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Comm. Appl. Anal., **7**, no.2-3 p. 209-223, 2003.

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ABSTRACT : We study the asymptotic behaviour, as h tends to $+\infty$, of the nonlinear system:

$$\begin{cases} -\Delta u_h - u_h + |u_h|^2 u_h = f \quad \text{in} \quad \Omega_h, \\ Du_h \cdot \nu = 0 \quad \text{on} \quad \partial\Omega_h, \\ u_h : \Omega_h \to \mathbb{R}^2, \end{cases}$$

in a varying domain Ω_h in $\mathbb{R}^2$. The boundary $\partial\Omega_h$ contains an oscillating part like a comb with fine teeth periodically distributed in the first direction $0x_1$ with period h^{-1} and thickness λh^{-1}, $0 < \lambda < 1$.

We identify the limit problem where the operator $-\Delta$ is reduced to $-\dfrac{\partial^2}{\partial x_2^2}$ in the domain corresponding to the oscillating boundary.

AMS (MOS) Subject Classification. 35B27, 35Q55, 58E50.

1. INTRODUCTION

In this paper we consider the Ginzburg-Landau equation (GL_h):

$$\begin{cases} -\Delta u_h - u_h + |u_h|^2 u_h = f \text{ in } \Omega_h, \\ \quad u_h : \Omega_h \longrightarrow \mathbb{R}^2, \end{cases}$$

with homogeneous Neumann boundary condition, in a varying domain Ω_h in $\mathbb{R}^2$. We are interested in a class of domains Ω_h which have the shape of a comb with fine teeth periodically distributed in the first direction $0x_1$ with period $\dfrac{1}{h}$ and thickness $\dfrac{\lambda}{h}$, $0 < \lambda < 1$ (see Fig. 1). The goal is to study the asymptotic behaviour of such problem when h tends to $+\infty$ (see Theorem 2.1).

For general references about homogenization, we refer to [2], [3], [4], [13], [24] and [25]. In the scalar case, for this kind of domains with crenelated part of the boundary, the limit problem has been studied for the Laplace operator in [9], [10], [15] and for quasilinear operator, more generally for a monotone operator in [12] and [7]. For reinforcement problems by a layer with oscillating thickness see [11], for problems related to the asymptotic behaviour of thin cylinders see [19] and [20].

An extensive study of Ginzburg-Landau equations is developed by Bethuel-Brezis and Hélein in [5] and [6]. The limit behaviour of the Ginzburg-Landau equation in a perforated domain in $\mathbb{R}^3$ with holes along a plane is studied in [17].

In order to identify the limit problem of (GL_h), as h tends to $+\infty$, the main steps are to establish a uniform estimate of u_h in $(L^\infty(\Omega_h))^2$ and to obtain an extension of u_h on a fixed domain; this is the object of Section 3 and 4. Then, in Section 5, we find the limit problem where the operator $-\Delta$ is reduced to $-\frac{\partial^2}{\partial x_2^2}$ in the domain corresponding to the oscillating boundary. The main difficulty is to pass to the limit in the nonlinear term of the Ginzburg-Landau equation (see Remark 5.1).

For sake of completeness, we give also the limit behaviour of the previous Ginzburg-Landau equation with homogeneous Dirichlet boundary condition (for scalar problems with Dirichlet boundary condition in a domain with oscillating boundary see [7], [9], [10], [14],[18], [21], [22] and [23]).

2. STATEMENT OF THE PROBLEM AND MAIN RESULT

Let a, b_1, b_2, α be in $]0,+\infty[$ such that $0 < \alpha < \dfrac{a}{2}$ and let us introduce the following domains in $\mathbb{R}^2$ (see Fig.1):

$$(2.1)\qquad \begin{cases} \Omega =]0,a[\times]-b_1,b_2[, \\ \Omega^- =]0,a[\times]-b_1,0[, \quad \Omega^+ =]0,a[\times]0,b_2[, \\ \Sigma =]0,a[\times\{0\}, \\ \Omega_h = \Omega^- \cup \left(\displaystyle\bigcup_{k=0}^{h-1} \left(\frac{1}{h}]\alpha, a-\alpha[+\frac{ak}{h} \right) \times [0,b_2[\right) \quad h \in \mathbb{N}, \\ \Omega_h^+ = \Omega^+ \cap \Omega_h \quad h \in \mathbb{N}. \end{cases}$$

In the sequel, $x = (x_1, x_2)$ denotes the generic point of $\mathbb{R}^2$, χ_A the characteristic function of a subset A of Ω and $\tilde{v}$ the zero-extension to Ω of any (vector) function v defined on a subset of Ω.

We recall that

$$(2.2)\qquad \chi_{\Omega_h^+} \rightharpoonup \theta = \frac{a-2\alpha}{a} \text{ weakly} \star \text{ in } L^\infty(\Omega^+)$$

and

$$(2.3)\qquad \chi_{\Omega_h \cap \Sigma} \rightharpoonup \theta \text{ weakly} \star \text{ in } L^\infty(\Sigma)$$

as h diverges.

The aim of this paper is to study the asymptotic behaviour, as h tends to $+\infty$, of the following homogeneous Neumann problem:

$$(2.4)\qquad \begin{cases} -\Delta u_h - u_h + |u_h|^2 u_h = f \text{ in } \Omega_h, \\ Du_h \cdot \nu = 0 \text{ on } \partial\Omega_h, \end{cases}$$

where $f = (f_1, f_2)$ is a given function in $\left(L^2(\Omega)\right)^2$ and ν denotes the exterior unit normal to Ω_h.

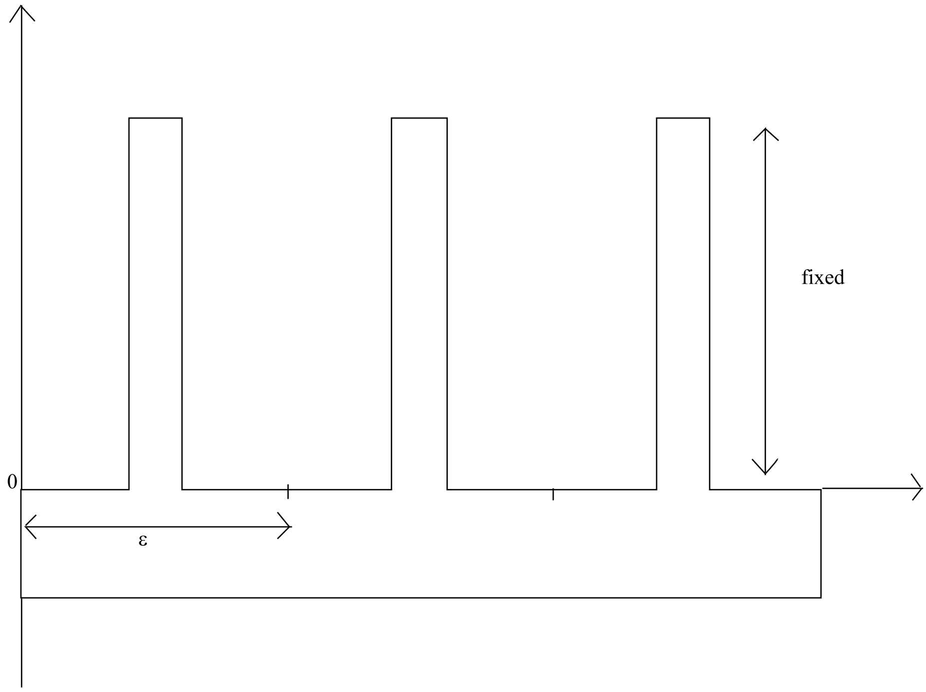

Figure 1: the middle surface of our three-dimensional plate.

Problem (2.4) admits a weak solution $u_h \in \left(H^1(\Omega_h)\right)^2$. In fact, it is easy to see that a minimizing sequence of the following problem:

$$\inf\left\{\int_{\Omega_h}\left(|Dv|^2+\frac{1}{2}(1-|v|^2)^2-2fv\right)dx \ : \ v\in\left(H^1(\Omega_h)\right)^2\right\} \tag{2.5}$$

is bounded in $\left(L^4(\Omega_h)\right)^2$ and $\left(H^1(\Omega_h)\right)^2$ and then the infimum in (2.5) is achieved by u_h satisfying the following variational equation :

$$\begin{cases}\displaystyle\int_{\Omega_h}\left(Du_hDv-u_hv+|u_h|^2u_hv\right)dx=\int_{\Omega_h}fv\,dx \quad \forall v\in\left(H^1(\Omega_h)\right)^2,\\ u_h=\left(u_h^{(1)},u_h^{(2)}\right)\in\left(H^1(\Omega_h)\right)^2.\end{cases} \tag{2.6}$$

The main result of this paper is given by the following theorem:

Theorem 2.1. *For every h in $\mathbb{N}$, let u_h be a solution of Problem (2.4) with f in $\left(H^1(\Omega)\right)^2\cap(L^\infty(\Omega))^2$ and let θ be defined by (2.2).*

Then, for every h in $\mathbb{N}$, there exists a linear extension - operator $P_h\in\mathcal{L}\left(\left(H^1(\Omega_h^+)\right)^2,\left(H^1(\Omega^+)\right)^2\right)$, a strictly increasing sequence of positive integer numbers $\{h_k\}_{k\in\mathbb{N}}$ and u in $\left(H^1(\Omega)\right)^2\cap(L^\infty(\Omega))^2$ (depending possibly on the selected subsequence) such that

$$\begin{cases}P_{h_k}u_{h_k}\rightharpoonup u \text{ weakly in } \left(H^1(\Omega^+)\right)^2,\\ u_{h_k}\rightharpoonup u \text{ weakly in } \left(H^1(\Omega^-)\right)^2,\end{cases} \tag{2.7}$$

as k tends to $+\infty$ and u is a solution of the following problem:

$$
(2.8) \qquad \begin{cases} -\dfrac{\partial^2 u}{\partial x_2^2} - u + |u|^2 u = f \ \textit{in} \ \Omega^+, \\ -\Delta u - u + |u|^2 u = f \ \textit{in} \ \Omega^-, \\ \theta \dfrac{\partial u^+}{\partial x_2} = \dfrac{\partial u^-}{\partial x_2} \ \textit{on} \ \Sigma, \\ \dfrac{\partial u}{\partial x_2} = 0 \ \textit{on} \ ]0, a[\times\{b_2\}, \\ Du \cdot \nu = 0 \ \textit{on} \ \partial\Omega^- \backslash \Sigma. \end{cases}
$$

Moreover the energies converge in the sense that

$$
(2.9) \qquad \begin{aligned} &\lim_{k\to+\infty} \int_{\Omega_{h_k}} \left(|Du_{h_k}|^2 - |u_{h_k}|^2 + |u_{h_k}|^4\right) dx \\ &= \theta \int_{\Omega^+} \left(\left|\frac{\partial u}{\partial x_2}\right|^2 - |u|^2 + |u|^4 \right) dx + \int_{\Omega^-} \left(|Du|^2 - |u|^2 + |u|^4\right) dx. \end{aligned}
$$

The variational formulation of Problem (2.8) is given by

$$
(2.10) \qquad \begin{cases} \theta \displaystyle\int_{\Omega^+} \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2} - uv + |u|^2 uv \, dx + \int_{\Omega^-} DuDv - uv + |u|^2 uv \, dx = \\ = \theta \displaystyle\int_{\Omega^+} fv \, dx + \int_{\Omega^-} fv \, dx \quad \forall v \in \left(H^1(\Omega)\right)^2, \\ u \in \left(H^1(\Omega)\right)^2 \cap (L^\infty(\Omega))^2. \end{cases}
$$

Remark 2.2. If Problem (2.10) admits a unique solution, then convergences (2.7) and (2.9) hold true for the whole sequence.

Remark 2.3. If in Problem (2.4), instead of the Neumann boundary condition, we assume the Dirichlet boundary condition $u_h = 0$ on $\partial\Omega_h$, it is easy to show that every sequence of zero-extension to Ω of solutions of the Dirichlet problem admits a subsequence which strongly converges in $\left(H_0^1(\Omega)\right)^2$ to a solution of the following problem:

$$
\begin{cases} u = 0 \text{ a.e. in } \Omega^+, \\ -\Delta u - u + |u|^2 u = f \text{ in } \Omega^-, \\ u = 0 \text{ on } \partial\Omega^-. \end{cases}
$$

Moreover the convergence of the energies holds.

3. A PRIORI L^∞ ESTIMATE

In this section, we establish an a priori norm-estimate for the solution u_h of problem (2.4).

Let A be a bounded open set of $\mathbb{R}^n$. We shall denote by $C_b(A)$ the Banach space defined by $C_b(A) = \{v \in C(A) \, : \, v \text{ is bounded}\}$ provided with the L^∞- norm.

The goal of this section is to prove the following result:

Proposition 3.1. *For every h in $\mathbb{N}$, let u_h be a solution of Problem (2.4) with f in $(L^\infty(\Omega))^2$. Then u_h is in $(C_b(\Omega_h))^2$ and there exists a constant c (independent of h) such that*

$$
\|u_h\|_{(L^\infty(\Omega_h))^2} \leqslant c \quad \forall h \in \mathbb{N}.
$$

To this aim, we begin by giving some preliminary results:

Lemma 3.2. *Let u_h, h in $\mathbb{N}$, be a solution of Problem (2.4) with f in $\left(L^2(\Omega)\right)^2$. Then, u_h is in $\left(C_b(\Omega_h)\right)^2$.*

Proof. Let us fix h in N and let us set

$$f_h = u_h - |u_h|^2 u_h + f \text{ a.e. in } \Omega_h.$$

Since u_h belongs to $\left(H^1(\Omega_h)\right)^2$ and $H^1(\Omega_h)$ is embedded into $L^p(\Omega_h)$ for every p in $[1, +\infty[$, it turns out that f_h belongs to $\left(L^2(\Omega_h)\right)^2$. Moreover, from (2.4) it follows that u_h is a solution of

$$\begin{cases} -\Delta u_h = f_h \text{ in } \Omega_h, \\ Du_h \cdot \nu = 0 \text{ on } \partial\Omega_h. \end{cases}$$

Consequently there exists s in $\left]\frac{3}{2}, \frac{5}{3}\right[$ such that u_h belongs to $(H^s(\Omega_h))^2$ (see [16], Lemma 5.1 and Theorem 5.2). Finally, the thesis follows from the embeddings of $H^s(\Omega_h)$, with $s > 1$, into $C_b(\Omega_h)$ (see [1], 7.57). □

We recall the following well-known classical variational inequality and we give the proof for the reader's convenience.

Lemma 3.3. *Let A be an open subset of $\mathbb{R}^n$ satisfying the segment property, γ a positive constant and v a function in $H^1(A)$ such that*

$$\int_A \big(DvD\varphi + \gamma v\varphi\big)dx \geqslant 0 \quad \forall\varphi \in C^1(\mathbb{R}^n) \text{ with } \varphi \geqslant 0.$$

Then, it results that

$$v \geqslant 0 \text{ a.e. in } A.$$

Proof. By the assumptions it follows that

$$\int_A \big(DvD\varphi + \gamma v\varphi\big)dx \geqslant 0 \quad \forall\varphi \in H^1(A) \text{ with } \varphi \geqslant 0 \text{ a.e. in } A. \tag{3.1}$$

By choosing $\varphi = v^- = -\min\{v, 0\}$ in (3.1), it results

$$-\int_A \big(|Dv^-|^2 + \gamma|v^-|^2\big)dx \geqslant 0.$$

This inequality provides

$$v^- = 0 \text{ a.e. in } A$$

and consequently

$$v \geqslant 0 \text{ a.e. in } A.$$

□

Now we can prove the main result of this section.

Proof of Proposition 3.1. Let us fix h in $\mathbb{N}$.

Lemma 3.2 provides that

$$|u_h|^2 \in H^1(\Omega_h) \tag{3.2}$$

and consequently

$$\int_{\Omega_h} D(|u_h|^2) D\varphi \, dx = 2 \int_{\Omega_h} u_h D u_h D\varphi \, dx \quad \forall \varphi \in C^1(\mathbb{R}^2). \tag{3.3}$$

On the other hand, by choosing $v = \varphi u_h$ in (2.6) with φ in $C^1(\mathbb{R}^2)$, it results

$$\begin{aligned} &\int_{\Omega_h} |Du_h|^2 \varphi \, dx + \int_{\Omega_h} u_h D u_h D\varphi \, dx = \\ &= \int_{\Omega_h} \big(|u_h|^2 (1 - |u_h|^2) \varphi + f u_h \varphi \big) dx \quad \forall \varphi \in C^1(\mathbb{R}^2). \end{aligned} \tag{3.4}$$

By combining (3.3) with (3.4) it follows that

$$\begin{aligned} &\int_{\Omega_h} D(|u_h|^2) D\varphi \, dx = \\ &= 2 \int_{\Omega_h} \big(- |Du_h|^2 + |u_h|^2 (1 - |u_h|^2) + f u_h \big) \varphi \, dx \quad \forall \varphi \in C^1(\mathbb{R}^2). \end{aligned} \tag{3.5}$$

If we set

$$v_h = 1 - |u_h|^2 \ \text{ a.e. in } \Omega_h, \tag{3.6}$$

equation (3.5) provides that

$$\int_{\Omega_h} D v_h D\varphi \, dx = 2 \int_{\Omega_h} \big(|Du_h|^2 - v_h (1 - v_h) - f u_h \big) \varphi \, dx \quad \forall \varphi \in C^1(\mathbb{R}^2)$$

and consequently

$$\begin{aligned} &\int_{\Omega_h} \big(D v_h D\varphi + 2 v_h \varphi \big) dx = 2 \int_{\Omega_h} \big(|Du_h|^2 + |v_h|^2 - f u_h \big) \varphi \, dx \geqslant \\ &\geqslant -2 \int_{\Omega_h} f u_h \varphi \, dx \quad \forall \varphi \in C^1(\mathbb{R}^2) \text{ with } \varphi \geqslant 0. \end{aligned} \tag{3.7}$$

Now let us fix η in $]0, \frac{1}{2}[$.

By applying the Young inequality it results

$$u_h f = u_h^{(1)} f_1 + u_h^{(2)} f_2 \leqslant \eta |u_h|^2 + \frac{1}{\eta} |f|^2 = \eta (1 - v_h) + \frac{1}{\eta} |f|^2 \ \text{ a.e. in } \Omega_h$$

and consequently

$$-2 u_h f \geqslant -2\eta + 2\eta v_h - \frac{2}{\eta} \|f\|^2_{(L^\infty(\Omega))^2} \ \text{ a.e. in } \Omega_h. \tag{3.8}$$

By combining (3.7) with (3.8) we obtain

$$\begin{aligned} &\int_{\Omega_h} \big(D v_h D\varphi + 2(1-\eta) v_h \varphi \big) dx \geqslant \\ &\geqslant -2 \left(\eta + \frac{\|f\|^2_{(L^\infty(\Omega))^2}}{\eta} \right) \int_{\Omega_h} \varphi \, dx \quad \forall \varphi \in C^1(\mathbb{R}^2) \text{ with } \varphi \geqslant 0. \end{aligned} \tag{3.9}$$

If we set

$$c = -2\left(\eta + \frac{\|f\|^2_{(L^\infty(\Omega))^2}}{\eta}\right)$$

and

$$w_h = v_h - c \text{ a.e. in } \Omega_h, \tag{3.10}$$

since $0 < \eta < \dfrac{1}{2}$ and $c < 0$, from (3.9) it follows that

$$\begin{aligned} &\int_{\Omega_h} \big(Dw_h D\varphi + 2(1-\eta) w_h \varphi\big) dx \geqslant \\ &\geqslant c\,(2\eta - 1) \int_{\Omega_h} \varphi \, dx \geqslant 0 \quad \forall \varphi \in C^1(\mathbb{R}^2) \text{ with } \varphi \geqslant 0. \end{aligned} \tag{3.11}$$

Now observe that, by virtue of (3.10), (3.6) and (3.2), w_h belongs to $H^1(\Omega_h)$. Consequently, by virtue Lemma 3.3, from (3.11) we deduce that

$$w_h \geqslant 0 \text{ a.e. in } \Omega_h. \tag{3.12}$$

By recalling the definitions (3.10) and (3.6), the last inequality provides

$$|u_h|^2 \leqslant 1 - c \text{ a.e. in } \Omega_h.$$

Since c does not depend on h, the thesis holds. □

Corollary 3.4. *For every h in $\mathbb{N}$, let u_h be a solution of Problem (2.4) with f in $(L^\infty(\Omega))^2$. Then, there exists a constant c (independent of h) such that*

$$\|u_h\|_{(H^1(\Omega_h))^2} \leqslant c \quad \forall h \in \mathbb{N}.$$

Proof. By choosing $v = u_h$ in (2.6) and by using Hölder's inequality, it results

$$\begin{aligned} &\int_{\Omega_h} |Du_h|^2 \, dx \leqslant \int_{\Omega_h} \big(|Du_h|^2 + |u_h|^4\big) dx = \int_{\Omega_h} f u_h \, dx + \int_{\Omega_h} |u_h|^2 \, dx \leqslant \\ &\leqslant \|f\|_{(L^2(\Omega))^2} \|u_h\|_{(L^2(\Omega_h))^2} + \|u_h\|^2_{(L^2(\Omega_h))^2} \quad \forall h \in \mathbb{N} \end{aligned}$$

from which, by virtue of Proposition 3.1, the thesis follows. □

Remark 3.5. For every h in $\mathbb{N}$, let u_h be a solution of Problem (2.4). By assuming f only in $\big(L^2(\Omega)\big)^2$ (and without making use of Proposition 3.1), it is easy to prove the existence of a constant c (independent of h) such that

$$\|u_h\|_{(L^4(\Omega_h))^2} \leqslant c \quad \forall h \in \mathbb{N}.$$

$$\|u_h |u_h|^2\|_{L^{\frac{4}{3}}(\Omega_h)^2} \leqslant c \quad \forall h \in \mathbb{N}.$$

$$\|u_h\|_{(H^1(\Omega_h))^2} \leqslant c \quad \forall h \in \mathbb{N}.$$

4. EXTENSION RESULT

This section is devoted to prove the following result:

Proposition 4.1. *For every h in $\mathbb{N}$, let u_h be a solution of Problem (2.4) with f in $(L^\infty(\Omega))^2 \cap \left(H^1(\Omega)\right)^2$. Then, for every h in $\mathbb{N}$, there exists a linear extension-operator $P_h \in \mathcal{L}\left(\left(H^1(\Omega_h^+)\right)^2, \left(H^1(\Omega^+)\right)^2\right)$ such that*

$$\|P_h u_h\|_{(H^1(\Omega^+))^2} \leqslant c \quad \forall h \in \mathbb{N}, \tag{4.1}$$

where c is a constant independent of h.

In this section, if $\phi(x_1, x_2)$ is a real function defined on Ω_h, we assume ϕ extended on $T_h = \bigcup\limits_{k\in Z} ((ka, 0) + \Omega_h)$ in the following way: first we extend ϕ on $(-a, 0) + \Omega_h$ by reflection, then we extend this function on T_h by $2a$-periodicity in the variable x_1. Moreover, we define

$$\tau_h\phi \,:\, (x_1, x_2) \in T_h \longrightarrow \phi(x_1, x_2) - \phi(x_1 + \frac{a}{h}, x_2). \tag{4.2}$$

We recall the following extension result proved in [10] Lemma 2.2:

Lemma 4.2. [10] *Let τ_h, h in $\mathbb{N}$, be defined by (4.2). Then, for every h in $\mathbb{N}$, there exists a linear extension-operator $Q_h \in \mathcal{L}\left(H^1(\Omega_h^+), H^1(\Omega^+)\right)$ such that*

$$\|Q_h\phi\|^2_{H^1(\Omega^+)} \leqslant c\left(\|\phi\|^2_{H^1(\Omega_h^+)} + \|\tau_h\phi\|^2_{H^1(\Omega_h^+)} + h^2\|\tau_h\phi\|^2_{L^2(\Omega_h^+)}\right) \ \forall \phi \in H^1(\Omega_h^+) \ \forall h \in \mathbb{N},$$

where c is a constant independent of ϕ and h.

Proposition 3.1 allows us to adapt the proof of Lemma 2.4 in [10] to our nonlinear case and obtain the following estimate result:

Lemma 4.3. *For every h in $\mathbb{N}$, let τ_h be defined by (4.2) and u_h, be a solution of Problem (2.4) with f in $(L^\infty(\Omega))^2 \cap \left(H^1(\Omega)\right)^2$. Then there exists a constant c (independent of h) such that*

$$\|\tau_h u_h^{(i)}\|_{H^1(\Omega_h^+)} \leqslant c\,\frac{1}{h} \quad \forall h \in \mathbb{N} \quad \forall i \in \{1, 2\}.$$

Proof. Let us fix i in $\{1, 2\}$.
For every h in $\mathbb{N}$ let us define

$$f_h^{(i)} = 2u_h^{(i)} - |u_h|^2 u_h^{(i)} + f_i \ \text{ a.e. in } \Omega_h.$$

Let us observe that $f_h^{(i)}$ is in $H^1(\Omega_h)$ since f is in $(H^1(\Omega))^2$ and u_h is in $(H^1(\Omega_h))^2 \cap (L^\infty(\Omega_h))^2$ by virtue of Lemma 3.2. Moreover, by using Proposition 3.1 and Corollary 3.4 we obtain the existence of a constant $\overline{c}$, independent of h, such that

$$\left\|\frac{\partial f_h^{(i)}}{\partial x_1}\right\|_{L^2(\Omega_h)} \leqslant \overline{c} \quad \forall h \in \mathbb{N}. \tag{4.3}$$

Since $u_h^{(i)}$ is the solution of

$$\begin{cases} -\Delta u_h^{(i)} + u_h^{(i)} = f_h^{(i)} \text{ in } \Omega_h, \\ Du_h^{(i)} \cdot \nu = 0 \text{ on } \partial\Omega_h, \end{cases}$$

it turn out that $\tau_h u_h^{(i)}$ is the solution of

$$\begin{cases} -\Delta \tau_h u_h^{(i)} + \tau_h u_h^{(i)} = \tau_h f_h^{(i)} \text{ in } \bigcup_{k=-1,0} ((ak,0)+\Omega_h), \\ \dfrac{\partial\left(\tau_h u_h^{(i)}\right)}{\partial\nu} = 0 \text{ on } \partial\left(\bigcup_{k=-1,0} ((ak,0)+\Omega_h)\right) - (\{-a,a\}\times]-b_1,0[), \\ \tau_h u_h^{(i)}(\cdot,x_2) \text{ is } 2a\text{-periodic in } x_1 \quad \forall x_2 \in]-b_1,0[. \end{cases} \tag{4.4}$$

By choosing $\tau_h u_h^{(i)}$ as test function in (4.4) and making use of Prop. IX.3 in [8] and (4.3), it results

$$\begin{aligned} \left\|\tau_h u_h^{(i)}\right\|_{H^1(\Omega_h)} &\leqslant \left\|\tau_h u_h^{(i)}\right\|_{H^1\left(\bigcup_{k=-1,0}((ak,0)+\Omega_h)\right)} \leqslant \\ &\leqslant \left\|\tau_h f_h^{(i)}\right\|_{L^2\left(\bigcup_{k=-1,0}((ak,0)+\Omega_h)\right)} \leqslant \frac{1}{h}\left\|\frac{\partial f_h^{(i)}}{\partial x_1}\right\|_{L^2\left(\bigcup_{k=-2}^{1}((ak,0)+\Omega_h)\right)} \leqslant \\ &\leqslant \frac{4}{h}\left\|\frac{\partial f_h^{(i)}}{\partial x_1}\right\|_{L^2(\Omega_h)} \leqslant \frac{4\bar{c}}{h} \quad \forall h \in \mathbb{N} \end{aligned}$$

and the thesis holds. □

Proof of Proposition 4.1. By choosing $\phi = u_h^{(i)}$, $i = 1,2$, in Lemma 4.2 and making use of Lemma 4.3 and Corollary 3.4 we obtain

$$\|Q_h u_h^{(i)}\|_{H^1(\Omega^+)} \leqslant c \quad \forall h \in \mathbb{N} \quad \forall i \in \{1,2\}, \tag{4.5}$$

where c is a constant independent of h. The estimate (4.1) follows from (4.5), by setting

$$P_h = (Q_h, Q_h) \quad \forall h \in \mathbb{N}.$$

□

5. PROOF OF THEOREM 2.1.

By virtue of Proposition 3.1, Corollary 3.4 and Proposition 4.1 there exist a strictly increasing sequence of positive integer numbers $\{h_k\}_{k\in\mathbb{N}}$, u^+ in $\left(H^1(\Omega^+)\right)^2$, u^- in $\left(H^1(\Omega^-)\right)^2\cap(L^\infty(\Omega^-))^2$, u^* in $(L^\infty(\Omega^+))^2$, z in $(L^\infty(\Omega))^2$ and d, e in $\left(L^2(\Omega^+)\right)^2$ such that

$$P_{h_k}u_{h_k}\rightharpoonup u^+ \quad \text{weakly in } \left(H^1(\Omega^+)\right)^2 \text{ and strongly in } (L^p(\Omega^+))^2 \quad \forall p\in[1,+\infty[, \tag{5.1}$$

$$u_{h_k}\rightharpoonup u^- \text{ weakly in } \left(H^1(\Omega^-)\right)^2 \text{ and weakly} \star \text{ in } \left(L^\infty(\Omega^-)\right)^2, \tag{5.2}$$

$$\widetilde{u_{h_k}}\rightharpoonup \theta u^* \text{ weakly} \star \text{ in } \left(L^\infty(\Omega^+)\right)^2, \tag{5.3}$$

$$\widetilde{u_{h_k}}|\widetilde{u_{h_k}}|^2\rightharpoonup z \text{ weakly} \star \text{ in } (L^\infty(\Omega))^2, \tag{5.4}$$

$$\widetilde{\frac{\partial u_{h_k}}{\partial x_1}}\rightharpoonup d \text{ weakly in } \left(L^2(\Omega^+)\right)^2, \tag{5.5}$$

$$\widetilde{\frac{\partial u_{h_k}}{\partial x_2}}\rightharpoonup e \text{ weakly in } \left(L^2(\Omega^+)\right)^2, \tag{5.6}$$

as k diverges.

Since

$$\chi_{\Omega_{h_k}\cap\Sigma}P_{h_k}u_{h_k}=\chi_{\Omega_{h_k}\cap\Sigma}u_{h_k} \text{ a.e. in } \Sigma \quad \forall k\in\mathbb{N},$$

from (2.3), (5.1) and (5.2) it follows that

$$u^+=u^- \text{ a.e. in } \Sigma$$

and consequently

$$u=\begin{cases} u^+ & \text{a.e. in } \Omega^+ \\ u^- & \text{a.e. in } \Omega^- \end{cases} \in \left(H^1(\Omega)\right)^2. \tag{5.7}$$

On the other hand, since

$$\chi_{\Omega^+_{h_k}}P_{h_k}u_{h_k}=\widetilde{u_{h_k}} \text{ a.e. in } \Omega^+ \quad \forall k\in\mathbb{N},$$

from (2.2), (5.1) and (5.3) it follows that

$$u^+=u^* \text{ a.e. in } \Omega^+ \tag{5.8}$$

and consequently

$$u\in(L^\infty(\Omega))^2, \tag{5.9}$$

where u is defined in (5.7).
By combining (5.1) and (5.2) with (5.7) and (5.9) we obtain (2.7).

Now let us prove that

$$z = \begin{cases} \theta |u|^2 u & \text{a.e. in } \Omega^+, \\ |u|^2 u & \text{a.e. in } \Omega^-. \end{cases} \tag{5.10}$$

To this purpose, first let us verify that

$$|P_{h_k} u_{h_k}|^2 \to |u|^2 \text{ strongly in } L^2(\Omega^+), \tag{5.11}$$

as k diverges. In fact, Hölder's inequality provides that

$$\| \, |P_{h_k} u_{h_k}|^2 - |u|^2 \|^2_{L^2(\Omega^+)} \leqslant 2 \int_{\Omega^+} |P_{h_k} u_{h_k} - u|^2 \left(|P_{h_k} u_{h_k}|^2 + |u|^2 \right) dx \leqslant$$
$$\leqslant 2 \|P_{h_k} u_{h_k} - u\|^2_{(L^4(\Omega^+))^2} \left(\|P_{h_k} u_{h_k}\|^2_{(L^4(\Omega^+))^2} + \|u\|^2_{(L^4(\Omega^+))^2} \right) \quad \forall k \in \mathbb{N},$$

from which, by virtue of (5.1) and definition (5.7), convergence (5.11) follows.

Since

$$\widetilde{u_{h_k}} |\widetilde{u_{h_k}}|^2 = \widetilde{u_{h_k}} |P_{h_k} u_{h_k}|^2 \text{ a.e. in } \Omega^+ \quad \forall k \in \mathbb{N},$$

by making use of (5.4), (5.3), (5.8), (5.7) and (5.11), we obtain $z = \theta |u|^2 u$ a.e. in Ω^+. The identification of z on Ω^- is similar.

By following arguments identical to those used in [12] Proposition 2.2, it is easy to prove that

$$e = \theta \frac{\partial u}{\partial x_2} \text{ a.e. in } \Omega^+. \tag{5.12}$$

Arguing as in [10], let us prove that

$$d = 0 \text{ a.e. in } \Omega^+. \tag{5.13}$$

Let $\{w_h\}_{h \in \mathbb{N}}$ be a sequence in $H^1(\Omega^+) \cap L^\infty(\Omega^+)$ such that

$$w_h \to x_1 \text{ strongly in } L^\infty(\Omega^+) \tag{5.14}$$

as h diverges and

$$D w_h = 0 \text{ a.e. in } \Omega_h^+ \quad \forall h \in \mathbb{N}.$$

The existence of such sequence is proved in [10] (see also [12], Lemma 4.3).

¿From (2.6) it follows that

$$\int_\Omega \left(\widetilde{D u_{h_k}} D v - \widetilde{u_{h_k}} v + |\widetilde{u_{h_k}}|^2 \widetilde{u_{h_k}} v \right) dx =$$
$$= \int_\Omega \chi_{\Omega_{h_k}} f v \, dx \ \ \forall v \in \left(H^1(\Omega) \right)^2 \ \ \forall k \in \mathbb{N}. \tag{5.15}$$

By choosing $v = w_{h_k} \varphi$, with φ in $\left(C_0^\infty(\Omega^+) \right)^2$, as test function in (5.15) it results

$$\int_{\Omega^+} \left(w_{h_k} \widetilde{D u_{h_k}} D\varphi - w_{h_k} \widetilde{u_{h_k}} \varphi + w_{h_k} |\widetilde{u_{h_k}}|^2 \widetilde{u_{h_k}} \varphi \right) dx =$$
$$= \int_{\Omega^+} \chi_{\Omega_{h_k}} w_{h_k} f \varphi \, dx \quad \forall \varphi \in \left(C_0^\infty(\Omega^+) \right)^2 \quad \forall k \in \mathbb{N}. \tag{5.16}$$

By passing to the limit, as k diverges, in (5.16) and by making use of (5.14), (5.5), (5.6), (5.12), (5.3), (5.8), (5.7), (5.4), (5.10) and (2.2), it results

$$\begin{aligned} &\int_{\Omega^+}\left(x_1 d\frac{\partial\varphi}{x_1}+x_1\theta\frac{\partial u}{x_2}\frac{\partial\varphi}{x_2}-x_1\theta u\varphi+x_1\theta|u|^2u\varphi\right)dx= \\ &=\int_{\Omega^+}\theta x_1 f\varphi\,dx \quad \forall\varphi\in\left(C_0^\infty(\Omega^+)\right)^2. \end{aligned} \tag{5.17}$$

On the other hand, by passing to the limit in (5.15) with $v = x_1\varphi$ as test function and by making use of (5.5), (5.6), (5.12), (5.3), (5.8), (5.7) (5.4), (5.10) and (2.2) it results

$$\begin{aligned} &\int_{\Omega^+}\left(x_1 d\frac{\partial\varphi}{x_1}+d\varphi+x_1\theta\frac{\partial u}{x_2}\frac{\partial\varphi}{x_2}-x_1\theta u\varphi+x_1\theta|u|^2u\varphi\right)dx= \\ &=\int_{\Omega^+}\theta x_1 f\varphi\,dx \quad \forall\varphi\in\left(C_0^\infty(\Omega^+)\right)^2. \end{aligned} \tag{5.18}$$

By comparing (5.17) with (5.18), we obtain

$$\int_{\Omega^+} d\varphi\,dx=0 \quad \forall\varphi\in\left(C_0^\infty(\Omega^+)\right)^2,$$

which implies (5.13).

By passing to the limit, as k diverges, in (5.15) and by making use of (5.2), (5.5), (5.13), (5.6), (5.12), (5.3), (5.8), (5.7) (5.4), (5.10) and (2.2) we obtain that u is a solution of (2.10).

To prove the convergence of the energies, let us choose $v = u_h$ as test function in (2.6). Then, by virtue of (2.7) and (2.2), we obtain

$$\lim_{k\to+\infty}\int_{\Omega_{h_k}}\left(|Du_{h_k}|^2-|u_{h_k}|^2+|u_{h_k}|^4\right)dx=\theta\int_{\Omega^+}fu\,dx+\int_{\Omega^-}fu\,dx. \tag{5.19}$$

On the other hand, by choosing $v = u$ as test function in (2.10), it results

$$\begin{aligned} &\theta\int_{\Omega^+}\left(\left|\frac{\partial u}{\partial x_2}\right|^2-|u|^2+|u|^4\right)dx+\int_{\Omega^-}\left(|Du|^2-|u|^2+|u|^4\right)dx= \\ &=\theta\int_{\Omega^+}fu\,dx+\int_{\Omega^-}fu\,dx. \end{aligned} \tag{5.20}$$

By comparing (5.19) with (5.20), the convergence of the energies (2.9) holds. □

Remark 5.1. Let us observe that the assumption f in $(L^\infty(\Omega))^2$ is used to obtain the uniform estimate in Proposition 3.1. This estimate together with the assumption f in $(H^1(\Omega))^2$ allows us to obtain (4.3) in order to prove Lemma 4.3 and consequently Proposition 4.1. Thanks to Proposition 4.1, we have the strong convergence (5.1) in $\left(L^p(\Omega^+)\right)^2$ for every p in $[1,+\infty[$, which allows us to pass to the limit in the nonlinear term of the Ginzburg-Landau equation (see (5.11)).

ACKNOWLEDGMENTS

Part of this work was done during the visit of A. Gaudiello at the University of Picardie, Amiens (F), where he was visiting Professor (May 1998) and during a visit of R. Hadiji at the University of Napoli (I).

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