Junction of One-Dimensional Minimization Problems involving S^2 Valued Maps

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Abstract

This paper is composed of two parts. In the first part, via a reduction dimension method, we derive a one-dimensional minimization problem involving S^2 valued maps for a thin T-shaped multidomain. In the second one, we analyze this limit model.

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1 Introduction

This paper, composed of two parts, carries on the research we started in [9]. In the first part, via a reduction dimension method, we derive a one-dimensional minimization problem involving S^2 valued maps for a thin T-shaped multidomain. In the second one, we analyze this limit model.

Let $\Omega_n \subset \mathbb{R}^3$, $n \in \mathbb{N}$, be a thin multidomain union of two joined orthogonal cylinders: $r_n\Theta \times [0,1[$ and $]-\frac{1}{2},\frac{1}{2}[\times r_n\left(]-\frac{1}{2},\frac{1}{2}[\times]-1,0[\right)$, where $(0,0) \in \Theta \subseteq]-\frac{1}{2},\frac{1}{2}[\times]-\frac{1}{2},\frac{1}{2}[$ and r_n is a vanishing positive parameter (see Figure 1). We point out that the first cylinder has constant height along the direction x_3, the second one has constant height along the direction x_1, while both of them have a small cross section and are joined by the surface $\{0\} \times r_n\Theta$.

For every $n \in \mathbb{N}$ and $\lambda \in [0,+\infty[$, we consider the following minimization problem:

$$E_{n,\lambda} := \min\left\{ \int_{\Omega_n} |DV(x_1,x_2,x_3)|^2 d(x_1,x_2,x_3) + \right.$$
$$\left. +\lambda \int_{\Omega_n} |V(x_1,x_2,x_3) - G_n(x_1,x_2,x_3)|^2 d(x_1,x_2,x_3) \ : \ V \in H^1(\Omega_n, S^2) \right\}, \tag{1.1}$$

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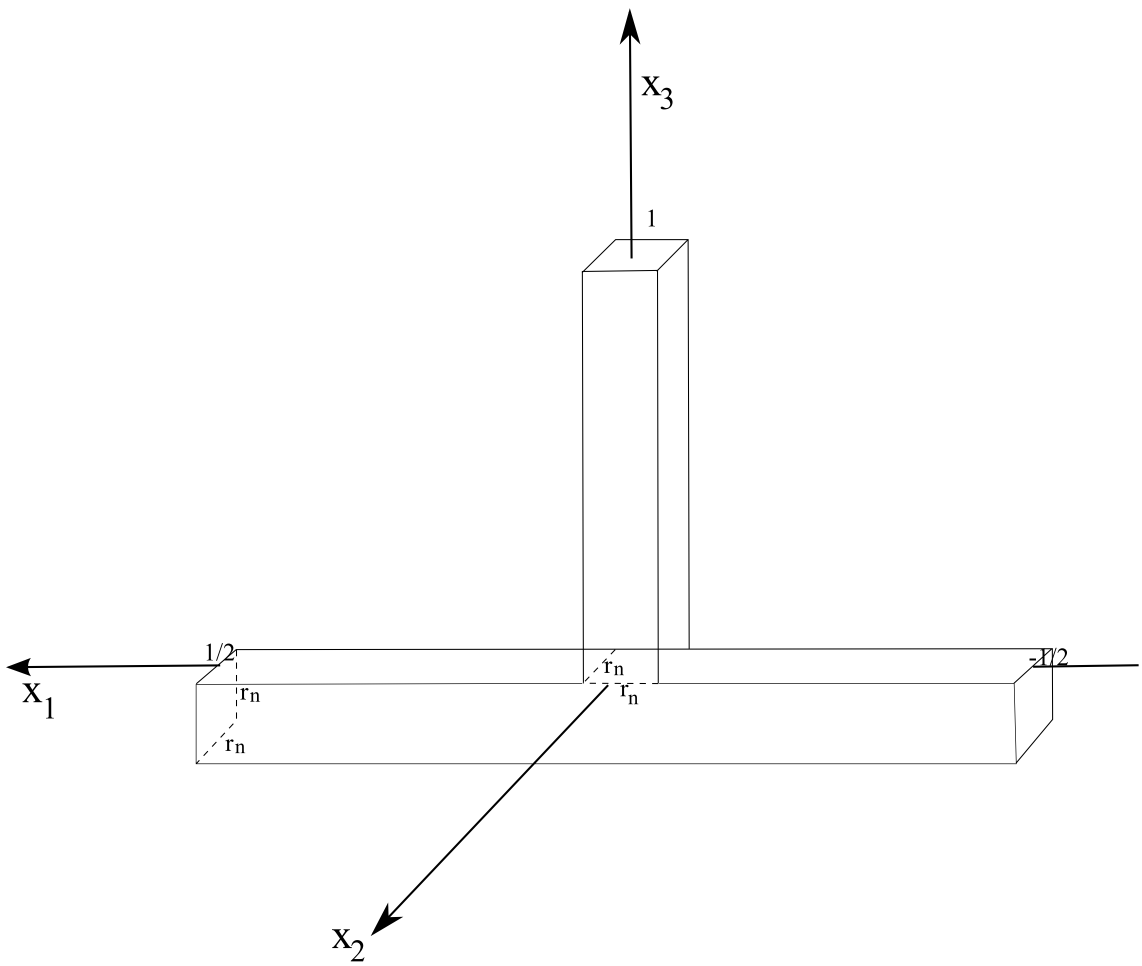

Figure 1:

where $F_n \in L^2(\Omega_n, S^2)$, and S^2 denotes the unit sphere of $\mathbb{R}^3$. Problem (1.1) comes from the classical 3D system for the static isotropic Heisenberg model (see [19]), where V is the spin-density with finite spin magnitude (i.e., $|V| = V_1^2 + V_2^2 + V_3^2 = 1$) and G_n is an external magnetic field. We recall that the Euler system associated to Problem (1.1) is

$$\Delta V + |DV|^2 V + \lambda G_n - < V, \lambda G_n > V = 0. \tag{1.2}$$

System (1.2) is equivalent to the time independent spin equation of motion (see [14]). The time dependent spin equation of motion was first derived by Landau and Lifshitz (see [16]). We refer the reader to [12] and [14] about links between harmonic maps and the Landau-Lifshitz equation of the spin chain.

For n fixed, in [13] it was proved that, for λ large enough and for every function $G_n \in H^1(\Omega_n, S^2)$ which can not be approximated by smooth maps, every minimizer V_λ of (1.1) is not regular, and energy $E_{n,\lambda}$ is bounded. In this case, near each singularity x_0, a minimizer of (1.1) is of the type: $R\frac{x-x_0}{|x-x_0|}$, where R is a rotation. This description was first given in [4] for minimizing harmonic maps. In [6], it was proved that, for λ small enough and for every function $G_n \in L^2(\Omega_n, S^2)$, every minimizer V_λ of (1.1) is regular. Problems of this type were also studied in [2].

The aim of our paper is twofold. Firstly, passing to the limit in (1.1), as n diverges, we derive a one-dimensional static isotropic Heisenberg model for a thin T-shaped domain. Secondly, we study the dependence on λ of the limit model. Precisely, in the first part of this paper, we prove that

$$\lim_n \frac{E_{n,\lambda}}{r_n^2} =$$

$$\begin{aligned} E_\lambda^{Lim} := \min \Bigg\{ & |\Theta| \int_0^1 |w'(x_3)|^2 \, dx_3 - 2\lambda \int_0^1 w(x_3) \left(\int_\Theta f^a(x_1,x_2,x_3) d(x_1,x_2) \right) dx_3 \\ & + \int_{-\frac{1}{2}}^{\frac{1}{2}} |\zeta'(x_1)|^2 \, dx_1 - 2\lambda \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta(x_1) \left(\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[} f^b(x_1,x_2,x_3) d(x_2,x_3) \right) dx_1 + \\ & +2\left(|\Theta| + \lambda\right) \quad : \quad w \in H^1(]0,1[, S^2), \quad \zeta \in H^1\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[, S^2\right), \quad w(0) = \zeta(0) \Bigg\}, \end{aligned} \tag{1.3}$$

where w' and ζ' stand for the derivative of w and ζ, respectively, and (f^a, f^b) is the L^2-weak limit of the rescaled exterior field (see (2.5) and (2.9)). Moreover, we derive strong H^1-convergences for the rescaled minimizers (see Theorem 2.1 and Corollary 2.2).

The proof of this result is developed in several steps. After having rescaled the problem on two fixed domains in the wake of [5], appropriate convergence assumptions on the rescaled exterior fields enable us to obtain *a priori* estimates on rescaled minimizers. The first difficulty arises in deriving $w(0) = \zeta(0)$ for the limit of rescaled minimizers. This limit junction condition lies essentially on the compact embedding of $H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right)$ into $C^0\left(\left[-\frac{1}{2},\frac{1}{2}\right]\right)$, and on the fact that the small cross sections of the two cylinders scale down with same rate r_n. Then, next steps of the proof are based on the main ideas of Γ-convergence method introduced in [7]. Precisely, as in [9] (see also [1] and [3]), working with a particular projection from $\mathbb{R}^3$ into S^2 and using the Sard's Lemma, we construct a recovery sequence for smooth functions with values in S^2. Finally, developing a suitable density result approximating functions of our limit space with more regular functions, and using l.s.c arguments, we achieve the proof. Other scalings are discussed in Remark 2.4.

We recall that in [9] we treated the same minimization problem in a thin multidomain composed of two cylinders attached together that shrink respectively to a one-dimensional segment and to a bidimensional disc, but in this situation the limit problem is uncoupled, i.e., without junction conditions.

If f^a is independent of (x_1, x_2), f^b is independent of (x_2, x_3), $|f^a| = 1$ a.e. in $]0,1[$ and

$|f^b| = 1$ a.e. in $]-\frac{1}{2}, \frac{1}{2}[$, then the limit energy in (1.3) may be rewritten in the following way:

$$E_\lambda^{Lim} := \min \left\{ |\Theta| \int_0^1 \left(|w'(x_3)|^2 + \lambda \, |w(x_3) - f^a(x_3)|^2 \right) dx_3 + \right.$$

$$+ \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(|\zeta'(x_1)|^2 + \lambda \left| \zeta(x_1) - f^b(x_1) \right|^2 \right) dx_1 : \tag{1.4}$$

$$\left. w \in H^1(]0,1[, S^2), \quad \zeta \in H^1\left(\left] -\tfrac{1}{2}, \tfrac{1}{2} \right[, S^2 \right), \quad w(0) = \zeta(0) \right\}.$$

In the second part of this paper, we study the dependence on λ of the limit problem E_λ^{Lim} given in (1.4). We recall that in [9] we have studied the asymptotic behavior both of 2-dimensional and of 1-dimensional problem of the kind (1.3), but without junction conditions.

If $\lambda = 0$, $E_0^{Lim} = 0$. Moreover it is easy to see that the function $\lambda \in [0, +\infty[\to E_\lambda^{Lim}$ is increasing and $\frac{d E_\lambda^{Lim}}{d\lambda} = |\Theta| \int_0^1 |w_\lambda - f^a|^2 \, dx_3 + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \zeta_\lambda - f^b \right|^2 dx_1$, for λ a.e. in $]0, +\infty[$, where $(w_\lambda, \zeta_\lambda)$ is a minimizer of (1.4). Then, it remains to study the asymptotic behavior, as λ diverges, of E_λ^{Lim}. If $f^a \in H^1(]0,1[, S^2)$, $f^b \in H^1\left(\left] -\frac{1}{2}, \frac{1}{2} \right[, S^2 \right)$ and $f^a(0) = f^b(0)$, it is easy to see that $\lim_{\lambda \to +\infty} E_\lambda^{Lim} = |\Theta| \, \|(f^a)'\|^2_{(L^2(]0,1[))^3} + \left\|(f^b)'\right\|^2_{\left(L^2\left(-\frac{1}{2}, \frac{1}{2}[\right)\right)^3}$, and every sequence of minimizers converges to (f^a, f^b) weakly in $H^1(]0,1[, S^2) \times H^1\left(\left] -\frac{1}{2}, \frac{1}{2} \right[, S^2 \right)$. In all remaining cases, the energies diverge, as λ diverges. Then, we examine some particular, but significant situations. For instance, we consider the case where $f^a = (1,0,0)$ and $f^b = (0,1,0)$, or $f^a = \left(\frac{x_3 - \gamma}{|x_3 - \gamma|}, 0, 0 \right)$ and $f^b = \left(\frac{x_1 - \delta}{|x_1 - \delta|}, 0, 0 \right)$, and we prove that energy E_λ^{Lim} is of order of $\sqrt{\lambda}$, for λ large enough. Consequently, in these cases every sequence of minimizers converges to (f^a, f^b) strongly-L^2 (but not weakly-$H^1 \times H^1$), as λ diverges. To prove this result, we find sharp lower and upper estimates. For obtaining the lower bound we introduce a suitable scalar problem. For obtaining the upper bound we use particular test functions which take into account the junction condition $w(0) = \zeta(0)$. In the case $\delta \leq 0$, the building of test functions satisfying the junction condition is more complicated and, to do that, we introduce more sophisticated arguments (see Proposition 3.3) which make use of the same projection from $\mathbb{R}^3$ into S^2 utilized in the recovery sequence.

For the study of rod structures and multi-structures we refer the reader to [15], [17], [18], [20] and the references quoted therein. Results on T-shaped domain may be also found in [8], [10] and [11]. Precisely, a quasilinear Neumann second order scalar problem was considered in [8]. A fourth order problem was examined in [11]. The spectrum of a Laplace operator was considered in [10].

2 First part: derivation of the limit model

In the sequel, $x = (x_1, x_2, x_3)$ denotes the generic point of $\mathbb{R}^3$. If $a, b, c \in \mathbb{R}^3$, then $(a|b|c)$ denotes the 3×3 real matrix having a^T as first column, b^T as second column, and c^T as third column. In according to this notation, if $v \in H^1(A,\mathbb{R}^3)$ with A open subset of $\mathbb{R}^3$, then $Dv := (D_{x_1}v|D_{x_2}v|D_{x_3}v)$, where $D_{x_i}v$, i=1,2,3, stands for the derivative of v with respect to x_i.

Let $\Theta \subseteq]-\frac{1}{2},\frac{1}{2}[\times]-\frac{1}{2},\frac{1}{2}[$ be an open connected set with smooth boundary such that the origin in $\mathbb{R}^2$ belongs to Θ, and let $\{r_n\}_{n\in\mathbb{N}} \subset]0,1[$ be a sequence such that

$$\lim_n r_n = 0. \tag{2.1}$$

For every $n \in \mathbb{N}$, let $\Omega_n^a := r_n\Theta \times [0,1[$, $\Omega_n^b := \left]-\frac{1}{2},\frac{1}{2}\right[\times r_n \left(\left]-\frac{1}{2},\frac{1}{2}\right[\times]-1,0[\right)$ and $\Omega_n := \Omega_n^a \cup \Omega_n^b$ (see Figure 1).

For every $n \in \mathbb{N}$, let $F_n \in L^2(\Omega_n,\mathbb{R}^3)$ and

$$J_n : U \in H^1(\Omega_n,S^2) \longrightarrow \int_{\Omega_n} |DU(x)|^2 dx - 2\int_{\Omega_n} U(x)F_n(x)dx, \tag{2.2}$$

where $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$. By applying the Direct Method of Calculus of Variations, for every $n \in \mathbb{N}$ there exists a solution $U_n \in H^1(\Omega_n,S^2)$ of the following problem:

$$J_n(U_n) = \min\left\{J_n(U) : U \in H^1(\Omega_n,S^2)\right\}. \tag{2.3}$$

Remark that energy (2.2) is more general of that considered in the Introduction. In particular, if $F_n = \lambda G_n$, with $G_n \in L^2(\Omega_n,S^2)$, problem (2.3) is equal to problem (1.1), up the additive constant $2\lambda|\Omega_n|$.

As it is usual (see [5]), problem (2.3) can be reformulated on a fixed domain through appropriate rescalings mapping the interior of Ω_n^a into $\Omega^a := \Theta\times]0,1[$ and Ω_n^b into $\Omega^b := \left]-\frac{1}{2},\frac{1}{2}\right[\times\left]-\frac{1}{2},\frac{1}{2}\right[\times]-1,0[$. Namely, for every $n \in \mathbb{N}$ by setting

$$u_n(x) := \begin{cases} u_n^a(x) = U_n(r_nx_1,r_nx_2,x_3), & x \text{ a.e. in } \Omega^a, \\ u_n^b(x) = U_n(x_1,r_nx_2,r_nx_3), & x \text{ a.e. in } \Omega^b, \end{cases} \tag{2.4}$$

$$f_n(x) := \begin{cases} f_n^a(x) = F_n(r_nx_1,r_nx_2,x_3), & x \text{ a.e. in } \Omega^a, \\ f_n^b(x) = F_n(x_1,r_nx_2,r_nx_3), & x \text{ a.e. in } \Omega^b, \end{cases} \tag{2.5}$$

$$\begin{aligned} V_n := \Big\{ \quad &(v^a,v^b) \in H^1(\Omega^a,S^2)\times H^1(\Omega^b,S^2) : \\ &v^a(x_1,x_2,0) = v^b(r_nx_1,x_2,0), \text{ for } (x_1,x_2) \text{ a.e. in } \Theta\Big\}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} j_n : v = (v^a,v^b) \in V_n \longrightarrow &\int_{\Omega^a} \left(\left| \left(\frac{1}{r_n}D_{x_1}v^a|\frac{1}{r_n}D_{x_2}v^a|D_{x_3}v^a \right) \right|^2 - 2v^af_n^a \right) dx + \\ &+ \int_{\Omega^b} \left(\left| \left(D_{x_1}v^b|\frac{1}{r_n}D_{x_2}v^b|\frac{1}{r_n}D_{x_3}v^b \right) \right|^2 - 2v^bf_n^b \right) dx, \end{aligned} \tag{2.7}$$

it results that $u_n \in V_n$ solves the following problem:

$$j_n(u_n) = \min\left\{j_n(v) : v \in V_n\right\}. \tag{2.8}$$

Remark that we have also multiplied the rescaled functional by $\dfrac{1}{r_n^2}$.

For studying the asymptotic behavior of problem (2.8), as $n \to +\infty$, assume that

$$f_n^a \rightharpoonup f^a \text{ weakly in } L^2(\Omega^a, \mathbb{R}^3), \qquad f_n^b \rightharpoonup f^b \text{ weakly in } L^2(\Omega^b, \mathbb{R}^3). \tag{2.9}$$

Moreover, set

$$\begin{aligned} V := \{&(w,\zeta) \in H^1(\Omega^a, S^2) \times H^1(\Omega^b, S^2) : \\ &w \text{ is independent of } (x_1,x_2), \quad \zeta \text{ is independent of } (x_2,x_3), \quad w(0)=\zeta(0)\} \\ \simeq &\left\{(w,\zeta) \in H^1(]0,1[, S^2) \times H^1\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[, S^2\right) : w(0)=\zeta(0)\right\}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} &j^a : w \in H^1(]0,1[, S^2) \longrightarrow \\ &\qquad |\Theta| \int_0^1 |w'(x_3)|^2\, dx_3 - 2\int_0^1 w(x_3) \left(\int_\Theta f^a(x_1,x_2,x_3) d(x_1,x_2)\right) dx_3 \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &j^b : \zeta \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[, S^2\right) \longrightarrow \\ &\int_{-\frac{1}{2}}^{\frac{1}{2}} |\zeta'(x_1)|^2\, dx_1 - 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta(x_1) \left(\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[} f^b(x_1,x_2,x_3) d(x_2,x_3)\right) dx_1, \end{aligned} \tag{2.12}$$

where w' and ζ' stand for the derivative of w and ζ, respectively.

2.1 Convergence results when $n \to +\infty$

The following result describes the asymptotic behavior of problem (2.8) when $n \to +\infty$.

Theorem 2.1. *For every $n \in \mathbb{N}$, let $u_n = (u_n^a, u_n^b)$ be a solution of problem (2.6)-(2.7)-(2.8), under assumptions (2.1) and (2.9).*

Then, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and $(u^a, u^b) \in V$ (depending on the selected subsequence) such that

$$u_{n_i}^a \to u^a \text{ strongly in } H^1(\Omega^a, S^2), \qquad u_{n_i}^b \to u^b \text{ strongly in } H^1(\Omega^b, S^2), \tag{2.13}$$

as $i \to +\infty$, and (u^a, u^b) solves the following problem:

$$j^a(u^a) + j^b(u^b) = \min\left\{j^a(w) + j^b(\zeta) : (w,\zeta) \in V\right\}, \tag{2.14}$$

where V, j^a and j^b are defined in (2.10), (2.11) and (2.12), respectively. Moreover,

$$\begin{cases} \dfrac{1}{r_n} D_{x_1} u_n^a \to 0, \quad \dfrac{1}{r_n} D_{x_2} u_n^a \to 0 \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3), \\ \dfrac{1}{r_n} D_{x_2} u_n^b \to 0, \quad \dfrac{1}{r_n} D_{x_3} u_n^b \to 0 \text{ strongly in } L^2(\Omega^b, \mathbb{R}^3), \end{cases} \tag{2.15}$$

as $n \to +\infty$. Furthermore, the energies converge in the sense that

$$\lim_n j_n(u_n) = j^a(u^a) + j^b(u^b). \tag{2.16}$$

As regard as the asymptotic behavior of original problem (2.3), as $n \to +\infty$, from the rescaling (2.4)-(2.5) and Theorem 2.1, the result below follows immediately.

Corollary 2.2. *For every $n \in \mathbb{N}$, let U_n be a solution of problem (2.2)-(2.3), under assumptions (2.1) and (2.9) with $\{f_n\}_{n\in\mathbb{N}}$ defined by (2.5).*

Then, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and $(u^a, u^b) \in V$ (depending on the selected subsequence) such that

$$\lim_i \left[\frac{1}{r_{n_i}^2} \int_{r_{n_i}\Theta\times]0,1[} \left(|U_{n_i} - u^a|^2 + |D_{x_1}U_{n_i}|^2 + |D_{x_2}U_{n_i}|^2 + |D_{x_3}U_{n_i} - D_{x_3}u^a|^2\right) dx \right] = 0,$$

$$\lim_i \left[\frac{1}{r_{n_i}^2} \int_{]-\frac{1}{2},\frac{1}{2}[\times]-\frac{r_{n_i}}{2},\frac{r_{n_i}}{2}[\times]-r_{n_i},0[} \left(\left|U_{n_i} - u^b\right|^2 + \left|D_{x_1}U_{n_i} - D_{x_1}u^b\right|^2 + |D_{x_2}U_{n_i}|^2 + |D_{x_3}U_{n_i}|^2\right) dx \right] = 0,$$

$$\lim_n \frac{J_n(U_n)}{r_n^2} = j^a(u^a) + j^b(u^b),$$

and (u^a, u^b) solves problem (2.14).

Remark 2.3. *If problem (2.14) admits a unique solution, then all previous convergences hold true for the whole sequence.*

Remark 2.4. *We have assumed that the small cross sections of the two cylinders scale down with same rate r_n. Well, if one scales down the cross section of the second cylinder with a different parameter h_n, i.e. $\Omega_n^b := \left]-\frac{1}{2},\frac{1}{2}\right[\times h_n \left(\left]-\frac{1}{2},\frac{1}{2}\right[\times]-1,0[\right)$, then it is not difficult to show that (compare Theorem 2.2 and Theorem 2.3 in [9])*

$$\begin{cases} \displaystyle\lim_n \frac{J_n(U_n)}{h_n^2} = \min\left\{ j^b(\zeta) : \zeta \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[, S^2\right)\right\}, & \text{if } \displaystyle\lim_n \frac{h_n}{r_n} = +\infty, \\ \displaystyle\lim_n \frac{J_n(U_n)}{r_n^2} = \min\left\{ j^a(w) : w \in H^1(]0,1[, S^2)\right\}, & \text{if } \displaystyle\lim_n \frac{h_n}{r_n} = 0. \end{cases}$$

Proof of Theorem 2.1. The proof of Theorem 2.1 will be performed in several steps.

1) A priori estimates. Being $((0,0,1),(0,0,1)) \in V_n$ for every $n \in \mathbb{N}$, by virtue of (2.9), there exists a constant c, independent of n, such that

$$j_n(u_n) \leq -2\int_{\Omega^a} (0,0,1)f_n^a dx - 2\int_{\Omega^b} (0,0,1)f_n^b dx \leq c, \quad \forall n \in \mathbb{N}. \tag{2.17}$$

Consequently, by taking into account that $|u_n| = 1$ a.e. in $\Omega^a \bigcup \Omega^b$ for every $n \in \mathbb{N}$ and (2.9), there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$, $u^a \in H^1(\Omega^a, S^2)$

independent of (x_1, x_2), $u^b \in H^1(\Omega^b, S^2)$ independent of (x_2, x_3), $\xi^a = (\xi_1^a, \xi_2^a) \in (L^2(\Omega^a, \mathbb{R}^3))^2$ and $\xi^b = (\xi_2^b, \xi_3^b) \in (L^2(\Omega^b, \mathbb{R}^3))^2$ such that

$$u_{n_i}^a \rightharpoonup u^a \text{ weakly in } H^1(\Omega^a, S^2), \qquad u_{n_i}^b \rightharpoonup u^b \text{ weakly in } H^1(\Omega^b, S^2), \tag{2.18}$$

$$\begin{cases} \dfrac{1}{r_{n_i}} D_{x_1} u_{n_i}^a \rightharpoonup \xi_1^a, \quad \dfrac{1}{r_{n_i}} D_{x_2} u_{n_i}^a \rightharpoonup \xi_2^a \text{ weakly in } L^2(\Omega^a, \mathbb{R}^3), \\ \\ \dfrac{1}{r_{n_i}} D_{x_2} u_{n_i}^b \rightharpoonup \xi_2^b, \quad \dfrac{1}{r_{n_i}} D_{x_3} u_{n_i}^b \rightharpoonup \xi_3^b \text{ weakly in } L^2(\Omega^b, \mathbb{R}^3), \end{cases} \tag{2.19}$$

as $i \to +\infty$. Remark that $u^a \in H^1(]0,1[, S^2)$ and $u^b \in H^1(]-\frac{1}{2}, \frac{1}{2}[, S^2)$.

2) Limit junction condition. For asserting that $(u^a, u^b) \in V$, it remains to prove that

$$u^a(0) = u^b(0). \tag{2.20}$$

The proof of (2.20) will be performed in three steps. The first step is devoted to prove the existence of three constants $c \in]0, +\infty[$, $\overline{x}_3 \in]-1, 0[$ and $\overline{x}_2 \in]-\frac{1}{2}, \frac{1}{2}[$, and of an increasing sequence of positive integer numbers $\{i_k\}_{k \in \mathbb{N}}$ such that

$$\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left| \frac{1}{r_{n_{i_k}}} D_{x_2} u_{n_{i_k}}^b (x_1, x_2, \overline{x}_3) \right|^2 d(x_1, x_2) \leq c, \quad \forall k \in \mathbb{N}, \tag{2.21}$$

and

$$u_{n_{i_k}}^b(\cdot, \overline{x}_2, \overline{x}_3) \to u^b \text{ strongly in } C^0\left(\left[-\frac{1}{2}, \frac{1}{2}\right], S^2\right), \tag{2.22}$$

as $k \to +\infty$. To this aim, for every $i \in \mathbb{N}$, set

$$\rho_i : x_3 \in]-1, 0[\longrightarrow$$

$$\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left(\left| D_{x_1} u_{n_i}^b (x_1, x_2, x_3) \right|^2 + \left| \frac{1}{r_{n_i}} D_{x_2} u_{n_i}^b (x_1, x_2, x_3) \right|^2 + \left| u_{n_i}^b (x_1, x_2, x_3) \right|^2 \right) d(x_1, x_2).$$

From Fatou Lemma and (2.18)-(2.19), it follows that

$$\int_{-1}^0 \liminf_i \rho_i(x_3) dx_3 \leq \liminf_i \int_{-1}^0 \rho_i(x_3) dx_3 < +\infty.$$

Consequently, there exist two constants $c \in]0, +\infty[$ and $\overline{x}_3 \in]-1, 0[$, and an increasing sequence of positive integer numbers $\{i_k\}_{k \in \mathbb{N}}$ such that

$$\rho_{i_k}(\overline{x}_3) < c \quad \forall k \in \mathbb{N},$$

i,e., estimate (2.21) holds true and, by virtue of the second convergence in (2.18), it results that

$$u_{n_{i_k}}^b(\cdot, \cdot, \overline{x}_3) \rightharpoonup u^b \text{ weakly in } H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2, S^2\right), \tag{2.23}$$

as $k \to +\infty$.

Now, for every $k \in \mathbb{N}$, let

$$\sigma_k : x_2 \in \left]-\frac{1}{2}, \frac{1}{2}\right[\to \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left| D_{x_1} u^b_{n_{i_k}}(x_1, x_2, \overline{x}_3) \right|^2 + \left| u^b_{n_{i_k}}(x_1, x_2, \overline{x}_3) \right|^2 \right) dx_1.$$

From Fatou Lemma and (2.23), it follows that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \liminf_k \sigma_k(x_2) dx_2 \leq \liminf_k \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_k(x_2) dx_2 < +\infty.$$

Consequently, there exist two constants $c \in]0, +\infty[$ and $\overline{x}_2 \in]-\frac{1}{2}, \frac{1}{2}[$, and a subsequence of $\{i_k\}_{k \in \mathbb{N}}$ (not relabelled) such that

$$\sigma_{i_k}(\overline{x}_2) < c \quad \forall k \in \mathbb{N}.$$

Hence, taking into account (2.23), one derives that

$$u^b_{n_{i_k}}(\cdot, \overline{x}_2, \overline{x}_3) \rightharpoonup u^b \text{ weakly in } H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[, S^2 \right),$$

as $k \to +\infty$, which provides (2.22).

The second step is devoted to prove that

$$\lim_k \int_\Theta u^b_{n_{i_k}}(r_{n_{i_k}} x_1, x_2, 0) d(x_1, x_2) = |\Theta| u^b(0). \tag{2.24}$$

To this aim, the integral in (2.24) will be split in the following way:

$$\begin{aligned}
&\int_\Theta u^b_{n_{i_k}}(r_{n_{i_k}} x_1, x_2, 0) d(x_1, x_2) = \\
&\int_\Theta \left(u^b_{n_{i_k}}(r_{n_{i_k}} x_1, x_2, 0) - u^b_{n_{i_k}}(r_{n_{i_k}} x_1, x_2, \overline{x}_3) \right) d(x_1, x_2) + \\
&\int_\Theta \left(u^b_{n_{i_k}}(r_{n_{i_k}} x_1, x_2, \overline{x}_3) - u^b_{n_{i_k}}(r_{n_{i_k}} x_1, \overline{x}_2, \overline{x}_3) \right) d(x_1, x_2) + \\
&\int_\Theta \left(u^b_{n_{i_k}}(r_{n_{i_k}} x_1, \overline{x}_2, \overline{x}_3) - u^b(r_{n_{i_k}} x_1) \right) d(x_1, x_2) + \\
&\int_\Theta u^b(r_{n_{i_k}} x_1) d(x_1, x_2), \quad \forall k \in \mathbb{N},
\end{aligned} \tag{2.25}$$

and one will pass to the limit, as k diverges, in each term of this decomposition.

By virtue of the last convergence in (2.19), there exists a constant $c \in]0,+\infty[$ such that

$$
\begin{aligned}
&\limsup_k \left| \int_\Theta \left(u^b_{n_{i_k}}(r_{n_{i_k}} x_1, x_2, 0) - u^b_{n_{i_k}}(r_{n_{i_k}} x_1, x_2, \overline{x}_3) \right) d(x_1, x_2) \right| = \\
&\limsup_k \left| \int_\Theta \left(\int_{\overline{x}_3}^0 D_{x_3} u^b_{n_{i_k}}(r_{n_{i_k}} x_1, x_2, x_3) dx_3 \right) d(x_1, x_2) \right| \leq \\
&|\Omega^b|^{\frac{1}{2}} \limsup_k \left(\int_{\Omega^b} |D_{x_3} u^b_{n_{i_k}}(r_{n_{i_k}} x_1, x_2, x_3)|^2 dx \right)^{\frac{1}{2}} \leq \qquad (2.26) \\
&|\Omega^b|^{\frac{1}{2}} \limsup_k \left(\frac{1}{r_{n_{i_k}}} \int_{\Omega^b} |D_{x_3} u^b_{n_{i_k}}(x_1, x_2, x_3)|^2 dx \right)^{\frac{1}{2}} \leq \\
&|\Omega^b|^{\frac{1}{2}} c \lim_k r^{\frac{1}{2}}_{n_{i_k}} = 0.
\end{aligned}
$$

By virtue of (2.21), there exists a constant $c \in]0,+\infty[$ such that

$$
\begin{aligned}
&\limsup_k \left| \int_\Theta \left(u^b_{n_{i_k}}(r_{n_{i_k}} x_1, x_2, \overline{x}_3) - u^b_{n_{i_k}}(r_{n_{i_k}} x_1, \overline{x}_2, \overline{x}_3) \right) d(x_1, x_2) \right| = \\
&\limsup_k \left| \int_\Theta \left(\int_{\overline{x}_2}^t D_{x_2} u^b_{n_{i_k}}(r_{n_{i_k}} x_1, x_2, \overline{x}_3) dx_2 \right) d(x_1, t) \right| \leq \\
&\limsup_k \left(\int_{]-\frac{1}{2},\frac{1}{2}[^2} \left| D_{x_2} u^b_{n_{i_k}}(r_{n_{i_k}} x_1, x_2, \overline{x}_3) \right|^2 d(x_1, x_2) \right)^{\frac{1}{2}} \leq \qquad (2.27) \\
&\limsup_k \left(\frac{1}{r_{n_{i_k}}} \int_{]-\frac{1}{2},\frac{1}{2}[^2} \left| D_{x_2} u^b_{n_{i_k}}(x_1, x_2, \overline{x}_3) \right|^2 d(x_1, x_2) \right)^{\frac{1}{2}} \leq \\
&c \lim_k r^{\frac{1}{2}}_{n_{i_k}} = 0.
\end{aligned}
$$

By virtue of (2.22), it results that

$$\begin{aligned}&\limsup_k \left| \int_\Theta \left(u^b_{n_{i_k}}(r_{n_{i_k}} x_1, \overline{x}_2, \overline{x}_3) - u^b(r_{n_{i_k}} x_1) \right) d(x_1,x_2) \right| \leq \\ &\limsup_k \int_{]-\frac12,\frac12[^2} \left| \left(u^b_{n_{i_k}}(r_{n_{i_k}} x_1, \overline{x}_2, \overline{x}_3) - u^b(r_{n_{i_k}} x_1) \right) \right| d(x_1,x_2) = \\ &\limsup_k \left(\frac{1}{r_{n_{i_k}}} \int_{]-\frac{r_{n_{i_k}}}{2}, \frac{r_{n_{i_k}}}{2}[\times]-\frac12,\frac12[} \left| \left(u^b_{n_{i_k}}(x_1, \overline{x}_2, \overline{x}_3) - u^b(x_1) \right) \right| d(x_1,x_2) \right) \leq \\ &\lim_k \| u^b_{n_{i_k}}(\cdot, \overline{x}_2, \overline{x}_3) - u^b(\cdot) \|_{L^\infty(]-\frac12,\frac12[^2)} = 0.\end{aligned} \tag{2.28}$$

Since $u^b \in C^0\left(\left[-\frac12,\frac12\right], S^2\right)$, it results that

$$\lim_k \int_\Theta u^b(r_{n_{i_k}} x_1) d(x_1,x_2) = |\Theta| u^b(0). \tag{2.29}$$

By passing to the limit in (2.25), as k diverges, and taking into account (2.26)-(2.29), one obtains (2.24).

Finally, junction condition (2.20) is obtained by passing to the limit, as k diverges, in

$$\int_\Theta u^a(x_1,x_2,0) d(x_1,x_2) = \int_\Theta u^b(r_n x_1, x_2, 0) d(x_1,x_2),$$

and using the first convergence in (2.18) and (2.24).

3) Recovery sequence. Let $(w,\zeta) \in C^1([0,1],S^2) \times C^1([-\frac12,\frac12],S^2)$ such that $w(0) = \zeta(0)$. This step is devoted to prove the existence of a sequence $\{v_n\}_{n\in\mathbb{N}}$ with $v_n \in V_n$ such that

$$\lim_n j_n(v_n) = j^a(w) + j^b(\zeta). \tag{2.30}$$

Since the proof of (2.30) is very similar to the proof of (2.31) in [9], we recall its framework for the sake of clarity, and we refer the reader to [9] for the details.

For every $n \in \mathbb{N}$, let

$$g_n(x) = \begin{cases} w(x_3), & \text{if } x \in \Theta \times]r_n, 1[, \\ w(r_n)\dfrac{x_3}{r_n} + \zeta(r_n x_1)\dfrac{r_n - x_3}{r_n}, & \text{if } x \in \Theta \times [0, r_n], \\ \zeta(x_1), & \text{if } x \in \Omega^b. \end{cases} \tag{2.31}$$

Of course, $g^a_n \in H^1(\Omega^a)$, $g^b_n \in H^1(\Omega^b)$, and $g^a_n(x_1,x_2,0) = g^b_n(r_n x_1, x_2, 0)$ a.e. in Θ; but $|g_n(x)| < 1$ in $\Theta\times]0,r_n[$. Then, g_n is not an admissible test function for problem (2.6)-(2.8). To overcome this difficulty, for $y \in B_{\frac12}(0) = \{x \in \mathbb{R}^3 : |x| \leq \frac12\}$, introduce the function

$$\pi_y : x \in B_1(0) \setminus \{y\} \to y + \frac{y(y-x) + \sqrt{(y(x-y))^2 + |x-y|^2(1-|y|^2)}}{|x-y|^2}(x-y) \in S^2 \tag{2.32}$$

projecting $x \in B_1(0) \setminus \{y\} = \{x \in \mathbb{R}^3 : |x| \leq 1\} \setminus \{y\}$ on S^2 along the direction $x - y$ (see [3] and [1]). The idea is to choose $y \in B_{\frac{1}{2}}(0)$ opportunely, and to define $v_n = \pi_y \circ g_n$. To do that, one has to be careful that the set $\{x : g_n(x) = y\}$ is "sufficiently small". By setting $G = \bigcup_{n\in\mathbb{N}} \left\{ y \in B_{\frac{1}{2}}(0) : \exists x \in \Theta \times]0, r_n[\text{ with } g_n(x) = y \text{ and } rank((Dg_n)(x)) < 3 \right\}$, Sard's Lemma assures that meas$(G) = 0$. Moreover, for every $n \in \mathbb{N}$ and for every $y \in B_{\frac{1}{2}}(0) \setminus G$, the set $G_{n,y} = \{x \in \Theta \times]0, r_n[: g_n(x) = y\}$ has dimension 0. Consequently, for every $n \in \mathbb{N}$ and for every $y \in B_{\frac{1}{2}}(0) \setminus G$, the function $\pi_y \circ (g_{n|_{\Omega\setminus G_{n,y}}})$ is well defined. By arguing as in the proof of (2.36) in [9], one can prove the existence of a sequence $\{y_n\}_{n\in\mathbb{N}} \subset B_{\frac{1}{2}}(0) \setminus G$ such that (crucial point!)

$$\lim_n \int_{(\Theta\times]0,r_n[)\setminus G_{n,y_n}} \left| \left(\frac{1}{r_n} D_{x_1} \left(\pi_{y_n}(g_n(x))\right) |0| D_{x_3} \left(\pi_{y_n}(g_n(x))\right) \right) \right|^2 dx = 0. \tag{2.33}$$

Now, for every $n \in \mathbb{N}$ set $v_n = \pi_{y_n} \circ (g_{n|_{\Omega\setminus G_{n,y_n}}})$. Then, by virtue of (2.31) and of the fact that $\pi_y(x) = x$, $\forall x \in S^2$, it results that

$$v_n(x) = \begin{cases} w(x_3), & \text{if } x \in \Theta \times]r_n, 1[, \\ \pi_{y_n}\left(w(r_n)\dfrac{x_3}{r_n} + \zeta(r_n x_1)\dfrac{r_n - x_3}{r_n} \right) & \text{if } x \in (\Theta \times [0, r_n]) \setminus G_{n,y_n} \\ \zeta(x_1), & \text{if } x \in \Omega^b. \end{cases} \tag{2.34}$$

It is easy to see that $v_n \in V_n$. Moreover, $j_n(v_n)$ can be split in the following way:

$$\begin{aligned} j_n(v_n) = & \int_{\Omega^a} \left(|D_{x_3} w|^2 - 2w f_n^a \right) dx - \int_{\Theta\times]0,r_n[} \left(|D_{x_3} w|^2 - 2w f_n^a \right) dx + \\ & \int_{(\Theta\times]0,r_n[)\setminus G_{n,y_n}} \left(\left| \left(\frac{1}{r_n} D_{x_1} (\pi_{y_n} \circ g_n) |0| D_{x_3} (\pi_{y_n} \circ g_n) \right) \right|^2 - 2(\pi_{y_n} \circ g_n) f_n^a \right) dx + \\ & \int_{\Omega^b} \left(|D_{x_1} \zeta|^2 - 2\zeta f_n^b \right) dx, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{2.35}$$

Finally, passing to the limit, as n diverges, in (2.35) and using (2.9) and (2.33), one obtains (2.30).

4) Density result. Let $(w, \zeta) \in V$. This step is devoted to prove the existence of a sequence $\{(w_k, \zeta_k)\}_{k\in\mathbb{N}} \subset C^1([0, 1], S^2) \times C^1([-\frac{1}{2}, \frac{1}{2}], S^2)$, with $w_k(0) = \zeta_k(0)$ for every $k \in \mathbb{N}$, such that

$$(w_k, \zeta_k) \to (w, \zeta) \text{ strongly in } H^1(]0, 1[, S^2) \times H^1\left(\left] -\frac{1}{2}, \frac{1}{2} \right[, S^2 \right). \tag{2.36}$$

Let $\{(\tilde{w}_k, \tilde{\zeta}_k)\}_{k\in\mathbb{N}} \subset C^1([0, 1], \mathbb{R}^3) \times C^1([-\frac{1}{2}, \frac{1}{2}], \mathbb{R}^3)$ be a sequence such that

$$(\tilde{w}_k, \tilde{\zeta}_k) \to (w, \zeta) \text{ strongly in } H^1(]0, 1[, \mathbb{R}^3) \times H^1\left(\left] -\frac{1}{2}, \frac{1}{2} \right[, \mathbb{R}^3 \right), \tag{2.37}$$

and, for every $k \in \mathbb{N}$, set $\overline{w}_k = \tilde{w}_k - \tilde{w}_k(0) + w(0) \in C^1([0,1], \mathbb{R}^3)$ and $\overline{\zeta}_k = \tilde{\zeta}_k - \tilde{\zeta}_k(0) + \zeta(0) \in C^1([-\frac{1}{2}, \frac{1}{2}], \mathbb{R}^3)$. Then, convergence (2.37) provides that

$$(\overline{w}_k, \overline{\zeta}_k) \to (w, \zeta) \text{ strongly in } H^1(]0,1[, \mathbb{R}^3) \times H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[, \mathbb{R}^3\right), \tag{2.38}$$

and consequently, since $|w(x_3)| = 1$ for every $x_3 \in [0,1]$ and $|\zeta(x_1)| = 1$ for every $x_1 \in [-\frac{1}{2}, \frac{1}{2}]$, it follows that

$$\lim_k \|\overline{w}_k\|_{L^\infty[0,1]} = 1 \quad \lim_k \|\overline{\zeta}_k\|_{L^\infty[-\frac{1}{2},\frac{1}{2}]} = 1. \tag{2.39}$$

Then, by setting $\pi : x \in \mathbb{R}^3 - \{0\} \to \frac{x}{|x|} \in \mathbb{R}^3 - \{0\}$, it is evident that, for $k \in \mathbb{N}$ sufficiently large, the functions $w_k = \pi \circ \overline{w}_k$ and $\zeta_k = \pi \circ \overline{\zeta}_k$ are well defined, $(w_k, \zeta_k) \in C^1([0,1], S^2) \times C^1([-\frac{1}{2}, \frac{1}{2}], S^2)$ and $w_k(0) = \zeta_k(0)$. Moreover, it is obvious that

$$(w_k, \zeta_k) \to (w, \zeta) \text{ strongly in } L^2(]0,1[, S^2) \times L^2\left(\left]-\frac{1}{2}, \frac{1}{2}\right[, S^2\right).$$

For obtaining (2.36), it remains to prove that

$$(w_k', \zeta_k') \to (w', \zeta') \text{ strongly in } L^2(]0,1[, \mathbb{R}^3) \times L^2\left(\left]-\frac{1}{2}, \frac{1}{2}\right[, \mathbb{R}^3\right). \tag{2.40}$$

By virtue of (2.38) and (2.39), there exist $c \in]0, +\infty[$, $g_1 \in L^1]0,1[$ and $g_2 \in L^1] - \frac{1}{2}, \frac{1}{2}[$ such that, passing eventually to a subsequence, it results that

$$\begin{cases} \lim_k w_k'(x_3) = \lim_k \left(D\pi(\overline{w}_k) \cdot \overline{w}_k'\right)(x_3) = (D\pi(w) \cdot w')(x_3) = w'(x_3), \text{ a.e. in }]0,1[, \\ |w_k'(x_3)|^2 = |(D\pi(\overline{w}_k) \cdot \overline{w}_k')(x_3)|^2 \leq c|\overline{w}_k'(x_3)|^2 \leq c g_1(x_3), \text{ a.e. in }]0,1[\\ \qquad \text{and for } k \in \mathbb{N} \text{ sufficiently large}, \\ \lim_k \zeta_k'(x_1) = \lim_k \left(D\pi(\overline{\zeta}_k) \cdot \overline{\zeta}_k'\right)(x_1) = (D\pi(\zeta) \cdot \zeta')(x_1) = \zeta'(x_1) \text{ a.e. in } \left]-\frac{1}{2}, \frac{1}{2}\right[, \\ |\zeta_k'(x_1)|^2 = \left|(D\pi(\overline{\zeta}_k) \cdot \overline{\zeta}_k')(x_1)\right|^2 \leq c|\overline{\zeta}_k'(x_1)|^2 \leq c g_2(x_1), \text{ a.e. in } \left]-\frac{1}{2}, \frac{1}{2}\right[\\ \qquad \text{and for } k \in \mathbb{N} \text{ sufficiently large}. \end{cases}$$

Consequently, using the dominated convergence Theorem, one obtains (2.40).

5) <u>*Conclusion*</u>. By using a l.s.c argument, from (2.9), (2.18) and (2.19) it follows that

$$\int_{\Omega^a} \left(|\xi_1^a|^2 + |\xi_2^a|^2\right) dx + j^a(u^a) + j^b(u^b) + \int_{\Omega^b} \left(|\xi_2^b|^2 + |\xi_3^b|^2\right) dx \leq \liminf_i j_{n_i}(u_{n_i}). \tag{2.41}$$

On the other hand, by virtue of step 3, for every $(w, \zeta) \in C^1([0,1], S^2) \times C^1([-\frac{1}{2}, \frac{1}{2}], S^2)$ with $w(0) = \zeta(0)$, there exists a sequence $\{v_n\}_{n \in \mathbb{N}}$ with $v_n \in V_n$ such that

$$\limsup_i j_{n_i}(u_{n_i}) \leq \limsup_i j_{n_i}(v_{n_i}) = \lim_n j_n(v_n) = j^a(w) + j^b(\zeta). \tag{2.42}$$

Then, by combining (2.41) with (2.42), one obtains that

$$\begin{aligned}&\int_{\Omega^a}\left(|\xi_1^a|^2+|\xi_2^a|^2\right)dx+j^a(u^a)+j^b(u^b)+\int_{\Omega^b}\left(|\xi_2^b|^2+|\xi_3^b|^2\right)dx\le\liminf_i j_{n_i}(u_{n_i})\le\\&\limsup_i j_{n_i}(u_{n_i})\le j^a(w)+qj^b(\zeta),\end{aligned}\tag{2.43}$$

for every $(w,\zeta)\in C^1([0,1],S^2)\times C^1([-\frac{1}{2},\frac{1}{2}],S^2))$ such that $w(0)=\zeta(0)$.

Step 4 provides that inequality (2.43) holds true for every $(w,\zeta)\in V$. Consequently, it results that

$$\xi^a=0,\quad \xi^b=0,\tag{2.44}$$

(u^a,u^b) solves problem (2.14) and

$$\lim_i j_{n_i}(u_{n_i})=j^a(u^a)+j^b(u^b).\tag{2.45}$$

Really, convergence (2.45) holds true for the whole sequence (so (2.16) is proved), since $j^a(u^a)+j^b(u^b)$ is independent of the selected subsequence, being the minimum of problem (2.14).

Finally, by combining (2.9), (2.18), (2.19) and (2.44) with (2.45), and by using the Rellich-Kondrachov compact embedding Theorem and the uniform convexity of the space L^2, it is easy to see that convergences (2.18) and (2.19) occur in the strong sense, i.e., (2.13) and (2.15) hold true. □

3 Second part: analysis of the limit model

For every $n\in\mathbb{N}$ and $\lambda\in[0,+\infty[$, consider the following problem:

$$J_{n,\lambda}:U\in H^1(\Omega_n,S^2)\longrightarrow\int_{\Omega_n}|DU(x)|^2dx+\lambda\int_{\Omega_n}|U(x)-F_n(x)|^2dx,\tag{3.1}$$

where $F_n:\Omega_n\to S^2$ is a measurable function.

Remark that $J_{n,\lambda}$ has the same minimum points of the functional:

$$\widetilde{J}_{n,\lambda}:U\in H^1(\Omega_n,S^2)\longrightarrow\int_{\Omega_n}|DU(x)|^2dx-2\lambda\int_{\Omega_n}U(x)F_n(x)dx,$$

since $J_{n,\lambda}(U)=\widetilde{J}_{n,\lambda}(U)+2\lambda|\Omega_n|$, for every $U\in H^1(\Omega_n,S^2)$. Consequently, after a rescaling as in Section 2, by passing to the limit as $n\to+\infty$, one obtains all the results of Subsection 2.1 with

$$\begin{aligned}j^a_\lambda(w)=\ &|\Theta|\int_0^1|w'(x_3)|^2dx_3-2\lambda\int_0^1 w(x_3)\left(\int_\Theta f^a(x_1x_2,x_3)d(x_1,x_2)\right)dx_3+\\&+2\lambda|\Theta|,\quad\forall w\in H^1(]0,1[,S^2),\end{aligned}\tag{3.2}$$

$$j^b_\lambda(\zeta) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\zeta'(x_1)|^2\, dx_1 - 2\lambda \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta(x_1) \left(\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[} f^b(x_1,x_2,x_3) d(x_2,x_3) \right) dx_1 +$$

$$+2\lambda, \quad \forall \zeta \in H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[, S^2 \right), \tag{3.3}$$

where f^a and f^b are given by (2.5) and (2.9). Remark that, since $|f^a_n(x)| = 1$ a.e. in Ω^a and $|f^b_n(x)| = 1$ a.e. in Ω^b for every $n \in \mathbb{N}$, weak convergences in (2.9) are always satisfied by a subsequence.

If $|f^a(x)| = 1$ a.e. in Ω^a, f^a is independent of (x_1, x_2), $|f^b(x)| = 1$ a.e. in Ω^b and f^b is independent of (x_2, x_3), then functionals (3.2) an (3.3) can be rewritten as follows:

$$j^a_\lambda(w) = |\Theta| \int_0^1 \left(|w'(x_3)|^2 + \lambda\, |w(x_3) - f^a(x_3)|^2 \right) dx_3, \quad \forall w \in H^1(]0,1[, S^2), \tag{3.4}$$

$$j^b_\lambda(\zeta) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(|\zeta'(x_1)|^2 + \lambda \left|\zeta(x_1) - f^b(x_1)\right|^2 \right) dx_1, \quad \forall \zeta \in H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[, S^2 \right). \tag{3.5}$$

In the sequel, $(w_\lambda, \zeta_\lambda) \in V$ denotes a solution of the following problem:

$$\begin{aligned} j^a_\lambda(w_\lambda) + j^b_\lambda(\zeta_\lambda) = \min \Bigg\{ & |\Theta| \int_0^1 \left(|w'(x_3)|^2 + \lambda\, |w(x_3) - f^a(x_3)|^2 \right) dx_3 + \\ & + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(|\zeta'(x_1)|^2 + \lambda \left|\zeta(x_1) - f^b(x_1)\right|^2 \right) dx_1 \, : \, (w,\zeta) \in V \Bigg\}, \end{aligned} \tag{3.6}$$

where V is the space defined in (2.10).

Remark that, if $\lambda = 0$, the solutions of problem (3.6) are the constants $(c, c) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $|c| = 1$ and $j^a_0(w_0) + j^b_0(\zeta_0) = 0$. Moreover (compare the proof of (3.16) in [9]) the function $\lambda \in [0, +\infty[\to j^a_\lambda(w_\lambda) + j^b_\lambda(\zeta_\lambda)$ is increasing and

$$\frac{d\left(j^a_\lambda(w_\lambda) + j^b_\lambda(\zeta_\lambda)\right)}{d\lambda} = |\Theta| \int_0^1 |w_\lambda(x_3) - f^a(x_3)|^2\, dx_3 + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\zeta_\lambda(x_1) - f^b(x_1)\right|^2 dx_1,$$

for λ a.e. in $]0, +\infty[$. Then, it remains to study the asymptotic behavior, as $\lambda \to +\infty$, of problem (3.6).

3.1 Convergence results when $\lambda \to +\infty$

If $(f^a, f^b) \in V$, choosing $(w, \zeta) = (f^a, f^b)$ as test function in (3.6), it is easy to see that

$$(w_{\lambda_\iota}, \zeta_{\lambda_\iota}) \rightharpoonup (f^a, f^b) \text{ weakly in } H^1(]0,1[, S^2) \times H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[, S^2 \right),$$

for any diverging sequence of positive numbers $\{\lambda_\iota\}_{\iota \in \mathbb{N}}$. Consequently, using a l.s.c. argument, it follows that (compare Subsection 3.1 in [9])

$$\lim_{\lambda \to +\infty} \left(j^a_\lambda(w_\lambda) + j^b_\lambda(\zeta_\lambda) \right) = |\Theta|\, \|(f^a)'\|^2_{(L^2(]0,1[))^3} + \left\|(f^b)'\right\|^2_{\left(L^2\left(-\frac{1}{2},\frac{1}{2}[\right)\right)^3}.$$

Interesting situations occur when $(f^a, f^b) \notin V$, since in this case it results that

$$\lim_{\lambda \to +\infty} \left(j^a_\lambda(w_\lambda) + j^b_\lambda(\zeta_\lambda) \right) = +\infty. \tag{3.7}$$

In fact, by arguing by contradiction, if (3.7) does not hold true, then there exists $c \in]0, +\infty[$ and a diverging sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ such that

$$j^a_{\lambda_k}(w_{\lambda_k}) + j^b_{\lambda_k}(\zeta_{\lambda_k}) \leq c, \quad \forall k.$$

Consequently, it follows that

$$(w_{\lambda_k}, \zeta_{\lambda_k}) \rightharpoonup (f^a, f^b) \text{ weakly in } H^1(]0,1[, S^2) \times H^1 \left(\left] -\frac{1}{2}, \frac{1}{2} \right[, S^2 \right),$$

as λ diverges, and, in particular, one obtains that $(f^a, f^b) \in H^1(]0,1[, S^2) \times H^1 \left(\left] -\frac{1}{2}, \frac{1}{2} \right[, S^2 \right)$ and, by virtue of the Rellich Theorem, $f^a(0) = f^b(0)$. But this statement is false, since $(f^a, f^b) \notin V$.

Now, we examine some particular, but significant cases. At first, consider the case $f^a = (1,0,0)$ and $f^b = (0,1,0)$. Remark that $(f^a, f^b) \in H^1(]0,1[, S^2) \times H^1 \left(\left] -\frac{1}{2}, \frac{1}{2} \right[, S^2 \right)$, but $(f^a, f^b) \notin V$ since $f^a(0) \neq f^b(0)$. In this case, the following *a priori* estimates hold true:

Proposition 3.1. *For every $\lambda \in [0, +\infty[$, let $(w_\lambda, \zeta_\lambda)$ be a solution of problem (3.6) with $f^a = (1,0,0)$ and $f^b = (0,1,0)$.*

Then, there exist two constants $c_1, c_2 \in]0, +\infty[$ such that

$$c_1 \sqrt{\lambda} \leq j^a_\lambda(w_\lambda) + j^b_\lambda(\zeta_\lambda) \leq c_2 \sqrt{\lambda}, \qquad \textit{for } \lambda \textit{ sufficiently large.} \tag{3.8}$$

Proof. We adapt, to our coupled problem, a technique we introduced in [9].

For every $t \in]0, +\infty[$, let (w_t, ζ_t) be the couple of functions defined by

$$w_t : x_3 \in]0,1[\to \frac{1}{\sqrt{x_3^2 + t^2}} (x_3, t, 0) \in S^2, \quad \zeta_t : x_1 \in \left] -\frac{1}{2}, \frac{1}{2} \right[\to (0,1,0) \in S^2.$$

Since $(w_t, \zeta_t) \in V$, it results that

$$j^a_\lambda(w_\lambda) + j^b_\lambda(\zeta_\lambda) \leq j^a_\lambda(w_t) + j^b_\lambda(\zeta_t) = j^a_\lambda(w_t) \quad \forall t \in]0, +\infty[, \quad \forall \lambda \in]0, +\infty[. \tag{3.9}$$

Consequently, being

$$j^a_\lambda(w_t) = |\Theta| \left[\frac{1}{t} \left(\frac{t}{2(t^2+1)} + \frac{\arctan(t)}{2} \right) + \lambda t \left(\frac{2}{t} - \frac{2\sqrt{1+t^2}}{t} + 2 \right) \right],$$

$$\forall t \in]0, +\infty[, \quad \forall \lambda \in]0, +\infty[,$$

$$\lim_{t \to 0^+} \left(\frac{t}{2(t^2+1)} + \frac{\arctan(t)}{2} \right) = \frac{\pi}{4}, \quad \lim_{t \to 0^+} \left(\frac{2}{t} - \frac{2\sqrt{1+t^2}}{t} + 2 \right) = 2,$$

$$\frac{d}{dt}\left(\frac{t}{2(t^2+1)}+\frac{\arctan(t)}{2}\right)<0, \quad \frac{d}{dt}\left(\frac{2}{t}-\frac{2\sqrt{1+t^2}}{t}+2\right)<0, \quad \forall t\in]0,+\infty[,$$

one derives that

$$j^a_\lambda(w_\lambda)+j^b_\lambda(\zeta_\lambda)\le|\Theta|\left(\frac{\pi}{4}\frac{1}{t}+2\lambda t\right), \quad \forall t\in]0,+\infty[, \quad \forall\lambda\in]0,+\infty[,$$

which provides the upper bound in (3.8).

To prove the lower bound in (3.8), at first remark that

$$\begin{aligned} j^a_\lambda(w_\lambda)+j^b_\lambda(\zeta_\lambda)\ge\min\Bigg\{&|\Theta|\int_0^1\left((v'(x_3))^2+\lambda\left(v(x_3)-1\right)^2\right)dx_3+\\ &+\int_{-\frac{1}{2}}^{\frac{1}{2}}\left((z'(x_1))^2+\lambda\left(z(x_1)\right)^2\right)dx_1\ : \qquad (3.10)\\ &(v,z)\in H^1(]0,1[,\mathbb{R})\times H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[,\mathbb{R}\right),\ v(0)=z(0)\Bigg\}. \end{aligned}$$

For every $\lambda\in]0,+\infty[$, the last minimum is attained in the solution $(v_\lambda,\zeta_\lambda)\in H^1(]0,1[,\mathbb{R})\times H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[,\mathbb{R}\right)$ of the following problem:

$$\begin{cases} v''_\lambda-\lambda v_\lambda=-\lambda, & \text{in }]0,1[,\\ z''_\lambda-\lambda z_\lambda=0, & \text{in } \left]-\frac{1}{2},0\right[,\\ z''_\lambda-\lambda z_\lambda=0, & \text{in } \left]0,\frac{1}{2}\right[,\\ v'_\lambda(1)=z'_\lambda\left(-\frac{1}{2}\right)=z'_\lambda\left(\frac{1}{2}\right)=0, \\ v_\lambda(0)=z_\lambda(0), \\ |\Theta|v'_\lambda(0)=z'_\lambda(0^-)-z'_\lambda(0^+), \end{cases} \tag{3.11}$$

i.e., in $(v_\lambda,\zeta_\lambda)$ given by

$$v_\lambda(x_3)=-\frac{2}{|\Theta|\left(1+e^{\sqrt{\lambda}}\right)^2+2(1+e^{2\sqrt{\lambda}})}(e^{2\sqrt{\lambda}}e^{-x_3\sqrt{\lambda}}+e^{x_3\sqrt{\lambda}})+1, \quad \text{in }]0,1[, \tag{3.12}$$

$$z_\lambda(x_1)=\begin{cases}\dfrac{|\Theta|\left(1+e^{\sqrt{\lambda}}\right)}{|\Theta|\left(1+e^{\sqrt{\lambda}}\right)^2+2(1+e^{2\sqrt{\lambda}})}(e^{-x_1\sqrt{\lambda}}+e^{\sqrt{\lambda}}e^{x_1\sqrt{\lambda}}), & \text{in } \left]-\dfrac{1}{2},0\right[,\\[2ex] \dfrac{|\Theta|\left(1+e^{\sqrt{\lambda}}\right)}{|\Theta|\left(1+e^{\sqrt{\lambda}}\right)^2+2(1+e^{2\sqrt{\lambda}})}(e^{\sqrt{\lambda}}e^{-x_1\sqrt{\lambda}}+e^{x_1\sqrt{\lambda}}), & \text{in } \left]0,\dfrac{1}{2}\right[.\end{cases}$$

Then, combining (3.10) with (3.11) and (3.12), it follows that

$$\begin{aligned}j^a_\lambda(w_\lambda)+j^b_\lambda(\zeta_\lambda)\geq \min\Bigg\{&|\Theta|\int_0^1\left((v'(x_3))^2+\lambda\left(v(x_3)-1\right)^2\right)dx_3+\\&+\int_{-\frac{1}{2}}^{\frac{1}{2}}\left((z'(x_1))^2+\lambda\left(z(x_1)\right)^2\right)dx_1\ :\\&(v,z)\in H^1(]0,1[,\mathbb{R})\times H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[,\mathbb{R}\right),\ v(0)=z(0)\Bigg\}=\\&-|\Theta|\lambda\int_0^1 v_\lambda dx_3+|\Theta|\lambda=\frac{2|\Theta|(e^{2\sqrt{\lambda}}-1)}{|\Theta|\left(1+e^{\sqrt{\lambda}}\right)^2+2(1+e^{2\sqrt{\lambda}})}\sqrt{\lambda}.\end{aligned}$$

Consequently, taking into account that

$$\lim_{\lambda\to+\infty}\frac{2|\Theta|(e^{2\sqrt{\lambda}}-1)}{|\Theta|\left(1+e^{\sqrt{\lambda}}\right)^2+2(1+e^{2\sqrt{\lambda}})}=\frac{2|\Theta|}{|\Theta|+2},$$

one derives the lower bound in (3.8). □

Remark 3.2. *The proof of Proposition 3.1 gives also an estimate of c_1 and c_2.*

Proposition 3.1 holds again true if one assumes that f^a and f^b have the unit on the same component. For instance, if one assumes $f^a=(1,0,0)$ and $f^b=(-1,0,0)$, one obtains the upper bound by performing previous proof with

$$w_t: x_3\in]0,1[\to\frac{1}{\sqrt{x_3^2+t^2}}(x_3,t,0)\in S^2,\quad \zeta_t: x_1\in\left]-\frac{1}{2},\frac{1}{2}\right[\to\frac{1}{\sqrt{x_1^2+t^2}}(-|x_1|,t,0)\in S^2.$$

While the estimate of the lower bound is obtained by performing previous computations with $\lambda(z_1(x_1))^2$ replaced by $\lambda(z_1(x_1)+1)^2$ in (3.10), and the second line and third line of (3.11) replaced by $z''_\lambda-\lambda z_\lambda=\lambda$.

Consider, now, the case: $f^a=\left(\dfrac{x_3-\gamma}{|x_3-\gamma|},0,0\right)$ and $f^b=\left(\dfrac{x_1-\delta}{|x_1-\delta|},0,0\right)$, where $\gamma\in]0,1[$ and $\delta\in]-\frac{1}{2},\frac{1}{2}[$.

Proposition 3.3. *For every $\lambda \in]0,+\infty[$, let $(w_\lambda, \zeta_\lambda)$ be a solution of problem (3.6) with $f^a = \left(\dfrac{x_3-\gamma}{|x_3-\gamma|}, 0, 0\right)$ and $f^b = \left(\dfrac{x_1-\delta}{|x_1-\delta|}, 0, 0\right)$, where $\gamma \in]0,1[$ and $\delta \in]-\frac{1}{2}, \frac{1}{2}[$.*

Then, there exist two constants $c_1, c_2 \in]0,+\infty[$ such that

$$c_1\sqrt{\lambda} \le j^a_\lambda(w_\lambda) + j^b_\lambda(\zeta_\lambda) \le c_2\sqrt{\lambda}, \quad \text{for } \lambda \text{ sufficiently large.} \tag{3.13}$$

Proof. To prove the lower bound in (3.13), it is enough to remark that

$$j^a_\lambda(w_\lambda) + j^b_\lambda(\zeta_\lambda) \ge |\Theta| \min \left\{ \int_0^1 |v'(x_3)|^2 dx_3 + \lambda \int_0^1 \left| v(x_3) - \frac{x_3-\gamma}{|x_3-\gamma|}\right|^2 dx_3 : \right.$$
$$\left. v \in H^1(]0,1[,\mathbb{R}) \right\}, \quad \forall \lambda \in]0,+\infty[. \tag{3.14}$$

and to use the estimate of the lower bound of the right hand side of (3.29) given in [9].

To prove the upper bound in (3.13), first we consider the case $\delta > 0$ and then the general case.

If $\delta > 0$, for every $t \in]0,+\infty[$, let (w_t, ζ_t) be the couple of functions defined by

$$\begin{cases} w_t : x_3 \in]0,1[\to \dfrac{1}{\sqrt{(x_3-\gamma)^2 + t^2}} (x_3-\gamma, 0, t) \in S^2, \\[2ex] \zeta_t : x_1 \in \left]-\dfrac{1}{2}, \dfrac{1}{2}\right[\to \dfrac{1}{\sqrt{\gamma^2(x_1-\delta)^2 + (t\delta)^2}} (\gamma(x_1-\delta), 0, t\delta) \in S^2. \end{cases} \tag{3.15}$$

Since $(w_t, \zeta_t) \in V$, it results that

$$j^a_\lambda(w_\lambda) + j^b_\lambda(\zeta_\lambda) \le j^a_\lambda(w_t) + j^b_\lambda(\zeta_t) \quad \forall t \in]0,+\infty[, \quad \forall \lambda \in]0,+\infty[. \tag{3.16}$$

Then, arguing as in the proof of (3.13) in [9], one obtains an upper bound of $j^a_\lambda(w_t)$ and $j^b_\lambda(\zeta_t)$ which provide the upper bound in (3.13).

If $\delta \in]-\frac{1}{2}, 0]$, it is not possible to use test function (3.15), since it does not satisfy the junction condition. Then we have to use a more sophisticated argument whic works also for δ positive.

Let $\lambda \in]0,+\infty[$ be sufficiently large (it is enough to choose $\lambda > \frac{1}{\gamma^2}$), and set

$$v_\lambda(x_3) = \begin{cases} \dfrac{1}{\sqrt{(x_3-\gamma)^2 + \lambda^{-1}}} \left(x_3-\gamma, 0, \lambda^{-\frac{1}{2}}\right), & \text{if } x_3 \in \left]\lambda^{-\frac{1}{2}}, 1\right[, \\[2ex] \dfrac{x_3}{\lambda^{-\frac{1}{2}}} \dfrac{1}{\sqrt{(\lambda^{-\frac{1}{2}}-\gamma)^2 + \lambda^{-1}}} \left(\lambda^{-\frac{1}{2}}-\gamma, 0, \lambda^{-\frac{1}{2}}\right) + \\ \qquad + \left(1 - \dfrac{x_3}{\lambda^{-\frac{1}{2}}}\right) \dfrac{1}{\sqrt{\delta^2 + \lambda^{-1}}} (-\delta, \lambda^{-\frac{1}{2}}, 0), & \text{if } x_3 \in \left]0, \lambda^{-\frac{1}{2}}\right[, \end{cases}$$

$$z_\lambda(x_1) = \frac{1}{\sqrt{(x_1-\delta)^2+\lambda^{-1}}}(x_1-\delta, \lambda^{-\frac{1}{2}}, 0), \qquad \text{if } x_1 \in \left]-\frac{1}{2}, \frac{1}{2}\right[.$$

At first, remark that

$$\int_0^{\lambda^{-\frac{1}{2}}} |v'_\lambda(x_3)|^2 \, dx_3 \leq 4\lambda^{\frac{1}{2}}. \tag{3.17}$$

Of course, $v_\lambda \in H^1(]0,1[,\mathbb{R}^3)$, $z_\lambda \in H^1(]-\frac{1}{2},\frac{1}{2}[,S^2)$, and $v_\lambda(0) = z_\lambda(0)$; but $|v_\lambda(x_3)| < 1$ for every $x_3 \in]0,\lambda^{-\frac{1}{2}}[$. Then, (v_λ, z_λ) is not an admissible test function for problem (3.6). To overcome this difficulty, for $y \in B_{\frac{1}{2}}(0) = \{x \in \mathbb{R}^3 : |x| \leq \frac{1}{2}\}$, let π_y be the function introduced in (2.32) projecting $x \in B_1(0) \setminus \{y\} = \{x \in \mathbb{R}^3 : |x| \leq 1\} \setminus \{y\}$ on S^2 along the direction $x-y$. Since

$$\exists \overline{c} \in]0,+\infty[\quad : \quad |D\pi_y(x)|^2 \leq \frac{\overline{c}}{|x-y|^2}, \quad \forall y \in B_{\frac{1}{2}}(0), \quad \forall x \in B_1(0) \setminus \{y\},$$

from (3.17) it follows that

$$\int_{B_{\frac{1}{2}}(0)\setminus v_\lambda(]0,\lambda^{-\frac{1}{2}}[)} \int_0^{\lambda^{-\frac{1}{2}}} \left|(\pi_y(v_\lambda(x_3)))'\right|^2 dx_3 dy \leq$$

$$\overline{c}\int_0^{\lambda^{-\frac{1}{2}}} \left(\int_{B_{\frac{1}{2}}(0)\setminus v_\lambda(]0,\lambda^{-\frac{1}{2}}[)} \frac{1}{|v_\lambda(x_3)-y|^2} dy \right) |v'_\lambda(x_3)|^2 \, dx_3 \leq \overline{c} \left(\int_{B_{\frac{3}{2}}(0)} \frac{1}{|z|^2} dz \right) 4\lambda^{\frac{1}{2}},$$

where $\int_{B_{\frac{3}{2}}(0)} |z|^{-2} dz < +\infty$. Consequently, there exist a constant $C > 0$ and $y_\lambda \in B_{\frac{1}{2}}(0) \setminus v_\lambda(]0,\lambda^{-\frac{1}{2}}[)$ such that

$$\int_0^{\lambda^{-\frac{1}{2}}} \left|\left(\pi_{y_\lambda}(v_\lambda(x_3))\right)'\right|^2 dx_3 \leq C\lambda^{\frac{1}{2}}. \tag{3.18}$$

Point out that C is independent of λ! For instance, since $|v_\lambda(]0,\lambda^{-\frac{1}{2}}[)| = 0$, one can choose $C = \frac{1+\overline{c}4\int_{B_{3\setminus 2}(0)} \frac{1}{|z|^2}dz}{|B_{1\setminus 2}(0)|}$.

Finally, set $u_\lambda = \pi_{y_\lambda} \circ v_\lambda$. That is, being $\pi_{y_\lambda}(x) = x$ for every $x \in S^2$,

$$u_\lambda(x_3) = \begin{cases} \dfrac{1}{\sqrt{(x_3-\gamma)^2+\lambda^{-1}}}\left(x_3-\gamma, 0, \lambda^{-\frac{1}{2}}\right), & \text{if } x_3 \in \left]\lambda^{-\frac{1}{2}}, 1\right[, \\ \pi_{y_\lambda}\left(\dfrac{x_3}{\lambda^{-\frac{1}{2}}} \dfrac{1}{\sqrt{(\lambda^{-\frac{1}{2}}-\gamma)^2+\lambda^{-1}}}\left(\lambda^{-\frac{1}{2}}-\gamma, 0, \lambda^{-\frac{1}{2}}\right) + \right. \\ \left. \qquad + \left(1-\dfrac{x_3}{\lambda^{-\frac{1}{2}}}\right)\dfrac{1}{\sqrt{\delta^2+\lambda^{-1}}}(-\delta, \lambda^{-\frac{1}{2}}, 0)\right), & \text{if } x_3 \in \left]0, \lambda^{-\frac{1}{2}}\right[. \end{cases}$$

Since, now, $(u_\lambda, z_\lambda) \in V$ it results that

$$
\begin{aligned}
& j^a_\lambda(w_\lambda) + j^b_\lambda(\zeta_\lambda) \le j^a_\lambda(u_\lambda) + j^b_\lambda(z_\lambda) = \\
& |\Theta| \int_0^{\lambda^{-\frac{1}{2}}} |u'_\lambda(x_3)|^2\, dx_3 + |\Theta|\lambda \int_0^{\lambda^{-\frac{1}{2}}} |u_\lambda(x_3) + 1|^2\, dx_3 + \\
& |\Theta| \int_{\lambda^{-\frac{1}{2}}}^1 |u'_\lambda(x_3)|^2\, dx_3 + |\Theta|\lambda \int_{\lambda^{-\frac{1}{2}}}^1 |u_\lambda(x_3) - f^a(x_3)|^2\, dx_3 + j^b_\lambda(z_\lambda).
\end{aligned} \tag{3.19}
$$

By virtue of (3.18) and of the fact that $|u_\lambda(x_3)| = 1$, one has that

$$
\int_0^{\lambda^{-\frac{1}{2}}} |u'_\lambda(x_3)|^2\, dx_3 + \lambda \int_0^{\lambda^{-\frac{1}{2}}} |u_\lambda(x_3) + 1|^2\, dx_3 \le C\lambda^{\frac{1}{2}} + 4\lambda^{\frac{1}{2}} \tag{3.20}
$$

Moreover, in [9] we proved the existence of a positive constant c, independent of λ, such that

$$
\int_{\lambda^{-\frac{1}{2}}}^1 |u'_\lambda(x_3)|^2\, dx_3 + \lambda \int_{\lambda^{-\frac{1}{2}}}^1 |u_\lambda(x_3) - f^a(x_3)|^2\, dx_3 + j^b_\lambda(z_\lambda) \le c\lambda^{\frac{1}{2}}. \tag{3.21}
$$

By combining (3.19) with (3.20) and (3.21), one obtains the upper bound in (3.13). □

Remark 3.4. *The proof of Proposition 3.3 gives also an estimate of c_1 and c_2.*

Proposition 3.3 holds again true if one assumes that f^a and f^b have the singularity on different components. For instance, if one assumes $f^a = \left(\dfrac{x_3-\gamma}{|x_3-\gamma|}, 0, 0\right)$ and $f^b = \left(0, \dfrac{x_1-\delta}{|x_1-\delta|}, 0\right)$, one obtains the lower bound as before. While the estimate of the upper bound is obtained by performing previous computations with

$$
v_\lambda(x_3) = \begin{cases} \dfrac{1}{\sqrt{(x_3-\gamma)^2+\lambda^{-1}}}\left(x_3-\gamma, 0, \lambda^{-\frac{1}{2}}\right), & \text{if } x_3 \in \left]\lambda^{-\frac{1}{2}}, 1\right[, \\[2ex] \dfrac{x_3}{\lambda^{-\frac{1}{2}}} \dfrac{1}{\sqrt{(\lambda^{-\frac{1}{2}}-\gamma)^2+\lambda^{-1}}}\left(\lambda^{-\frac{1}{2}}-\gamma, 0, \lambda^{-\frac{1}{2}}\right) + & \\ \qquad + \left(1 - \dfrac{x_3}{\lambda^{-\frac{1}{2}}}\right) \dfrac{1}{\sqrt{\delta^2+\lambda^{-1}}} (\lambda^{-\frac{1}{2}}, -\delta, 0), & \text{if } x_3 \in \left]0, \lambda^{-\frac{1}{2}}\right[, \end{cases}
$$

$$
z_\lambda(x_1) = \frac{1}{\sqrt{(x_1-\delta)^2+\lambda^{-1}}}(\lambda^{-\frac{1}{2}}, x_1-\delta, 0), \quad \text{if } x_1 \in \left]-\frac{1}{2}, \frac{1}{2}\right[.
$$

The last results immediately provide the following convergence result:

Corollary 3.5. *For every $\lambda \in [0, +\infty[$, let $(w_\lambda, \zeta_\lambda)$ be a solution of problem (3.6) with f^a and f^b satisfying the assumptions in Proposition 3.1 (see also Remark 3.2) or Proposition 3.3 (see also Remark 3.4).*

Then, it results that

$$
\int_0^1 |w_\lambda(x_3) - f^a(x_3)|^2\, dx_3 \le \frac{c_2}{|\Theta|\sqrt{\lambda}}, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\zeta(x_1) - f^b(x_1)\right|^2 dx_1 \le \frac{c_2}{\sqrt{\lambda}},
$$

for λ sufficiently large.

Obviously, if $\{\lambda_\iota\}_{\iota\in\mathbb{N}}$ is a diverging sequence of positive numbers, $\{w_{\lambda_\iota}, \zeta_{\lambda_\iota}\}_{\iota\in\mathbb{N}}$ does not converge weakly in $H^1(]0,1[, S^2) \times H^1((-\frac{1}{2}, \frac{1}{2}), S^2)$, since $(f^a, f^b) \notin V$.

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