Minimization of a Ginzburg-Landau type energy with a particular potential

Rejeb Hadiji, Itai Shafrir

Minimization of a Ginzburg-Landau type energy with a particular potential

Rejeb Hadiji[(1)]-Itai Shafrir[(2)]

[(1)] Université Paris-Est,
Laboratoire d'Analyse et de Mathématiques Appliquées,
CNRS UMR 8050, UFR des Sciences et Technologie,
61, Avenue du Général de Gaulle Bât. P3, 4e étage,
94010 Créteil Cedex , France.
E-mail: hadiji@univ-paris12.fr

and

[(2)] Department of Mathematics,
Technion, Israel Institute of Technology,
32000 Haifa, Israel.
E-mail: shafrir@math.technion.ac.il

Abstract. The energy of the Ginzburg-Landau is given by

$$E_\varepsilon(u) = \int_G |\nabla u|^2 dx + \frac{1}{\varepsilon^2}\int_G J(1-|u|^2)\, dx$$

We study the case where the potential J has a zero of infinite order. A significant example is $J(t) = \exp(-1/t^k)$ for $t > 0$ and $J(t) = 0$ for $t \leq 0$. We show that the energy cost of a degree-one vortex may be much less than the cost of $2\pi \log(\frac{1}{\varepsilon})$ for the classical Ginzburg-Landau functional. In fact, we shall show that this cost is

$$2\pi(\log\frac{1}{\varepsilon} - I(\frac{1}{\varepsilon}))$$

where $I(R)$ is a positive function satisfying $I(R) = o(\log R)$ as $R \to \infty$.

AMS Subject Classification (2000) 35B40; 82D55

1 Introduction

Let G be a bounded and smooth, simply connected domain in $\mathbb{R}^2$ and let $g : \partial G \to S^1$ be a boundary condition of degree $\deg(g, \partial G) = d \geq 0$ (as we may assume without loss of generality). Consider a C^2 functional $J : \mathbb{R} \to [0, \infty)$ satisfying the following conditions:

(H_1) $J(0) = 0$ and $J(t) > 0$ on $(0, \infty)$,
(H_2) $J'(t) > 0$ on $(0, 1]$,
(H_3) There exists $\eta_0 > 0$ such that $J''(t) > 0$ on $(0, \eta_0)$.
For $\varepsilon > 0$ consider the energy functional

$$E_\varepsilon(u) = \int_G |\nabla u|^2 dx + \frac{1}{\varepsilon^2} \int_G J(1 - |u|^2)\, dx \tag{1.1}$$

over

$$H_g^1(G, \mathbb{C}) := \{u \in H^1(G, \mathbb{C}) \text{ s.t. } u = g \text{ on } \partial G\}\,. \tag{1.2}$$

It is easy to see that $\min_{u \in H_g^1(G,\mathbb{C})} E_\varepsilon(u)$ is achieved by some smooth u_ε which satisfies:

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} j(1 - |u_\varepsilon|^2) u_\varepsilon \ \text{ in } G, \\ \quad u_\varepsilon = g \ \text{ on } \partial G, \end{cases} \tag{1.3}$$

where $j(t) := J'(t)$. The case $J(u) = (1 - |u|^2)^2$, corresponding to the Ginzburg-Landau (GL) energy, was studied by Bethuel, Brezis and Hélein [1, 2] (see also Struwe [6]), where it was shown that:

(i) For a subsequence $\varepsilon_n \to 0$ we have, $u_{\varepsilon_n} \to u_* = e^{i\phi} \prod_{j=1}^{d} \frac{z - a_j}{|z - a_j|}$ in $C^{1,\alpha}(\bar{G} \setminus \{a_1, ..., a_d\})$, where $a_1, ..., a_d$ are distinct points in G and ϕ is a smooth harmonic function determined by the requirement $u_* = g$ on ∂G.
(ii) $E_\varepsilon(u_\varepsilon) = 2\pi d |\log \varepsilon| + O(1)$ as $\varepsilon \to 0$.

The method of [1, 2, 6] can be adapted without difficulty to the case of J satisfying $(H_1) - (H_3)$ with a zero of *finite order* at $t = 0$. This applies for example to $J(t) = |t|^k$, $\forall k \geq 2$. The main objective of the current paper is to treat the case of J with zero of *infinite order* at $t = 0$, having in mind the examples

$$J_k(t) = \begin{cases} \exp(-1/t^k) & \text{for } t > 0\,, \\ 0 & \text{for } t \leq 0\,, \end{cases} \tag{1.4}$$

for any $k>0$. It turns out that a convergence result, as in (i) above, holds for such J's as well. The main difference with respect to the usual GL-energy is in the energy asymptotics. For J with a zero of infinite order the "energy cost" of a degree-one vortex may be much less than the cost of $2\pi\log\frac{1}{\varepsilon}$ for the GL-functional (see (ii) above). In fact, we shall see that this cost equals

$$2\pi(\log\frac{1}{\varepsilon}-\bar{I}(\frac{1}{\varepsilon}))\,,$$

where $\bar{I}(R)$ is a positive function satisfying $\bar{I}(R)=o(\log R)$ as $R\to\infty$,if $j(0)=0$ and $I(R)=O(\log R)$ as $R\to\infty$ which is determined by the particular functional J. More precisely, the function $\bar{I}(R)$ satisfies

$$\bar{I}(R)=\frac{1}{2}\int_{1/R^2}^{j(\eta_0)} j^{-1}(t)\frac{dt}{t}+O(1)\,,\quad \text{as } R\to\infty\quad \text{(see Lemma 2.2)}\,. \tag{1.5}$$

So for example, for J_1 in (1.4) we find $\bar{I}(R)=\frac{1}{2}\log\log R+O(1)$ (see Proposition 4.1 in the Appendix), and the asymptotics for the energies in this case reads:

$$E_\varepsilon(u_\varepsilon)=2\pi d\big(\log\frac{1}{\varepsilon}-\frac{1}{2}\log\log\frac{1}{\varepsilon}\big)+O(1).$$

Somewhat surprisingly, it turns out that we may have $\bar{I}(R)=O(1)$ also for J with a zero of infinite order, as is the case for $k\in(0,1)$ in (1.4), see Proposition 4.1.

Our first main theorem describes the asymptotic behavior of the minimizers and their energies.

Theorem 1. *For each $\varepsilon>0$, let u_ε be a minimizer for the energy E_ε over $H^1_g(G,\mathbb{C})$ with G,g (of degree $d\geq 0$) as above and J satisfying $(H_1)-(H_3)$. Then:*
(i) For a subsequence $\varepsilon_n\to 0$ we have

$$u_{\varepsilon_n}\to u_*=e^{i\phi}\prod_{j=1}^{d}\big(\frac{z-a_j}{|z-a_j|}\big)\quad in\ C^{1,\alpha}(\bar{G}\setminus\{a_1,...,a_d\}),$$

where $a_1,...,a_d$ are distinct points in G and ϕ is a smooth harmonic function determined by the requirement $u_*=g$ on ∂G.
(ii) Setting, for $R>\frac{1}{\sqrt{j(\eta_0)}}$,

$$I_0(R)=\frac{1}{2}\int_{1/R^2}^{j(\eta_0)} j^{-1}(t)\frac{dt}{t}\,,$$

we have

$$E_\varepsilon(u_\varepsilon)=2\pi d\Big(\log\frac{1}{\varepsilon}-I_0\big(\frac{1}{\varepsilon}\big)\Big)+O(1). \tag{1.6}$$

We show in Lemma 2.2 below that if $j^{-1}(0) = 0$ then the function I_0 satisfies $I_0(R) = o(\log R)$ otherwise $I_0(R) = O(\log R)$. A significant example in the first case is given in (1.4), while in the second case we can take $J(t) = t\exp(t)$. This implies that the leading term in the energy is always of the order $|\log\varepsilon|$. It is easy to see that $I_0(R)$ is a positive, monotone increasing, concave function of $\log R$ (for large R). It is natural to ask whether every function with these properties can appear in the second order term of the energy expansion, for some potential J. The answer to this "inverse problem" turns out to positive, as shown by our second theorem.

Theorem 2. *Let $h \in C^2[0,\infty)$ satisfy, for some $T > 0$,*

$$h'(t) > 0\,,\ \ h''(t) < 0\,,\quad \text{for } t \geq T > 0\,, \tag{1.7}$$

and

$$\lim_{t\to\infty} h'(t) = 0\,. \tag{1.8}$$

Then, there exists a functional J satisfying $(H_1) - (H_3)$, such that the minimizers $\{u_\varepsilon\}$ over $H^1_g(G,\mathbb{C})$, for E_ε defined by (1.1) and g of degree d as above, satisfy

$$E_\varepsilon(u_\varepsilon) = 2\pi d\Big(\log\frac{1}{\varepsilon} - h\big(\log\frac{1}{\varepsilon}\big)\Big) + O(1)\,.$$

2 A study of an auxiliary optimization problem

Let us begin by explaining the main idea of the proof of Theorem 1 and by showing how it leads to a certain optimization problem which is the object of the current section. It is natural to estimate first the energy cost of a degree-one "vortex" in a disc, say the unit disc $B_1 = B_1(0)$. In the case of the Ginzburg-Landau energy, it is easy to guess the energy cost, by taking $v_\varepsilon(r^{i\theta}) = f_\varepsilon(r)e^{i\theta}$ with f_ε given by:

$$f_\varepsilon(r) = \begin{cases} \frac{r}{\varepsilon} & \text{for } 0 \leq r < \varepsilon, \\ 1 & \text{for } \varepsilon \leq r \leq 1. \end{cases}$$

A simple computation gives

$$\int_{B_1} |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2 = 2\pi\log\frac{1}{\varepsilon} + O(1),$$

which turns out to be the optimal estimate, up to an additive constant, although the proof of this fact is far from trivial (see [2]). When looking for the right upper bound for the energy in the general case, we keep the ansatz $v_\varepsilon(r) = f_\varepsilon(r)e^{i\theta}$, and try to optimize over the function f_ε (since we do not know a priori what form should it take, for our particular J). What we can assume a priori on that function is that it satisfies

$$E_\varepsilon(v_\varepsilon, B_\varepsilon) = O(1), \tag{2.1}$$

and

$$\frac{1}{\varepsilon^2}\int_{B_1} J(1-|v_\varepsilon|^2) = O(1)\,. \tag{2.2}$$

Indeed, for a *minimizer* both (2.1) and (2.2) should hold, thanks to the estimates (**??**) and (2.21) that we shall verify below. Assuming then that f_ε is chosen in such a way that (2.1)–(2.2) are satisfied, we get for the energy of v_ε:

$$\begin{aligned} E_\varepsilon(v_\varepsilon) &= 2\pi\int_0^1 \big((f_\varepsilon')^2 + \frac{f_\varepsilon^2}{r^2} + \frac{1}{\varepsilon^2}J(1-f_\varepsilon^2)\big)\,rdr \\ &= 2\pi\log\frac{1}{\varepsilon} - 2\pi\int_\varepsilon^1 \frac{1-f_\varepsilon^2}{r}\,dr + \int_\varepsilon^1 (f_\varepsilon')^2\,rdr + O(1). \end{aligned} \tag{2.3}$$

In order to get minimal energy (up to an $O(1)$-term), we shall look for f_ε which *maximizes* the term $\int_\varepsilon^1 \frac{1-f_\varepsilon^2}{r}\,dr$ (representing the gain of energy w.r.t. the "usual" cost of $2\pi\log\frac{1}{\varepsilon}$) under the constraint $\int_\varepsilon^1 J(1-f_\varepsilon^2)\,rdr \le C_0$. Here we did not take into account the contribution of the term $\int_\varepsilon^1 (f_\varepsilon')^2\,rdr$, but as we shall see below, this term is bounded for the solution of our optimization problem.

Rescaling by a factor of ε, we are led naturally to define the following quantity:

$$I(R,c) = \sup\Big\{\int_1^R \frac{1-f^2}{r}dr\ :\ \int_1^R J(1-f^2)r\,dr \le c\Big\}, \tag{2.4}$$

for any $R>1$ and $c>0$.

Lemma 2.1. *For every $R>1$ and $c>0$, there exists a maximizer $f_0 = f_0^{(R)}$ in (2.4) satisfying $0 \le f_0(r) \le 1, \forall r$, such that $f_0(r)$ is nondecreasing. Moreover, if $r_0 = r_0(c)$ is defined by the equation*

$$c = J(1)\Big(\frac{r_0^2-1}{2}\Big)\,, \tag{2.5}$$

then there exists $\tilde r_0 = \tilde r_0(c,R) \in [1,r_0]$ such that

$$f_0(r)\begin{cases} = 0 & \text{if } r\in[1,R] \text{ and } r<\tilde r_0\,,\\ > 0 & \text{if } r>\tilde r_0\,. \end{cases} \tag{2.6}$$

Furthermore,

$$\int_1^R J(1-f_0^2) r\, dr = c\,, \quad \textit{for } R > r_0\,, \tag{2.7}$$

$$j(1-f_0^2(r)) = \frac{1}{\lambda r^2}\,, \qquad r > \tilde{r}_0\,, \tag{2.8}$$

for some $\lambda = \lambda(R,c) > 0$. *and*

There exist two constants $0 < a(c) < b(c)$ *such that*

$$a(c) \leq \lambda \leq b(c), \;\; R \geq r_0 + 1.$$

The proof of this Lemma is contained in [5], so we omit it.

Remark 2.1. *The proof of the last Lemma actually shows that the bounds for* λ *are uniform for* c *lying in a bounded interval.*

Using [5], Lemma 2.3., we have for every $c > 0$ there exists a constant $C = C(c)$ such that for every $0 < c_1, c_2 \leq c$ we have

$$|I(R,c_1) - I(R,c_2)| \leq C, \;\; \forall R \geq 1\,.$$

It is then natural to set:

$$I(R) := I(R,1)\,. \tag{2.9}$$

For any fixed $c_0 > 0$ we have then:

$$|I(R,c) - I(R)| \leq C(c_0)\,, \quad \forall c \leq c_0, \forall R \geq 1\,. \tag{2.10}$$

Next we prove, by the method of proof of Lemma 2;3. an explicit estimate for $I(R)$. In the sequel we shall denote by f_0 be a maximizer for $I(R) = I(R,1)$ as given by Lemma 2.1.

Lemma 2.2. *We have*

$$I(R) = \frac{1}{2}\int_{\frac{1}{R^2}}^{j(\eta_0)} j^{-1}(t)\,\frac{dt}{t} + O(1), \;\; \forall R > \frac{1}{\sqrt{j(\eta_0)}}\,. \tag{2.11}$$

In particular,

$$\lim_{R\to\infty} \frac{I(R) - j^{-1}(0)}{\log R} = 0\,. \tag{2.12}$$

Proof. By Lemma 2.1 we have $j(1-f_0^2(r)) = \frac{1}{\lambda r^2}$ for $r > r_0(1)$ and by Lemma 2.2. we have

$$\lambda = \lambda(R) \in [a,b], \quad \text{for } R \geq r_0(1)+1\,, \tag{2.13}$$

for some two positive constants a and b. Using hypothesis (H_3) we conclude that

$$1 - f_0^2(r) = j^{-1}\Big(\frac{1}{\lambda r^2}\Big)\,, \quad \text{for } R \geq r \geq \mu_0 := \max\Big(r_0(1), \frac{1}{\sqrt{a j(\eta_0)}}\Big)\,. \tag{2.14}$$

It follows that

$$I(R) = \int_{\mu_0}^{R} j^{-1}\Big(\frac{1}{\lambda r^2}\Big)\frac{dr}{r} + O(1) = \frac{1}{2}\int_{\frac{1}{\lambda R^2}}^{j(\eta_0)} j^{-1}(t)\frac{dt}{t} + O(1)\,.$$

In order to get (2.11) it suffices to notice that

$$\begin{aligned}\left|\int_{\frac{1}{\lambda R^2}}^{j(\eta_0)} j^{-1}(t)\frac{dt}{t} - \int_{\frac{1}{R^2}}^{j(\eta_0)} j^{-1}(t)\frac{dt}{t}\right| &\leq \left|\int_{\frac{1}{R^2}}^{\frac{1}{\lambda R^2}} j^{-1}(t)\frac{dt}{t}\right| \\ &\leq C|\log\lambda| \leq C\max(|\log b|, |\log a|) = O(1)\,.\end{aligned}$$

Finally we note that (2.12) follows easily from (2.11). □

As announced in the introduction, the next lemma provides an estimate that we shall use in the proof of the upper-bound for the energy.

Lemma 2.3. *We have*

$$\int_{\mu_0}^{R} (f_0')^2\, r dr \leq C, \quad \forall R > \mu_0\,, \tag{2.15}$$

for a as in (2.13) and μ_0 as defined in (2.14).

Proof. Differentiating the equality (2.14) yields for $r \geq \mu_0$,

$$-2f_0 f_0' = (j^{-1})'\Big(\frac{1}{\lambda r^2}\Big)\cdot\Big(-\frac{2}{\lambda r^3}\Big)\,,$$

which implies

$$f_0'(r) \leq C(j^{-1})'\Big(\frac{1}{br^2}\Big)\cdot\frac{1}{r^3}\,,$$

with b given by (2.13). Therefore, denoting by C different positive constants, we get

$$\begin{aligned}\int_{\mu_0}^{R} (f_0')^2\, r dr &\leq C\int_{\mu_0}^{R}\Big[(j^{-1})'\Big(\frac{1}{br^2}\Big)\Big]^2\frac{dr}{r^5} = C\int_{\frac{1}{bR^2}}^{\frac{1}{b\mu_0^2}} \big[(j^{-1})'(\alpha)\big]^2\alpha\, d\alpha \\ &= C\int_{\frac{1}{bR^2}}^{\frac{1}{b\mu_0^2}} \frac{\alpha d\alpha}{\big(j'(j^{-1}(\alpha))\big)^2} = C\int_{j^{-1}(\frac{1}{bR^2})}^{j^{-1}(\frac{1}{b\mu_0^2})} \frac{j(\beta)}{j'(\beta)}\, d\beta\,.\end{aligned} \tag{2.16}$$

It is elementary to verify that

$$\lim_{\beta\to 0^+} \frac{j(\beta)}{j'(\beta)} = 0\,. \tag{2.17}$$

Indeed, if $J''(0) = j'(0) > 0$ then

$$\lim_{\beta\to 0^+} \frac{j(\beta)}{j'(\beta)} = \lim_{\beta\to 0^+} \frac{J'(\beta)}{J''(\beta)} = 0\,,$$

since $J'(0) = 0$ by (H_1), while if $J''(0) = 0$ then by L'hôpital rule

$$\lim_{\beta\to 0^+} \frac{j(\beta)}{j'(\beta)} = \lim_{\beta\to 0^+} \frac{J'(\beta)}{J''(\beta)} = \lim_{\beta\to 0^+} \frac{J(\beta)}{J'(\beta)} = 0\,,$$

since by convexity $J(\beta) = \int_0^\beta J'(s)\,ds \le \beta J'(\beta)$ for $\beta \le \eta_0$. Therefore, (2.15) follows from (2.16) and (2.17). □

We next study a similar functional to that of (2.4). It will serve in the proof of the lower-bound of the energy. For any $R > 1$ and $c > 0$ set

$$\widetilde{I}(R,c) = \sup\left\{\int_1^R \left(\frac{1-f^2}{r} + 4\frac{(1-f^2)^2}{r}\right) dr \;:\; \int_1^R J(1-f^2) r\,dr = c\right\}. \tag{2.18}$$

By using the above arguments we also obtain the following result.

Lemma 2.4. *For every $c_0, \alpha > 0$ there exists a constant $C_1(c_0,\alpha)$ such that*

$$\begin{cases} |I(\alpha R, c) - I(R)| \le C_1(c,\alpha) \\ |\widetilde{I}(\alpha R, c) - I(R)| \le C_1(c,\alpha) \end{cases} \quad \textit{for } R > \max(1, \frac{1}{\alpha}) \textit{ and } c \in (0, c_0]\,. \tag{2.19}$$

2.1 Some basic estimates for u_ε

The next lemma provides L^∞-estimates for u_ε and its gradient.

Lemma 2.5. *Any solution u_ε of satisfies:*

$$\|u_\varepsilon\|_{L^\infty(G)} \le 1 \quad \textit{and} \quad \|\nabla u_\varepsilon\|_{L^\infty(G)} \le \frac{C}{\varepsilon}. \tag{2.20}$$

Proof. The first estimate follows easily form the observation that replacing $u_\varepsilon(x)$ by $u_\varepsilon(x)/|u_\varepsilon(x)|$ on the set $\{x \in G : |u_\varepsilon(x)| > 1\}$ strictly decreases the energy if the latter set has a positive measure. The second estimate in follows from a simple rescaling argument and standard elliptic estimates as in [1, 6]. □

In the case of a starshaped G the following Pohozaev identity holds for u_ε (actually it is valid for any solution of problem. The proof is identical to the one for the GL-energy in [2], so we omit it.

Lemma 2.6. *If G is starshaped then*

$$\frac{1}{\varepsilon^2}\int_G J(1-|u_\varepsilon|^2) \le C_0,\ \forall \varepsilon > 0\,. \tag{2.21}$$

We shall show later that the assumption of starshapeness of the domain can be dropped, by applying an argument of del Pino and Felmer [4].

3 Proof of the main results

For the proof of Theorem 1 we need a sharp upper bound and a adequate lower bound for the energy. Recall that u_ε is a minimizer for E_ε over $H^1_g(G,\mathbb{C})$. We assume without loss of generality that $d \ge 0$.

Proposition 3.1. *We have*

$$E_\varepsilon(u_\varepsilon) \le 2\pi d\Big(\log\Big(\frac{1}{\varepsilon}\Big) - I\Big(\frac{1}{\varepsilon}\Big)\Big) + O(1),\ \forall \varepsilon > 0\,. \tag{3.1}$$

Proposition 3.2. *Let $x_1,\dots,x_m$ be m points in $B_\sigma(0)$ satisfying*

$$|x_i - x_j| \ge 4\delta,\ \forall i \ne j \quad \textit{and} \quad |x_i| < \frac{\sigma}{4},\ \forall i\,,$$

with $\delta \le \sigma/32$. Set $\Omega = B_\sigma(0) \setminus \bigcup_{j=1}^m B_\delta(x_j)$ and let u be a C^1-map from Ω into $\mathbb{C}$, which is continuous on $\partial\Omega$, satisfying

$$\frac{1}{2} \le |u| \le 1 \ \textit{in}\ \Omega \quad \textit{and} \quad \deg(u, \partial B_\sigma(x_j)) = d_j,\ \ \forall j,$$

and

$$\frac{1}{\delta^2}\int_\Omega J(1-|u|^2) \le K\,.$$

Then, denoting $d = \sum_{j=1}^m d_j$, we have

$$\int_\Omega |\nabla u|^2 \ge 2\pi|d|\Big(\log\frac{\sigma}{\delta} - I\big(\frac{\sigma}{\delta}\big)\Big) - C, \tag{3.2}$$

with $C = C(K, m, \sum_{j=1}^m |d_j|)$.

The proof of Theorem 1 uses an argument of del Pino and Felmer [4] can now be used to show that (2.21) holds without the assumption on the starshapeness of G. Having the estimate (2.21) on our hands see [5], we can now follow the bad-discs construction of [2] and complete the convergence assertion of Theorem 1. Since the arguments are identical to those of [2], we omit the details.

4 Appendix

In this Appendix we compute the energy cost of a degree one vortex for the functionals J_k, $k>0$, that were defined in (1.4) or for the case where $J_{1,k}(t)=exp(-exp(\frac{1}{t^k}))$ for $t>0$ and 0 where $t\le 0$ with $k>0$. In view of Theorem 1 it suffices to compute for each $k>0$:

$$I_{0,k}(R):=\frac{1}{2}\int_{1/R^2}^{j_k(\eta_k)} j_k^{-1}(t)\frac{dt}{t}\,, \tag{4.1}$$

with $j_k=J_k'$, a simple computation shows that $J_k''>0$ on $(0,\eta_k)$.

Proposition 4.1. *As R goes to the infinity, we have*

1. *In the case where J_k is defined by* (1.4)*, we have*

$$I_{0,k}(R)=\begin{cases} O(1), & 0<k<1\,,\\ \frac{1}{2}\log\log R+O(1), & k=1\,,\\ 2^{-\frac{1}{k}}\frac{k}{k-1}\big(\log(R)\big)^{\frac{k-1}{k}}+O(1), & k>1\,. \end{cases} \tag{4.2}$$

2. *For $J_{1,k}$, we have*

$$I_{0,k}(R)=\frac{1}{2k}(\ln\ln(R^2))^{\frac{-(k+1)}{k}}\ln(R^2)+O(1).$$

Proof. The change of variable $s=j_k^{-1}(t)$ gives

$$I_{0,k}(R)=\frac{1}{2}\int_{j_k^{-1}(1/R^2)}^{\eta_k} s\frac{j_k'(s)}{j_k(s)}\,ds=\frac{1}{2}\int_{j_k^{-1}(1/R^2)}^{\eta_k}\Big(\frac{k}{s^k}-(k+1)\Big)\,ds\,. \tag{4.3}$$

If $k<1$ then it follows immediately from (4.3) that $I_{0,k}(R)=O(1)$.

For $k>1$ we obtain from (4.3) that

$$I_{0,k}(R)=\frac{k}{2(k-1)}\Big((j_k)^{-1}(\frac{1}{R^2})\Big)^{1-k}+O(1)\,. \tag{4.4}$$

Set $\alpha = \alpha(R) = j_k^{-1}\big(\frac{1}{R^2}\big)$. Since $j_k(\alpha) = \big(\frac{k}{\alpha^{k+1}}\big)\exp(-1/\alpha^k)$, we have

$$\frac{1}{R^2} = \big(\frac{k}{\alpha^{k+1}}\big)\exp(-1/\alpha^k)\,.$$

Taking the logarithm of both sides gives

$$-2\log R = \log k - (k+1)\log\alpha - \frac{1}{\alpha^k}\,, \quad \text{for } k > 0\,. \tag{4.5}$$

By (4.5) we have $\lim_{R\to\infty} 2\alpha^k \log R = 1$, which we plug in (4.4) to obtain the case $k > 1$ in (4.2).

Finally, if $k = 1$ then by (4.3) we have

$$I_{0,1}(R) = \frac{1}{2}\int_{j_1^{-1}(1/R^2)}^{\eta_1} \big(\frac{1}{s} - 2\big)\,ds = -\frac{1}{2}\log\Big(j_1^{-1}\big(\frac{1}{R^2}\big)\Big) + O(1) = -\frac{1}{2}\log\alpha + O(1)\,, \tag{4.6}$$

with $\alpha = j_1^{-1}\big(\frac{1}{R^2}\big)$, as above. In our case (4.5) gives $\lim_{R\to\infty} 2\alpha\log R = 1$, which implies that $\log\alpha = \log\Big(\frac{1}{2\log R}\Big) + o(1)$. Plugging it in (4.6) gives the result (4.2) for $k = 1$.

For the proof of 2. We have $j_k(t) = \frac{k}{t^{k+1}} e^{\frac{1}{t^k}} J_{1,k}(t)$. Set $\alpha = j_{1,k}^{-1}(\frac{1}{R^2})$, integrate by part (4.1) we obtain

$$2I_{0,k}(R) = \alpha\ln(\frac{1}{R^2}) + \frac{1}{2}\int_\alpha^{j_k^{-1}(\eta_k)} [\ln(k) - (k+1)\ln(t) + \frac{1}{t^k} - e^{\frac{1}{t^k}}]dt.$$

Set

$$I_{0,k}(R) = \frac{-1}{k}\int_\alpha^{j^{-1}(\eta_k)} t^{k+1}(\frac{-k}{t^{k+1}} e^{\frac{1}{t^k}})dt + O(1)$$

Integrate by part two times we obtain,

$$I_{0,k}(R) = \frac{1}{k}\alpha^{k+1} e^{\frac{1}{\alpha^k}} + \frac{k+1}{k^2}\alpha^{2k+1} e^{\frac{1}{\alpha^k}} + \frac{(k+1)(2k+1)}{k}\int_\alpha^{j_k^{-1}(\eta_k)} t^{2k} e^{\frac{1}{t^k}}\,dt + O(1).$$

Hence, since $t \to t^{k+1} e^{\frac{1}{t^k}}$ is decreasing, we have

$$I_{0,k}(R) = \frac{1}{k}\alpha^{k+1} e^{\frac{1}{\alpha^k}}[1 + O(\alpha^k)].$$

On the other hand we have,

$$\ln(k) - (k+1)\ln(\alpha) - \frac{1}{\alpha^k} - e^{\frac{1}{\alpha^k}} = \ln\frac{1}{R^2}$$

then

$$\lim_{R\to\infty} \frac{e^{\frac{1}{\alpha^k}}}{\ln(R)} = 1.$$

Thus we find

$$2I_{0,k}(R) = \frac{1}{k}(\ln\ln(R^2))^{\frac{-(k+1)}{k}} \ln(R^2) + O(1).$$

□

References

[1] F. Bethuel, H. Brezis and F. Hélein, *Asymptotics for the minimization of a Ginzburg-Landau functional*, Calc. Var. Partial Differential Equations **1** (1993), 123–148.

[2] F. Bethuel, H. Brezis and F. Hélein, Ginzburg-Landau Vortices, Birkhäuser, 1994.

[3] H. Brezis, F. Merle and T. Rivière, *Quantization effects for* $-\Delta u = u(1-|u|^2)$ *in* $\mathbb{R}^2$, Arch. Rational Mech. Anal. **126** (1994), 35–58.

[4] M. del Pino and P. L. Felmer, *Local minimizers for the Ginzburg-Landau energy*, Math. Z. **225** (1997), 671–684.

[5] R. Hadiji and I. Shafrir, Minimization of a Ginzburg-Landau type energy decreasing with potential having a zero of infinite order, en *Differential Integral Equations* **19**, no.10, (2006), p. 1157–1176.

[6] M. Struwe, *On the asymptotic behavior of minimizers of the Ginzburg-Landau model in 2 dimensions*, Differential Integral Equations **7** (1994), 1613–1624; erratum, loc. cit. **8** (1995), 124.