mathbb{R}$.

We now give a result which explains why a $WJ_{V,F}$-inequality has consequences on the long-time behavior of McKean-Vlasov diffusions (I). It is similar to [BGG12b, Proposition 1.1].

Proposition 2.2. *Let V and F be two functions satisfying Hypotheses (A-1)–(A-8). Let μ_0 and ν_0 be two probability measures on $\mathbb{R}^d$ absolutely continuous with respect to the Lebesgue measure. Set $(X_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$ two McKean-Vlasov diffusions (I) starting with law μ_0 and ν_0. By μ_t (respectively ν_t), we denote the law of X_t (respectively Y_t).*
Therefore, we have the inequality

$$\frac{1}{2}\frac{d}{dt}\mathbb{W}_2^2\left(\mu_t\,;\,\nu_t\right) \leq -J_{V,F}\left(\nu_t\,|\,\mu_t\right)\,. \tag{2.3}$$

Consequently, if μ^σ is an invariant probability of Diffusion (I) and if μ^σ satisfies a $WJ_{V,F}(C)$-inequality, by combining Ineq. (2.2) and Ineq. (2.3), we obtain

$$\frac{1}{2}\frac{d}{dt}\mathbb{W}_2^2\left(\mu_t\,;\,\mu^\sigma\right) \leq -J_{V,F}\left(\mu_t\,|\,\mu^\sigma\right) \leq -C\mathbb{W}_2^2\left(\mu_t\,;\,\mu^\sigma\right)\,,$$

for any μ_0 absolutely continuous with respect to the Lebesgue measure. Hence, by integration in time, $\mathbb{W}_2\left(\mu_t;\mu^\sigma\right) \leq e^{-Ct}\mathbb{W}_2\left(\mu_0;\mu^\sigma\right)$.

In [BGG12b], Bolley, Gentil and Guillin suggested a method to obtain a $WJ_{V,F}$-inequality in the non-convex case. But, we proceed in a slightly different way. We first use their result which provides a $WJ_{V_0,0}\left(\mathcal{C}^\sigma\right)$-inequality. Then, we prove that $\mathcal{C}^\sigma$ goes to infinity as σ goes to infinity. Finally, we remark that $J_{V,F}\left(\mu\,|\,\mu^\sigma\right) \geq J_{V_0,0}\left(\mu\,|\,\mu^\sigma\right) - \left(\max\left\{\alpha\,;\,0\right\}+\theta\right)\mathbb{W}_2^2\left(\mu\,;\mu^\sigma\right)$ for any measure μ.

In the following, μ^σ denotes an invariant probability of Diffusion (I). We know that such a measure exists, see [Tug12, Proposition 2.1]. Moreover, the measure satisfies the following implicit equation

$$\mu^\sigma(dx) := \frac{\exp\left\{-\frac{2}{\sigma^2}W^\sigma(x)\right\}}{\int_{\mathbb{R}^d}\exp\left\{-\frac{2}{\sigma^2}W^\sigma(y)\right\}dy}\,dx$$

with $W^\sigma(x) := V(x) + F*\mu^\sigma(x)$. Let us now give a $WJ_{V_0,0}$-inequality on the measure μ^σ.

Proposition 2.3. *We assume that V, F and μ_0 satisfy the set of Hypotheses (A). Thus, the measure μ^σ satisfies a $WJ_{V_0,0}\left(\mathcal{C}^\sigma\right)$-inequality where the constant $\mathcal{C}^\sigma$ is defined by*

$$\mathcal{C}^\sigma := \max_{R>0}\ \mathcal{C}^\sigma(R) > 0$$

$$\textit{with}\quad \mathcal{C}^\sigma(R) := \min\left\{\frac{K(R)}{3}\,;\,\frac{\sigma^2}{72R^2}e^{-\frac{2}{\sigma^2}S(R)}\,;\,\frac{K(R)}{3}\frac{3^d-2^d}{2^d}e^{\frac{2}{\sigma^2}(I(R)-S(R))}\right\}\,,$$

$$K(R) := \inf_{||x||\geq R}\nabla^2V_0(x)\,,\ I(R) := \inf_{||x||\leq 2R}W^\sigma(x)\ \textit{and}\ S(R) := \sup_{||x||\leq 3R}W^\sigma(x)\,.$$

The proof is left to the reader and consists in a simple adaptation of the proof of [BGG12a, Proposition 3.4] that is to say [BGG12a, Section 5]. Let us mention that we do not need to apply the whole set of assumptions. Indeed, to prove this result, we simply use Hypotheses (A-1)-(A-2)-(A-5). More precisely, we need the potential W^σ to be $\mathcal{C}^1$-continuous (which is an immediate consequence of (A-1) and (A-5)). We also need the function V_0 to be convex at infinity, which is proven by (A-2).

Corollary 2.4. *We assume that V, F and μ satisfy the set of Hypotheses (A). Therefore, we have the following inequality:*

$$\left(\mathcal{C}^\sigma - \max\left\{\alpha\,;\,0\right\} - \theta\right)\mathbb{W}_2^2\left(\mu\,;\,\mu^\sigma\right) \leq J_{V,F}\left(\mu\mid\mu^\sigma\right)\,. \tag{2.4}$$

Particularly, if $\mathcal{C}^\sigma - \max\left\{\alpha\,;\,0\right\} - \theta > 0$, Diffusion (I) admits a unique invariant probability μ^σ and for any μ_0 satisfying (A-9)-(A-10), we have

$$\mathbb{W}_2\left(\mu_t\,;\,\mu^\sigma\right) \leq \exp\left[-\left(\mathcal{C}^\sigma - \max\left\{\alpha\,;\,0\right\} - \theta\right)t\right]\mathbb{W}_2\left(\mu_0\,;\,\mu^\sigma\right)\,, \tag{2.5}$$

for any $t \geq 0$.

Like with Proposition 2.3, we do not need the whole set of assumptions. We assume V and F to verify (A-1)-(A-2)-(A-5). And, in order to apply Proposition 2.2, we assume that the initial law μ_0 satisfy (A-9)-(A-10).

Proof. By Proposition 2.3, we have

$$\mathcal{C}^\sigma\mathbb{W}_2^2\left(\mu\,;\,\mu^\sigma\right) \leq J_{V_0,0}\left(\mu\mid\mu^\sigma\right)\,. \tag{2.6}$$

However, by definition, the quantity $J_{V,F}\left(\mu\mid\mu^\sigma\right)$ is equal to

$$\begin{aligned}J_{V,F}\left(\mu\mid\mu^\sigma\right) &= J_{V_0,0}\left(\mu\mid\mu^\sigma\right) - \theta\int_{\mathbb{R}^d}||\nabla\tau(x)-x||^2\,\mu^\sigma(dx)\\ &- \frac{\alpha}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}||(\nabla\tau(x)-\nabla\tau(y))-(x-y)||^2\,\mu^\sigma(dx)\mu^\sigma(dy)\\ &+ \frac{1}{2}\iint_{\mathbb{R}^{2d}}\left\langle\nabla F_0\left(Z(x,y)\right)-\nabla F_0(x-y)\,;\,Z(x,y)-(x-y)\right\rangle\mu(dx)\mu(dy)\,,\end{aligned}$$

with $Z(x,y) := \nabla\tau(x) - \nabla\tau(y)$. However, F_0 is a convex function. Consequently, we have

$$J_{V,F}\left(\mu \mid \mu^\sigma\right) \geq J_{V_0,0}\left(\mu \mid \mu^\sigma\right) - \left(\max\left\{\alpha\,;\,0\right\} + \theta\right) \int_{\mathbb{R}^d} \left|\left|\nabla\tau(x) - x\right|\right|^2 \mu^\sigma(dx)\,.$$

By Brenier's theorem, we obtain

$$J_{V,F}\left(\mu \mid \mu^\sigma\right) \geq J_{V_0,0}\left(\mu \mid \mu^\sigma\right) - \left(\max\left\{\alpha\,;\,0\right\} + \theta\right) \mathbb{W}_2^2\left(\mu\,;\,\mu^\sigma\right)\,,$$

which with (2.6) gives (2.4). Here, the convex function τ is defined by $\mu =: \nabla\tau\#\mu^\sigma$. The uniqueness of the stationary measure if $\mathcal{C}^\sigma - \alpha - \theta > 0$ and the exponential decay in (2.5) are consequences of Proposition 2.2. □

Let us note that the inequality

$$\sup_{\mathbb{R}^d} -\nabla^2 V < \inf_{\mathbb{R}^d} \nabla^2 F < 0$$

implies the uniqueness of the stationary measure μ^σ and the exponential convergence toward μ^σ for any $\sigma > 0$. Such a result has already been proven in [CMV03].

We now give the proof of Theorem A.

Proof. In order to prove it, we first admit the following limit

$$\lim_{\sigma\to+\infty} \frac{1}{\sigma^2} \int_{\mathbb{R}^d} \left|\left|x\right|\right|^{2n} \mu^\sigma(dx) = 0\,, \tag{2.7}$$

for any family $\{\mu^\sigma\,;\,\sigma \geq 1\}$ of invariant probabilities of Diffusion (I). In a first step, we prove that Limit (2.7) implies the statement of Theorem A. In a second step, we prove (2.7).

Step 1. We admit the limit (2.7). We remind the reader the following equality

$$W^\sigma(x) = V(x) + F * \mu^\sigma(x)\,.$$

Moreover, Hypothesis (A-5) on F implies

$$\left|F * \mu^\sigma(x)\right| \leq C\left(1 + \left|\left|x\right|\right|^{2n}\right)\left(1 + \int_{\mathbb{R}^d} \left|\left|y\right|\right|^{2n} \mu^\sigma(dy)\right)$$

so that, for any $R > 0$, we have the limit

$$\lim_{\sigma\to+\infty} \frac{1}{\sigma^2} \sup_{||x||\leq 3R} \left|\left|W^\sigma(x)\right|\right| = 0\,,$$

thanks to Limit (2.7). Therefore, for any $R > 0$, the quantities $\exp\left[-\frac{2}{\sigma^2}S(R)\right]$ and $\exp\left[\frac{2}{\sigma^2}\left(I(R) - S(R)\right)\right]$ go to 1 as σ goes to infinity. We remind the reader that $I(R)$ and $S(R)$ are defined in Proposition 2.3. We obtain the following limit for any $R>0$:

$$\lim_{\sigma\to\infty}\mathcal{C}^\sigma(R)=\frac{K(R)}{3}\min\left\{1\,;\,\frac{3^d-2^d}{2^d}\right\}\,,$$

where $K(R):=\inf_{||x||\geq R}\nabla^2V_0(x)$. By Assumption (A-2), the quantity $K(R)$ goes to infinity as R goes to infinity. We take R_0 such that

$$\frac{K(R_0)}{3}\min\left\{1\,;\,\frac{3^d-2^d}{2^d}\right\}>4(|\alpha|+|\theta|)\,.$$

Then, we take σ_c large enough such that $\mathcal{C}^\sigma(R_0)>\frac{1}{2}\lim_{\xi\to\infty}\mathcal{C}^\xi(R_0)$ for any $\sigma\geq\sigma_c$. Thus, we have the inequality

$$\mathcal{C}^\sigma-\max\{\alpha;0\}-\theta\geq\mathcal{C}^\sigma(R_0)-\max\{\alpha;0\}-\theta>|\alpha|+|\theta|$$

for any $\sigma\geq\sigma_c$. Consequently, if Limit (2.7) is satisfied, the statement of the theorem is proven.

Step 2. We now achieve the proof by establishing Limit (2.7). *It is in this step that we use the hypothesis $p>2n$.* We proceed a *reducto ad absurdum*. Let us assume the existence of a positive constant C and an increasing sequence $(\sigma_k)_{k\in\mathbb{N}}$ which goes to infinity such that for any $k\in\mathbb{N}$, Diffusion (I) admits an invariant probability μ^{σ_k} satisfying

$$\eta_{2n}(k):=\int_{\mathbb{R}^d}||x||^{2n}\mu^{\sigma_k}(dx)\geq C\sigma_k^2\,.$$

In particular, we deduce that the sequence $(\eta_{2n}(k))_{k\in\mathbb{N}}$ goes to infinity as k goes to infinity. Since μ^{σ_k} is an invariant probability, we have

$$\eta_{2n}(k)=\frac{\int_{\mathbb{R}^d}||x||^{2n}\exp\left\{-\frac{2}{\sigma_k^2}\left[V(x)+F*\mu^{\sigma_k}(x)\right]\right\}dx}{\int_{\mathbb{R}^d}\exp\left\{-\frac{2}{\sigma_k^2}\left[V(x)+F*\mu^{\sigma_k}(x)\right]\right\}dx}\,.$$

By making the transformation $x:=(\eta_{2n}(k))^{\frac{1}{2n}}y$, we obtain

$$1=\frac{\int_{\mathbb{R}^d}||y||^{2n}\exp\left\{-\frac{2}{\widehat{\sigma_k}^2}\left[\frac{V\left((\eta_{2n}(k))^{\frac{1}{2n}}y\right)}{(\eta_{2n}(k))^{\frac{p}{n}}}+\frac{F*\mu^{\sigma_k}\left((\eta_{2n}(k))^{\frac{1}{2n}}y\right)}{(\eta_{2n}(k))^{\frac{p}{n}}}\right]\right\}dy}{\int_{\mathbb{R}^d}\exp\left\{-\frac{2}{\widehat{\sigma_k}^2}\left[\frac{V\left((\eta_{2n}(k))^{\frac{1}{2n}}y\right)}{(\eta_{2n}(k))^{\frac{p}{n}}}+\frac{F*\mu^{\sigma_k}\left((\eta_{2n}(k))^{\frac{1}{2n}}y\right)}{(\eta_{2n}(k))^{\frac{p}{n}}}\right]\right\}dy}\,,\tag{2.8}$$

with $\widehat{\sigma_k}:=\frac{\sigma_k}{\sqrt{\eta_{2n}(k)}}(\eta_{2n}(k))^{-\frac{p-n}{2n}}\leq\frac{1}{\sqrt{C}}(\eta_{2n}(k))^{-\frac{p-n}{2n}}\to0$ as k goes to infinity. For any $y\in\mathbb{R}^d$, Hypothesis (A-8) implies

$$\lim_{k\to+\infty}\frac{F*\mu^{\sigma_k}\left((\eta_{2n}(k))^{\frac{1}{2n}}y\right)}{(\eta_{2n}(k))^{\frac{p}{n}}}=0\,.$$

And, Assumption (A-7) yields

$$\lim_{k\to+\infty}\frac{V\left(\left(\eta_{2n}(k)\right)^{\frac{1}{2n}}y\right)}{\left(\eta_{2n}(k)\right)^{\frac{p}{n}}}=\Theta(y)\,,$$

the function Θ being strictly convex and such that $\Theta(y)>\Theta(0)=0$ for any $y\neq 0$. Consequently, by applying [Tug12, Lemma A.2], the right hand term in (2.8) goes to 0 as k goes to infinity. Nevertheless, the left hand term is equal to 1. The initial assumption of Step 2 is absurd. This achieves the proof. $\square$

Let us remark that Theorem A goes further than the results in [Tug13c] concerning the uniqueness of the invariant probability for sufficiently large σ. Moreover, it could provide, with Corollary 2.4 a method for simulating a lower-bound of the critical value above which there is a unique invariant probability. Nevertheless, this method needs more computation than those described in [Tug13c] and is not really tractable.

Let us mention that the difference with the results obtained in [BGG12b] is that the confinement potential V is not assumed to be convex.

3 Propagation of chaos

We now give the proof of Proposition B. This result is not the classical propagation of chaos because the initial random variables are not supposed to be independent. However, we have the same inequality and this is one of the main tools of the proof of the main theorem.

Proof. By μ_t, we denote the law $\mathcal{L}\left(X_t^1\right)=\cdots=\mathcal{L}\left(X_t^N\right)$. By definition, for any $1\leq i\leq N$, we have

$$Z_t^i-X_t^i=-\int_0^t\left\{\nabla V(Z_s^i)-\nabla V(X_s^i)+\sum_{j=1}^N\frac{1}{N}\nabla F(Z_s^i-Z_s^j)-\nabla F*\eta_s^N(X_s^i)\right\}ds\,.$$

We apply Itô formula to $Z_t^i-X_t^i$ with the function $x\mapsto||x||^2$. By using the notation $\xi_i(t):=\left|\left|Z_t^i-X_t^i\right|\right|^2$, we obtain

$$\begin{aligned}d\xi_i(t)=&-2\left\langle Z_t^i-X_t^i\,;\,\nabla V(Z_t^i)-\nabla V\left(X_t^i\right)\right\rangle\\&-\frac{2}{N}\left\langle Z_t^i-X_t^i\,;\,\sum_{j=1}^N\left[\nabla F(Z_t^i-Z_t^j)-\nabla F*\eta_t^N\left(X_t^i\right)\right]\right\rangle\,.\end{aligned}$$

By taking the sum on the integer i running between 1 and N, we get

$$
\begin{aligned}
d\sum_{i=1}^{N}\xi_i(t) &= -2\sum_{i=1}^{N}\left\langle Z_t^i - X_t^i\,;\,\nabla V(Z_t^i) - \nabla V\left(X_t^i\right)\right\rangle dt\\
&\quad -\frac{2}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\Big(\Delta_2(i,j,t)+\Delta_3(i,j,t)\Big)dt
\end{aligned}
$$

$$
\text{with} \qquad \Delta_2(i,j,t) := \left\langle \nabla F(Z_t^i - Z_t^j) - \nabla F(X_t^i - X_t^j)\,;\, Z_t^i - X_t^i\right\rangle
$$

$$
\text{and} \qquad \Delta_3(i,j,t) := \left\langle \nabla F(X_t^i - X_t^j) - \nabla F * \eta_t^N\left(X_t^i\right)\,;\, Z_t^i - X_t^i\right\rangle .
$$

According to the definition of the function F_0 in Hypothesis (A-6), it is convex. This implies $\langle x-y\,;\,\nabla F_0(x-y)\rangle \geq 0$ for any $x,y\in\mathbb{R}^d$. This inequality yields

$$
\mathbb{E}\left\{-\frac{2}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\Delta_2(i,j,t)\right\} \leq 4\alpha\sum_{i=1}^{N}\left|\left|Z_t^i - X_t^i\right|\right|^2 . \tag{3.1}
$$

By definition of θ, for any $x,y\in\mathbb{R}^d$ we have $\langle\nabla V(x)-\nabla V(y)\,;\,x-y\rangle \geq -\theta\left|\left|x-y\right|\right|^2$. This implies

$$
-2\sum_{i=1}^{N}\left\langle Z_t^i - X_t^i\,;\,\nabla V(Z_t^i)-\nabla V\left(X_t^i\right)\right\rangle \leq 2\theta\sum_{i=1}^{N}\xi_i(t)\,. \tag{3.2}
$$

We now deal with the sum containing $\Delta_3(i,j,t)$. We apply Cauchy-Schwarz inequality:

$$
-\mathbb{E}\left[\sum_{j=1}^{N}\Delta_3(i,j,t)\right] \leq \left\{\mathbb{E}\left[\left|\left|Z_t^i - X_t^i\right|\right|^2\right]\right\}^{\frac{1}{2}}\left\{\sum_{j=1}^{N}\sum_{k=1}^{N}\mathbb{E}\left[\left\langle\rho_j^i(t)\,;\,\rho_k^i(t)\right\rangle\right]\right\}^{\frac{1}{2}}
$$

$$
\text{with} \quad \rho_j^i(t) := \nabla F(X_t^i - X_t^j) - \nabla F * \eta_t^N\left(X_t^i\right) .
$$

The idea now is to prove an inequality of the form

$$
\sum_{j=1}^{N}\sum_{k=1}^{N}\mathbb{E}\left[\left\langle\rho_j^i(t)\,;\,\rho_k^i(t)\right\rangle\right] \leq CN\,,
$$

where C is a positive constant. We use the following conditioning:

$$
\mathbb{E}\left[\left\langle\rho_j^i(t)\,;\,\rho_k^i(t)\right\rangle\right] = \mathbb{E}\left\{\mathbb{E}\left[\left\langle\rho_j^i(t)\,;\,\rho_k^i(t)\right\rangle \mid X_0^1,\cdots,X_0^i,\cdots,X_0^N,X_t^i\right]\right\} .
$$

The particles X^r, $1\leq r\leq N$, are not independent but they are independent conditionally to the knowledge of the initial random variables $X_0^1,\cdots,X_0^i,\cdots,X_0^N$. Therefore, we have the equality

$$
\begin{aligned}
&\mathbb{E}\left[\left\langle\rho_j^i(t)\,;\,\rho_k^i(t)\right\rangle\right]\\
&= \mathbb{E}\left\{\left\langle\mathbb{E}\left[\rho_j^i(t) \mid X_0^1,\cdots,X_0^i,\cdots,X_0^N,X_t^i\right]\,;\,\mathbb{E}\left[\rho_k^i(t) \mid X_0^1,\cdots,X_0^i,\cdots,X_0^N,X_t^i\right]\right\rangle\right\} ,
\end{aligned}
$$

if $j \neq k$. Consequently, for any $1 \leq j \leq N$, we have

$$\begin{aligned}
&\sum_{k=1}^{N} \mathbb{E}\left[\left\langle \rho_j^i(t)\,;\,\rho_k^i(t)\right\rangle\right]\\
=&\mathbb{E}\left\{\left\langle \mathbb{E}\left[\rho_j^i(t)\mid X_0^1,\cdots,X_0^i,\cdots,X_0^N,X_t^i\right]\,;\,\mathbb{E}\left[\sum_{k=1}^{N}\rho_k^i(t)\mid X_0^1,\cdots,X_0^i,\cdots,X_0^N,X_t^i\right]\right\rangle\right\}\\
&+\mathbb{E}\left[\left|\left|\rho_j^i(t)\right|\right|^2\right]-\mathbb{E}\left\{\left|\left|\mathbb{E}\left[\rho_j^i(t)\mid X_0^1,\cdots,X_0^i,\cdots,X_0^N,X_t^i\right]\right|\right|^2\right\}\\
\leq&\mathbb{E}\left\{\left\langle \mathbb{E}\left[\rho_j^i(t)\mid X_0^1,\cdots,X_0^i,\cdots,X_0^N,X_t^i\right]\,;\,\mathbb{E}\left[\sum_{k=1}^{N}\rho_k^i(t)\mid X_0^1,\cdots,X_0^i,\cdots,X_0^N,X_t^i\right]\right\rangle\right\}\\
&+\mathbb{E}\left[\left|\left|\rho_j^i(t)\right|\right|^2\right].
\end{aligned}$$

We now take the sum over j:

$$\begin{aligned}
\sum_{j=1}^{N}\sum_{k=1}^{N}\mathbb{E}\left[\left\langle \rho_j^i(t)\,;\,\rho_k^i(t)\right\rangle\right]\leq&\sum_{j=1}^{N}\mathbb{E}\left[\left|\left|\rho_j^i(t)\right|\right|^2\right]\\
&+\mathbb{E}\left\{\left|\left|\mathbb{E}\left[\sum_{k=1}^{N}\rho_k^i(t)\mid X_0^1,\cdots,X_0^i,\cdots,X_0^N,X_t^i\right]\right|\right|^2\right\}.
\end{aligned}$$

Now, we will prove that

$$\mathbb{E}\left[\sum_{k=1}^{N}\rho_k^i(t)\mid X_0^1,\cdots,X_0^i,\cdots,X_0^N,X_t^i\right]$$

is equal to $-\nabla F * \nu_t^{X_0^i,\frac{1}{N}\sum_{l=1}^N\delta_{X_0^l}}\left(X_t^i\right)$ where $\nu_t^{x_0,\mu_0}$ is the law of the diffusion

$$Y_t := x_0+\sigma B_t-\int_0^t\nabla V\left(Y_s\right)ds-\int_0^t\left(\nabla F*\mu_0P_s\right)\left(Y_s\right)ds\,.$$

Indeed, for any $1\leq k\leq N$ with $k\neq i$, we have

$$\mathbb{E}\left[\nabla F(X_t^i-X_t^k)\mid X_0^1,\cdots,X_0^N,X_t^i\right]=\nabla F*\nu_t^{X_0^k,\frac{1}{N}\sum_{l=1}^N\delta_{X_0^l}}\left(X_t^i\right)\,,$$

We remark that

$$\frac{1}{N}\sum_{k=1}^{N}\nu_t^{X_0^k,\frac{1}{N}\sum_{l=1}^N\delta_{X_0^l}}=\left(\frac{1}{N}\sum_{l=1}^{N}\delta_{X_0^l}\right)P_t\,.$$

The right-hand side of the previous equality being η_t^N, we consequently get

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k=1}^{N}\nabla F(X_t^i-X_t^k)\mid X_0^1,\cdots,X_0^i,\cdots,X_0^N,X_t^i\right]\\
&=N\nabla F*\eta_t^N\left(X_t^i\right)-\nabla F*\nu_t^{X_0^i,\frac{1}{N}\sum_{l=1}^N\delta_{X_0^l}}\left(X_t^i\right)\,.
\end{aligned}$$

Let us remark that the term for $k = i$ is equal to zero in the left-hand side of the previous equality. This yields

$$\mathbb{E}\left[\sum_{k=1}^{N}\rho_k^i(t) \mid X_0^1, \cdots, X_0^i, \cdots, X_0^N, X_t^i\right] = -\nabla F * \nu_t^{X_0^i, \frac{1}{N}\sum_{l=1}^N \delta_{X_0^l}}\left(X_t^i\right).$$

We obtain immediately:

$$\sum_{j=1}^{N}\sum_{k=1}^{N}\mathbb{E}\left[\langle\rho_j^i(t)\,;\,\rho_k^i(t)\rangle\right] \leq \sum_{j=1}^{N}\mathbb{E}\left[\left|\left|\rho_j^i(t)\right|\right|^2\right]+\mathbb{E}\left\{\left|\left|\nabla F * \nu_t^{X_0^i, \frac{1}{N}\sum_{l=1}^N \delta_{X_0^l}}\left(X_t^i\right)\right|\right|^2\right\}.$$

Let us now compute $\mathbb{E}\left[\left|\left|\rho_j^i(t)\right|\right|^2\right]$. The diffusions X^i and X^j are not independent but they are independent conditionally to the initial random variables. However, according to Hypothesis (A-5), we have $F(x) = G\left(||x||\right)$ where G is a polynomial function of degree $2n$, we have the following inequality:

$$\begin{aligned}&\mathbb{E}\left[\left|\left|\nabla F(X_t^i - X_t^j) - \nabla F * \eta_t^N(X_t^i) \mid X_0^1, \cdots, X_0^N\right|\right|^2\right]\\
&\leq C\left(1 + \mathbb{E}\left[\left|\left|X_t^i\right|\right|^{4n-2} \mid X_0^1, \cdots, X_0^i, \cdots, X_0^N\right] + \mathbb{E}\left[\left|\left|X_t^j\right|\right|^{4n-2} \mid X_0^1, \cdots, X_0^i, \cdots, X_0^N\right]\right).\end{aligned}$$

Then, we use the control of the moments obtained in [HIP08, Theorem 2.13] and we obtain the following majoration:

$$\begin{aligned}&\sup_{t\geq 0}\mathbb{E}\left[\left|\left|\nabla F(X_t^i - X_t^j) - \nabla F * \eta_t^N(X_t^i)\right|\right|^2 \mid X_0^1, \cdots, X_0^i, \cdots, X_0^N\right]\\
&\leq K\left(1 + \frac{1}{N}\sum_{k=1}^{N}\left|\left|X_0^k\right|\right|^{8q^2}\right),\end{aligned}$$

K being a positive constant. Consequently, we have

$$\sup_{t\geq 0}\mathbb{E}\left[\left|\left|\nabla F(X_t^i - X_t^j) - \nabla F * \eta_t^N(X_t^i)\right|\right|^2\right] \leq K\left(1 + \int_{\mathbb{R}^d}||x||^{8q^2}\,\mu_0(dx)\right).$$

We have a similar control on $\mathbb{E}\left\{\left|\left|\nabla F * \nu_t^{X_0^i, \frac{1}{N}\sum_{l=1}^N \delta_{X_0^l}}\left(X_t^i\right)\right|\right|^2\right\}$. Therefore, we deduce the following inequality:

$$-\mathbb{E}\left[\sum_{j=1}^{N}\Delta_3(i,j,t)\right] \quad \leq \sqrt{C(\mu_0)}\sqrt{N\mathbb{E}\left[\xi_i(t)\right]}\,. \tag{3.3}$$

By combining (3.1), (3.2) and (3.3), we obtain

$$\frac{d}{dt}\sum_{i=1}^{N}\mathbb{E}\left[\xi_i(t)\right] \leq 2\sum_{i=1}^{N}\left\{\left(\theta + 2\alpha\right)\mathbb{E}\left[\xi_i(t)\right] + \frac{\sqrt{C(\mu_0)}}{\sqrt{N}}\sqrt{\mathbb{E}\left[\xi_i(t)\right]}\right\}. \tag{3.4}$$

However, the particles are exchangeable. Consequently, for any $1 \leq i \leq N$, we have

$$\frac{d}{dt}\mathbb{E}\left\{\xi_i(t)\right\} \leq 2\left(\theta+2\alpha\right)\mathbb{E}\left\{\xi_i(t)\right\} + \frac{2\sqrt{C(\mu_0)}}{\sqrt{N}}\sqrt{\mathbb{E}\left[\xi_i(t)\right]}\,.$$

By introducing $\tau_i(t) := \sqrt{\mathbb{E}\left\{\xi_i(t)\right\}}$, we obtain

$$\tau_i'(t) \leq (\theta+2\alpha)\left\{\tau_i(t) + \frac{\sqrt{C(\mu_0)}}{(\theta+2\alpha)\sqrt{N}}\right\}\,.$$

The application of Grönwall lemma yields

$$\mathbb{E}\left\{\left|\left|Z_t^i - X_t^i\right|\right|^2\right\} \leq \frac{C(\mu_0)}{N\left(\theta+2\alpha\right)^2}\exp\left[2\left(\theta+2\alpha\right)t\right]\,.$$

We obtain the statement of Proposition B by taking the supremum for t running between 0 and T. $\square$

4 Uniform propagation of chaos

In this paragraph, we prove that there is uniform (with respect to the time) propagation of chaos with sufficiently large σ, that is Theorem C. In all this section, we assume the inequality of simple propagation of chaos.

We consider an additional hypothesis, that is we are in the synchronized case:

$$\inf_{x\in\mathbb{R}} G''(x) = \alpha_0 > 0 \quad \text{and} \quad \alpha_0 - \theta > 0\,,$$

θ being defined between Assumption (A-2) and Assumption (A-3).

Before giving the proof of the main theorem, we give the following result to control the moments.

Proposition 4.1. *We assume that the potentials V and F and the probability measure μ_0 satisfy the set of Assumptions (A). Let $Z_0^1, \cdots, Z_0^N$ be N i.i.d. random variables with common law μ_0. We consider the following particle system:*

$$Z_t^i = Z_0^i + \sigma B_t^i - \int_0^t \nabla V\left(Z_s^i\right)ds - \frac{1}{N}\sum_{j=1}^N\int_0^t \nabla F\left(Z_s^i - Z_s^j\right)ds\,, \tag{4.1}$$

$B^1, \cdots, B^N$ being N independent Brownian motions (and independent from the initial random variables). Then, there exists a constant $M(\mu_0)$ such that

$$\max_{1\leq k\leq 8q^2}\sup_{t\geq 0}\mathbb{E}\left\{\left|\left|Z_t^i\right|\right|^k\right\} \leq M(\mu_0)\,, \tag{4.2}$$

for any $N \in \mathbb{N}$.

The proof is classical and can be adapted from [CGM08, Section 2.1] so it is left to the reader. Let us just mention that the only hypotheses that we need on the potentials V and F are (A-1), (A-2), (A-4) and (A-6) and the law to satisfy assumption (A-9). Indeed, these hypotheses are sufficient to ensure the convexity at infinity of the drift $V + F * \mu_t$. We now give the proof of Theorem C.

Proof. **Step 1.** Let t be a positive real. The idea is to consider a nonlinear diffusion. Let T be a positive real.
Step 2. The triangular inequality implies

$$\mathbb{W}_2\left(\mu_{T+t}\,;\,\mu_{T+t}^{1,N}\right) \leq \mathbb{W}_2\left(\mu_{T+t}\,;\,\mu^\sigma\right) + \mathbb{W}_2\left(\mu^\sigma\,;\,\rho_{T,t}^N\right) + \mathbb{W}_2\left(\rho_{T,t}^N\,;\,\eta_{T,t}^N\right) + \mathbb{W}_2\left(\eta_{T,t}^N\,;\,\mu_{T+t}^{1,N}\right)\,,$$

where $\eta_{T,t}^N$ is the law at time T of the diffusion

$$Z_{s;t}^{i,N} = X_t^{i,N} + \sigma\left(B_{t+s}^i - B_t^i\right) - \int_0^s \left[\nabla V + \nabla F * \nu_u^N\right]\left(Z_{u;t}^{i,N}\right)du\,,$$

where $\nu_u^N := \left(\frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}\right)P_u$, and $\rho_{T,t}^N$ is the law of the diffusion

$$Y_s = Y_0 + \sigma\left(B_{t+s} - B_t\right) - \int_0^s \left(\nabla V + \nabla F * \mathcal{L}\left(Y_u\right)\right)(Y_u)du\,,$$

where Y_0 follows the law $\frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}$. More precisely, we put $Y_0(\omega) := X_t^{\zeta(\omega),N}$ where ζ is a random variable which follows the equiprobability on the set $\{1,\cdots,N\}$. We also assume that ζ is independent from $X_t^{i,N}$ and B^i for any $1 \leq i \leq N$.
Step 3. Let us bound each of the four terms.
Step 3.1. We can bound easily the last term. By definition and by assumption, we have:

$$\mathbb{W}_2\left(\eta_{T,t}^N\,;\,\mu_{T+t}^{1,N}\right) \leq \frac{\exp\left[\psi(T)\right]}{\sqrt{N}}\,.$$

Indeed, we remark that

$$\mathbb{W}_2\left(\eta_{T,t}^N\,;\,\mu_{T+t}^{1,N}\right)^2 \leq \mathbb{E}\left\{\left|\left|Z_{T;t}^{i,N} - X_{T+t}^i\right|\right|^2\right\} \leq \frac{\exp\left[2\psi(T)\right]}{N}\,,$$

by Proposition B.
Step 3.2. The first term can be bounded like so:

$$\mathbb{W}_2\left(\mu_{T+t}\,;\,\mu^\sigma\right) \leq e^{-C(\sigma)(T+t)}\mathbb{W}_2\left(\mu_0\,;\,\mu^\sigma\right)\,.$$

Step 3.3. We proceed in a similar way with the second term. We introduce the McKean-Vlasov diffusion starting from the law $\frac{1}{N}\sum_{j=1}^N \delta_{x_0^j}$. We have

$$\mathbb{W}_2\left(\mu^\sigma\,;\,\rho_{T,t}^N\right)^2 = \inf\mathbb{E}\left\{||X_\sigma - X_2||^2\right\}$$

where the infimum runs for X_σ which follows the law μ^σ and for X_2 which has the same law as the diffusion Y_t. We can write

$$\mathbb{W}_2\left(\mu^\sigma\,;\,\eta^N_{T,t}\right)^2 \leq \mathbb{E}\left\{\mathbb{W}_2\left(\mu^\sigma\,;\,\left(\frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}\right)P_T\right)^2\right\}\,. \tag{4.3}$$

Indeed, the application $\mu \mapsto \mathbb{W}_2^2(\mu^\sigma,\mu)$ is convex so that we obtain (4.3) since $\rho^N_{T,t}$ is the expectation of the random measure $\left(\frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}\right)P_T$. We deduce immediately:

$$\begin{aligned}
&\mathbb{W}_2\left(\mu^\sigma\,;\,\rho^N_{T,t}\right)^2\\
&\leq \mathbb{E}\left\{e^{-2C(\sigma)T}\mathbb{W}_2\left(\mu^\sigma\,;\,\frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}\right)^2\right\}\\
&\leq 2e^{-2C(\sigma)T}\mathbb{W}_2\left(\mu^\sigma\,;\,\mu_t^{1,N}\right)^2 + \mathbb{E}\left\{2e^{-2C(\sigma)T}\mathbb{W}_2\left(\mu_t^{1,N}\,;\,\frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}\right)^2\right\}\\
&\leq 2e^{-2C(\sigma)T}\mathbb{W}_2\left(\mu^\sigma\,;\,\mu_t^{1,N}\right)^2 + Me^{-2C(\sigma)T}\,.
\end{aligned}$$

Here, M is a positive constant which depends on the supremum of the second moment of $\mu_t^{1,N}$. We thus have:

$$\begin{aligned}
\mathbb{W}_2\left(\mu^\sigma\,;\,\rho^N_{T,t}\right) &\leq 2e^{-C(\sigma)T}\mathbb{W}_2\left(\mu^\sigma\,;\,\mu_t^{1,N}\right) + Me^{-C(\sigma)T}\\
&\leq 2e^{-C(\sigma)T}\left(\mathbb{W}_2\left(\mu^\sigma\,;\,\mu_t\right) + \mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right)\right) + Me^{-C(\sigma)T}\\
&\leq 2e^{-C(\sigma)(T+t)}\mathbb{W}_2\left(\mu^\sigma\,;\,\mu_0\right) + 2e^{-C(\sigma)T}\mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right) + Me^{-C(\sigma)T}\,.
\end{aligned}$$

Step 3.4 We now bound easily the term $\mathbb{W}_2\left(\rho^N_{T,t}\,;\,\eta^N_{T,t}\right)$. Indeed, since we are in the synchronized case, we know that $\nabla^2\left(V + F * \mu\right) \geq \alpha - \theta > 0$. So, if we take the same Brownian motion for Y and for $Z^{1,N}_{\cdot;t}$, we obtain:

$$\frac{d}{dt}\left|\left|Z_t^{1,N} - Y_t\right|\right|^2 \leq -\left(\alpha-\theta\right)\left|\left|Z_t^{1,N} - Y_t\right|\right|^2\,.$$

As a consequence, we have:

$$\mathbb{W}_2\left(\rho^N_{T,t}\,;\,\eta^N_{T,t}\right) \leq e^{-(\alpha-\theta)T}\mathbb{W}_2\left(\rho^N_{0,t}\,;\,\eta^N_{0,t}\right)\,.$$

By proceeding like in the Step 3.3, we can prove that there exists a constant M which depend on the supremum of the second moment of $\mu_t^{1,N}$ such that $\mathbb{W}_2\left(\rho^N_{T,t}\,;\,\eta^N_{T,t}\right) \leq Me^{-(\alpha-\theta)T}$.

Step 4. Consequently, we have:

$$\begin{aligned}\mathbb{W}_2\left(\mu_{T+t}\,;\,\mu^{1,N}_{T+t}\right)\leq&2e^{-C(\sigma)T}\mathbb{W}_2\left(\mu_t\,;\,\mu^{1,N}_t\right)+3e^{-C(\sigma)(T+t)}\mathbb{W}_2\left(\mu_0\,;\,\mu^\sigma\right)\\&+\frac{\exp\left[\psi(T)\right]}{\sqrt{N}}+Me^{-C(\sigma)T}+Me^{-(\alpha-\theta)T}\,.\end{aligned}$$

We now consider the quantity $\min\left\{\alpha-\theta;C(\sigma)\right\}$. Without any loss of generality, we denote this quantity by $C(\sigma)$. We deduce that

$$\begin{aligned}\mathbb{W}_2\left(\mu_{T+t}\,;\,\mu^{1,N}_{T+t}\right)\leq&2e^{-C(\sigma)T}\mathbb{W}_2\left(\mu_t\,;\,\mu^{1,N}_t\right)+3e^{-C(\sigma)(T+t)}\mathbb{W}_2\left(\mu_0\,;\,\mu^\sigma\right)\\&+\frac{\exp\left[\psi(T)\right]}{\sqrt{N}}+Me^{-C(\sigma)T}\,.\end{aligned}$$

where the constant M in the above displayed formula corresponds to two times the previous constant M. We now take the supremum for t running between $(k-1)T$ and kT, we denote $\lambda_k(T):=\sup_{kT\leq t\leq(k+1)T}\mathbb{W}_2\left(\mu_t\,;\,\mu^{1,N}_t\right)$ and $\gamma:=3\mathbb{W}_2\left(\mu_0\,;\,\mu^\sigma\right)+M$. We obtain:

$$\lambda_k(T)\leq 2e^{-C(\sigma)T}\lambda_{k-1}(T)+\gamma e^{-C(\sigma)T}+\frac{\exp\left[\psi(T)\right]}{\sqrt{N}}\,.$$

Step 5. By elementary computations, we have:

$$\lambda_k(T)\leq\frac{1}{1-2e^{-C(\sigma)T}}\frac{\exp\left[\psi(T)\right]}{\sqrt{N}}+\left(2e^{-C(\sigma)T}\right)^k\frac{\exp\left[\psi(T)\right]}{\sqrt{N}}e^{-C(\sigma)T}+\gamma\frac{e^{-C(\sigma)T}}{1-2e^{-C(\sigma)T}}\,.$$

This follows from the bound $\lambda_0(T)\leq\frac{e^{\psi(T)}}{\sqrt{N}}$ which is a consequence of classical coupling between the system of particles and the McKean-Vlasov diffusion.

By taking T large enough, we deduce

$$\sup_{t\geq0}\mathbb{W}_2\left(\mu_t\,;\,\mu^{1,N}_t\right)=\sup_{k\geq0}\lambda_k(T)\leq\frac{\exp\left[\psi(T)\right]}{1-2e^{-C(\sigma)T}}\frac{2}{\sqrt{N}}+2\gamma e^{-C(\sigma)T}\,.\tag{4.4}$$

Let $\epsilon>0$ be arbitrarily small. We take $T>\frac{1}{C}\log\left(\frac{4\gamma}{\epsilon}\right)$ so that $2\gamma e^{-C(\sigma)T}<\frac{\epsilon}{2}$. Then, by taking N large enough, we have $\frac{\exp[\psi(T)]}{1-2e^{-C(\sigma)T}}\frac{2}{\sqrt{N}}<\frac{\epsilon}{2}$. This implies $\sup_{t\geq0}\mathbb{W}_2\left(\mu_t\,;\,\mu^{1,N}_t\right)<\epsilon$ if N is large enough. This proves Limit (1.2).

Step 6. We now prove the rate of convergence result. Let $\delta>0$ be arbitrarily small.

Step 6.1. We look at the first case. Inequality (4.4) holds for any $T>0$. We take $T_N:=\frac{1}{C(\sigma)}\frac{1}{2(1+1/\lambda)}\log(N)$. We immediately deduce $\frac{1}{1-2e^{-C(\sigma)T_N}}\leq 2$ for N large enough. For N large enough, the quantity $\frac{\psi(T_N)}{C(\sigma)T_N}$ is less than $\frac{1}{\lambda}+\delta\left(1+\frac{1}{\lambda}\right)$ so that the quantity $\frac{\exp[\psi(T_N)]}{\sqrt{N}}$ is less than $N^{-\left(\frac{1}{2(1+1/\lambda)}-\frac{\delta}{2}\right)}$. We deduce

$$N^{\frac{1}{2(1+1/\lambda)}-\delta}\frac{\exp\left[\psi\left(T_N\right)\right]}{1-e^{-\varphi(T_N)}}\frac{1}{\sqrt{N}}\leq 2N^{-\frac{\delta}{2}}\longrightarrow 0\,,$$

as N goes to infinity. The second term, $\gamma e^{-C(\sigma)T_N}$, is equal to $\gamma N^{-\frac{1}{2(1+1/\lambda)}}$ so

$$N^{\frac{1}{2(1+1/\lambda)}-\delta}\gamma e^{-C(\sigma)T_N}=\gamma N^{-\delta}\longrightarrow 0\,,$$

as N goes to infinity. This achieves the proof of Limit (1.3).
Step 6.2. We now look at the second case. Here, we obtain

$$\sup_{t\geq0}\mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right)\leq\frac{N^{-\frac{\delta}{2}}}{1-2e^{-C(\sigma)T_N}}+\gamma e^{-C(\sigma)\psi^{-1}\left(\frac{1}{2}(1-\delta)\log(N)\right)}\,,$$

by taking $T_N:=\psi^{-1}\left(\frac{1}{2}\left(1-\delta\right)\log(N)\right)$. This implies Limit (1.4). □

The second case, that is to say when $\lim_{t\to+\infty}\frac{\psi(t)}{t}=0$, does not hold with McKean-Vlasov diffusion. However, the current work aims to be applied for more general diffusions.

In [CGM08, Theorem 3.2], the authors obtain a uniform propagation of chaos of the form

$$\sup_{t\geq0}\mathbb{E}\left\{\left|\left|X_t-X_t^1\right|\right|^2\right\}\leq\frac{K}{N^{-(1-\rho)}}\,,$$

with $0<\rho<1$. However, by using a method similar to the one of the proof of Theorem C, we obtain a better inequality with the Wasserstein distance. This is Corollary D, that we now give the proof.

Proof. By proceeding exactly like in [BRTV98, Lemma 5.4], there exists $K>0$ such that the following inequality holds:

$$\sup_{0\leq t\leq T}\mathbb{E}\left\{\left|\left|X_t-X_t^1\right|\right|^2\right\}\leq\frac{KT^2}{N}\tag{4.5}$$

for any $T>0$. Here, there are two differences with the proof in [BRTV98]. First, here, there is the presence of a confinement potential but since this potential is convex, we can proceed similarly. And, in [BRTV98], the initial random variables are assumed to be independent. However, we need here to relax this independence hypothesis (like in the proof of Theorem C). We use the same technic than the one in Proposition B by conditioning with respect to the initial random variables and we have the result. Inequality (4.5) implies

$$\sup_{0\leq t\leq T}\mathbb{W}_2\left(\mu_t\,;\eta_t^N\right)\leq\frac{\exp\left[\psi(T)\right]}{\sqrt{N}}$$

with $\psi(T):=\frac{1}{2}\log(K)+\log(T)$.
Now, since $\alpha\leq0$ and $\theta\leq0$, any invariant probability μ^σ satisfies a $WJ_{V,F}$-inequality with a constant $C(\sigma)>0$. Consequently, we have

$$\lim_{t\to+\infty}\frac{C(\sigma)t}{\psi(t)}=+\infty\,.$$

We apply Theorem C and we obtain the statement for any $\delta>0$. □

5 Creation of chaos

From now on, $X_0^1 = \cdots = X_0^N = X_0$.

5.1 Creation of chaos in the hydrodynamical limit

In the following, we look at the quantity $\mathbb{E}\left\{f_1\left(X_t^1\right)f_2\left(X_t^2\right)\right\}$. We remark that

$$\begin{aligned}\mathbb{E}\left\{f_1\left(X_t^1\right)f_2\left(X_t^2\right)\right\} &= \mathbb{E}\left\{f_1\left(X_t^1\right)\mathbb{E}\left[f_2\left(X_t^2\right)\ \mid\ X_0^1,\left(B_s^1\right)_{0\leq s\leq t}\right]\right\}\\ &= \mathbb{E}\left\{f_1\left(X_t^1\right)\mathbb{E}\left[f_2\left(X_t^2\right)\ \mid\ X_0^1\right]\right\}.\end{aligned}$$

Consequently, to study the long-time behaviour of $\mathbb{E}\left\{f_1\left(X_t^1\right)f_2\left(X_t^2\right)\right\}$ requires to study $\mathbb{E}\left[f_2\left(X_t^2\right)\ \mid\ X_0^1\right]$. Since X_0^1 and X_0^2 are not independent, we do not have $\mathbb{E}\left[f_2\left(X_t^2\right)\ \mid\ X_0^1\right] = \mathbb{E}\left[f_2\left(X_t^2\right)\right]$.

According to previous results, see [BCCP98, BGG12b, BRV98, CGM08, CMV03] for the convex case and [Tug13a, Tug13b] for the general case, we know that the measure μ_t converges weakly to μ as t goes to infinity, under the assumptions of the article. However, we do not know anything about the convergence of $\mathbb{E}\left[f_2\left(X_t^2\right)\ \mid\ X_0^1\right]$ as t goes to infinity. This is the purpose of next proposition.

Proposition 5.1. *Let f be a Lipschitz function from $\mathbb{R}$ to itself. Under the sets of assumptions (A) and (B), we have:*

$$\mathbb{E}\left\{f_2\left(X_t^2\right)\ \mid X_0^1\right\}\longrightarrow\int_{\mathbb{R}}f_2(x)\mu(dx)\,, \tag{5.1}$$

and the convergence holds almost surely, as t goes to infinity.

Proof. For any x_0, we can write

$$\mathbb{E}\left\{f\left(X_t^2\right)\ \mid\ X_0\right\}\mathbb{1}_{\{X_0=x_0\}} = \mathbb{E}\left\{f\left(Y_t^{x_0}\right)\right\}\mathbb{1}_{\{X_0=x_0\}}$$

Consequently, for any random variable X which follows the law μ, we have

$$\begin{aligned}&\left|\mathbb{E}\left\{f\left(X_t^2\right)\ \mid\ X_0\right\}\mathbb{1}_{\{X_0=x_0\}} - \mathbb{E}\left\{f\left(X\right)\right\}\mathbb{1}_{\{X_0=x_0\}}\right|\\ &\leq \mathbb{E}\left\{\left|f\left(Y_t^{x_0}\right)-f\left(X\right)\right|\right\}\mathbb{1}_{\{X_0=x_0\}} \leq C\mathbb{E}\left\{\left|Y_t^{x_0}-X\right|\right\}\mathbb{1}_{\{X_0=x_0\}}\,.\end{aligned}$$

By taking X which minimizes $\mathbb{W}_2\left(\mathcal{L}\left(Y_t^{x_0}\right);\mu\right)$, we find

$$\begin{aligned}&\left|\mathbb{E}\left\{f\left(X_t^2\right)\ \mid\ X_0\right\}\mathbb{1}_{\{X_0=x_0\}} - \int f(x)\mu(dx)\mathbb{1}_{\{X_0=x_0\}}\right|\\ &\leq C\mathbb{W}_2\left(\mathcal{L}\left(Y_t^{x_0}\right);\mu\right)\mathbb{1}_{\{X_0=x_0\}}\\ &\leq Ce^{-C_\sigma t}\mathbb{W}_2\left(\delta_{x_0};\mu\right)\mathbb{1}_{\{X_0=x_0\}}\\ &\leq Ce^{-C_\sigma t}\sqrt{\int_{\mathbb{R}}\left(x-x_0\right)^2\mu(dx)\mathbb{1}_{\{X_0=x_0\}}}\,.\end{aligned}$$

This tends to 0 as t goes to infinity which achieves the proof.

∎

Let us remark that we have obtained better: the convergence is exponential.

Now, we can prove that, as time t goes to infinity, the law of the couple (X_t^1, X_t^2) becomes the tensorial product of the marginal laws.

Proposition 5.2. *Let f_1 and f_2 be two Lipschitz functions from $\mathbb{R}$ to itself. Under the sets of assumptions (A) and (B), we have:*

$$\mathbb{E}\left\{f_1\left(X_t^1\right) f_2\left(X_t^2\right)\right\} \longrightarrow \left(\int_{\mathbb{R}} f_1(x)\mu(dx)\right)\left(\int_{\mathbb{R}} f_2(x)\mu(dx)\right) . \tag{5.2}$$

The convergence holds as t goes to infinity.

Proof. We observe that

$$\mathbb{E}\left\{f_1\left(X_t^1\right) f_2\left(X_t^2\right)\right\} = \mathbb{E}\left\{f_1\left(X_t^1\right)\mathbb{E}\left[f_2\left(X_t^2\right) \mid X_0^1\right]\right\} = \left(\int_{\mathbb{R}} f_2(x)\mu(dx)\right)\mathbb{E}\left\{f_1\left(X_t^1\right)\right\} + \mathbb{E}\left\{f_1\left(X_t^1\right) A_t\right\} ,$$

with

$$A_t := \mathbb{E}\left\{f_2\left(X_t^2\right) \mid X_0^1\right\} - \int_{\mathbb{R}} f_2(x)\mu(dx) .$$

The limit in (5.1) gives us the convergence almost surely of the random variable A_t to 0 as t goes to infinity.

Furthermore, since f_2 is Lipschitz-continuous and according to the boundedness of the moments of X_t^2, we have the following inequality:

$$\mathbb{E}\left(||A_t||^2\right) \leq 2\mathbb{E}\left[\left|\left|f_2\left(X_t^2\right)\right|\right|^2\right] + 2\left(\int_{\mathbb{R}} f_2(x)\mu(dx)\right)^2 \leq C\left\{1 + \mathbb{E}\left[\left|\left|X_t^2\right|\right|^2\right]\right\} \leq C_\sigma .$$

By Lebesgue theorem, we deduce the following limit:

$$\lim_{t\to\infty} \mathbb{E}\left\{f_1\left(X_t^1\right) A_t\right\} = 0 .$$

Moreover, due to the set of assumptions on the initial random variable, we have the following convergence as t goes to infinity:

$$\mathbb{E}\left[f_1\left(X_t^1\right)\right] \longrightarrow \int_{\mathbb{R}} f_1(x)\mu(dx) .$$

This achieves the proof. □

Let us remark that the convergence is exponential.

In fact, we could have obtained a more general result by proceeding similarly.

Remark 5.1. Let $f_1, \cdots, f_k$ be k functions Lipschitz-continuous. Then, under the hypotheses of Proposition 5.2, we have the convergence almost surely of

$$\mathbb{E}\left\{\prod_{i=1}^k f_i\left(X_t^i\right)\right\} \quad \text{toward} \quad \prod_{i=1}^k \int_{\mathbb{R}} f_i(x)\mu(dx) .$$

By observing that $\prod_{i=1}^{k}\mathbb{E}\left[f_i\left(X_t^i\right)\right]$ converges to $\prod_{i=1}^{k}\int_{\mathbb{R}}f_i(x)\mu(dx)$, we immediately obtain the following theorem.

Theorem 5.3. *Let f_1 and f_2 be two Lipschitz functions from $\mathbb{R}$ to itself. Under the sets of assumptions (A) and (B), we have:*

$$\mathrm{Cov}\left(f_1\left(X_t^1\right)\,;\,f_2\left(X_t^2\right)\right)\longrightarrow 0\,, \tag{5.3}$$

as t goes to infinity.

More generally, let any $k\geq 2$ and let $f_1,\cdots,f_k$ be k Lipschitz-continuous functions. Thus, we have the following convergence almost surely as t goes to infinity:

$$\mathbb{E}\left\{\prod_{i=1}^{k}f_i\left(X_t^i\right)\right\}-\prod_{i=1}^{k}\mathbb{E}\left\{f_i\left(X_t^i\right)\right\}\longrightarrow 0\,.$$

Let us point out that to obtain this result, we only use the convergence in long-time. We do not need to know anything about the rate of convergence.

However, we know that this convergence is exponential.

5.2 Creation of chaos in the mean-field system

We first provide a coupling result.

Proposition 5.4. *We assume that V, F and μ_0 satisfy the set of Hypotheses (A) and (B). Let X_0 be a random variable which follows the law μ_0. Then, for any $T>0$, we have the following inequality:*

$$\sup_{t\in[0;T]}\mathbb{E}\left\{\left|\left|X_t^i-X_t^{i,N}\right|\right|^2\right\}\leq\frac{C(\mu_0)}{N}\exp\left[2CT\right]\,, \tag{5.4}$$

where $C(\mu_0)$ is a positive function of $\int_{\mathbb{R}}||x||^{8q^2}\mu_0(dx)$ and C is a positive constant.

Proof. The proof is an adaptation of the proof of Proposition B. By $\nu_t^{X_0}$, we denote the solution of the granular media equation starting from δ_{X_0}. By definition, for any $1\leq i\leq N$, we have

$$X_t^{i,N}-X_t^i=-\int_0^t\left\{\nabla V(X_s^{i,N})-\nabla V(X_s^i)\right\}ds-\int_0^t\left\{\frac{1}{N}\sum_{j=1}^{N}\nabla F(X_s^{i,N}-X_s^{j,N})-\nabla F*\nu_s^{X_0}(X_s^i)\right\}ds\,.$$

We apply Itô formula to $X_t^{i,N}-X_t^i$ with the function $x\mapsto||x||^2$. By introducing the notation $\xi_i(t):=\left|\left|X_t^{i,N}-X_t^i\right|\right|^2$ and by taking the sum on the integer i

running between 1 and N, we get

$$d\sum_{i=1}^N \xi_i(t) = -2\Delta_1(t)dt - \frac{2}{N}\sum_{i=1}^N\sum_{j=1}^N\Big(\Delta_2(i,j,t)+\Delta_3(i,j,t)\Big)dt$$

$$\text{with} \qquad \Delta_1(t) := \sum_{i=1}^N \Delta_1(i,t)\ ,$$

$$\Delta_2(i,j,t) := \Big\langle \nabla F(X_t^{i,N}-X_t^{j,N}) - \nabla F(X_t^i - X_t^j)\,;\, X_t^{i,N}-X_t^i\Big\rangle$$

$$\text{and} \qquad \Delta_3(i,j,t) := \Big\langle \nabla F(X_t^i - X_t^j) - \nabla F * \nu_t^{X_0}\left(X_t^i\right)\,;\, X_t^{i,N}-X_t^i\Big\rangle\ .$$

According to the definition of the function F_0 in Hypothesis (A-6), it is convex. This implies $\langle x-y\,;\,\nabla F_0(x-y)\rangle \geq 0$ for any $x,y\in\mathbb{R}$. This inequality yields

$$\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N\left(\Delta_2(i,j,t)+\Delta_2(j,i,t)\right) \geq -4\alpha\sum_{i=1}^N \left|\left|X_t^{i,N}-X_t^i\right|\right|^2\ .$$

Consequently, we have

$$\mathbb{E}\left\{-\frac{2}{N}\sum_{i=1}^N\sum_{j=1}^N\Delta_2(i,j,t)\right\} = \frac{1}{2}\mathbb{E}\left\{-\frac{2}{N}\sum_{1\leq i,j\leq N}^N\Big(\Delta_2(i,j,t)+\Delta_2(j,i,t)\Big)\right\} \leq 4\alpha\sum_{i=1}^N\left|\left|X_t^{i,N}-X_t^i\right|\right|^2\ . \tag{5.5}$$

By definition of θ, for any $x,y\in\mathbb{R}$ we have the inequality $\langle\nabla V(x)-\nabla V(y)\,;\,x-y\rangle \geq -\theta\left|\left|x-y\right|\right|^2$. This implies

$$-2\sum_{i=1}^N\Delta_1(i,t) \leq 2\theta\sum_{i=1}^N\xi_i(t)\,. \tag{5.6}$$

We conclude as in the proof of Proposition B:

$$\mathbb{E}\left\{\left\langle\rho_j^i(t)\,;\,\rho_k^i(t)\right\rangle\right\} = 0\,,$$

for any $j\neq k$. And, if $j=k$, we have

$$\mathbb{E}\left\{\left|\left|\rho_j^i(t)\right|\right|^2\right\} = \mathbb{E}\left\{\left|\left|\nabla F\left(X_t^i - X_t^j\right) - \nabla F * \nu_t^{X_0}\left(X_t^i\right)\right|\right|^2\right\}\ .$$

The diffusions X^i and X^j are not independent but they are independent conditionally to the initial random variables. However, according to Hypothesis (A-4) and (A-5), we have $F(x) = G\left(||x||\right)$ where G is a polynomial function of degree $2n$, we have the following inequality:

$$\mathbb{E}\left[\left|\left|\nabla F(X_t-Y_t) - \nabla F*\nu_t^{X_0}(X_t)\right|\right|^2\right] \leq C\left(1+\mathbb{E}\left[\left|\left|X_t\right|\right|^{4n-2}\right]\right)\,,$$

X_t and Y_t being two independent random variables with common law μ_t and C is a positive constant. Then, we use the control of the moments obtained in [HIP08, Theorem 2.13] and we obtain the following majoration:

$$\sup_{t\geq 0}\mathbb{E}\left[\left|\left|\nabla F(X_t-Y_t)-\nabla F*\mu_t(X_t)\right|\right|^2\right]\leq C(\mu_0)\,,$$

$C(\mu_0)$ being a function of the $8q^2$ moment of the law μ_0. Consequently, we have

$$\mathbb{E}\left\{\left|\left|\rho_j^i(t)\right|\right|^2\,\mid\,X_0\right\}\leq C(\mu_0)\,,$$

for any $1\leq i,j\leq N$. By taking the expectation, we obtain

$$\mathbb{E}\left\{\left|\left|\rho_j^i(t)\right|\right|^2\right\}\leq C(\mu_0)\,,$$

for any $1\leq i,j\leq N$. Therefore, we deduce the following inequality:

$$-\mathbb{E}\left[\sum_{j=1}^N\Delta_3(i,j,t)\right]\quad\leq\sqrt{C(\mu_0)}\sqrt{N\mathbb{E}\left[\xi_i(t)\right]}\,. \tag{5.7}$$

By combining (5.5), (5.6) and (5.7), we obtain

$$\frac{d}{dt}\sum_{i=1}^N\mathbb{E}\left[\xi_i(t)\right]\leq 2\sum_{i=1}^N\left\{(\theta+2\alpha)\,\mathbb{E}\left[\xi_i(t)\right]+\frac{\sqrt{C(\mu_0)}}{\sqrt{N}}\sqrt{\mathbb{E}\left[\xi_i(t)\right]}\right\}\,.$$

However, the particles are exchangeable. Consequently, for any $1\leq i\leq N$, we have

$$\frac{d}{dt}\mathbb{E}\left\{\xi_i(t)\right\}\leq 2\left(\theta+2\alpha\right)\mathbb{E}\left\{\xi_i(t)\right\}+\frac{2\sqrt{C(\mu_0)}}{\sqrt{N}}\sqrt{\mathbb{E}\left[\xi_i(t)\right]}\,.$$

By introducing $\tau_i(t):=\sqrt{\mathbb{E}\left\{\xi_i(t)\right\}}$, we obtain

$$\tau_i'(t)\leq(\theta+2\alpha)\left\{\tau_i(t)+\frac{\sqrt{C(\mu_0)}}{(\theta+2\alpha)\sqrt{N}}\right\}$$

The application of Grönwall lemma yields

$$\mathbb{E}\left\{\left|\left|X_t^{i,N}-X_t^i\right|\right|^2\right\}\leq\frac{C(\mu_0)}{N\left(\theta+2\alpha\right)^2}\exp\left[2\left(\theta+2\alpha\right)t\right]\,.$$

We obtain (5.4) by taking the supremum for t running between 0 and T. $\square$

5.2.1 Decorrelation for two particles

We now are able to provide the proof of Theorem E.

Proof. Let T and ϵ be any positive reals. Set $0 \leq t$.

Step 1. We use the following decomposition

$$\begin{aligned}
&\operatorname{Cov}\left(f_1\left(X_t^{1,N}\right) ; f_2\left(X_t^{2,N}\right)\right) = \mathbb{E}\left\{f_1\left(X_t^{1,N}\right)\left[f_2\left(X_t^{2,N}\right) - f_2\left(X_t^2\right)\right]\right\} \\
&+ \mathbb{E}\left\{f_2\left(X_t^2\right)\left[f_1\left(X_t^{1,N}\right) - f_1\left(X_t^1\right)\right]\right\} + \operatorname{Cov}\left(f_1\left(X_t^1\right) ; f_2\left(X_t^2\right)\right) \\
&+ \mathbb{E}\left\{f_1\left(X_t^1\right)\right\}\left[\mathbb{E}\left(f_2\left(X_t^2\right)\right) - \mathbb{E}\left(f_2\left(X_t^{2,N}\right)\right)\right] \\
&+ \mathbb{E}\left\{f_2\left(X_t^{2,N}\right)\right\}\left[\mathbb{E}\left(f_1\left(X_t^1\right)\right) - \mathbb{E}\left(f_1\left(X_t^{1,N}\right)\right)\right] \\
=&: T_1 + T_2 + T_3 + T_4 + T_5 .
\end{aligned}$$

Step 2. We can control T_1 in the following way.

$$|T_1| \leq \sqrt{\mathbb{E}\left\{\left|\left|f_1\left(X_t^{1,N}\right)\right|\right|^2\right\}}\sqrt{\mathbb{E}\left\{\left|\left|f_2\left(X_t^{2,N}\right) - f_2\left(X_t^2\right)\right|\right|^2\right\}}$$

by Cauchy-Schwarz inequality. The triangular inequality provides us:

$$\left|\left|f_1\left(X_t^{1,N}\right)\right|\right|^2 \leq 3\,||f_1(0)||^2 + 3\,||f_1\left(X_t^1\right) - f_1(0)||^2 + 3\left|\left|f_1\left(X_t^{1,N}\right) - f_1\left(X_t^1\right)\right|\right|^2$$

Since f_1 is a Lipschitz-continuous function, there exists $\rho > 0$ such that

$$\left|\left|f_1\left(X_t^{1,N}\right)\right|\right|^2 \leq 3\,||f_1(0)||^2 + 3\rho^2\,||X_t^1||^2 + 3\rho^2\left|\left|X_t^{1,N} - X_t^1\right|\right|^2 .$$

Due to the inequalities (1.1) and (5.4), we have

$$\mathbb{E}\left\{\left|\left|f_1\left(X_t^{1,N}\right)\right|\right|^2\right\} \leq 3\rho^2\left(M_0 + ||f_1(0)||^2 + K^2\frac{e^{2Ct}}{N}\right)$$

Still by using the coupling result (5.4), we have

$$\mathbb{E}\left\{\left|\left|f_2\left(X_t^{2,N}\right) - f_2\left(X_t^2\right)\right|\right|^2\right\} \leq \rho^2 K^2\frac{e^{2Ct}}{N} ,$$

so that the term T_1 is bounded like so

$$|T_1| \leq 2\rho\sqrt{M_0 + ||f_1(0)||^2 + K^2\frac{e^{2Ct}}{N}}\,\rho K\frac{e^{Ct}}{\sqrt{N}} \leq 2\rho^2 K\sqrt{K_0 + K^2\frac{e^{2Ct}}{N}}\,\frac{e^{Ct}}{\sqrt{N}} .$$

Step 3. By proceeding similarly, we have the following boundedness of the fifth term.

$$|T_5| \leq 2\rho^2 K\sqrt{K_0 + K^2\frac{e^{2Ct}}{N}}\,\frac{e^{Ct}}{\sqrt{N}} .$$

Step 4. By using the uniform boundedness of the moments (1.1), Jensen's inequality and the coupling result (5.4), we obtain the following control:

$$\max\left\{|T_2| ; |T_4|\right\} \leq \rho^2 K\sqrt{M_0}\,\frac{e^{Ct}}{\sqrt{N}} .$$

Step 5. Finally, the limit (5.3) provides us the existence of a decreasing function φ which limit at infinity is 0 such that

$$|T_3| \le \varphi(t)\,.$$

Step 6. Let $t_0(\epsilon)$ be a positive real such that $\varphi(t_0(\epsilon)) < \frac{\epsilon}{2}$. Then, we take $N_0(\epsilon)$ large enough such that we have $|T_1| + |T_2| + |T_4| + |T_5| \le \frac{\epsilon}{2}$. We deduce that for any $t \in [t_0(\epsilon); t_0(\epsilon) + T]$, for any $N \ge N_0(\epsilon)$, we have

$$\left|\mathrm{Cov}\left[f_1\left(X_t^{1,N}\right)\,;\,f_2\left(X_t^{2,N}\right)\right]\right| \le \epsilon\,.$$

□

This theorem means that, for a time and a number of particles sufficiently large, two particles are as independent as we desire. Moreover, the convergence in time is exponential.
We do not need neither V nor F to be convex. Nevertheless, if both potentials V and F are convex, we know that we have a uniform coupling between the particles and the inequality (5.4) becomes

$$\sup_{t\ge 0}\mathbb{E}\left\{\left|\left|X_t^{i,N} - X_t^i\right|\right|\right\} \le \frac{K(\mu_0)^2}{N}\,, \tag{5.8}$$

so that the four terms T_1, T_2, T_4 and T_5 (defined in the proof of Theorem E) are bounded like so

$$\sup_{t\ge 0}\max\left\{|T_1|;|T_2|;|T_4|;|T_5|\right\} \le \frac{\lambda}{4\sqrt{N}}\,.$$

In the previous (uniform) inequality, λ is a positive constant. Immediately, we have the majoration:

$$\left|\mathrm{Cov}\left(f_1\left(X_t^{1,N}\right)\,;\,f_2\left(X_t^{2,N}\right)\right)\right| \le \varphi(t) + \lambda\frac{1}{\sqrt{N}}\,.$$

Taking t and N sufficiently large yields

$$\left|\mathrm{Cov}\left(f_1\left(X_t^{1,N}\right)\,;\,f_2\left(X_t^{2,N}\right)\right)\right| \le \epsilon\,,$$

since the function φ is decreasing. This ends the proof of Theorem C.

5.2.2 Creation of chaos for the empirical measure

When the initial random variables $X_0^1, \cdots, X_0^N$ are independent, the empirical measure $\eta_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}$ converges as N goes to infinity toward the deterministic measure μ_t (the law at time t of the McKean-Vlasov diffusion).
However, due to Theorem E, we have the Theorem G.

Proof. By definition of $\eta_t^N(f)$, we have

$$\mathrm{Cov}\left(\eta_t^N(f_1)\,;\,\eta_t^N(f_2)\right)=\frac{1}{N^2}\sum_{1\leq i,j\leq N}\mathrm{Cov}\left(f_1\left(X_t^{i,N}\right)\,;\,f_2\left(X_t^{j,N}\right)\right)\,.$$

Consequently, if $N\geq N_0(\epsilon)$ (where the integer $N_0(\epsilon)$ has been defined in Theorem E), we have by triangular inequality:

$$\sup_{t\in[t_0(\epsilon);t_0(\epsilon)+T]}\left|\mathrm{Cov}\left(\eta_t^N(f_1)\,;\,\eta_t^N(f_2)\right)\right|$$
$$\leq\left(1-\frac{1}{N_0(\epsilon)}\right)\epsilon+\frac{1}{N^2}\sum_{i=1}^N\left|\mathrm{Cov}\left(f_1\left(X_t^{i,N}\right)\,;\,f_2\left(X_t^{i,N}\right)\right)\right|\,.$$

Nevertheless, due to the hypotheses, we have the convergence of the quantity

$$\mathrm{Cov}\left(f_1\left(X_t^i\right)\,;\,f_2\left(X_t^i\right)\right)$$

to

$$\int_{\mathbb{R}}f_1(x)f_2(x)\mu(dx)-\left(\int_{\mathbb{R}}f_1(x)\mu(dx)\right)\left(\int_{\mathbb{R}}f_2(x)\mu(dx)\right)\,,$$

as t goes to infinity. Then, since f_1 and f_2 are Lipschitz-continuous functions, thanks to the coupling inequality (5.4), we obtain that for all $\epsilon>0$, the quantity

$$\left|\mathrm{Cov}\left(f_1\left(X_t^{i,N}\right)\,;\,f_2\left(X_t^{i,N}\right)\right)-\left[\int_{\mathbb{R}}f_1f_2\mu-\left(\int_{\mathbb{R}}f_1\mu\right)\left(\int_{\mathbb{R}}f_2\mu\right)\right]\right|$$

is less than ϵ if t and N are large enough. Particularly, we deduce the boundedness of $\left|\mathrm{Cov}\left(f_1\left(X_t^{i,N}\right)\,;\,f_2\left(X_t^{i,N}\right)\right)\right|$:

$$\sup_{N\geq1}\sup_{t\geq0}\left|\mathrm{Cov}\left(f_1\left(X_t^{i,N}\right)\,;\,f_2\left(X_t^{i,N}\right)\right)\right|\leq M\,,$$

M being a positive constant.
Taking $N_1(\epsilon):=\max\left\{N_0(\epsilon)\,;\,\frac{MN_0(\epsilon)}{\epsilon}\right\}$ yields

$$\sup_{t\in[t_0(\epsilon);t_0(\epsilon)+T]}\left|\mathrm{Cov}\left(\eta_t^N(f_1)\,;\,\eta_t^N(f_2)\right)\right|\leq\epsilon$$

The second part of the theorem can be proved in a similar way so it is left to the reader. □

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