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Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

ASYMPTOTIC ANALYSIS FOR MICROMAGNETICS OF THIN FILMS GOVERNED BY INDEFINITE MATERIAL COEFFICIENTS

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(Communicated by Itai Shafrir)

Abstract. In this paper, a class of minimization problems, associated with the micromagnetics of thin films, is dealt with. Each minimization problem is distinguished by the thickness of the thin film, denoted by $0 < h < 1$, and it is considered under spatial indefinite and degenerative setting of the material coefficients. On the basis of the fundamental studies of the governing energy functionals, the existence of minimizers, for every $0 < h < 1$, and the 3D-2D asymptotic analysis for the observing minimization problems, as $h \searrow 0$, will be demonstrated in the main theorem of this paper.

1. **Introduction.** Let $S \subset \mathbb{R}^2$ be a two-dimensional bounded domain with a smooth boundary, and let $\Omega \subset \mathbb{R}^3$ be a three-dimensional cylindrical domain, given by $\Omega := S \times (0, 1)$. Also, let us set $\Omega^{(h)} := S \times (0, h)$, for any $h > 0$. Let $\alpha : \overline{\Omega} \longrightarrow [0, \infty)$ be a given continuous function, and let $A_0 := \alpha^{-1}(0)$ be the set of zero-points of α on $\overline{\Omega}$.

In this paper, we suppose that $0 < h < 1$, and deal with the following minimization problem, denoted by $(\mathrm{MP})^{(h)}_{\mathrm{org}}$.

$(\mathrm{MP})^{(h)}_{\mathrm{org}}$ Find a vectorial function $m^{(h)}_{\mathrm{org}} = (m^{(h)}_{\mathrm{org}\,1}, m^{(h)}_{\mathrm{org}\,2}, m^{(h)}_{\mathrm{org}\,3}) \in L^2(\Omega^{(h)}; \mathbb{R}^3)$ of three variables, which minimizes the following functional:

2000 *Mathematics Subject Classification.* Primary: 74G65, 35J70; Secondary: 74K35, 82D40.
Key words and phrases. Micromagnetics of thin film, indefinite and degenerative material coefficient, 3D-2D asymptotic analysis.
The second author is supported by Grant-in-Aid for Encouragement of Young Scientists (B) (No. 21740120) JSPS.

$$\mathcal{E}_{\rm org}^{(h)}(m) := \begin{cases} \dfrac{1}{\mathcal{L}^3(\Omega^{(h)})} \left(\displaystyle\int_{\Omega^{(h)} \setminus A_0} \alpha |\nabla m|^2 \, d\mathcal{L}^3 + \int_{\Omega^{(h)}} \varphi(m) \, d\mathcal{L}^3 \right. \\ \qquad\qquad\qquad\qquad \left. + \dfrac{1}{2} \displaystyle\int_{\Omega^{(h)}} \nabla \zeta_{\rm mag} \cdot m \, d\mathcal{L}^3 \right), \\ \quad \text{if } m \in W^{1,2}_{\rm loc}(\Omega^{(h)} \setminus A_0; \mathbb{R}^3) \text{ and } \sqrt{\alpha}\, \nabla m \in L^2(\Omega^{(h)} \setminus A_0; \mathbb{R}^{3 \times 3}), \\ \infty, \ \text{ otherwise}, \end{cases} \tag{1}$$

for any $m = (m_1, m_2, m_3) \in L^2(\Omega^{(h)}; \mathbb{R}^3)$;

subject to:

$$\operatorname{div}(-\nabla \zeta_{\rm mag} + \overline{{}^0 m}) = 0, \quad \text{in } \mathbb{R}^3, \tag{2}$$

$$|m| = m_s, \quad \mathcal{L}^3\text{-a.e. in } \Omega. \tag{3}$$

In (1), the functional $\mathcal{E}_{\rm org}^{(h)}$ is supposed to be the free energy, per unit volume, in a ferromagnetic thin film (cf. Brown [9]). In the context, the index $0 < h < 1$ and $\Omega^{(h)}$ denote the thickness and the distribution region of the magnetic thin film, respectively, and the unknown $m = (m_1(x), m_2(x), m_3(x))$ $(x = (x_1, x_2, x_3) \in \Omega^{(h)})$ is a vectorial function of three variables, which describes the magnetization in $\Omega^{(h)}$. The given continuous function $\alpha = \alpha(x)$ $(x \in \overline{\Omega})$ is the so-called material coefficient, and here, it is supposed that this coefficient may degenerate somewhere on $\overline{\Omega}$. $\varphi : \mathbb{R}^3 \longrightarrow [0, \infty)$ is a given continuous and even function, which represents the magnetization anisotropy.

Equation (2) is a simplified version of the Maxwell equation, and hence its solution $\zeta_{\rm mag} : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is supposed to be the potential of the magnetic field. Besides, the notation "$\overline{{}^0 \ }$" denotes the zero-extension of functions. Equation (3) is the constraint for the magnetization strength, and m_s is a given positive constant of the magnetization saturation.

Hereafter, for the sake of simplicity, let us set:

$$\mathcal{L}^2(S) = 1 \text{ (and hence } \mathcal{L}^3(\Omega) = 1), \text{ and } m_s = 1;$$

and let us denote by $T^{(h)}$ the diffeomorphism, defined as:

$$T^{(h)} : x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto (x_1, x_2, h x_3) \in \mathbb{R}^3.$$

Also, let us put

$$\alpha^{(h)} := \alpha \circ T^{(h)} \in C(\overline{\Omega}) \quad \text{and} \quad A_0^{(h)} := (\alpha^{(h)})^{-1}(0).$$

Then, by fundamental calculations with use of the area formula, it will be seen that the minimization problem $(\mathrm{MP})_{\rm org}^{(h)}$, for any $0 < h < 1$, has the following equivalent form, denoted by $(\mathrm{MP})^{(h)}$.

$(\mathrm{MP})^{(h)}$ Find a vectorial function $m^{(h)} = (m_1^{(h)}, m_2^{(h)}, m_3^{(h)}) \in L^2(\Omega; \mathbb{R}^3)$ of three variables, which minimizes the following functional:

$$\mathcal{E}^{(h)}(m) := \begin{cases} \Phi_\alpha^{(h)}(m) + \displaystyle\int_\Omega \varphi(m) \, d\mathcal{L}^3 \\ \qquad + \dfrac{1}{2} \displaystyle\int_\Omega \left(\nabla_{\rm P} \zeta \cdot m_{\rm P} + \frac{1}{h} \partial_3 \zeta \, m_3 \right) d\mathcal{L}^3, \\ \qquad \text{if } m \in L^2(\Omega; \mathbb{S}^2), \\ \infty, \ \text{ otherwise}, \end{cases} \tag{4}$$

for any $m = (m_1, m_2, m_3) \in L^2(\Omega; \mathbb{R}^3)$;

subject to:

$$\nabla_{\mathrm{P}} \cdot (-\nabla_{\mathrm{P}} \zeta + \overline{{}^0 m_{\mathrm{P}}}) + \frac{1}{h}\,\partial_3 \left(-\frac{1}{h} \partial_3 \zeta + \overline{{}^0 m_3} \right) = 0, \quad \text{in } \mathbb{R}^3; \tag{5}$$

where the subscript " $_{\mathrm{P}}$ " denotes the restriction of the situation onto the two-dimensional plane $\mathbb{R}^2$, e.g.:

$$y_{\mathrm{P}} := (y_1, y_2), \quad \text{for } y = (y_1, y_2, y_3) \in \mathbb{R}^3,$$

$$\mu_{\mathrm{P}} := (\mu_1, \mu_2) \in L^2(\Omega; \mathbb{R}^2), \quad \text{for } \mu = (\mu_1, \mu_2, \mu_3) \in L^2(\Omega; \mathbb{R}^3),$$

and the distributional gradient

$$\nabla_{\mathrm{P}} \mu := \begin{pmatrix} \partial_1 \mu_1 & \partial_2 \mu_1 \\ \partial_1 \mu_2 & \partial_2 \mu_2 \\ \partial_1 \mu_3 & \partial_2 \mu_3 \end{pmatrix}, \text{ for } \mu = (\mu_1, \mu_2, \mu_3) \in L^2(\Omega; \mathbb{R}^3);$$

and $\Phi_\alpha^{(h)}$ is a convex function on $L^2(\Omega; \mathbb{R}^3)$, defined as:

$$\Phi_\alpha^{(h)}(m) := \begin{cases} \displaystyle\int_{\Omega \setminus A_0^{(h)}} \alpha^{(h)} \left(|\nabla_{\mathrm{P}} m|^2 + \frac{1}{h^2} |\partial_3 m|^2 \right) d\mathcal{L}^3, \\ \qquad \text{if } m \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus A_0^{(h)}; \mathbb{R}^3), \\ \infty, \ \text{otherwise}, \end{cases} \tag{6}$$
$$\text{for any } m = (m_1, m_2, m_3) \in L^2(\Omega; \mathbb{R}^3);$$

Additionally, for any $0 < h < 1$, the equality:

$$m^{(h)} = m^{(h)}_{\mathrm{org}} \circ T^{(h)} \quad \text{in } L^2(\Omega; \mathbb{R}^3); \tag{7}$$

holds between the minimizers $m^{(h)}$ and $m^{(h)}_{\mathrm{org}}$ of the respective problems $(\mathrm{MP})^{(h)}$ and $(\mathrm{MP})^{(h)}_{\mathrm{org}}$. In either case, the minimizers, as in (7), are supposed to represent the most probable profile of the magnetization in the observing thin film. However, under the very thin situation of the thickness h, the problem $(\mathrm{MP})^{(h)}/(\mathrm{MP})^{(h)}_{\mathrm{org}}$ is often reduced to some simpler one.

For the detailed description of this matter, let us first set:

$$\alpha^\circ(x_1, x_2) := \alpha(x_1, x_2, 0) \text{ for any } (x_1, x_2) \in \overline{S}, \ \text{ and } \ A_0^\circ := (\alpha^\circ)^{-1}(0).$$

Here, if we consider the nondegenerate case of the material coefficient α, namely the case that:

$$A_0^{(h)} = A_0^\circ = \emptyset \text{ for } 0 < h < 1, \ \text{ and } \ \alpha_* := \min_{x \in \overline{\Omega}} \alpha(x) > 0;$$

then the convex part $\Phi_\alpha^{(h)}$ of the energy $\mathcal{E}^{(h)}$ satisfies the following coercivity condition:

$$\Phi_\alpha^{(h)}(m) \geq \alpha_* |\nabla m|^2_{L^2(\Omega; \mathbb{R}^{3\times 3})}, \quad \text{for all } m \in L^2(\Omega; \mathbb{R}^3); \tag{8}$$

and hence we can apply the theories, studied in [3, 6, 7, 8, 14, 17, 18, 19, 20], to find a definite association between the limiting profile of $(\mathrm{MP})^{(h)}$ as $h \searrow 0$, and the following minimization problem, denoted by $(\mathrm{MP})^\circ$, for the magnetization on the two-dimensional domain S.

$(\mathrm{MP})^\circ$ Find a vectorial function $m^\circ = (m_1^\circ, m_2^\circ, m_3^\circ) \in L^2(S; \mathbb{R}^3)$ of two variables, which minimizes the following functional:

$$\mathcal{E}^\circ(m) := \begin{cases} \Phi_\alpha^\circ(m) + \displaystyle\int_S \varphi(m)\, d\mathcal{L}^2 + \frac{1}{2} \int_S |m_3|^2 \, d\mathcal{L}^2, \\ \qquad \text{if } m \in L^2(S; \mathbb{S}^2), \\ \infty, \ \text{otherwise}, \end{cases} \tag{9}$$
$$\text{for any } m = (m_1, m_2, m_3) \in L^2(S; \mathbb{R}^3);$$

where Φ_α° is a convex function on $L^2(S; \mathbb{R}^3)$, defined as:

$$\Phi_\alpha^\circ(m) := \begin{cases} \displaystyle\int_{S \setminus A_0^\circ} \alpha^\circ |\nabla m|^2 \, d\mathcal{L}^2, & \text{if } m \in W^{1,2}_{\mathrm{loc}}(S \setminus A_0^\circ; \mathbb{R}^3), \\ \infty, \ \text{otherwise}, \end{cases} \tag{10}$$
$$\text{for any } m = (m_1, m_2, m_3) \in L^2(S; \mathbb{R}^3).$$

Up to the present date, the proof of the above fact has been performed by relying on the compactness of the sublevel sets of $\mathcal{E}^{(h)}$, that has been derived from the coercivity condition (8).

Now, the main theme of this study is to verify whether some analogous conclusion can be obtained even under degenerative situations of α, or not. So, as the first step of the research, we here impose the following two conditions for the material coefficient α:

(a1) $\mathcal{L}^3(A_0) = 0$, and hence $\mathcal{L}^3(A_0^{(h)}) = 0$, for $0 < h < 1$;
(a2) there exists a constant $C_\alpha \geq 1$, such that

$$\alpha^\circ(x_{\mathrm{P}}) \leq \alpha(x) \leq C_\alpha \alpha^\circ(x_{\mathrm{P}}), \quad \text{for all } x = (x_1, x_2, x_3) \in \overline{\Omega}.$$

Consequently, some positive conclusions for our theme will be shown in the main theorem, stated as follows.

Main Theorem. *(I) Let us assume the condition (a1). Then, for any $0 < h < 1$, the minimization problem $(MP)^{(h)}$ admits at least one solution (minimizer) $m^{(h)}$, and hence the same holds for the problem $(MP)^{(h)}_{\mathrm{org}}$.*

(II) Under the conditions (a1)-(a2), there exist a sequence $\{h_i \,|\, i = 1, 2, 3, \cdots\} \subset (0, 1)$ and a limiting function $m^\circ \in L^2(S; \mathbb{R}^3)$ of two variables, such that:

(i) $h_i \searrow 0$, $m^{(h_i)} \to m^\circ$ in $L^2(\Omega; \mathbb{R}^3)$, $\mathcal{E}^{(h_i)}(m^{(h_i)}) \to \mathcal{E}^\circ(m^\circ)$, and

$$\begin{cases} \nabla_{\mathrm{P}} m^{(h_i)}(x_1, x_2, x_3) \to \nabla_{\mathrm{P}} m^\circ(x_1, x_2) \ (= \nabla m^\circ(x_1, x_2)), \\ \dfrac{1}{h_i} \partial_3 m^{(h_i)}(x_1, x_2, x_3) \to 0, \end{cases} \tag{11}$$
$$\text{for } \mathcal{L}^2\text{-a.e. } (x_1, x_2) \in S \text{ and } \mathcal{L}^1\text{-a.e. } x_3 \in (0, 1),$$

as $i \to \infty$;

(ii) the limit m° solves the problem $(MP)^\circ$;

where $\{m^{(h)} \,|\, 0 < h < 1\}$ is the sequence of minimizers $m^{(h)}$, $0 < h < 1$, obtained in (I).

The content of this paper is as follows. In the next Section 2, the mathematical treatment of the coupled Maxwell equation (5) is described, with the references of foregoing works [18, 20]. In Sections 3-4, the key-properties of the energy functionals $\mathcal{E}^{(h)}$ and $\Phi_\alpha^{(h)}$ are shown, including the compactness of sublevel sets, without help from (8), and the limiting observation (Γ-convergence) for the functionals as $h \searrow 0$. Finally, Section 5 is devoted to the proof of Main Theorem.

Notation. For any dimension $n \in \mathbb{N}$, the n-dimensional Lebesgue measure is denoted by $\mathcal{L}^n$, and for any Borel set $E \subset \mathbb{R}^n$, the characteristic function on E is denoted by χ_E.

For any abstract Banach space, the norm of X is denoted by $|\cdot|_X$. However, when X is an Euclidean space, the norm is simply denoted by $|\cdot|$. Also, we denote by $\mathrm{dist}_X(\xi, Y)$ the distance between any point $\xi \in X$ and any subset $Y \subset X$, that is defined as $\mathrm{dist}_X(\xi, Y) := \inf_{\eta \in Y} |\xi - \eta|_X$. Additionally, for any $r > 0$ and any functional $F : X \longrightarrow [-\infty, \infty]$, we denote by $L(r; F)$ the sublevel set of F, more precisely:

$$L(r; F) := \left\{ \ \xi \in X \ \middle| \ F(\xi) \leq r \ \right\}.$$

For any abstract Hilbert space H, the inner product of H is denoted by $(\cdot, \cdot)_H$. However, when H is an Euclidean space, the inner product between two vectors $\xi, \eta \in H$ is simply denoted by $\xi \cdot \eta$. Besides, for arbitrary $k, \ell \in \mathbb{N}$ and arbitrary $k \times \ell$-matrices $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{k \times \ell}$, the scalar product between these two matrices is denoted by $A : B$, more precisely, $A : B := \sum_{i=1}^{k} \sum_{j=1}^{\ell} a_{ij} b_{ij}$.

2. **Mathematical treatment of the Maxwell equation.** In this section, we focus on the coupled Maxwell equation (5), to recall its rigorous mathematical treatment, studied in [18, 20].

Hereafter, let us fix any (three-dimensional) open ball B_Ω, which contains the cylindrical domain Ω. Then, the phase space for the Maxwell equation (5) is settled as the following functional space, denoted by V:

$$V := \left\{ \ v \in H^1_{\mathrm{loc}}(\mathbb{R}^3) \ \middle| \ \nabla v \in L^2(\mathbb{R}^3; \mathbb{R}^3) \text{ and } \int_{B_\Omega} v \, d\mathcal{L}^3 = 0 \ \right\}.$$

As it is easily checked (cf. [5, Theorem 5.4.3]), this functional space is a Hilbert space, endowed with the inner product:

$$(z, v)_V := \int_{\mathbb{R}^3} \left(\nabla_{\mathrm{P}} z \cdot \nabla_{\mathrm{P}} v + \frac{1}{h^2} \partial_3 z \partial_3 v \right) d\mathcal{L}^3, \quad \text{for all } z, v \in V;$$

where $0 < h < 1$ is the same constant as in (5). Additionally, the Hilbert space V is compactly embedded into the space $L^2(B_\Omega)$.

On the basis of the above notation, the solution of the Maxwell equation (5) is prescribed as follows.

Definition 2.1. Let us fix any constant $0 < h < 1$, and any function $m = (m_1, m_2, m_3) \in L^2(\Omega; \mathbb{R}^3)$. Then, the solution of the equation (5) is defined as a function $\zeta \in V$, which solves the following variational identity:

$$(\zeta, v)_V = \int_\Omega \left(m_{\mathrm{P}} \cdot \nabla_{\mathrm{P}} v + \frac{1}{h} m_3 \partial_3 v \right) d\mathcal{L}^3, \quad \text{for any } v \in V. \tag{12}$$

The above definition method was proposed by James-Kinderlehrer [20], and in the cited paper, the authors also demonstrated the well-posedness for (5), summarized in the following proposition.

Proposition 1. *(Summary of [20, Lemma 3.1]) Let us fix any $0 < h < 1$. Then, for any $m \in L^2(\Omega; \mathbb{R}^3)$, the Maxwell equation (5) admits a unique solution ζ. Hence, the solution operator $\mathcal{S}^{(h)} : L^2(\Omega; \mathbb{R}^3) \longrightarrow V$, that maps any $m \in L^2(\Omega; \mathbb{R}^3)$ to the* *solution $\zeta \in V$ of (5), is well-defined as a single-valued mapping. Moreover, the solution operator $\mathcal{S}^{(h)}$ is a bounded linear operator, such that:*

$$|\mathcal{S}^{(h)} m|_V \leq |m|_{L^2(\Omega;\mathbb{R}^3)}, \quad \textit{for any } m \in L^2(\Omega;\mathbb{R}^3). \tag{13}$$

Next, let us look toward the limiting observation for (5), as $h \searrow 0$. As a ground-breaking work for this theme, we can refer to [18, Proposition 4.1], stated as follows.

Proposition 2. *(Summary of* [18, Proposition 4.1]*) Let $\{\tilde{m}^{(h)} \,|\, 0 < h < 1\} \subset L^2(\Omega;\mathbb{R}^3)$ be a fixed sequence, such that $\tilde{m}^{(h)} \to \tilde{m}$ in $L^2(\Omega;\mathbb{R}^3)$ as $h \searrow 0$, for some $\tilde{m} = (\tilde{m}_1, \tilde{m}_2, \tilde{m}_3) \in L^2(\Omega;\mathbb{R}^3)$. For any $0 < h < 1$, let $\zeta^{(h)}$ be the solution of the Maxwell equation (5) when $m = \tilde{m}^{(h)}$. Let $E_{\mathrm{mag}}^{(h)}$ and E_{mag}° be functionals on $L^2(\Omega;\mathbb{R}^3)$, which are respectively defined as:*

$$\begin{cases} E_{\mathrm{mag}}^{(h)}(m) := \dfrac{1}{2} \displaystyle\int_\Omega \left(\nabla_{\mathrm{P}} \zeta^{(h)} \cdot m_{\mathrm{P}} + \frac{1}{h} \partial_3 \zeta^{(h)} \, m_3 \right) d\mathcal{L}^3, \\ E_{\mathrm{mag}}^{\circ}(m) := \dfrac{1}{2} \displaystyle\int_\Omega |m_3|^2 \, d\mathcal{L}^3, \end{cases} \tag{14}$$
$$\textit{for any } m = (m_1, m_2, m_3) \in L^2(\Omega;\mathbb{R}^3).$$

Then,

$$\zeta^{(h)} \to 0 \textit{ in } V, \quad \textit{and} \quad \frac{1}{h} \partial_3 \zeta^{(h)} \to \tilde{m}_3 \textit{ in } L^2(\Omega), \quad \textit{as } h \searrow 0, \tag{15}$$

and hence

$$E_{\mathrm{mag}}^{(h)}(\tilde{m}^{(h)}) \to E_{\mathrm{mag}}^{\circ}(\tilde{m}), \quad \textit{as } h \searrow 0.$$

Remark 1. For any $0 < h < 1$, the functional $E_{\mathrm{mag}}^{(h)}$, given in (14), links to the part of the free energy $\mathcal{E}^{(h)}$, given in (4), that is involved in the coupled Maxwell equation (5). Moreover, in the light of Definition 2.1, we have:

$$E_{\mathrm{mag}}^{(h)}(\tilde{m}^{(h)}) = \frac{1}{2} |\zeta^{(h)}|_V^2 \geq 0, \quad \text{for any } 0 < h < 1; \tag{16}$$

under the same notations as in Proposition 2.

3. **Key-properties of energy functionals.** We start with the description of the discussion points, that are planning in Sections 3-4. In these sections, four theorems will be demonstrated with some corollaries.

The first theorem is concerned with a Hilbert space, associated with the effective domains of convex parts of energy functionals.

Theorem 3.1. *Let us set:*

$$\begin{cases} A_0^\dagger := A_0^\circ \times (0,1), \\ X_\alpha^\dagger := \left\{ m \in L^2(\Omega;\mathbb{R}^3) \;\middle|\; \begin{array}{l} m \in W_{\mathrm{loc}}^{1,2}(\Omega \setminus A_0^\dagger;\mathbb{R}^3), \\ \sqrt{\alpha^\circ}\, \nabla m \in L^2(\Omega \setminus A_0^\dagger;\mathbb{R}^{3\times 3}) \end{array} \right\}. \end{cases} \tag{17}$$

Then, $X_\alpha^\dagger$ is a Hilbert space, endowed with the inner product:

$$(\xi, \eta)_{X_\alpha^\dagger} := \int_\Omega \xi \cdot \eta \, d\mathcal{L}^3 + \int_{\Omega \setminus A_0^\dagger} \alpha^\circ \, \nabla \xi : \nabla \eta \, d\mathcal{L}^3, \quad \textit{for all } \xi, \eta \in X_\alpha^\dagger. \tag{18}$$

Hence, the functional $\Phi_\alpha^\dagger$, defined as:

$$\Phi_\alpha^\dagger(m) := \begin{cases} \displaystyle\int_{\Omega\setminus A_0^\dagger} \alpha^\circ |\nabla m|^2 \, d\mathcal{L}^2, & \text{if } m \in X_\alpha^\dagger, \\ \infty, & \text{otherwise}, \end{cases} \tag{19}$$
$$\text{for any } m = (m_1, m_2, m_3) \in L^2(\Omega;\mathbb{R}^3);$$

is proper l.s.c. and convex on $L^2(\Omega;\mathbb{R}^3)$.

Just as in the above theorem, we can prove the following corollary.

Corollary 1. *(I) Let us fix any $0 < h < 1$, and let us denote by $X_\alpha^{(h)}$ the effective domain of the convex function $\Phi_\alpha^{(h)}$, given in (6). Then, $X_\alpha^{(h)}$ is a Hilbert space, endowed with the inner product:*

$$(\xi,\eta)_{X_\alpha^{(h)}} := \int_\Omega \xi\cdot\eta \, d\mathcal{L}^3 + \int_{\Omega\setminus A_0^{(h)}} \alpha^{(h)} \left(\nabla_{\mathrm{P}}\xi : \nabla_{\mathrm{P}}\eta + \frac{1}{h^2}\partial_3\xi\cdot\partial_3\eta \right) d\mathcal{L}^3,$$
$$\text{for all } \xi,\eta \in X_\alpha^{(h)}.$$

Hence, the convex function $\Phi_\alpha^{(h)}$ turns out to be proper and l.s.c. on $L^2(\Omega;\mathbb{R}^3)$.

(II) Let us denote by X_α° the effective domain of the convex function Φ_α°, given in (10). Then, X_α° is a Hilbert space, endowed with the inner product:

$$(\xi,\eta)_{X_\alpha^\circ} := \int_S \xi\cdot\eta \, d\mathcal{L}^2 + \int_{S\setminus A_0^\circ} \alpha^\circ \, \nabla\xi : \nabla\eta \, d\mathcal{L}^2, \quad \text{for all } \xi,\eta \in X_\alpha^\circ.$$

Hence, the convex function Φ_α° turns out to be proper and l.s.c. on $L^2(S;\mathbb{R}^3)$.

Remark 2. As it is easily checked, the two convex functions $\Phi_\alpha^\dagger$ and Φ_α°, as in Theorem 3.1 and Corollary 1, coincide with as functionals on $L^2(S;\mathbb{R}^3)$, namely:

$$\Phi_\alpha^\dagger(m) = \Phi_\alpha^\circ(m), \quad \text{if } m \in L^2(S;\mathbb{R}^3).$$

The discussion point of the second theorem is in the compactness of the embedding, relative to the Hilbert spaces $X_\alpha^{(h)}$, $0 < h < 1$, and X_α°.

Theorem 3.2. *(Compactness) Let us assume the condition (a1), as in introduction, and let us take any $2 < p \le \infty$. Then, for any $0 < h < 1$, any bounded sequence in $X_\alpha^{(h)} \cap L^p(\Omega;\mathbb{R}^3)$ is relatively compact in $L^2(\Omega;\mathbb{R}^3)$. As well as, if we assume that:*

(a1)$^\circ$ $\mathcal{L}^2(A_0^\circ) = 0$;

then any bounded sequence in $X_\alpha^\circ \cap L^p(S;\mathbb{R}^3)$ is relatively compact in $L^2(S;\mathbb{R}^3)$.

Here is a corollary that is derived from the second theorem.

Corollary 2. *(I) Let us assume the condition (a1), as in introduction, and let us take any $1 \le p < 2$. Then, for any $0 < h < 1$, the Hilbert space $X_\alpha^{(h)}$ is compactly embedded into the Banach space $L^p(\Omega;\mathbb{R}^3)$. As well as, if we assume the condition (a1)$^\circ$, as in Theorem 3.2, then the Hilbert space X_α° is compactly embedded into the Banach space $L^p(S;\mathbb{R}^3)$.*

(II) Let us assume the condition (a1), as in introduction, then for any $0 < h < 1$ and any $r > 0$, the sublevel set:

$$L(r;\mathcal{E}^{(h)}) := \left\{ \ m \in L^2(\Omega;\mathbb{R}^3) \ \middle| \ \mathcal{E}^{(h)}(m) \le r \ \right\};$$

is compact in $L^2(\Omega;\mathbb{R}^3)$*. As well as, if we assume the condition (a1)*$^\circ$*, as in Theorem 3.2, then for any* $r > 0$*, the sublevel set:*

$$L(r;\mathcal{E}^\circ) := \left\{ \ m \in L^2(\Omega;\mathbb{R}^3) \ \middle| \ \mathcal{E}^\circ(m) \leq r \ \right\};$$

is compact in $L^2(S;\mathbb{R}^3)$*.*

In the third theorem, we focus on the limiting observation of the sequence $\{\Phi_\alpha^{(h)} \,|\, 0 < h < 1\}$ of convex functions, as $h \searrow 0$.

Theorem 3.3. *(Mosco convergence as* $h \searrow 0$*) Let us assume the conditions (a1)-(a2), as in introduction. Then, the sequence* $\{\Phi_\alpha^{(h)} \,|\, 0 < h < 1\}$ *of convex functions converges to (the infinity-extension of) the convex function* Φ_α°*, on* $L^2(\Omega;\mathbb{R}^3)$*, in the sense of Mosco (cf. [23]), as* $h \searrow 0$*. More precisely:*

(m1) $\liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\mu^{(h)}) \geq \Phi_\alpha^\circ(\mu)$*, if* $\{\mu^{(h)} \,|\, 0 < h < 1\} \subset L^2(\Omega;\mathbb{R}^3)$ *is a bounded sequence, in the topology of* $L^2(\Omega;\mathbb{R}^3)$*, having a weak limit* $\mu \in L^2(\Omega;\mathbb{R}^3)$*;*

(m2) for any $\nu \in X_\alpha^\circ$ $(\subset L^2(S;\mathbb{R}^3))$*, there exists a sequence* $\{\mu_\nu^{(h)} \,|\, 0 < h < 1\} \subset L^2(\Omega;\mathbb{R}^3)$*, such that* $\mu_\nu^{(h)} \to \nu$ *in* $L^2(\Omega;\mathbb{R}^3)$ *and* $\Phi_\alpha^{(h)}(\mu_\nu^{(h)}) \to \Phi_\alpha^\circ(\nu)$*, as* $h \searrow 0$*.*

Additionally, checking the above Mosco convergence from the theory of Γ- convergence (cf. [1, 11]), we conclude the following corollary.

Corollary 3. *(*Γ*-convergence as* $h \searrow 0$*) Under the same assumption as in Theorem 3.3, the sequence* $\{\mathcal{E}^{(h)} \,|\, 0 < h < 1\}$ *of energy functionals converges to (the infinity-extension of) the functional* $\mathcal{E}^\circ$*, on* $L^2(\Omega;\mathbb{R}^3)$*, in the sense of* Γ*-convergence, as* $h \searrow 0$*.*

Remark 3. According to [1, Lemma 2.3], the Γ-convergence, asserted in Corollary 3, can be concluded, if and only if:

(γ1) $\liminf_{h \searrow 0} \mathcal{E}^{(h)}(\mu^{(h)}) \geq \mathcal{E}^\circ(\mu)$, if $\{\mu^{(h)} \,|\, 0 < h < 1\} \subset L^2(\Omega;\mathbb{R}^3)$ is a convergent sequence, in the topology of $L^2(\Omega;\mathbb{R}^3)$, having a (strong) limit $\mu \in L^2(\Omega;\mathbb{R}^3)$;

(γ2) for any $\nu \in X_\alpha^\circ$ $(\subset L^2(S;\mathbb{R}^3))$, there exists a sequence $\{\mu_\nu^{(h)} \,|\, 0 < h < 1\} \subset L^2(\Omega;\mathbb{R}^3)$, such that $\mu_\nu^{(h)} \to \nu$ in $L^2(\Omega;\mathbb{R}^3)$ and $\mathcal{E}^{(h)}(\mu_\nu^{(h)}) \to \mathcal{E}^\circ(\nu)$, as $h \searrow 0$.

Here, with helps from Proposition 2 and the constraint onto $L^2(\Omega;\mathbb{S}^2)$ as in (4), we can derive the conditions (γ1)-(γ2) from the conditions (m1)-(m2) of Mosco convergence, mentioned in Theorem 3.3.

The final fourth theorem is concerned with a sort of uniform compactness of sublevel sets, with respect to $0 < h < 1$.

Theorem 3.4. *(Uniform compactness) Let us assume the conditions (a1)-(a2). Then, for arbitrary* $2 < p \leq \infty$ *and arbitrary* $r, R > 0$*,*

$$\bigcup_{0<h<1} L(r;\Phi_\alpha^{(h)}) \cap \left\{ \ m \in X_\alpha^{(h)} \ \middle| \ |m|_{L^p(\Omega;\mathbb{R}^3)} \leq R \ \right\} \subset X_\alpha^\dagger, \tag{20}$$

and the above union is compact in $L^2(\Omega;\mathbb{R}^3)$*. As well as,*

$$\bigcup_{0<h<1} L(r;\mathcal{E}^{(h)}) \subset X_\alpha^\dagger \cap L^2(\Omega;\mathbb{S}^2), \tag{21}$$

and the above union is compact in $L^2(\Omega;\mathbb{R}^3)$*.*

4. **Proofs of Theorems 3.1-3.4.** This section is devoted to the proofs of Theorems 3.1-3.4. The proofs of all theorems will be based on the following lemma.

Lemma 4.1. *(Approximating open sets)*

(I) For any $0 < h < 1$*, there exists a sequence* $\{\Omega_\ell^{(h)} \,|\, \ell = 1, 2, 3, \cdots\} \subset \mathbb{R}^3$ *of three-dimensional open sets, having Lipschitz boundaries, such that:*

$$\emptyset \neq \Omega_1^{(h)} \subset\subset \Omega_2^{(h)} \subset\subset \Omega_3^{(h)} \subset\subset \cdots \subset\subset \Omega_\ell^{(h)} \subset\subset \cdots \subset\subset \Omega \setminus A_0^{(h)} = \bigcup_{\ell=0}^{\infty} \Omega_\ell^{(h)}. \quad (22)$$

(II) There exists a sequence $\{S_\ell \,|\, \ell = 1, 2, 3, \cdots\} \subset \mathbb{R}^2$ *of two-dimensional open sets, having Lipschitz boundaries, such that:*

$$\emptyset \neq S_1 \subset\subset S_2 \subset\subset S_3 \subset\subset \cdots \subset\subset S_\ell \subset\subset \cdots \subset\subset S \setminus A_0^\circ = \bigcup_{\ell=0}^{\infty} S_\ell.$$

Proof of Lemma 4.1. In the proof of the assertion (I), the elements of the required sequence will be selected from a class of open sets $\Delta_{\varepsilon,\delta,\tau}$, $0 < \varepsilon, \delta, \tau < 1$, prescribed as follows.

$$\Delta_{\varepsilon,\delta,\tau} := \left\{ \ x \in \Omega \ \middle| \ (\rho_\varepsilon * \chi_{D_\delta})(x) > \tau \ \right\}, \quad \text{for all } 0 < \varepsilon, \delta, \tau < 1;$$

where the functions ρ_ε, $0 < \varepsilon < 1$, are the usual mollifiers, the notation "$*$" denotes the convolution between functions, and

$$D_\delta := \left\{ \ x \in \Omega \setminus A_0^{(h)} \ \middle| \ \mathrm{dist}_{\mathbb{R}^3}(y, A_0^{(h)} \cup \partial\Omega) > \delta \ \right\}, \quad \text{for any } 0 < \delta < 1.$$

Then, Sard's theorem enables to take appropriate sequences $\{\varepsilon_\ell, \delta_\ell, \tau_\ell \,|\, \ell = 1, 2, 3, \cdots\} \subset (0, 1)$, such that the open sets $\Omega_\ell^{(h)} := \Delta_{\varepsilon_\ell,\delta_\ell,\tau_\ell}$, $\ell = 1, 2, 3, \cdots$, satisfy the condition (22) with the smoothness of their boundaries.

On the other hand, the proof of the assertion (II) will be just a modified version of the above one, arranged for the two-dimensional situation. □

Remark 4. As a consequence of Lemma 4.1, we infer that:

$$\begin{cases} a_\ell^{(h)} := \min\limits_{x \in \overline{\Omega_\ell^{(h)}}} \alpha^{(h)}(x) > 0, \quad \ell = 1, 2, 3, \cdots, \\ a_\ell^{(h)} \searrow 0, \quad \text{as } \ell \to \infty, \end{cases} \qquad \text{for any } 0 < h < 1.$$

As well as, we may say that:

$$a_\ell^\circ := \min_{x \in \overline{S_\ell}} \alpha^\circ(x) > 0, \ \ell = 1, 2, 3, \cdots, \ \text{ and } \ a_\ell^\circ \searrow 0 \text{ as } \ell \to \infty.$$

Remark 5. If we additionally assume the condition (a2) in Lemma 4.1, then the sequences $\{\Omega_\ell^{(h)}\}$, $0 < h < 1$, can be taken independently of h. In fact, since the condition (a2) implies that:

(†) $A_0^{(h)} = A_0^\dagger$, for any $0 < h < 1$;

it is easily checked that all of open sets, given as:

$$\Omega_\ell^\dagger := S_\ell \times (0, 1), \quad \ell = 1, 2, 3, \cdots;$$

have Lipschitz boundaries, and the (h-independent) sequence $\{\Omega_\ell^\dagger \,|\, \ell = 1, 2, 3, \cdots\}$ fulfills (22), for any $0 < h < 1$.

Proof of Theorem 3.1. We can easily check that the set $X_\alpha^\dagger$ is a linear space. Also, by the definition formula (18),

$$(\xi, \xi)_{X_\alpha^\dagger} \geq |\xi|^2_{L^2(\Omega;\mathbb{R}^3)}, \quad \text{for any } \xi \in X_\alpha^\dagger. \tag{23}$$

It implies that the left hand side of (23) is always nonnegative, and it is equal to zero if and only if $\xi = 0$, $\mathcal{L}^3$-a.e. in Ω. After this, the bi-linearity, inherent in (18), guarantees that the pairing $(\cdot, \cdot)_{X_\alpha^\dagger}$ defines a certain inner product in $X_\alpha^\dagger$.

Now, all we have to do is to verify the completeness of the topology, provided by the inner product $(\cdot, \cdot)_{X_\alpha^\dagger}$. To this end, we take any Cauchy sequence $\{\xi^{(i)} \,|\, i = 1, 2, 3, \cdots\} \subset X_\alpha^\dagger$, namely for any $\varepsilon > 0$, we suppose the existence of the index number $n_\varepsilon \in \mathbb{N}$, such that:

$$|\xi^{(i)} - \xi^{(j)}|^2_{X_\alpha^\dagger} = \int_\Omega |\xi^{(i)} - \xi^{(j)}|^2 \, d\mathcal{L}^3 + \int_{\Omega \setminus A_0^\dagger} \alpha^\circ |\nabla(\xi^{(i)} - \xi^{(j)})|^2 \, d\mathcal{L}^3 \leq \varepsilon,$$
$$\text{for all } i, j \geq n_\varepsilon.$$

Then, in the light of Remark 4,

$$\begin{cases} \bullet \ \{\xi^{(i)}\} \text{ is a Cauchy sequence of } L^2(\Omega;\mathbb{R}^3); \\ \bullet \ \{\xi^{(i)}\} \text{ is a Cauchy sequence of } W^{1,2}(\Omega_\ell^\dagger;\mathbb{R}^3), \text{ for any } \ell \in \mathbb{N}; \\ \bullet \ \{\sqrt{\alpha^\circ}\, \nabla \xi^{(i)}\} \text{ is a Cauchy sequence in } L^2(\Omega \setminus A_0^\dagger;\mathbb{R}^{3\times 3}). \end{cases}$$

So, by the completeness in the Hilbert spaces, listed the above, we find a function $\xi \in L^2(\Omega;\mathbb{R}^3) \cap W^{1,2}_{\mathrm{loc}}(\Omega \setminus A_0^\dagger;\mathbb{R}^3)$, such that:

$$\begin{cases} \xi^{(i)} \to \xi \text{ in } L^2(\Omega;\mathbb{R}^3), \\ \xi^{(i)} \to \xi \text{ in } W^{1,2}(\Omega_\ell^\dagger;\mathbb{R}^3), \text{ for any } \ell \in \mathbb{N}, \\ \sqrt{\alpha^\circ}\, \nabla \xi^{(i)} \to \sqrt{\alpha^\circ}\, \nabla \xi \text{ in } L^2(\Omega \setminus A_0^\dagger;\mathbb{R}^{3\times 3}), \end{cases} \quad \text{as } i \to \infty.$$

From these convergences, it is deduced that:

$$\begin{aligned} |\xi^{(i)} - \xi|_{X_\alpha^\dagger} &= \lim_{j \to \infty} \left(\int_\Omega |\xi^{(i)} - \xi^{(j)}|^2 \, d\mathcal{L}^3 + \int_{\Omega \setminus A_0^\dagger} \alpha^\circ |\nabla(\xi^{(i)} - \xi^{(j)})|^2 \, d\mathcal{L}^3 \right) \\ &\leq \sup_{j \geq n_\varepsilon} |\xi^{(i)} - \xi^{(j)}|^2_{X_\alpha^\dagger} \leq \varepsilon, \quad \text{for all } i \geq n_\varepsilon; \end{aligned}$$

and hence $\xi = \xi^{(n_\varepsilon)} + (\xi - \xi^{(n_\varepsilon)}) \in X_\alpha^\dagger$. Thus, $X_\alpha^\dagger$ is a Hilbert space.

Next, with regard to the functional $\Phi_\alpha^\dagger$, its convexity is immediately seen from the quadratic form, as in (19). Also, noting that $X_\alpha^\dagger \supset W^{1,2}(\Omega;\mathbb{R}^3)$, we infer that the convex function $\Phi_\alpha^\dagger$ is proper on $L^2(\Omega;\mathbb{R}^3)$. Furthermore, the lower semi-continuity of $\Phi_\alpha^\dagger$ can be verified by checking the closedness of the sublevel sets:

$$L(r;\Phi_\alpha^\dagger) := \left\{ \ m \in X_\alpha^\dagger \ \middle| \ \Phi_\alpha^\dagger(m) \leq r \ \right\}, \quad \text{for } r > 0.$$

In fact, assuming that:

$$\begin{cases} \bullet \ r > 0, \ \ \{\eta^{(i)} \,|\, i = 1, 2, 3, \cdots\} \subset L(r;\Phi_\alpha^{(h)}), \ \ \eta \in L^2(\Omega;\mathbb{R}^3), \\ \bullet \ \eta^{(i)} \to \eta \text{ in } L^2(\Omega;\mathbb{R}^3) \text{ as } i \to \infty; \end{cases}$$

we immediately have:

$$\eta^{(i_k)} \to \eta \text{ weakly in } X_\alpha^\dagger \text{ as } k \to \infty, \ \text{ and hence } \ \eta \in X_\alpha^\dagger;$$

for some subsequence $\{\eta^{(i_k)} \,|\, k = 1, 2, 3, \cdots\} \subset \{\eta^{(i)}\}$. So, by the weak lower semi-continuity of the convex function $m \in X_\alpha^\dagger \mapsto |m|_{X_\alpha^\dagger}^2$, it is deduced that:

$$\begin{aligned}\Phi_\alpha^{(h)}(\eta) &= |\eta|_{X_\alpha^\dagger}^2 - |\eta|_{L^2(\Omega;\mathbb{R}^3)}^2 \leq \liminf_{k\to\infty} |\eta^{(i_k)}|_{X_\alpha^\dagger}^2 - \lim_{k\to\infty} |\eta^{(i_k)}|_{L^2(\Omega;\mathbb{R}^3)}^2 \\ &= \liminf_{k\to\infty} \left(|\eta^{(i_k)}|_{X_\alpha^\dagger}^2 - |\eta^{(i_k)}|_{L^2(\Omega;\mathbb{R}^3)}^2 \right) \leq \sup_{i\in\mathbb{N}} \Phi_\alpha^{(h)}(\eta^{(i)}) \leq r.\end{aligned}$$

Hence, $\eta \in L(r; \Phi_\alpha^{(h)})$. □

Proof of Theorem 3.2. We prove only the assertion for the space $X_\alpha^\circ \cap L^p(S;\mathbb{R}^3)$ with $2 < p \leq \infty$, since the demonstration method for the other one is essentially the same.

Let us assume the condition (a1)°, let us fix any $2 < q < p$, and let us set $r := \lim_{\tilde{p}\nearrow p}(\tilde{p}/(\tilde{p}-q))$. Besides, let us take any sequence $\{\xi_*^{(i)} \,|\, i = 1, 2, 3, \cdots\} \subset X_\alpha^\circ \cap L^p(S;\mathbb{R}^3)$, such that:

$$\sup_{i\in\mathbb{N}} |\xi_*^{(i)}|_{X_\alpha^\circ} \leq R_0 \quad \text{and} \quad \sup_{i\in\mathbb{N}} |\xi_*^{(i)}|_{L^p(S;\mathbb{R}^3)} \leq R_0; \tag{24}$$

for some constant R_0, independent of $i \in \mathbb{N}$. Then, noting that $\mathcal{L}^2(S) = 1$, and:

$$\sup_{i\in\mathbb{N}} \left| |\xi_*^{(i)}|^2 \right|_{L^{q/2}(S;\mathbb{R}^3)} = \sup_{i\in\mathbb{N}} |\xi_*^{(i)}|_{L^q(S;\mathbb{R}^3)}^2 \leq \sup_{i\in\mathbb{N}} |\xi_*^{(i)}|_{L^p(S;\mathbb{R}^3)}^2 \leq R_0^2;$$

we find a sequence $\{n_k^{(0)} \,|\, k = 1, 2, 3, \cdots\} \subset \mathbb{N}$, and functions $\xi_* \in X_\alpha^\circ$ and $\gamma_* \in L^{q/2}(S)$, such that:

$$\begin{cases} n_k^{(0)} \nearrow \infty, \;\; \xi_*^{(n_k^{(0)})} \to \xi_* \text{ weakly in } X_\alpha^\circ, \\ \text{and } \;\; |\xi_*^{(n_k^{(0)})}|^2 \to \gamma_* \text{ weakly in } L^{q/2}(S), \end{cases} \quad \text{as } k \to \infty. \tag{25}$$

The above convergence implies that:

$$\int_E |\xi_*^{(n_k^{(0)})}|^2 \, d\mathcal{L}^2 = \int_S |\xi_*^{(n_k^{(0)})}|^2 \chi_E \, d\mathcal{L}^2 \to \int_S \gamma \chi_E \, d\mathcal{L}^2 = \int_E \gamma \, d\mathcal{L}^2$$
$$\text{as } k \to \infty, \;\; \text{for any Borel subset } E \subset S.$$

So, applying the assumption (a1)° and Vitali-Hahn-Saks's theorem [2, Theorem 1.30], we infer that:

$$I_*^{(\ell)} := \sup_{j\in\mathbb{N}} \int_{S\setminus S_\ell} |\xi_*^{(n_j^{(0)})}|^2 \, d\mathcal{L}^2 \to 0, \quad \text{as } \ell \to \infty. \tag{26}$$

Next, due to Lemma 4.1 and Remark 4, the subsequence $\{\xi_*^{(n_k^{(0)})} \,|\, k = 1, 2, 3, \cdots\} \subset \{\xi_*^{(i)}\}$ turns out to be bounded in the space $W^{1,2}(S_\ell;\mathbb{R}^3)$, for any $\ell \in \mathbb{N}$. Hence, Sobolev's embedding theorem enables to construct a decreasing family of subsequences:

$$\{n_k^{(0)}\} \supset \{n_k^{(1)}\} \supset \{n_k^{(2)}\} \supset \{n_k^{(3)}\} \supset \cdots \supset \{n_k^{(\ell)}\} \supset \cdots;$$

to fulfill that:

$$\left.\begin{array}{l} \bullet \text{ the subsequence } \{\xi_*^{(n_k^{(\ell)})}\} \text{ admits a limit } \\ \quad \eta_*^{(\ell)} \in W^{1,2}(S_\ell;\mathbb{R}^3) \text{, in the (strong) topology} \\ \quad \text{of } L^2(S_\ell;\mathbb{R}^3) \text{, as } k \to \infty, \\ \bullet \; |\xi_*^{(n_k^{(\ell)})} - \eta_*^{(\ell)}|_{L^2(S_\ell;\mathbb{R}^3)}^2 \leq \dfrac{1}{\ell}, \;\; k = 1, 2, 3, \cdots, \end{array}\right\} \text{for any } \ell \in \mathbb{N}. \tag{27}$$

Now, let us set a function $\eta_* \in W^{1,2}_{\rm loc}(S \setminus A_0^\circ; \mathbb{R}^3)$, by putting:

$$\eta_*(x) := \eta_*^{(\ell)}(x), \text{ if } x \in S_\ell, \text{ for } \mathcal{L}^2\text{-a.e. } x \in S.$$

Then, by virtue of (24) and the monotone convergence theorem,

$$\begin{aligned} &\int_S |\eta_*|^2 \, d\mathcal{L}^2 = \lim_{\ell \to \infty} \int_S \chi_{S_\ell} |\eta_*|^2 \, d\mathcal{L}^2 \leq \sup_{\ell \in \mathbb{N}} \int_{S_\ell} |\eta_*^{(\ell)}|^2 \, d\mathcal{L}^2 \\ = \ & \sup_{\ell \in \mathbb{N}} \left(\lim_{k \to \infty} \int_{S_\ell} |\xi_*^{(n_k^{(\ell)})}|^2 \, d\mathcal{L}^2 \right) \leq \sup_{i \in \mathbb{N}} |\xi_*^{(i)}|^2_{X_\alpha^\circ} \leq R_0^2; \end{aligned} \tag{28}$$

therefore $\eta_* \in L^2(S; \mathbb{R}^3)$.

Subsequently, let us set a subsequence $\{\xi_{**}^{(k)} \,|\, k = 1, 2, 3, \cdots\} \subset \{\xi_*^{(i)}\}$, by putting:

$$\xi_{**}^{(k)} := \xi_*^{(n_k^{(k)})} \text{ in } L^2(S; \mathbb{R}^3), \text{ for } k = 1, 2, 3, \cdots.$$

Then, taking into account of the assumption (a1)°, and (26)-(28), we obtain that:

$$\begin{aligned} & |\xi_{**}^{(k)} - \eta_*|^2_{L^2(S;\mathbb{R}^3)} \\ \leq \ & |\xi_*^{(n_k^{(k)})} - \eta_*|^2_{L^2(S_k;\mathbb{R}^3)} + 2(|\xi_{**}^{(k)}|^2_{L^2(S \setminus S_k;\mathbb{R}^3)} + |\eta_*|^2_{L^2(S \setminus S_k;\mathbb{R}^3)}) \\ \leq \ & \frac{1}{k} + 2I_*^{(k)} + 2 \int_{S \setminus S_k} |\eta_*|^2 \, d\mathcal{L}^2 \to 0, \text{ as } k \to \infty. \end{aligned}$$

Thus, the subsequence $\{\xi_{**}^{(k)}\}$ is a convergent sequence in the topology of $L^2(S; \mathbb{R}^3)$, and the limit η_* must coincide with the weak limit ξ_* as in (25). □

Proof of Theorem 3.3. First, let us take into account of the assumptions (a1)-(a2), Theorem 3.1 and Remark 5, to check that:

(‡) $\mathcal{L}^3(A_0^{(h)}) = \mathcal{L}^3(A_0^\dagger) = \mathcal{L}^2(A_0^\circ) = 0$, $\Phi_\alpha^{(h)} \geq \Phi_\alpha^\dagger$ on $L^2(\Omega; \mathbb{R}^3)$, and hence $X_\alpha^{(h)} \subset X_\alpha^\dagger$, for any $0 < h < 1$.

Now, the proof is divided into two steps, which are concerned with the respective verifications of items (m1) and (m2).

<u>(Step 1) verification of (m1).</u> Let us take any sequence $\{\mu^{(h)} \,|\, 0 < h < 1\} \subset L^2(\Omega; \mathbb{R}^3)$ and any $\mu \in L^2(\Omega; \mathbb{R}^3)$, such that:

$$\mu^{(h)} \to \mu \text{ weakly in } L^2(\Omega; \mathbb{R}^3), \text{ as } h \searrow 0. \tag{29}$$

Then, it is enough to consider only the finite case of $\liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\mu^{(h)})$, since the other case is obvious. In this case, we find a sequence $\{\breve{h}_i \,|\, i = 1, 2, 3, \cdots\} \subset (0, 1)$ and a constant R_1, independent of the index $i \in \mathbb{N}$, such that:

$$\left\{ \begin{array}{l} \breve{h}_{i+1} < \breve{h}_i < \dfrac{1}{2^i}, \\ \dfrac{1}{\breve{h}_i^2} \displaystyle\int_{\Omega \setminus A_0^\dagger} \alpha^{(\breve{h}_i)} |\partial_3 \mu^{(\breve{h}_i)}|^2 \, d\mathcal{L}^3 \leq \Phi_\alpha^{(\breve{h}_i)}(\mu^{(\breve{h}_i)}) \leq R_1, \end{array} \right. \text{ for } i = 1, 2, 3, \cdots, \tag{30}$$

and

$$\lim_{i \to \infty} \Phi_\alpha^{(\breve{h}_i)}(\mu^{(\breve{h}_i)}) = \liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\mu^{(h)}) \ (< \infty). \tag{31}$$

By virtue of (29)-(31) and Remark 4,

$$\left\{ \begin{array}{l} |\partial_3 \mu^{(\breve{h}_i)}|^2_{L^2(\Omega_\ell^\dagger;\mathbb{R}^3)} \leq \dfrac{R_1}{a_\ell^{(\breve{h}_i)}} \breve{h}_i^2 \leq \dfrac{R_1}{a_\ell^{(1/2)}} \breve{h}_i^2 \to 0, \text{ as } i \to \infty, \\ \partial_3 \mu = 0 \text{ in } L^2(\Omega_\ell^\dagger; \mathbb{R}^3), \end{array} \right. \text{ for any } \ell \in \mathbb{N}.$$

Therefore, for any $\ell \in \mathbb{N}$, we find a function of two-variables $\mu_*^{(\ell)} \in L^2(S_\ell; \mathbb{R}^3)$, such that:

$$\mu(x_1, x_2, x_3) = \mu_*^{(\ell)}(x_1, x_2), \quad \text{for } \mathcal{L}^2\text{-a.e. } (x_1, x_2) \in S_\ell \text{ and } \mathcal{L}^1\text{-a.e. } x_3 \in (0,1).$$

Here, let us set:

$$\mu_*(x_1, x_2) := \begin{cases} \mu_*^{(\ell)}(x_1, x_2), & \text{if } \ell \in \mathbb{N} \text{ and } (x_1, x_2) \in S_\ell, \\ 0, & \text{otherwise}, \end{cases}$$
$$\text{for } \mathcal{L}^2\text{-a.e. } (x_1, x_2) \in S.$$

Then, with helps from (a1)-(a2), Fubini's theorem and the monotone convergence theorem, it is inferred that:

$$\int_S |\mu_*|^2 \, d\mathcal{L}^2 = \int_0^1 \int_{S \setminus A_0^\circ} |\mu_*|^2 d\mathcal{L}^2 d\mathcal{L}^1 = \lim_{\ell \to \infty} \int_0^1 \int_{S \setminus A_0^\circ} \chi_{S_\ell} |\mu_*|^2 d\mathcal{L}^2 d\mathcal{L}^1$$
$$= \lim_{\ell \to \infty} \int_0^1 \int_{S_\ell} |\mu_*|^2 d\mathcal{L}^2 d\mathcal{L}^1 = \lim_{\ell \to \infty} \int_{\Omega_\ell^\dagger} |\mu|^2 \, d\mathcal{L}^3 \leq |\mu|^2_{L^2(\Omega;\mathbb{R}^3)} < \infty,$$

and

$$\begin{aligned} \int_\Omega |\mu - \mu^*|^2 \, d\mathcal{L}^3 &= \lim_{\ell \to \infty} \int_0^1 \int_S \chi_{S_\ell} |\mu - \mu_*|^2 \, d\mathcal{L}^2 d\mathcal{L}^1 \\ &= \lim_{\ell \to \infty} \int_0^1 \int_{S_\ell} |\mu - \mu_*|^2 \, d\mathcal{L}^2 d\mathcal{L}^1 = 0. \end{aligned}$$

Hence, the limit μ can be regarded as the function $\mu_* \in L^2(S; \mathbb{R}^3)$ of two-variables.

Now, taking into account of (‡), Theorem 3.1 and Remark 2, we conclude that:

$$\liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\mu^{(h)}) \geq \liminf_{h \searrow 0} \Phi_\alpha^\dagger(\mu^{(h)}) \geq \Phi_\alpha^\dagger(\mu) = \Phi_\alpha^\circ(\mu).$$

(Step 2) verification of (m2). Let us take any $\nu \in X_\alpha^\circ$. Then, under (a1)-(a2), we can construct the required sequence $\{\mu_\nu^{(h)} \,|\, 0 < h < 1\}$, by putting:

$$\mu_\nu^{(h)} := \nu \in X_\alpha^\dagger \; (= X_\alpha^{(h)}), \quad \text{for any } 0 < h < 1. \tag{32}$$

In fact, since:

$$\begin{cases} \alpha^{(h)} \to \alpha^\circ \text{ in } C(\overline{\Omega}), \text{ as } h \searrow 0, \\ |\alpha^{(h)}| \; (= \alpha^{(h)}) \; \leq C_\alpha \, \alpha^\circ \text{ on } \overline{\Omega}, \text{ for any } 0 < h < 1; \end{cases}$$

we obtain that:

$$\Phi_\alpha^{(h)}(\mu_\nu^{(h)}) = \int_{\Omega \setminus A_0^\dagger} \alpha^{(h)} |\nabla \nu|^2 \, d\mathcal{L}^3$$
$$\to \int_{\Omega \setminus A_0^\dagger} \alpha^\circ |\nabla \nu|^2 \, d\mathcal{L}^3 = \int_{S \setminus A_0^\circ} \alpha^\circ |\nabla \nu|^2 \, d\mathcal{L}^2 = \Phi_\alpha^\circ(\nu), \quad \text{as } h \searrow 0; \tag{33}$$

by applying Lebesgue's dominated convergence theorem. □

Proof of Theorem 3.4. Let us assume the conditions (a1)-(a2), and let us take any $2 < p \leq \infty$. Then, the inclusions (20) and (21) will be direct consequences of (†)-(‡). In view of this, either compactness, asserted in Theorem 3.4, is reduced to that of the embedding $X_\alpha^\dagger \cap L^p(\Omega; \mathbb{R}^3)$ into $L^2(\Omega; \mathbb{R}^3)$.

However, the above compact embedding will be obtained, immediately, by applying Theorem 3.2 for the special situation that $C_\alpha = 1$ in the assumption (a2).

Eventually, under the assumption (a1)-(a2), it can be said that Theorem 3.4 is a corollary of Theorem 3.2. □

5. **Proof of Main Theorem.** This section is largely divided into two subsections, concerned with the proofs of the respective assertions (I) and (II) of Main Theorem.

5.1. **Proof of (I) of Main Theorem.** Let us assume the condition (a1), and let us fix any $0 < h < 1$. Then, the assertion (I) can be shown by applying standard technique, with the help from the compactness, as in Theorem 3.2 and Corollary 2.

Let us put $e^\dagger := [1, 0, 0] \in \mathbb{S}^2$. Then, by virtue of (4), (14) and (16),

$$0 \leq E_*^{(h)} := \inf_{m \in L^2(\Omega;\mathbb{R}^3)} \mathcal{E}^{(h)}(m) \leq \mathcal{E}(e^\dagger).$$

Since the above inequality implies the finiteness of the infimum $E_*^{(h)}$ of $\mathcal{E}^{(h)}$ in $L^2(\Omega;\mathbb{R}^3)$, we naturally find a sequence $\{\breve{m}^{(i)} \,|\, i = 1, 2, 3, \cdots\} \subset X_\alpha^{(h)}$, such that:

$$\mathcal{E}^{(h)}(\breve{m}^{(i)}) \searrow E_*^{(h)} \quad \text{as } i \to \infty. \tag{34}$$

Subsequently, by (II) of Corollary 2 and the constraint onto $L^2(\Omega;\mathbb{S}^2)$ as in (4), we further find a subsequence $\{\breve{m}^{(i_k)} \,|\, k = 1, 2, 3, \cdots\} \subset \{\breve{m}^{(i)}\}$ with a limiting function $\breve{m} \in L^2(\Omega;\mathbb{S}^2)$, such that:

$$\begin{cases} \breve{m}^{(i_k)} \to \breve{m} \ \text{ in } L^2(\Omega;\mathbb{R}^3), \\ \varphi(\breve{m}^{(i_k)}) \to \varphi(\breve{m}) \ \text{ in } L^1(\Omega), \end{cases} \quad \text{as } k \to \infty. \tag{35}$$

Besides, for any $k \in \mathbb{N}$, let us denote by $\breve{\zeta}^{(k)}$ the solution of the coupled Maxwell equation (5), when $m = \breve{m}^{(i_k)}$, and also, let us denote by $\breve{\zeta}$ the solution of (5), when $m = \breve{m}$. Then, in the light of Proposition 1,

$$\breve{\zeta}^{(k)} \to \breve{\zeta} \ \text{ in } V, \ \text{ as } k \to \infty. \tag{36}$$

Now, taking into account of (16), (34)-(36) and Theorem 3.1, we obtain that:

$$\begin{aligned} E_*^{(h)} &= \lim_{k\to\infty} \mathcal{E}^{(h)}(\breve{m}^{(i_k)}) \\ &= \liminf_{k\to\infty} \Phi_\alpha^{(h)}(\breve{m}^{(i_k)}) + \lim_{k\to\infty} \left(|\varphi(\breve{m}^{(i_k)})|_{L^1(\Omega)} + \frac{1}{2}|\breve{\zeta}^{(k)}|_V^2 \right) \\ &\geq \Phi_\alpha^{(h)}(\breve{m}) + |\varphi(\breve{m})|_{L^1(\Omega)} + \frac{1}{2}|\breve{\zeta}|_V^2 = \mathcal{E}^{(h)}(\breve{m}) \geq E_*^{(h)}. \end{aligned}$$

Therefore, the limit $\breve{m}$ is the minimizer, that is denoted by $m^{(h)}$ in the assertion (I) of Main Theorem. □

5.2. **Proof of (II) of Main Theorem.** Let us assume the conditions (a1)-(a2), and let us take a sequence $\{m^{(h)} \,|\, 0 < h < 1\}$ of minimizers $m^{(h)}$ of $\mathcal{E}^{(h)}$, for every $0 < h < 1$. Namely, we may say that:

$$\mathcal{E}^{(h)}(m^{(h)}) \leq \mathcal{E}^{(h)}(m), \quad \text{for all } m \in L^2(\Omega;\mathbb{R}^3) \text{ and all } 0 < h < 1. \tag{37}$$

For any $0 < h < 1$, let $\zeta_{e^\dagger}^{(h)}$ be the solution of the coupled Maxwell equation (5), when $m \equiv e^\dagger$, $\mathcal{L}^3$-a.e. in Ω. Then, by (12) and (16),

$$2E_{\rm mag}^{(h)}(e^\dagger) \leq |\zeta_{e^\dagger}^{(h)}|_V^2 \leq |e^\dagger|_{L^2(\Omega;\mathbb{R}^3)} |\zeta_{e^\dagger}^{(h)}|_V \leq \sqrt{2E_{\rm mag}^{(h)}(e^\dagger)}.$$

Since the above inequality implies that:

$$E_{\rm mag}^{(h)}(e^\dagger) \leq \frac{1}{2} < 1, \quad \text{for all } 0 < h < 1;$$

it is further seen that:

$$\begin{aligned}\Phi_\alpha^{(h)}(m^{(h)}) &\leq \mathcal{E}^{(h)}(m^{(h)}) \leq \mathcal{E}^{(h)}(e^\dagger) = \Phi_\alpha^{(h)}(e^\dagger) + |\varphi(e^\dagger)|_{L^1(\Omega)} + E_{\mathrm{mag}}^{(h)}(e^\dagger) \\ &\leq \varphi(e^\dagger) + 1, \quad \text{for all } 0 < h < 1. \end{aligned} \tag{38}$$

By virtue of (38), Theorem 3.4 and the constraint onto $L^2(\Omega; \mathbb{S}^2)$ as in (4), we find a sequence $\{\hat{h}_i \,|\, i = 1, 2, 3, \cdots\} \subset (0, 1)$ with a limiting function $m^\circ \in L^2(\Omega; \mathbb{S}^2)$, such that:

$$\begin{cases} \hat{h}_i \searrow 0, \ m^{(\hat{h}_i)} \to m^\circ \ \text{ in } L^2(\Omega; \mathbb{R}^3), \\ \varphi(m^{(\hat{h}_i)}) \to \varphi(m^\circ) \ \text{ in } L^1(\Omega), \end{cases} \quad \text{as } i \to \infty. \tag{39}$$

Also, taking into account of (38) and Corollary 3 and Remark 3, it will be observed that:

$$\Phi_\alpha^\circ(m^\circ) \leq \mathcal{E}^\circ(m^\circ) \leq \liminf_{i\to\infty} \mathcal{E}^{(\hat{h}_i)}(m^{(\hat{h}_i)}) \leq \varphi(e^\dagger) + 1; \tag{40}$$

and hence $m^\circ \in X_\alpha^\circ \cap L^2(S; \mathbb{S}^2)$. Furthermore, by (32)-(33), we obtain that:

$$\mathcal{E}^\circ(m^\circ) \leq \limsup_{i\to\infty} \mathcal{E}^{(\hat{h}_i)}(m^{(\hat{h}_i)}) \leq \lim_{i\to\infty} \mathcal{E}^{(\hat{h}_i)}(m) = \mathcal{E}^\circ(m), \quad \text{for any } m \in X_\alpha^\circ \cap L^2(S; \mathbb{S}^2),$$

and

$$\mathcal{E}^\circ(m^\circ) \leq \liminf_{i\to\infty} \mathcal{E}^{(\hat{h}_i)}(m^{(\hat{h}_i)}) \leq \limsup_{i\to\infty} \mathcal{E}^{(\hat{h}_i)}(m^{(\hat{h}_i)}) \leq \mathcal{E}^\circ(m^\circ).$$

Now, all we have to do is to show that the pointwise convergence, asserted in (11), is certainly realized by some subsequence $\{h_i \,|\, i = 1, 2, 3, \cdots\}$ of $\{\hat{h}_i\}$.

By (a2), (38) and (40),

$$\begin{aligned}\int_{\Omega\setminus A_0^\dagger} \alpha^\circ |\nabla_{\mathrm{P}}(m^{(h)} - m^\circ)|^2 \, d\mathcal{L}^3 &\leq 2 \int_{\Omega\setminus A_0^\dagger} \alpha^\circ \left(|\nabla_{\mathrm{P}} m^{(h)}|^2 + |\nabla_{\mathrm{P}} m^\circ|^2 \right) d\mathcal{L}^3 \\ \leq \ 2\left(\Phi_\alpha^{(h)}(m^{(h)}) + \Phi_\alpha^\circ(m^\circ) \right) &\leq 4(\varphi(e^\dagger) + 1), \quad \text{for all } 0 < h < 1. \end{aligned} \tag{41}$$

So, putting:

$$\beta^{(h)} := \sqrt{\alpha^\circ}\, \nabla_{\mathrm{P}}(m^{(h)} - m^\circ) \in L^2(\Omega \setminus A_0^\dagger; \mathbb{R}^{3\times 2}), \quad \text{for any } 0 < h < 1;$$

we infer from (41) the existence of a subsequence $\{\hat{h}_i^{(0)} \,|\, i = 1, 2, 3, \cdots\} \subset \{\hat{h}_i\}$ with a limiting function $\beta^\circ \in L^2(\Omega \setminus A_0^\dagger; \mathbb{R}^{3\times 2})$, such that:

$$\beta^{(\hat{h}_i^{(0)})} \to \beta^\circ \text{ weakly in } L^2(\Omega \setminus A_0^\dagger; \mathbb{R}^{3\times 2}), \ \text{ as } i \to \infty. \tag{42}$$

Besides, in the light of Remark 4, (39) and (41), we can construct a decreasing family of subsequences:

$$\{\hat{h}_i^{(0)}\} \supset \{\hat{h}_i^{(1)}\} \supset \{\hat{h}_i^{(2)}\} \supset \{\hat{h}_i^{(3)}\} \supset \cdots \supset \{\hat{h}_i^{(\ell)}\} \supset \cdots;$$

so that:

$$\nabla_{\mathrm{P}} m^{(\hat{h}_i^{(\ell)})} \to \nabla_{\mathrm{P}} m^\circ \text{ weakly in } L^2(\Omega_\ell; \mathbb{R}^{3\times 2}), \ \text{ as } i \to \infty, \ \text{ for any } \ell \in \mathbb{N}.$$

Here, for a subsequence $\{\check{h}_i := \hat{h}_i^{(i)} \,|\, i = 1, 2, 3, \cdots\} \subset \{\hat{h}_i\}$, we deduce that:

$$\begin{cases} \nabla_{\mathrm{P}} m^{(\check{h}_i)} \to \nabla_{\mathrm{P}} m^\circ, \\ \beta^{(\check{h}_i)} \to 0, \end{cases} \text{ in the distribution sense on } \Omega \setminus A_0^\dagger, \ \text{ as } i \to \infty. \tag{43}$$

As a consequence of (42) and (43), it can be said that:

$$\beta^\circ = 0 \ \text{ in } L^2(\Omega \setminus A_0^\dagger); \tag{44}$$

since the distributional limits should be unique.

In the meantime, from (a2) and (37), it is derived that:

$$\begin{aligned}
& \left(\Phi_\alpha^{(\check{h}_i)}(m^\circ) - \Phi_\alpha^\circ(m^\circ)\right) + |\varphi(m^{(\check{h}_i)}) - \varphi(m^\circ)|_{L^1(\Omega)} \\
& \qquad + |E_{\mathrm{mag}}^{(\check{h}_i)}(m^\circ) - E_{\mathrm{mag}}^\circ(m^\circ)| + |E_{\mathrm{mag}}^{(\check{h}_i)}(m^{(\check{h}_i)}) - E_{\mathrm{mag}}^\circ(m^\circ)| \\
\geq\ & \Phi_\alpha^{(\check{h}_i)}(m^{(\check{h}_i)}) - \Phi_\alpha^\circ(m^\circ) \geq \Phi_\alpha^\dagger(m^{(\check{h}_i)}) - \Phi_\alpha^\circ(m^\circ) \\
=\ & \int_{\Omega \setminus A_0^\dagger} [\sqrt{\alpha^\circ}\,\nabla_{\mathrm{P}} m^\circ + \beta^{(\check{h}_i)}] : [\sqrt{\alpha^\circ}\,\nabla_{\mathrm{P}} m^\circ + \beta^{(\check{h}_i)}]\, d\mathcal{L}^3 \\
& \qquad + \frac{1}{\check{h}_i^2} \int_{\Omega \setminus A_0^\dagger} \alpha^\circ |\partial_3 m^{(\check{h}_i)}|^2\, d\mathcal{L}^3 - \Phi_\alpha^\circ(m^\circ) \\
=\ & 2\left(\sqrt{\alpha^\circ}\,\nabla_{\mathrm{P}} m^\circ, \beta^{(\check{h}_i)}\right)_{L^2(\Omega \setminus A_0^\dagger; \mathbb{R}^{3\times 2})} \\
& \qquad + \left(|\beta^{(\check{h}_i)}|^2_{L^2(\Omega \setminus A_0^\dagger; \mathbb{R}^{3\times 2})} + \left| \frac{1}{\check{h}_i} \sqrt{\alpha^\circ}\, \partial_3 m^{(\check{h}_i)} \right|^2_{L^2(\Omega \setminus A_0^\dagger; \mathbb{R}^3)} \right), \qquad (45)
\end{aligned}$$

for $i = 1, 2, 3, \cdots$.

In view of (32)-(33), (39), (42), (44) and Proposition 2, letting $i \to \infty$ in (45) yields that:

$$\limsup_{i \to \infty} \left(|\beta^{(\check{h}_i)}|^2_{L^2(\Omega \setminus A_0^\dagger; \mathbb{R}^{3\times 2})} + \left| \frac{1}{\check{h}_i} \sqrt{\alpha^\circ}\, \partial_3 m^{(\check{h}_i)} \right|^2_{L^2(\Omega \setminus A_0^\dagger; \mathbb{R}^3)} \right) \leq 0.$$

Thus,

$$\begin{cases} \beta^{(\check{h}_i)} = \sqrt{\alpha^\circ}\,\nabla_{\mathrm{P}}(m^{(\check{h}_i)} - m^\circ) \to 0 \text{ in } L^2(\Omega \setminus A_0^\dagger; \mathbb{R}^{3\times 2}), \\ \dfrac{1}{\check{h}_i} \sqrt{\alpha^\circ}\, \partial_3 m^{(\check{h}_i)} \to 0 \text{ in } L^2(\Omega \setminus A_0^\dagger; \mathbb{R}^3), \end{cases} \text{as } i \to \infty. \qquad (46)$$

On account of (a1)-(a2), the above convergence (46) implies the existence of a subsequence $\{h_i \,|\, i = 1, 2, 3, \cdots\} \subset \{\check{h}_i\}$ ($\subset \{\hat{h}_i\}$), satisfying (11). □

Remark 6. (Further conclusion) From (32)-(33) and (46), the convergence:

$$\sqrt{\alpha^{(\check{h}_i)}}\,\nabla_{\mathrm{P}} m^{(\check{h}_i)} \to \sqrt{\alpha^\circ}\,\nabla_{\mathrm{P}} m^\circ \text{ in } L^2(\Omega \setminus A_0^\dagger; \mathbb{R}^{3\times 2}), \text{ as } i \to \infty;$$

is derived, as a result of the following calculation:

$$\begin{aligned}
& |\sqrt{\alpha^{(\check{h}_i)}}\,\nabla_{\mathrm{P}} m^{(\check{h}_i)} - \sqrt{\alpha^\circ}\,\nabla_{\mathrm{P}} m^\circ|^2_{L^2(\Omega \setminus A_0^\dagger; \mathbb{R}^{3\times 2})} \\
\leq\ & 2C_\alpha |\sqrt{\alpha^\circ}\,\nabla_{\mathrm{P}}(m^{(\check{h}_i)} - m^\circ)|^2_{L^2(\Omega \setminus A_0^\dagger; \mathbb{R}^{3\times 2})} \\
& \qquad + 2\int_{\Omega \setminus A_0^\dagger} \left(\sqrt{\alpha^{(\check{h}_i)}} - \sqrt{\alpha^\circ}\right)^2 |\nabla_{\mathrm{P}} m^\circ|^2\, d\mathcal{L}^3 \qquad (47) \\
\to\ & 0, \quad \text{as } i \to \infty.
\end{aligned}$$

Incidentally, the zero-convergence of the integral part of (47) is easily checked by applying Lebesgue's dominated convergence theorem, for the situation that:

$$\begin{cases} \left(\sqrt{\alpha^{(\check{h}_i)}} - \sqrt{\alpha^\circ}\right)^2 |\nabla_{\mathrm{P}} m^\circ|^2 \to 0 \\ \left(\sqrt{\alpha^{(\check{h}_i)}} - \sqrt{\alpha^\circ}\right)^2 |\nabla_{\mathrm{P}} m^\circ|^2 \leq C_\alpha \alpha^\circ |\nabla m^\circ|^2, \end{cases} \mathcal{L}^3\text{-a.e. in } \Omega \setminus A_0^\dagger.$$

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Received xxxx 20xx; revised xxxx 20xx.

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