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Habib Ammari, Hajer Bahouri, David dos Santos Ferreira, Isabelle Gallagher

STABILITY ESTIMATES FOR AN INVERSE SCATTERING PROBLEM AT HIGH FREQUENCIES

HABIB AMMARI, HAJER BAHOURI, DAVID DOS SANTOS FERREIRA, AND ISABELLE GALLAGHER

Abstract. We consider an inverse scattering problem and its near-field approximation at high frequencies. We first prove, for both problems, Lipschitz stability results for determining the low-frequency component of the potential. Then we show that, in the case of a radial potential supported sufficiently near the boundary, infinite resolution can be achieved from measurements of the near-field operator in the monotone case.

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Introduction

The first aim of this paper is to establish Lipschitz stability results for the inverse scattering problem of determining the low-frequency component (lower than the operating frequency) of the compactly supported potential from scattering or near-field measurements. It is known that, in general, the problem is exponentially unstable [4, 5, 6, 36, 40]. However, taking advantage of a priori information may improve stability and give accurate reconstruction algorithms [8, 12, 13]. The Lipschitz stability results proved in this paper together with the recent analysis of the local convergence of the nonlinear Landweber iteration in [33] show that the low-frequency component of the potential can be determined from the data in a linearly stable

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way. Moreover, they precisely quantify the resolution limit, which is defined as the characteristic size of the smallest oscillations in the potential that can be stably recovered from the data. Since Rayleigh's work, it has been admitted that the resolution limit in inverse scattering is of order π over the operating frequency [19]. This is nothing else than a direct application of the uncertainty principle in inverse scattering [18, 24, 28, 55]. It is well-known that if the support of the potential is a point support, then the reconstructed location of the point potential from the scattering data has finite size of order of the Rayleigh resolution limit [7, 18]. Having this in mind, the results of this paper prove that the Fourier transform of the potential can be reconstructed in a linearly stable way for all frequencies (dual variable to the space one) smaller than the operating frequency, and therefore, justify the notion of resolution limit. More intriguingly, again in view of [33], they prove that the stability of the reconstruction of the potential increases at high operating frequencies.

The second aim of the paper is to show that infinite resolution can be achieved from near-field measurements. Here, the near-field operator approximates Sommerfeld's radiation condition and is equivalent to the measurements of the Cauchy data at a finite distance. Moreover, if the potential is supported near the boundary, then infinite resolution can be achieved in the monotone case. In fact, a Lipschitz stability result holds for both the low and high frequency components of the potential. It should be noted that the scattering amplitude can be recovered from the near-field operator. However, approximating the near-field operator from the scattering amplitude is a severely ill-posed problem [37, 47, 48, 59] and therefore cannot be of any practical and realistic use. It was shown in [37] that in order to compute the near-field operator from the scattering amplitude one needs to differentiate the scattering amplitude an infinite number of times.

The results of this paper extend to medium scattering the recent results in [9, 10, 63], where a stability and resolution analysis was performed for linearized conductivity and wave imaging problems. They can be also used to justify the hopping (or continuation in the frequency) reconstruction algorithms proposed in [14, 24, 25, 26].

In connection with our results in this paper, we also refer to the works by Isakov [34] and Isakov and Kindermann [35], Bao, Lin, and Triki [15, 16], Nagayasu, Uhlmann and Wang [45, 46], as well as Derveaux, Papanicolaou, and Tsogka [27]. In [34, 35], an evidence of increasing stability in wave imaging when frequency is growing was given. In [15], stability estimates for the inverse source problem were established and the conversion of the logarithmic type stability to a Lispchitz one first proved. Numerical results to illustrate the stability of the source reconstruction problem were presented in [16]. In [17], Lipschitz stability estimates for the time-dependent wave equation were obtained. In [45], a stability estimate for a linearized conductivity problem was derived and its dependence on the depth of the inclusion highlighted. In [46], it is shown that the ill-posedness of the inverse acoustic problem decreases when the frequency increases and the stability estimate changes from logarithmic type for low frequencies to a Lipschitz type for large frequencies. In [27], the enhancement of resolution in the near-field was studied and numerically illustrated. Our results in this paper confirm these important observations in a quite general situation and precisely quantify them.

Our paper is organized as follows. Section 1 is devoted to the stability of the reconstruction of the potential from the scattering amplitude (called also far-field pattern) in the high frequency regime. Theorem 2 proves that the low-frequency component of the potential can be determined in a stable way from the scattering amplitude. The threshold frequency determines the resolution limit. Section 2 extends the results of Section 1 to the near-field measurements. Theorem 3 shows that the same results as those in Section 1 hold for reconstructing the potential from measurements of the near-field operator. In Section 3 we show that we gain infinite resolution for potentials supported near the boundary. If the potential is supported near the boundary, then infinite resolution can be achieved in the monotone case. Theorem 4 provides a Lipschitz stability result for both the low and high frequency components of the potential. Finally, in Appendix A, we provide useful results on Bessel's functions.

Finally, we mention that the letter C will be used to denote a universal constant which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant C. We also use the classical notation $\langle x\rangle = \sqrt{1+|x|^2}$.

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1. Far field pattern

1.1. **Definitions and notations.** Let Ω be a bounded domain in the Euclidean space $\mathbb{R}^d$ of dimension $d \geq 2$, let $q \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ be a real-valued potential supported in Ω. We use the classical notation $D = -i\partial$ for derivatives and consider the Helmholtz equation with potential

$$D^2 u - \lambda^2 u + qu = 0 \tag{1.1}$$

at frequency $\lambda \in \mathbb{R}_+^* := \mathbb{R}_+ \setminus \{0\}$. Plane waves $\mathrm{e}^{i\lambda x\cdot\omega}$ propagating along the direction ω in S^{d-1} are solutions of the free Helmholtz equation

$$D^2 u - \lambda^2 u = 0. \tag{1.2}$$

Here S^{d-1} denotes the unit sphere in $\mathbb{R}^d$. More generally, plane waves generate the set of solutions in the space of tempered distributions, $\mathcal{S}'(\mathbb{R}^d)$, of the free Helmholtz equation: all solutions of (1.2) with polynomial growth are superpositions of elementary plane waves $\mathrm{e}^{i\lambda x\cdot\omega}$ when ω varies on the sphere S^{d-1}. It will be useful later on to adopt Melrose's notation in [42] to designate these solutions: for $g \in \mathcal{C}^\infty(S^{d-1})$, we shall write

$$\Phi_0(\lambda)g(x) = \int_{S^{d-1}} \mathrm{e}^{i\lambda x\cdot\omega} g(\omega)\,\mathrm{d}\omega.$$

Obviously $\Phi_0(\lambda)g$ is a solution of (1.2) which belongs to $\mathcal{S}'(\mathbb{R}^d)$.

To guarantee the uniqueness of solutions to the Helmholtz equation (1.1), one can impose conditions on the behavior of solutions at infinity. More precisely, we are interested in solutions which can be decomposed

$$u = \mathrm{e}^{i\lambda x\cdot\omega} + u^{\mathrm{scat}} = u^{\mathrm{in}} + u^{\mathrm{scat}} \tag{1.3}$$

as the sum of an incoming planar wave and a scattered wave satisfying Sommerfeld's radiation condition

$$\left|\left(\frac{\partial}{\partial|x|} - i\lambda\right)u^{\mathrm{scat}}\right| = o\left(\frac{1}{|x|^{\frac{d-1}{2}}}\right) \quad \text{as } |x| \to +\infty, \tag{1.4}$$

uniformly with respect to the direction $\theta = \frac{x}{|x|}$ at fixed frequency $\lambda \in \mathbb{R}_+^*$. The following result from [42, Lemma 2.4] holds.

Proposition 1.1. *There exists a unique solution to the Helmholtz equation* (1.1) *with potential* $q \in \mathcal{C}_0^\infty(\Omega)$ *of the form*

$$\varphi_q(x,\omega,\lambda) = \mathrm{e}^{i\lambda x\cdot\omega} + \varphi_q^{\mathrm{scat}}(x,\omega,\lambda), \tag{1.5}$$

where the scattered wave $\varphi_q^{\mathrm{scat}}$ *satisfies Sommerfeld's radiation condition* (1.4) *and which is given by*

$$\varphi_q^{\mathrm{scat}} = -R_q(\lambda)(\mathrm{e}^{i\lambda x\cdot\omega} q). \tag{1.6}$$

Here R_q *denotes the meromorphic continuation of the perturbed resolvent. Furthermore* φ_q *depends smoothly on* $(x,\omega,\lambda) \in \mathbb{R}^d \times S^{d-1} \times \mathbb{R}_+^*$ *and is bounded.*

We choose to denote

$$\Phi_q(\lambda)g = \int_{S^{d-1}} \varphi_q(x,\omega,\lambda) g(\omega)\,\mathrm{d}\omega$$

the operator with kernel $\varphi_q(x,\omega,\lambda)$ given by (1.5).

Theorem 1. *The scattered wave in the solution* (1.3) *to the Helmholtz equation* (1.1) *given by Proposition* 1.1 *assumes the form*

$$\varphi_q^{\mathrm{scat}}(x) = \frac{\mathrm{e}^{i\lambda|x|}}{|x|^{\frac{d-1}{2}}}\, a_q\left(\frac{x}{|x|},\omega,\lambda\right) + \mathcal{O}\left(\frac{1}{|x|^{\frac{d+1}{2}}}\right) \quad \textit{as } |x| \to +\infty, \tag{1.7}$$

where a_q *is a smooth function on* $S^{d-1} \times S^{d-1} \times \mathbb{R}_+^*$ *and*

$$a_q\left(\frac{x}{|x|},\omega,\lambda\right) = -\frac{1}{2i\lambda}\left(\frac{\lambda}{2\pi i}\right)^{\frac{d-1}{2}} \int q(y)\varphi_q(y,\omega,\lambda)\mathrm{e}^{-i\lambda\frac{x}{|x|}\cdot y}\,\mathrm{d}y.$$

Proof. We consider the Green function G_λ corresponding to the free Helmholtz equation

$$(D_y^2 - \lambda^2) G_\lambda(x,y) = \delta(x-y), \tag{1.8}$$

subject to Sommerfeld's radiation condition (1.4), with δ being the Dirac delta function.

Let $R > 0$ be large enough so that the ball of radius R contains the support of q. By definition of the Green function, for all $|x| \le R$ we have

$$u(x) = \int_{|y|\le R} (D_y^2 - \lambda^2) G_\lambda(x,y) u(y)\,\mathrm{d}y,$$

and if u is a solution to the Helmholtz equation (1.1) we deduce by Green's formula that for any $|x| \le R$

$$\begin{aligned} u(x) = &- \int_{|y|\le R} G_\lambda(x,y) q(y) u(y)\,\mathrm{d}y \\ &- \int_{|y|=R} \left(\frac{\partial G_\lambda}{\partial r}(x,y)\, u(y) - G_\lambda(x,y) \frac{\partial u}{\partial r}(y) \right) \mathrm{d}\sigma(y). \end{aligned}$$

Along the same lines, it is possible to derive a similar identity for the plane wave $u^{\mathrm{in}}(x) = \mathrm{e}^{i\lambda\omega\cdot x}$

$$u^{\mathrm{in}}(x) = - \int_{|y|=R} \left(\frac{\partial G_\lambda}{\partial r}(x,y)\, u^{\mathrm{in}}(y) - G_\lambda(x,y) \frac{\partial u^{in}(y)}{\partial r}(y) \right) \mathrm{d}\sigma(y),$$

taking into account the fact that $(D^2 - \lambda^2) u^{\mathrm{in}} = 0$. Subtracting the two identities gives the following representation formula for the scattered wave $u^{\mathrm{scat}} = u - u^{\mathrm{in}}$

$$\begin{aligned} u^{\mathrm{scat}}(x) = &- \int_{|y|\le R} G_\lambda(x,y)\, q(y)\, u(y)\,\mathrm{d}y \\ &- \int_{|y|=R} \left(\frac{\partial G_\lambda}{\partial r}(x,y)\, u^{\mathrm{scat}}(y) - G_\lambda(x,y) \frac{\partial u^{\mathrm{scat}}}{\partial r}(y) \right) \mathrm{d}\sigma(y). \end{aligned}$$

The Green function of the free Helmholtz equation is explicitly given by

$$G_\lambda(x,y) = \frac{1}{4i} \left(\frac{\lambda}{2\pi} \right)^{\frac{d-2}{2}} |x-y|^{-\frac{d-2}{2}} H^{(1)}_{d/2-1}\big(\lambda|x-y|\big), \tag{1.9}$$

where $H^{(1)}_{d/2-1}$ is the Hankel function of first kind and order $d/2-1$ (cf. Appendix A). The asymptotic behavior of Hankel functions (A.3) implies that for $|y|$ large enough

$$G_\lambda(x,y) = \frac{1}{2i\lambda} \left(\frac{\lambda}{2\pi i} \right)^{\frac{d-1}{2}} \mathrm{e}^{i\lambda|x-y|} |x-y|^{-\frac{d-1}{2}} \left(1 + \mathcal{O}\left(\frac{1}{\lambda|x-y|} \right) \right). \tag{1.10}$$

Since we have

$$|x-y| = |y| - \frac{x\cdot y}{|y|} + \mathcal{O}\left(\frac{|x|^2}{|y|} \right)$$

we find for $|y|$ large enough and fixed λ:

$$\mathrm{e}^{i\lambda|x-y|} = \mathrm{e}^{i\lambda\,|y|}\mathrm{e}^{-i\lambda\frac{y}{|y|}\cdot x}\left(1+\mathcal{O}\left(\frac{|x|^2}{|y|}\right)\right).$$

We therefore obtain for fixed x, λ and large $R = |y|$

$$G_\lambda(x,y) = \frac{1}{2i\lambda}\left(\frac{\lambda}{2\pi i}\right)^{\frac{d-1}{2}} |y|^{-\frac{d-1}{2}}\mathrm{e}^{i\lambda\,|y|}\mathrm{e}^{-i\lambda\frac{y}{|y|}\cdot x}\left(1+\mathcal{O}\left(\frac{1}{R}\right)\right). \tag{1.11}$$

Analogously, we have

$$\frac{\partial G_\lambda}{\partial r}(x,y) = i\lambda c_d(\lambda)|y|^{-\frac{d-1}{2}}\mathrm{e}^{i\lambda|y|}\mathrm{e}^{-i\lambda\frac{y}{|y|}\cdot x}\left(1+\mathcal{O}\left(\frac{1}{R}\right)\right),$$

with $c_d(\lambda) = \frac{1}{2i\lambda}(\lambda/2\pi i)^{\frac{d-1}{2}}$. This leads to

$$\begin{aligned} u^{\mathrm{scat}}(x) = &- \int_{|y|\le R} G_\lambda(x,y)q(y)u(y)\,\mathrm{d}y \\ &+ \frac{c_d(\lambda)}{R^{\frac{d-1}{2}}}\int_{|y|=R}\left(\frac{\partial u^{\mathrm{scat}}}{\partial r} - i\lambda u^{\mathrm{scat}}\right)\mathrm{e}^{i\lambda R - i\lambda\frac{y}{|y|}\cdot x}\left(1+\mathcal{O}\left(\frac{1}{R}\right)\right)\mathrm{d}\sigma(y). \end{aligned}$$

Sommerfeld's radiation condition implies that the second right-hand side term tends to zero when R tends to infinity, so we get

$$u^{\mathrm{scat}}(x) = -\int G_\lambda(x,y)q(y)u(y)\,\mathrm{d}y\,.$$

Using once again the asymptotic formula (1.11) together with the fact that the Green function is symmetric we get

$$u^{\mathrm{scat}}(x) = -\frac{1}{2i\lambda}\left(\frac{\lambda}{2\pi i}\right)^{\frac{d-1}{2}}\frac{\mathrm{e}^{i\lambda|x|}}{|x|^{\frac{d-1}{2}}}\int q(y)u(y)\mathrm{e}^{-i\lambda\frac{x}{|x|}\cdot y}\left(1+\mathcal{O}\left(\frac{|y|^2}{|x|}\right)\right)\mathrm{d}y.$$

To summarize, we have that

$$u^{\mathrm{scat}}(x) = \frac{\mathrm{e}^{i\lambda|x|}}{|x|^{\frac{d-1}{2}}}a_q\left(\frac{x}{|x|},\omega,\lambda\right) + \mathcal{O}\left(\frac{1}{|x|^{\frac{d+1}{2}}}\right),$$

with

$$a_q\left(\frac{x}{|x|},\omega,\lambda\right) = -\frac{1}{2i\lambda}\left(\frac{\lambda}{2\pi i}\right)^{\frac{d-1}{2}}\int q(y)u(y)\mathrm{e}^{-i\lambda\frac{x}{|x|}\cdot y}\,\mathrm{d}y.$$

This proves that the scattered part of any solution of Helmholtz' equation subject to Sommerfeld's radiation condition takes the form announced in Theorem 1. □

Definition 1.2. *We define the scattering amplitude associated with the potential $q \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ by the smooth function $a_q : S^{d-1}\times S^{d-1}\times\mathbb{R}_+^* \to \mathbb{C}$ given by*

$$a_q(\theta,\omega,\lambda) = -\frac{1}{2i\lambda}\left(\frac{\lambda}{2\pi i}\right)^{\frac{d-1}{2}}\int q(y)\varphi_q(y,\omega,\lambda)\mathrm{e}^{-i\lambda\theta\cdot y}\,\mathrm{d}y. \tag{1.12}$$

We denote

$$A_q(\lambda)g(\theta) = \int_{S^{d-1}} a_q(\theta,\omega,\lambda)g(\omega)\,\mathrm{d}\omega, \quad \theta \in S^{d-1},$$

the corresponding operator with kernel $a_q(\theta,\omega,\lambda)$*.*

It is easy to get an asymptotic expansion of

$$\Phi_q(\lambda)g(x) = \int_{S^{d-1}} \big(\mathrm{e}^{i\lambda x\cdot\omega} + \varphi_q^{\mathrm{scat}}(x,\omega,\lambda)\big)g(\omega)\,\mathrm{d}\omega$$

as $|x| \to +\infty$ using the stationary phase and Theorem 1:

$$\Phi_q(\lambda)g = \left(\frac{2\pi}{\lambda|x|}\right)^{\frac{d-1}{2}} \Bigg(\mathrm{e}^{-i\lambda|x|}\mathrm{e}^{i(d-1)\frac{\pi}{4}}g(-\theta) + \mathrm{e}^{i\lambda|x|}\left(\mathrm{e}^{-i(d-1)\frac{\pi}{4}}g(\theta) + \left(\frac{\lambda}{2\pi}\right)^{\frac{d-1}{2}} A_q(\lambda)g(\theta)\right) + \mathcal{O}\left(\frac{1}{|x|}\right)\Bigg) \tag{1.13}$$

with $\theta = x/|x| \in S^{d-1}$. The operator which maps the coefficient of $\mathrm{e}^{-i\lambda|x|}$ into the coefficient of $\mathrm{e}^{i\lambda|x|}$ is given by

$$g(-\theta) \mapsto i^{-d+1}\left(g(\theta) + \left(\frac{\lambda i}{2\pi}\right)^{\frac{d-1}{2}} A_q(\lambda)g(\theta)\right).$$

This is, after renormalization and composition with the antipodal map, the scattering matrix [42].

Definition 1.3. *The scattering matrix is the operator given by*

$$S_q(\lambda) = \mathrm{Id} + \left(\frac{\lambda i}{2\pi}\right)^{\frac{d-1}{2}} A_q(\lambda).$$

Integration by parts allows to relate the values of the potential inside the domain with the scattering matrix $S_q(\lambda)$.

Lemma 1.4. *We have the following identities*

$$\int (q_1 - q_2)u_1\overline{u_2}\,\mathrm{d}x = -2i\lambda\left(\frac{2\pi}{\lambda}\right)^{d-1} \int_{S^{d-1}} g_1\,\overline{g_2} - S_{q_1}(\lambda)g_1\,\overline{S_{q_2}(\lambda)g_2}\,\mathrm{d}\omega,$$

$$\int (q_1 - q_2)u_1u_2\,\mathrm{d}x = -2i\lambda\left(\frac{2\pi}{\lambda i}\right)^{d-1} \int_{S^{d-1}} \check{g}_2\,S_{q_1}(\lambda)g_1 - \check{g}_1\,S_{q_2}(\lambda)g_2\,\mathrm{d}\omega,$$

for any pair of solutions[1]

$$u_1 = \Phi_{q_1}(\lambda)g_1, \quad u_2 = \Phi_{q_2}(\lambda)g_2,$$

to the Helmholtz equations (1.1) *related to the potentials* q_1, q_2*.*

[1] We use the notation $\check{g}(\omega) = g(-\omega) = Pg(\omega)$ for the antipodal map.

Proof. Let R be large enough so that the ball of radius R contains the support of both potentials q_1, q_2. By Green's formula, we have

$$\int_{|x|\le R} (q_1 - q_2) u_1 \overline{u_2}\, \mathrm{d}x = \int_{|x|=R} \partial_r u_1\, \overline{u_2} - u_1\, \overline{\partial_r u_2}\, \mathrm{d}\sigma(x)$$

and using the asymptotic formula (1.13) on the functions $u_1 = \Phi_{q_1}(\lambda) g_1$ and $u_2 = \Phi_{q_2}(\lambda) g_2$, we deduce

$$\begin{aligned}
&\partial_r u_1\, \overline{u_2} - u_1\, \overline{\partial_r u_2} = \\
&\quad - 2i\lambda \Big(\frac{2\pi}{\lambda R}\Big)^{d-1} \Big(g_1(-\theta)\, \overline{g_2(-\theta)} - S_{q_1}(\lambda) g_1(\theta)\, \overline{S_{q_2}(\lambda) g_2(\theta)}\Big) + \mathcal{O}\Big(\frac{1}{R^d}\Big),
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_{|x|\le R} (q_1 - q_2) u_1 \overline{u_2}\, \mathrm{d}x \\
&\qquad\qquad = -2i\lambda \Big(\frac{2\pi}{\lambda}\Big)^{d-1} \int_{S^{d-1}} \Big(g_1\, \overline{g_2} - S_{q_1} g_1\, \overline{S_{q_2} g_2}\Big)\, \mathrm{d}\theta + \mathcal{O}\Big(\frac{1}{R}\Big).
\end{aligned}$$

Letting R tend to infinity provides the first identity.

The proof of the second identity is similar, since

$$\begin{aligned}
\partial_r u_1\, u_2 - u_1\, \partial_r u_2 = -2i\lambda \Big(\frac{2\pi}{\lambda R}\Big)^{d-1} &\Big(g_1(-\theta)\, S_{q_2} g_2(\theta) - S_{q_1}(\lambda) g_1(\theta)\, g_2(-\theta)\Big) \\
&+ \mathcal{O}\Big(\frac{1}{R^d}\Big).
\end{aligned}$$

This completes the proof of the lemma. $\square$

Choosing $q_1 = q_2 = q$ in Lemma 1.4, we obtain the following properties of the scattering matrix:

$${}^t S_q(\lambda) = P \circ S_q(\lambda) \circ P, \quad S_q(\lambda)^* S_q(\lambda) = \mathrm{Id}$$

recalling that $Pg = \check{g}$ is the antipodal map. Here, t denotes the transpose and $*$ the transpose conjugate. Besides, using the first relation together with Lemma 1.4, we finally get

(1.14)

$$\int (q_1 - q_2) u_1 u_2\, \mathrm{d}x = 2i\lambda \Big(\frac{2\pi}{\lambda i}\Big)^{\frac{d-1}{2}} \int_{S^{d-1}} \check{g}_2 \left(A_{q_1}(\lambda) - A_{q_2}(\lambda)\right) g_1\, \mathrm{d}\omega,$$

for any pair of solutions

$$u_1 = \Phi_{q_1}(\lambda) g_1, \quad u_2 = \Phi_{q_2}(\lambda) g_2,$$

to the Helmholtz equations (1.1) with Sommerfeld's radiation condition (1.4) related to the potentials q_1, q_2.

1.2. **Stability estimates at high frequencies.** The first stability result we shall prove in this paper states that low frequencies (i.e. smaller than 2λ) may be recovered in a stable way from the scattering amplitude.

Theorem 2. *For all $\varepsilon, M, R > 0$ and all $\alpha > d$ there exist $C_\varepsilon, \lambda_0 > 0$ such that the following stability estimate holds true.*
Let $q_1, q_2 \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ be two potentials supported in the ball centered at 0 and of radius R such that $\|q_1\|_{L^\infty}, \|q_2\|_{L^\infty} \leq M$. Then for all $\lambda \geq \lambda_0$,

$$
\begin{aligned}
(1.15)\quad \int_{|\xi|\leq(2-\varepsilon)\lambda} \langle\xi\rangle^{-\alpha} \big|(\widehat{q_1 - q_2})(\xi)\big|^2 \,\mathrm{d}\xi \\
\leq C_\varepsilon \lambda^3 \|a_{q_1} - a_{q_2}\|^2_{L^2} + C_\varepsilon \lambda^{-2} \|q_1 - q_2\|^2_{L^\infty}.
\end{aligned}
$$

The proof of Theorem 2 relies on estimates on solutions to the Helmoltz equations stated below, as well as on the following lemma, stating that one can relate the Fourier transform of the potential and the scattering amplitude.

Lemma 1.5. *For all $M, R > 0$ there exist constants $C, \lambda_0 > 0$ such that for all potentials $q \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ supported in the ball $B(0,R)$ and satisfying $\|q\|_{L^\infty} \leq M$, we have the following approximation:*

$$
(1.16)\qquad \left| a_q(\theta,\omega,\lambda) + \frac{1}{2i\lambda}\left(\frac{\lambda}{2\pi i}\right)^{\frac{d-1}{2}} \widehat{q}\big(\lambda(\theta-\omega)\big) \right| \leq C\,\lambda^{\frac{d-5}{2}}\, \|q\|^2_{L^2}
$$

for all $\lambda \geq \lambda_0$.

Lemma 1.5 is based on the following classical estimate on the free resolvent $R_0(\lambda)$. We refer to [21] for an exposition by Burq of an elementary proof due to Zworski. This type of *a priori* estimates have a long history and go back to the work of Agmon [2] on weighted estimates on the resolvent and the limiting absorption principle.

Proposition 1.6. *Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, there exists a constant $C > 0$ such that for all $\lambda > 1$ we have*

$$
\|\chi R_0(\lambda)\chi\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C\lambda^{-1}.
$$

Proof of Lemma 1.5. We start with the expression defining the scattering amplitude (see Definition 1.2):

$$
a_q(\theta,\omega,\lambda) = -\frac{1}{2i\lambda}\left(\frac{\lambda}{2\pi i}\right)^{\frac{d-1}{2}} \int q(y)\varphi_q(y,\omega,\lambda)\mathrm{e}^{-i\lambda\theta\cdot y}\,\mathrm{d}y
$$

so by (1.5) we deduce that

$$
\begin{aligned}
a_q(\theta,\omega,\lambda) &+ \frac{1}{2i\lambda}\left(\frac{\lambda}{2\pi i}\right)^{\frac{d-1}{2}} \widehat{q}\big(\lambda(\theta-\omega)\big) \\
&= -\frac{1}{2i\lambda}\left(\frac{\lambda}{2\pi i}\right)^{\frac{d-1}{2}} \int q(x)\varphi_q^{\mathrm{scat}}(x,\omega,\lambda)\,\mathrm{e}^{-i\lambda x\cdot\theta}\,\mathrm{d}x.
\end{aligned}
$$

The resolvent identity reads (see for instance Formula (2.3) in [42])

$$R_q(\lambda) = R_0(\lambda) - R_0(\lambda)\, q\, R_q(\lambda) \tag{1.17}$$

which implies that if $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ is a cutoff function which equals 1 on the ball $B(0,R)$, then

$$\chi R_q(\lambda)\chi = \chi R_0(\lambda)\chi - \big(\chi R_0(\lambda)\chi\big) q \big(\chi R_q(\lambda)\chi\big).$$

If we apply this identity to $-q\mathrm{e}^{i\lambda x\cdot\omega}$, we get in view of (1.6)

$$\chi\varphi_q^{\text{scat}} = -\big(\chi R_0(\lambda)\chi\big)\big(q\mathrm{e}^{i\lambda x\cdot\omega}\big) + \big(\chi R_0(\lambda)\chi\big)\big(q\chi\varphi_q^{\text{scat}}\big)$$

and using the estimate of Proposition 1.6, we obtain

$$\|\chi\varphi_q^{\text{scat}}\|_{L^2} \leq C\lambda^{-1}\|q\|_{L^2} + C\lambda^{-1}\|q\|_{L^\infty}\|\chi\varphi_q^{\text{scat}}\|_{L^2}.$$

Taking $\lambda \geq 2CM$ we deduce

$$\|\chi\varphi_q^{\text{scat}}\|_{L^2} \leq 2C\lambda^{-1}\|q\|_{L^2}.$$

Using Cauchy-Schwarz's inequality, we get

$$\left|\int q\,\varphi_q^{\text{scat}}\mathrm{e}^{-i\lambda x\cdot\omega}\,\mathrm{d}x\right| \leq 2C\lambda^{-1}\|q\|_{L^2}^2$$

and this completes the proof of the lemma. □

Note that the main ingredient in the proof of Lemma 1.5 consists in combining the estimate for the non-perturbed resolvent (Proposition 1.6) with the resolvent identity (1.17), along with the fact that q is compactly supported.

This lemma is the basis of the reconstruction of the potential from the scattering amplitude at high frequencies, since one can choose $\theta,\omega \in S^{d-1}$ such that $\lambda(\theta-\omega) = \xi$ for any fixed frequency ξ with $|\xi| \leq 2\lambda$ and let λ tend to infinity. We use a similar approach to prove a stability estimate at high frequencies.

Proof of Theorem 2. Again, we start with the expression defining the scattering amplitude and then express the potential in terms of the scattering amplitude: for $j \in \{1,2\}$ we write

$$\widehat{q_j}\big(\lambda(\theta-\omega)\big) = -2i(2\pi i)^{\frac{d-1}{2}}\lambda^{-\frac{d-3}{2}}a_{q_j}(\theta,\omega,\lambda) - \int q_j(x)\varphi_{q_j}^{\text{scat}}(x,\omega,\lambda)\,\mathrm{e}^{-i\lambda x\cdot\theta}\,\mathrm{d}x.$$

Taking the difference of the two expressions yields

$$\begin{aligned}\big|\widehat{(q_1-q_2)}\big(\lambda(\theta-\omega)\big)\big| &\leq 2(2\pi)^{\frac{d-1}{2}}\lambda^{-\frac{d-3}{2}}\big|(a_{q_1}-a_{q_2})(\theta,\omega,\lambda)\big| \\ &\quad + \|q_1-q_2\|_{L^2}\,\|\varphi_{q_1}^{\text{scat}}\|_{L^2(|x|\leq R)} + \|q_2\|_{L^2}\,\|\varphi_{q_1}^{\text{scat}} - \varphi_{q_2}^{\text{scat}}\|_{L^2(|x|\leq R)}\end{aligned} \tag{1.18}$$

where we recall that the supports of q_1, q_2 are contained in the ball of center 0 and radius $R > 0$. As in the proof of Lemma 1.5 we have

$$\|\varphi_{q_1}^{\text{scat}}\|_{L^2(|x|\leq R)} \leq C\lambda^{-1}\|q_1\|_{L^2}. \tag{1.19}$$

In light of (1.6), the difference of the two scattered waves satisfies

$$\varphi_{q_1}^{\text{scat}} - \varphi_{q_2}^{\text{scat}} = -\big(R_{q_1}(\lambda) - R_{q_2}(\lambda)\big)\big(q_1\mathrm{e}^{i\lambda x\cdot\omega}\big) - R_{q_2}(\lambda)\big((q_1-q_2)\mathrm{e}^{i\lambda x\cdot\omega}\big)$$

and by the resolvent identity (1.17), the first term on the right-hand side reads

$$R_{q_2}(\lambda)(q_1-q_2)R_{q_1}(\lambda)(q_1 e^{i\lambda x\cdot\omega}) = -R_{q_2}(\lambda)(q_1-q_2)\,\varphi^{\rm scat}_{q_1}.$$

Let $\chi\in\mathcal{C}^\infty_0(\mathbb{R}^d)$ be a cutoff function which equals 1 on the ball $B(0,R)$, then

$$\chi(\varphi^{\rm scat}_{q_1}-\varphi^{\rm scat}_{q_2}) = -\big(\chi R_{q_2}(\lambda)\chi\big)\big((q_1-q_2)\varphi^{\rm scat}_{q_1}\big) - \big(\chi R_{q_2}(\lambda)\chi\big)\big((q_1-q_2)e^{i\lambda x\cdot\omega}\big).$$

Therefore along the same lines as in the proof of Lemma 1.5, the resolvent identity (1.17) and Proposition 1.6 give rise to

$$\|\varphi^{\rm scat}_{q_1}-\varphi^{\rm scat}_{q_2}\|_{L^2(|x|\le R)} \lesssim \lambda^{-1}\|q_1-q_2\|_{L^\infty}\|\varphi^{\rm scat}_{q_1}\|_{L^2(|x|\le R)} + \lambda^{-1}\|q_1-q_2\|_{L^2}. \tag{1.20}$$

Indeed invoking the resolvent identity (1.17), we infer that

$$\chi R_{q_2}(\lambda)\chi = \chi R_0(\lambda)\chi - \big(\chi R_0(\lambda)\chi\big)q_2\big(\chi R_{q_2}(\lambda)\chi\big).$$

Thus applying this identity to $(q_1-q_2)\varphi^{\rm scat}_{q_1}$, we obtain

$$\|\big(\chi R_{q_2}(\lambda)\chi\big)\big((q_1-q_2)\varphi^{\rm scat}_{q_1}\big)\|_{L^2} \le C\lambda^{-1}\|q_1-q_2\|_{L^\infty}\|\varphi^{\rm scat}_{q_1}\|_{L^2(|x|\le R)} + C\lambda^{-1}\|q_2\|_{L^\infty}\big\|\big(\chi R_{q_2}(\lambda)\chi\big)\big((q_1-q_2)\varphi^{\rm scat}_{q_1}\big)\big\|_{L^2},$$

which implies for $\lambda>2CM$

$$\|\big(\chi R_{q_2}(\lambda)\chi\big)\big((q_1-q_2)\varphi^{\rm scat}_{q_1}\big)\|_{L^2} \le 2C\lambda^{-1}\|q_1-q_2\|_{L^\infty}\|\varphi^{\rm scat}_{q_1}\|_{L^2(|x|\le R)}.$$

We have also

$$\|\big(\chi R_{q_2}(\lambda)\chi\big)\big((q_1-q_2)e^{i\lambda x\cdot\omega}\big)\|_{L^2} \lesssim \lambda^{-1}\|q_1-q_2\|_{L^2},$$

which achieves the proof of Estimate (1.20).

Taking into account those bounds, we obtain in view of (1.18) the estimate

$$\big|\widehat{(q_1-q_2)}\big(\lambda(\theta-\omega)\big)\big| \lesssim \lambda^{-\frac{d-3}{2}}\big|(a_{q_1}-a_{q_2})(\theta,\omega,\lambda)\big| + \lambda^{-1}\|q_1-q_2\|_{L^\infty}. \tag{1.21}$$

We denote $r(\xi)=\sqrt{1-|\xi|^2}$ when $|\xi|\le 1$. For $\xi\in\mathbb{R}^d$ with $|\xi|\le 2\lambda$, choose $\eta\in\xi^\perp$ with norm $|\eta|=r(\xi/2\lambda)$. The vectors

$$\theta=\eta+\frac{\xi}{2\lambda},\quad \omega=\eta-\frac{\xi}{2\lambda},$$

have length one, taking the square of (1.21) and integrating the resulting estimate with respect to η yields

$$\big|\widehat{(q_1-q_2)}(\xi)\big|^2 \lesssim \lambda^{-2}\,\|q_1-q_2\|^2_{L^\infty} + \lambda^{-d+3}r(\xi/2\lambda)^{-d+2}\int_{\substack{|\eta|=r(\xi/2\lambda)\\ \eta\perp\xi}}\Big|(a_{q_1}-a_{q_2})\Big(\eta+\frac{\xi}{2\lambda},\eta-\frac{\xi}{2\lambda},\lambda\Big)\Big|^2\,d\eta.$$

When $|\xi| \leq (2-\varepsilon)\lambda$ we have $r(\xi/2\lambda) \geq \sqrt{\varepsilon(4-\varepsilon)}/2$ therefore multiplying the previous inequality by $\langle\xi\rangle^{-\alpha} \leq 1$ and integrating with respect to ξ we get

$$\int_{|\xi|\leq(2-\varepsilon)\lambda} \langle\xi\rangle^{-\alpha} \big|(\widehat{q_1 - q_2})(\xi)\big|^2 \, \mathrm{d}\xi \leq C_\varepsilon \lambda^{-2} \|q_1 - q_2\|_{L^\infty}^2$$
$$+ C_\varepsilon \lambda^3 \int_{|\xi|\leq 1} \int_{\substack{|\eta|=r(\xi)\\ \eta\perp\xi}} \big|(a_{q_1} - a_{q_2})\big(\eta+\xi, \eta-\xi, \lambda\big)\big|^2 \, \mathrm{d}\eta \, \mathrm{d}\xi.$$

We consider the following $2d-2$ dimensional submanifold of S^{2d-1}

$$\Sigma = \big\{(\xi,\eta) \in S^{2d-1} : \langle\xi,\eta\rangle = 0\big\}$$

and the following diffeomorphism

$$\begin{aligned} \varphi : \quad \Sigma &\to S^{d-1} \times S^{d-1} \\ (\xi,\eta) &\mapsto (\xi+\eta, \eta-\xi), \end{aligned}$$

with inverse

$$\begin{aligned} \varphi^{-1} : S^{d-1} \times S^{d-1} &\to \Sigma \\ (\theta,\omega) &\mapsto \left(\frac{\theta-\omega}{2}, \frac{\theta+\omega}{2}\right) \end{aligned}$$

for which we have

$$\begin{aligned} \int_\Sigma F(\eta+\xi, \eta-\xi) \, \mathrm{d}\eta \wedge \mathrm{d}\xi &= \int_{S^{d-1}\times S^{d-1}} F(\theta,\omega) \, \varphi^{-1*}(\mathrm{d}\eta \wedge \mathrm{d}\xi) \\ &= 2^{-d} \int_{S^{d-1}\times S^{d-1}} F(\theta,\omega) \, \mathrm{d}\omega \wedge \mathrm{d}\theta. \end{aligned}$$

Then we finally get

$$\begin{aligned} \int_{|\xi|\leq(2-\varepsilon)\lambda} & \langle\xi\rangle^{-\alpha} \big|(\widehat{q_1 - q_2})(\xi)\big|^2 \, \mathrm{d}\xi \\ &\leq C_\varepsilon \lambda^3 \iint_{S^{d-1}\times S^{d-1}} \big|a_{q_1}\big(\theta,\omega,\lambda\big) - a_{q_2}\big(\theta,\omega,\lambda\big)\big|^2 \, \mathrm{d}\theta \, \mathrm{d}\omega \\ &\quad + C_\varepsilon \, \lambda^{-2} \|q_1 - q_2\|_{L^\infty}^2 \end{aligned}$$

and this completes the proof of our estimate. □

In particular, we recover the uniqueness of the potential from the scattering amplitude at high frequencies.

Corollary 1.7. *If for all (θ,ω,λ) belonging to $S^{d-1} \times S^{d-1} \times \mathbb{R}_+^*$, we have $a_{q_1}(\theta,\omega,\lambda) = a_{q_2}(\theta,\omega,\lambda)$ then $q_1 = q_2$.*

2. Near field pattern

2.1. **Definitions and notations.** Instead of considering the Helmholtz equation on the whole Euclidean space (with Sommerfeld's radiation condition) we focus on the Cauchy problem with Robin boundary condition on a bounded open set $\Omega \subset \mathbb{R}^d$ with smooth boundary

$$\begin{cases} (D^2 - \lambda^2 + q)u = 0 & \text{in } \Omega, \\ (\partial_\nu - i\lambda)u|_{\partial\Omega} = f \in L^2(\partial\Omega). \end{cases} \tag{2.1}$$

This problem has a unique solution $u \in H^1(\Omega)$ for all $f \in L^2(\partial\Omega)$. Writing a variational formulation of (2.1) and using a unique continuation argument shows the uniqueness of a solution to (2.1). The existence follows from Fredholm's alternative [41].

Remark 2.1. Other classical boundary conditions are either Dirichlet or Neumann boundary conditions. However, one has to make the additional assumption that λ^2 is not a Dirichlet (or Neumann) eigenvalue of $D^2 + q$, for the Dirichlet problem corresponding to (2.1) to have a unique solution for all $f \in H^{\frac{1}{2}}(\partial\Omega)$. Unfortunately, this does not make sense if one wants to take the high frequency limit $\lambda \to \infty$. To bypass this difficulty, we study the boundary problem (2.1). Indeed, this condition is natural. It approximates Sommerfeld's radiation condition at high frequencies [30, 38].

Remark 2.2. It follows from [41] that the unique solution $u \in H^1(\Omega)$ of (2.1) satisfies

$$\|Du\|_{L^2(\Omega)} + \lambda\|u\|_{L^2(\Omega)} \leq C\big(\|qu\|_{L^2(\Omega)} + \|f\|_{L^2(\partial\Omega)}\big)$$

for some constant C independent of λ. Therefore, as $\lambda \to \infty$, we have

$$\|Du\|_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}.$$

Definition 2.3. *The near field pattern is the map*

$$\begin{aligned} \mathcal{N}_q(\lambda) : L^2(\partial\Omega) &\to L^2(\partial\Omega) \\ f &\mapsto u|_{\partial\Omega}. \end{aligned}$$

The typical inverse problem on the near field pattern is whether it uniquely determines the potential q. This was solved (for smooth potentials) by Sylvester and Uhlmann [57] in dimension $d \geq 3$ in the case of the Dirichlet-to-Neumann map. Reconstruction methods were proposed by Nachmann in [43] and stability issues were studied by Alessandrini [6]. It was shown by Mandache in [40] that the logarithmic stability result of Alessandrini in [6] is optimal.

2.2. **Stability estimates at high frequencies.** The following result is the counterpart of Theorem 2 in the near-field context.

Theorem 3. *For all $M, R > 0$ and all $\alpha > d$ there exist $C, \lambda_0 > 0$ such that the following stability estimate holds true.*
Let $q_1, q_2 \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ be two potentials supported in the ball centered at 0 and of radius R such that $\|q_1\|_{L^\infty}, \|q_2\|_{L^\infty} \leq M$. Then for all $\lambda \geq \lambda_0$,

$$\int_{|\xi|\leq 2\lambda} \langle \xi \rangle^{-\alpha} \big|\widehat{q_1 - q_2}(\xi)\big|^2 \, \mathrm{d}\xi \leq C \| \mathcal{N}_{q_1} - \mathcal{N}_{q_2} \|^2 + C\lambda^{-2} \|q_1 - q_2\|^2_{L^\infty}. \tag{2.2}$$

Proof. We start with two solutions u_1, u_2 of the equations

$$(D^2 - \lambda^2 + q_j)u_j = 0$$

and computing

$$\begin{aligned}\int_\Omega (q_1 - q_2)u_1 u_2 \, \mathrm{d}x &= \int_\Omega \Delta u_1 u_2 \, \mathrm{d}x - \int_\Omega u_1 \Delta u_2 \, \mathrm{d}x \\ &= \int_{\partial\Omega} (\partial_\nu - i\lambda)u_1 u_2 \, \mathrm{d}\sigma - \int_{\partial\Omega} u_1 (\partial_\nu - i\lambda) u_2 \, \mathrm{d}\sigma\end{aligned}$$

yields the formula

$$\int_\Omega (q_1 - q_2)u_1 u_2 \, \mathrm{d}x = \int_{\partial\Omega} \big(u_2 \, \mathcal{N}_{q_1}(\lambda) u_1 - u_1 \, \mathcal{N}_{q_2}(\lambda) u_2\big) \, \mathrm{d}\sigma.$$

Choosing $q_1 = q_2$ shows that the application $\mathcal{N}_{q_j}$ is symmetric, and using this additional information, we get the formula

$$\int_\Omega (q_1 - q_2)u_1 u_2 \, \mathrm{d}x = \int_{\partial\Omega} u_2 \left(\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\right) u_1 \mathrm{d}\sigma.$$

Let us write $q = q_1 - q_2$ and choose

$$\begin{aligned} u_1 &= \varphi_{q_1} = \mathrm{e}^{i\lambda x\cdot\theta_1} + \varphi_{q_1}^{\mathrm{scat}} \\ \text{and} \quad u_2 &= \varphi_{q_2} = \mathrm{e}^{i\lambda x\cdot\theta_2} + \varphi_{q_2}^{\mathrm{scat}}, \end{aligned}$$

with $\theta_1, \theta_2 \in S^{d-1}$. Recall from the proof of Lemma 1.5 that the scattered waves satisfy the estimate

$$\|\varphi_{q_1}^{\mathrm{scat}}\|_{L^2(B(0,R))} + \|\varphi_{q_2}^{\mathrm{scat}}\|_{L^2(B(0,R))} \leq C\lambda^{-1}.$$

It follows that

$$\begin{aligned}\int_\Omega \mathrm{e}^{i\lambda x\cdot(\theta_1+\theta_2)} q(x) \, \mathrm{d}x &= \int_{\partial\Omega} \big(\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\big) u_1 u_2 \, \mathrm{d}\sigma \\ &\quad - \int_\Omega q(x)(\varphi_{q_1}^{\mathrm{scat}} + \varphi_{q_2}^{\mathrm{scat}} + \varphi_{q_1}^{\mathrm{scat}}\varphi_{q_2}^{\mathrm{scat}}) \, \mathrm{d}x\end{aligned}$$

and therefore

$$\begin{aligned}\big|\widehat{q}\big(\lambda(\theta_1 + \theta_2)\big)\big| &\leq \| \mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| \|u_1\|_{L^2(\partial\Omega)} \|u_2\|_{L^2(\partial\Omega)} + \frac{C}{\lambda}\|q\|_{L^\infty} \\ &\leq C\| \mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| + \frac{C}{\lambda}\|q\|_{L^\infty}\end{aligned}$$

for all $(\theta_1, \theta_2) \in S^{d-1} \times S^{d-1}$.

We notice that

$$\begin{aligned} S^{d-1} \times S^{d-1} &\to B(0,2) \\ (\theta_1,\theta_2) &\mapsto \theta_1+\theta_2 \end{aligned}$$

is a submersion when θ_1, θ_2 are not colinear. This implies that

$$\left|\widehat{q}(\xi)\right| \le C\| \mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| + \frac{C}{\lambda}\|q\|_{L^\infty}, \quad \xi \in B(0,2\lambda). \tag{2.3}$$

Multiplying this estimate by $\langle\xi\rangle^{-\alpha}/2$, taking the square and integrating on the ball $B(0,2\lambda)$ completes the proof of the theorem. □

3. The case of a potential located near the boundary

3.1. **Definitions and notations.** In this section we show, in the model case of the unit disk, that if the potential is supported close to the boundary of the disk, then a larger range of frequencies may be recovered by the near field $\mathcal{N}_q(\lambda)$ than in the general case treated in the previous two sections. More precisely, introducing radial coordinates $(x_1, x_2) = (r\cos\theta, r\sin\theta)$, with $(r,\theta) \in \mathbb{R}^+ \times [0, 2\pi[$, we consider the following model problem in two space dimensions:

$$\begin{cases} (D^2 - \lambda^2 + q_\lambda)u_n = 0 \quad \text{in} \quad B = \left\{x \in \mathbb{R}^2,\ |x| \le 1\right\}, \\ \big(\partial_r - i\lambda\big)u_{n|\partial B} = \mathrm{e}^{in\theta}. \end{cases} \tag{3.1}$$

We suppose that q_λ is a smooth, radial function, with support included in D^κ_λ for some fixed constant $\kappa > 0$, where

$$D^\kappa_\lambda = \left\{r \in [0,1],\ 1 - \kappa\lambda^{-1} < r < 1\right\}. \tag{3.2}$$

In the following for simplicity we shall drop the index λ in the notation of the potential.

The following result shows that in the monotone case, one can improve on the frequency band recovered in the general case (see Theorems 2 and 3). We recall that $\| \mathcal{N}_q(\lambda)\|$ denotes the operator norm of $\mathcal{N}_q(\lambda)$ in $\mathcal{L}(L^2)$.

Theorem 4. *Let q_1 and q_2 be two smooth radial potentials supported on D^κ_λ as defined in* (3.2) *and such that $q_1 \ge q_2$. There are positive constants λ_0 and C such that the following holds.*
Let $\lambda \mapsto K(\lambda)$ be any function such that $\lambda \le K(\lambda)$. Then the following stability estimate is valid for all $\lambda \ge \lambda_0$:

$$\int_{|\xi|\le K(\lambda)} \left|(\widehat{q_1 - q_2})(\xi)\right|^2 \mathrm{d}\xi \le CK^2(\lambda)\Big(\lambda^4\| \mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\|^2 + \frac{C^2_{q_1,q_2}}{\lambda^4}\|q_1 - q_2\|^2_{L^\infty}\Big),$$

where $C_{q_1,q_2} = \max\big(\|q_1\|_{L^\infty}, \|q_2\|_{L^\infty}, \|q_1\|_{L^\infty}\|q_2\|_{L^\infty}\big)$.

Remark 3.1. One has trivially that

$$\int_{|\xi|\leq K(\lambda)} \big|(\widehat{q_1 - q_2})(\xi)\big|^2 \,\mathrm{d}\xi \leq CK^2(\lambda)\|\widehat{q_1 - q_2}\|^2_{L^\infty}$$
$$\leq \frac{CK^2(\lambda)}{\lambda^2}\|q_1 - q_2\|^2_{L^\infty},$$

so the estimate provided in Theorem 4 is of a different nature.

Remark 3.2. The proof of Theorem 4 is presented in Section 3.2, as an immediate consequence of Lemma 3.3 proved in Section 3.3. That lemma relates the Laplace transform of a function to the near field operator. It holds in much more generality than Theorem 4, without the additional assumption that $q_1 - q_2 \geq 0$. However we are unable to relate the Fourier transform of a function to its Laplace transform in general (this fact is well-known to be difficult and in general very unstable); for nonnegative functions however the relation is very easy and enables us to conclude.

3.2. **Proof of Theorem 4.** We start by stating a lemma, proved in Section 3.3, which relates the Laplace transform of a function to the near field operator. It is stated in the framework of general, non radial functions.

We define the operator $T : g \mapsto Tg$ by

$$Tg(r) = g(1 - r)$$

as well as the Laplace transform $\mathcal{L}$:

$$\mathcal{L}(g)(t) = \int_{\mathbb{R}} g(s)\,\mathrm{e}^{-s\,t}\,\mathrm{d}s.$$

For any function $\theta \mapsto f(\theta)$ we call $c_k(f)$ its Fourier transform, for $k \in \mathbb{Z}$:

$$c_k(f)(r) = \int_0^{2\pi} f(r,\theta)\,\mathrm{e}^{-i\,k\,\theta}\,\mathrm{d}\theta.$$

Lemma 3.3. *Let q_1 and q_2 be two functions supported on D^κ_λ. There are three positive constants λ_0, K and C such that if $t \geq 2K\lambda > 2K\lambda_0$, then*

$$\Big|\mathcal{L}\big(c_k\big(T(r(q_1 - q_2))\big)\big)(t)\Big|$$
$$\leq C\left(\lambda^2\,\|\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| + \frac{C_{q_1,q_2}}{\lambda^2}\|q_1 - q_2\|_{L^\infty}\right)$$

where $C_{q_1,q_2} = \max\big(\|q_1\|_{L^\infty}, \|q_2\|_{L^\infty}, \|q_1\|_{L^\infty}\|q_2\|_{L^\infty}\big)$.

Proof of Theorem 4. Let $q = q_1 - q_2$. Since q is radial, we have $q(x) = Q(|x|)$ and

$$\widehat{q}(\xi) = \mathcal{Q}(|\xi|)$$

with for all $\rho > 0$,

$$\mathcal{Q}(\rho) = \int_0^{2\pi}\int_0^\infty e^{-ir\rho\cos\theta} Q(r)\,r\mathrm{d}r\mathrm{d}\theta.$$

In particular recalling that Q is nonnegative, we find that for all $\xi \in \mathbb{R}^2$,

$$
\begin{aligned}
|\widehat{q}(\xi)| &\leq 2\pi \int_0^\infty Q(r)\, r\mathrm{d}r \\
&\leq 2\pi \int_0^\infty e^{\zeta_0(1-r)} e^{-\zeta_0(1-r)} Q(r)\, r\mathrm{d}r
\end{aligned}
$$

for any $\zeta_0 \in \mathbb{R}$. Then we can apply Lemma 3.3 to the particular case of a radial function, so choosing $k = 0$ and $\zeta_0 = 3K\lambda_0$ we infer that

$$
\begin{aligned}
(3.3)\, |\widehat{q}(\xi)| &\leq Ce^{\zeta_0\kappa/\lambda} \mathcal{L}\big(T(rQ)\big)(\zeta_0) \\
&\leq Ce^{3K\kappa} \left(\lambda^2 \left\| \mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda) \right\| + \frac{C_{q_1,q_2}}{\lambda^2} \|q_1 - q_2\|_{L^\infty} \right) .
\end{aligned}
$$

The end of the proof of Theorem 4 follows easily by taking the L^2 norm in ξ. □

3.3. **Proof of Lemma 3.3.** The method of proof follows the ideas developed in the proofs of Theorems 2 and 3, adapting the estimates to our special situation where the potentials are located near the boundary. The heart of the matter consists to approximate the solutions to Helmholtz equation (3.1) by separable solutions in radial coordinates involving Bessel functions, which allows in light of Debye's formula to relate the Laplace transform of the potential to its near field operator.

More precisely, we shall look for solutions to (3.1) under the following form, for $n \in \mathbb{Z}$:

$$
\begin{aligned}
u_n(r,\theta) &= \frac{J_{|n|}(\lambda r)}{\lambda(J'_{|n|}(\lambda) - iJ_{|n|}(\lambda))} \,\mathrm{e}^{in\theta} + v_n \\
&= z_n(r,\lambda)\,\mathrm{e}^{in\theta} + v_n,
\end{aligned}
$$

where J_n is a Bessel function of the first kind (see Appendix A), solution to

$$
\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{n^2}{r^2} + 1 \right) J_n(r) = 0.
$$

Suppose that for $\ell \in \{1,2\}$, u_n^ℓ solves (3.1) with potential q_ℓ. Then writing

$$
u_n^\ell(r,\theta) = z_n(r,\lambda)\,\mathrm{e}^{in\theta} + v_n^\ell(r,\theta),
$$

we find that

$$
\begin{cases}
(D^2 - \lambda^2 + q_\ell) v_n^\ell = -q_\ell\, z_n\, \mathrm{e}^{in\theta} \quad \text{in} \quad B, \\
(\partial_r - i\lambda) v^\ell_{n|\partial B} = 0.
\end{cases}
$$

Due to Property (A.6) we have if $1 - \lambda^{-1} \leq r \leq 1$ and $|n| \geq K\lambda$ for K large, that $\big|\lambda\, z_n(r,\lambda)\big|$ is bounded, for $\lambda \geq \lambda_0$, by a constant depending only on K and λ_0.

For now on we shall denote by $C(\lambda_0, K)$ such a constant, which may change from line to line.

Therefore, arguing as in the proof of Lemma 1.5 and taking advantage of the fact that on the support of q_ℓ, r varies in an interval of size λ^{-1}, we find as soon as λ is large enough compared to $\|q_\ell\|_{L^\infty}$

$$\|v_n^\ell\|_{L^2} \leq \frac{C(\lambda_0, K)}{\lambda}\,\|q_\ell(r,\theta)\, z_n(r,\lambda)\|_{L^2} \leq \frac{C(\lambda_0, K)}{\lambda^{\frac{5}{2}}}\|q_\ell\|_{L^\infty}. \tag{3.4}$$

Now going back to the computations of Section 2 we consider two positive integers n and m such that $n, m \geq K\lambda$, and we write

$$\int_B (q_1 - q_2) u_n^1 u_{-m}^2 \,\mathrm{d}x = \int_{\partial B} (\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda) \mathrm{e}^{in\theta}\mathrm{e}^{-im\theta}\,\mathrm{d}\theta.$$

Therefore decomposing

$$\begin{aligned} u_n^1(r,\theta) &= z_n(r,\lambda)\,\mathrm{e}^{in\theta} + v_n^1(r,\theta) \quad \text{ and} \\ u_{-m}^2(r,\theta) &= z_m(r,\lambda)\,\mathrm{e}^{-im\theta} + v_{-m}^2(r,\theta), \end{aligned}$$

we get

$$\begin{aligned} &\int_0^{2\pi} \big(\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\big)\mathrm{e}^{in\theta}\mathrm{e}^{-im\theta}\mathrm{d}\theta \\ &\quad = \int_B (q_1 - q_2)\mathrm{e}^{i(n-m)\theta} z_n(r,\lambda)\, z_m(r,\lambda)\, r\mathrm{d}r\,\mathrm{d}\theta \\ &\qquad + \int_B (q_1 - q_2)\mathrm{e}^{in\theta} z_n(r,\lambda)\, v_{-m}^2\, r\mathrm{d}r\,\mathrm{d}\theta \\ &\qquad + \int_B (q_1 - q_2)\mathrm{e}^{-im\theta} z_m(r,\lambda)\, v_n^1\, r\mathrm{d}r\,\mathrm{d}\theta \\ &\qquad + \int_B (q_1 - q_2) v_n^1 v_{-m}^2\, r\mathrm{d}r\,\mathrm{d}\theta. \end{aligned} \tag{3.5}$$

As $\big|\lambda\, z_n(r,\lambda)\big|$ is bounded by a constant $C(\lambda_0, K)$, we have (recalling that q_1 and q_2 are compactly supported in an interval in r of size λ^{-1})

$$\left|\int_B (q_1 - q_2)\mathrm{e}^{in\theta}\, z_n(r,\lambda)\, v_{-m}^2\, r\,\mathrm{d}r\,\mathrm{d}\theta\right| \leq \frac{C(\lambda_0, K)}{\lambda^{\frac{3}{2}}}\|q_1 - q_2\|_{L^\infty}\|v_{-m}^2\|_{L^2}$$

and similarly

$$\left|\int_B (q_1 - q_2)\mathrm{e}^{-im\theta}\, z_m(r,\lambda)\, v_n^1\, r\,\mathrm{d}r\,\mathrm{d}\theta\right| \leq \frac{C(\lambda_0, K)}{\lambda^{\frac{3}{2}}}\|q_1 - q_2\|_{L^\infty}\|v_n^1\|_{L^2}.$$

Finally

$$\left|\int_B (q_1 - q_2) v_n^1 v_{-m}^2\, r\mathrm{d}r\,\mathrm{d}\theta\right| \leq \|q_1 - q_2\|_{L^\infty}\|v_n^1\|_{L^2}\|v_{-m}^2\|_{L^2}.$$

Thus by (3.4), we get from (3.5), for $n, m \geq K\lambda > K\lambda_0$,

$$(3.6)\quad \left|\int_B (q_1 - q_2) \mathrm{e}^{i(n-m)\theta}\, z_n(r,\lambda)\, z_m(r,\lambda)\, r\mathrm{d}r\, \mathrm{d}\theta\right| \\ \leq \left|\int_0^{2\pi} \big(\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\big)\mathrm{e}^{in\theta}\, \mathrm{e}^{-im\theta}\, \mathrm{d}\theta\right| + \frac{C_{q_1,q_2}}{\lambda^4}\|q\|_{L^\infty},$$

where $q = q_1 - q_2$ and

$$C_{q_1,q_2} = C(\lambda_0, K) \max\big(\|q_1\|_{L^\infty}, \|q_2\|_{L^\infty}, \|q_1\|_{L^\infty}\|q_2\|_{L^\infty}\big).$$

This gives rise to

$$(3.7)\quad \left|\int_B (q_1 - q_2) \mathrm{e}^{i(n-m)\theta}\, z_n(r,\lambda)\, z_m(r,\lambda)\, r\mathrm{d}r\, \mathrm{d}\theta\right| \\ \leq \|\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| + \frac{C_{q_1,q_2}}{\lambda^4}\|q\|_{L^\infty},$$

which leads in light of (A.6) to

$$\left|\int_B (q_1 - q_2) \mathrm{e}^{i(n-m)\theta} \mathrm{e}^{-(n+m)(1-r)}\, r\mathrm{d}r\, \mathrm{d}\theta\right| \\ \leq C(\lambda_0, K)\Big(\lambda^2\, \|\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| + \frac{C_{q_1,q_2}}{\lambda^2}\|q\|_{L^\infty}\Big),$$

for all $n, m \geq K\lambda$. In conclusion we have for any $k \in \mathbb{Z}$, any $\ell \in \mathbb{N}$ and any $j \geq 2K\lambda > 2K\lambda_0$,

$$(3.8)\quad \left|\int_B (q_1 - q_2) \mathrm{e}^{-i\,k\theta} \mathrm{e}^{-j(1-r)}\, r\mathrm{d}r\, \mathrm{d}\theta\right| \\ \leq C(\lambda_0, K)\Big(\lambda^2\, \|\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| + \frac{C_{q_1,q_2}}{\lambda^2}\|q\|_{L^\infty}\Big).$$

The conclusion follows from (3.3). Lemma 3.3 is proved. □

4. Concluding remarks

In this paper we have shown that the low-frequency component of the potential can be determined in a stable way from the scattering measurements and justified the resolution limit. We have also proved that in the near-field we have in the monotone case infinite resolution in reconstructing the potential near the boundary. We think that the result holds in the general case. However, its proof seems to be out of reach. In fact, even though a sampling (or interpolation) formula for the Laplace transform does exist [20, 54], making norm-estimates similar to those in Theorem 4 is very challenging. Our results can be extended in many directions. It would be very interesting to study the limited-view case and show, as in [11], that we recover infinite resolution from near-field measurements on the overlap of the source and receiver apertures. Another challenging problem is to understand how probe interaction can improve local resolution by converting evanescent modes of the potential to propagating ones [23]. These problems will be the subject of forthcoming works.

Appendix A. Bessel functions

Bessel's equation arises when finding separable solutions to the Helmholtz equation in spherical coordinates, and writes as follows:

$$\left(\partial_z^2 + \frac{1}{z}\partial_z + \left(1 - \frac{n^2}{z^2}\right)\right) u = 0, \quad n \in \mathbb{Z}. \tag{A.1}$$

It is well known (see for instance [39, 49] and the references therein) that one of the solutions of Bessel's equation is the entire function $J_n(z)$ known as the Bessel function of the first kind of order n, and defined for arbitrary $z \in \mathbb{C}$ by the convergent series

$$J_n(z) = \sum_{k=0}^{k=\infty} \frac{(-1)^k}{k!\,(n+k)!}\,(z/2)^{n+2k},$$

in the case where $n \in \mathbb{N}$ and by $J_{-n}(z) = (-1)^n\, J_n(z)$.

To find a general solution of Bessel's equation (A.1), we need a second solution of (A.1) which is linearly independent of $J_n(z)$. For such a solution, we usually choose $Y_n(z)$ the Bessel function of the second kind which is entire in the complex plane cut along the segment $]-\infty, 0]$ and defined for $n \in \mathbb{N}$ by

$$\begin{aligned} Y_n(z) &= \frac{2}{\pi} J_n(z) \log\frac{z}{2} - \frac{1}{\pi}\sum_{k=0}^{k=n-1} \frac{(n-k-1)!}{k!}\,(\frac{z}{2})^{2k-n} \\ &- \frac{1}{\pi}\sum_{k=0}^{k=\infty} \frac{(-1)^k\,(z/2)^{n+2k}}{k!\,(n+k)!}[\psi(k+1)+\psi(k+n+1)], \end{aligned}$$

where $\psi(m+1) = -\gamma + 1 + \frac{1}{2} + ... + \frac{1}{m}$, γ being the Euler constant. We also define $Y_{-n}(z) = (-1)^n\, Y_n(z)$.

Since J_n and Y_n are linearly independent, the general expression for solutions of (A.1) is a linear combination of Bessel functions of the first and second kinds, i.e,

$$u(z) = A\, J_n(z) + B\, Y_n(z),$$

where A and B are constants.

Another basis of solutions to the differential equation (A.1) is given by the Bessel functions of the third kind or Hankel functions, denoted by $H_n^{(1)}$ and $H_n^{(2)}$. These functions are defined by the formulas

$$H_n^{(1)}(z) = J_n(z) + iY_n(z) \quad \text{and} \quad H_n^{(2)}(z) = J_n(z) - iY_n(z), \tag{A.2}$$

where z is any point of the complex plane cut along the segment $]-\infty, 0]$. The motivation for introducing the Hankel functions is that the linear combination of $J_n(z)$ and $Y_n(z)$ have very simple asymptotic expansions for large $|z|$: it is thus well-known that

$$\text{(A.3)} \qquad H_n^{(1)}(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \mathrm{e}^{i\,(z-\frac{n\pi}{2}-\frac{\pi}{4})}\Big(1+\mathcal{O}\Big(\frac{1}{|z|}\Big)\Big) \quad \text{and}$$

$$\text{(A.4)} \qquad H_n^{(2)}(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \mathrm{e}^{-i\,(z-\frac{n\pi}{2}-\frac{\pi}{4})}\Big(1+\mathcal{O}\Big(\frac{1}{|z|}\Big)\Big)$$

as $|z| \to \infty$.

Furthermore, we have the following Debye formulas whose proof can be found for instance in [49, Chapter 9.4] and [1, Chapter 9]:

$$J_n(n\ \mathrm{sech}\,\alpha) \sim \frac{\mathrm{e}^{-n(\alpha-\tanh\alpha)}}{(2\pi n\tanh\alpha)^{\frac{1}{2}}}\Big(1+(\frac{1}{8}\coth\alpha-\frac{5}{24}(\coth\alpha)^3)\frac{1}{n}+...\Big)$$

and particulary

$$\text{(A.5)} \qquad \begin{aligned} J_n(n\ \ \mathrm{sech}\,\alpha) &= \frac{\mathrm{e}^{-n(\alpha-\tanh\alpha)}}{(2\pi n\tanh\alpha)^{\frac{1}{2}}}\Big(1+\mathcal{O}\Big(\frac{1}{n}\Big)\Big), \\ J_n'(n\ \ \mathrm{sech}\,\alpha) &= \frac{(\tanh\alpha)^{\frac{1}{2}}\mathrm{e}^{-n(\alpha-\tanh\alpha)}}{(4\pi n)^{\frac{1}{2}}}\Big(1+\mathcal{O}\Big(\frac{1}{n}\Big)\Big) \end{aligned}$$

as $n \to \infty$, where $\mathrm{sech}\,z$ denotes the hyperbolic secant of z defined by

$$\mathrm{sech}\,z = \frac{1}{\cosh z}$$

with $\cosh z$ the hyperbolic cosine.

Debye's formula gives rise to the following asymptotic behavior for the function introduced in Section 3:

$$z_n(\lambda, r) = \frac{J_{|n|}(\lambda r)}{\lambda(J_{|n|}'(\lambda) - iJ_{|n|}(\lambda))}$$

defined for $n \in \mathbb{Z}$. Let us prove that for $|n| \gg \lambda$ and $r \sim 1$,

$$\text{(A.6)} \qquad \begin{aligned} \mathrm{Re}\, z_n(\lambda, r) &\sim \frac{C}{\lambda}\mathrm{e}^{-n(1-r)}\Big(1+\mathcal{O}\Big(\frac{1}{n}\Big)\Big), \\ \text{and} \quad \mathrm{Im}\, z_n(\lambda, r) &\sim \frac{C}{\lambda}\mathrm{e}^{-n(1-r)}\Big(1+\mathcal{O}\Big(\frac{1}{n}\Big)\Big). \end{aligned}$$

Without loss of generality we may assume that $n \in \mathbb{N}$, then defining

$$\cosh\alpha_1 = \frac{n}{\lambda r} \text{ and } \cosh\alpha_2 = \frac{n}{\lambda},$$

it is easy to see that under the above assumptions ($\lambda \ll n$ and $r \sim 1$), we have necessarily $\cosh \alpha_i \gg 1$ for $i \in \{1, 2\}$, hence $\alpha_i \gg 1$. This implies $\cosh \alpha_i \sim \mathrm{e}^{\alpha_i}$, $\sinh \alpha_i \sim \mathrm{e}^{\alpha_i}$, and $\tanh \alpha_i \sim 1$, which gives rise to

$$\mathrm{Re}\, z_n(\lambda, r) = \frac{J_n(n \,\mathrm{sech}\, \alpha_1) J_n'(n \,\mathrm{sech}\, \alpha_2)}{\lambda(J_n'^2(n \,\mathrm{sech}\, \alpha_2) + J_n^2(n \,\mathrm{sech}\, \alpha_2))}$$

and

$$\mathrm{Im}\, z_n(\lambda, r) = \frac{J_n(n \,\mathrm{sech}\, \alpha_1) J_n(n \,\mathrm{sech}\, \alpha_2)}{\lambda(J_n'^2(n \,\mathrm{sech}\, \alpha_2) + J_n^2(n \,\mathrm{sech}\, \alpha_2))} \cdot$$

Finally, taking advantage of (A.5) we get

$$\mathrm{Re}\, z_n(\lambda, r) \sim \frac{\mathrm{e}^{-n(\alpha_1 - \alpha_2)}}{\lambda} \sim \frac{\mathrm{e}^{-n \log(\frac{1}{r})}}{\lambda} \cdot$$

The computation is identical for $\mathrm{Im}\, z_n(\lambda, r)$, so using the fact that r is near to 1, we obtain the desired conclusion.

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Department of Mathematics and Applications, École Normale Supérieure, 45 Rue d'Ulm, F-75005 Paris, France
E-mail address: habib.ammari@ens.fr

Université Paris Est-Créteil, Umr 8050 Lama, 61 avenue du Général de Gaulle, F-94010 Créteil Cedex, France
E-mail address: hbahouri@math.cnrs.fr

Université Paris 13, Cnrs, Umr 7539 Laga, 99 avenue Jean-Baptiste Clément, F-93430 Villetaneuse, France
E-mail address: ddsf@math.univ-paris13.fr

Institut de Mathématiques Umr 7586, Université Paris VII, 175 rue du Chevaleret, F-75013 Paris, France
E-mail address: Isabelle.Gallagher@math.jussieu.fr