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3D-2D ASYMPTOTIC OBSERVATION FOR MINIMIZATION PROBLEMS ASSOCIATED WITH DEGENERATIVE ENERGY-COEFFICIENTS

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Abstract. In this paper, a class of minimization problems, labeled by an index $0 < h < 1$, is considered. Each minimization problem is for a *free-energy,* motivated by the magnetics in 3D-ferromagnetic thin film, and in the context, the index h denotes the thickness of the observing film. The Main Theorem consists of two themes, which are concerned with the study of the solvability (existence of minimizers) and the 3D-2D asymptotic analysis for our minimization problems. These themes will be discussed under degenerative setting of the material coefficients, and such degenerative situation makes the energy-domain be variable with respect to h. In conclusion, assuming some restrictive conditions for the domain-variation, a definite association between our 3D-minimization problems, for very thin h, and a 2D-limiting problem, as $h \searrow 0$, will be demonstrated with helps from the theory of Γ-convergence.

1. **Introduction.** Let $S \subset \mathbb{R}^2$ be a two-dimensional bounded domain with a smooth boundary, and let $\Omega \subset \mathbb{R}^3$ be a three-dimensional cylindrical domain, given by $\Omega := S \times (0,1)$. Let $\alpha : \overline{\Omega} \longrightarrow [0,\infty)$ be a given nonnegative and continuous function.

In this paper, let us imagine the situation that a ferromagnetic thin film is applied on a thin region $\Omega^{(h)} := S \times (0,h)$ with a (small) thickness $0 < h < 1$. As a possible free-energy for the magnetic study in such situation, the following functional, denoted by $\mathcal{E}_\alpha^{(h)}$:

$$\begin{gathered}\mathcal{E}_\alpha^{(h)}(m) := \Psi_\alpha^{(h)}(m) + \fint_{\Omega^{(h)}} \left(\varphi(m) + \frac{1}{2}\nabla\zeta \cdot m \right) d\mathcal{L}^3, \\ \text{for any } m = (m_1, m_2, m_3) \in L^2(\Omega^{(h)}; \mathbb{R}^3);\end{gathered} \tag{1}$$

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subject to:

$$\mathrm{div}\,(-\nabla\zeta + \overline{{}^0 m}) = 0, \quad \text{in } \mathbb{R}^3, \tag{2}$$

$$|m| = m_s, \quad \mathcal{L}^3\text{-a.e. in } \Omega; \tag{3}$$

was proposed by Brown [7] (1963), where Ψ_α is the lower semi-continuous envelopment of a functional:

$$\psi \in W^{1,2}(\Omega^{(h)}; \mathbb{R}^3) \cap L^2(\Omega^{(h)}; \mathbb{S}^2) \mapsto \fint_{\Omega^{(h)}} \alpha |\nabla\psi|^2 \, d\mathcal{L}^3;$$

onto the space $L^2(\Omega^{(h)}; \mathbb{R}^3)$.

In (1), the value of $\mathcal{E}_\alpha^{(h)}$ denotes an energy quantity, per unit volume in $\Omega^{(h)}$, and the variable $m = (m_1, m_2, m_3)$ denotes the magnetization in the region $\Omega^{(h)}$ of magnetic thin film. In this light, the minimizers of $\mathcal{E}_\alpha^{(h)}$ are supposed to represent the most probable profile of the magnetization distribution applied on $\Omega^{(h)}$. Here, the given function α is the so-called material coefficient, and this coefficient is supposed to be degenerative somewhere in Ω. $\varphi : \mathbb{R}^3 \longrightarrow [0, \infty)$ is a given continuous function, which is involved in the magnetization anisotropy.

The function $\zeta : \mathbb{R}^3 \longrightarrow \mathbb{R}$ as in (1)-(2) denotes the magnetic field potential, and hence, it is prescribed as the solution of the simplified Maxwell equation (2). Here, the notation "$\overline{{}^0\ \ }$" denotes the zero-extension of functions. In addition to the above, let us note that the free-energy $\mathcal{E}_\alpha^{(h)}$ is considered under the constrained condition (3), by a positive constant m_s of the magnetization saturation.

In this paper, we set:

$$\mathcal{L}^2(S) = 1 \text{ (and hence } \mathcal{L}^3(\Omega) = 1\text{), and } m_s = 1;$$

for simplicity. On that basis, let us denote by $T^{(h)}$ the scale transform, defined as:

$$T^{(h)} : x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto (x_1, x_2, hx_3) \in \mathbb{R}^3;$$

to consider a rescaled minimization problem, denoted by $(\mathrm{MP})^{(h)}$.

$(\mathrm{MP})^{(h)}$ Find a vectorial function $m^{(h)} = (m_1^{(h)}, m_2^{(h)}, m_3^{(h)}) \in L^2(\Omega; \mathbb{R}^3)$ of three variables, which minimizes the following functional on $L^2(\Omega; \mathbb{R}^3)$:

$$\begin{aligned} \mathcal{F}_\alpha^{(h)}(m) := \Phi_\alpha^{(h)}(m) &+ \int_\Omega \varphi(m) \, d\mathcal{L}^3 \\ &+ \frac{1}{2} \int_\Omega \left(\nabla_{\mathrm{P}}\zeta \cdot m_{\mathrm{P}} + \frac{1}{h} \partial_3\zeta \, m_3 \right) d\mathcal{L}^3, \\ &\text{for any } m = (m_1, m_2, m_3) \in L^2(\Omega; \mathbb{R}^3); \end{aligned} \tag{4}$$

subject to:

$$\nabla_{\mathrm{P}} \cdot (-\nabla_{\mathrm{P}}\zeta + \overline{{}^0 m_{\mathrm{P}}}) + \frac{1}{h}\,\partial_3 \left(-\frac{1}{h}\partial_3\zeta + \overline{{}^0 m_3} \right) = 0, \quad \text{in } \mathbb{R}^3; \tag{5}$$

where the subscript "$_{\mathrm{P}}$" denotes the restriction of the situation onto the two-dimensional plane $\mathbb{R}^2$, e.g.:

$$y_{\mathrm{P}} := (y_1, y_2), \quad \text{for } y = (y_1, y_2, y_3) \in \mathbb{R}^3,$$

$$\mu_{\mathrm{P}} := (\mu_1, \mu_2) \in L^2(\Omega; \mathbb{R}^2), \quad \text{for } \mu = (\mu_1, \mu_2, \mu_3) \in L^2(\Omega; \mathbb{R}^3),$$

and the distributional gradient

$$\nabla_{\mathrm{P}}\mu := \begin{pmatrix} \partial_1\mu_1 & \partial_2\mu_1 \\ \partial_1\mu_2 & \partial_2\mu_2 \\ \partial_1\mu_3 & \partial_2\mu_3 \end{pmatrix}, \text{ for } \mu = (\mu_1, \mu_2, \mu_3) \in L^2(\Omega; \mathbb{R}^3);$$

and $\Phi_\alpha^{(h)}$ is the rescaled version of the lower semi-continuous envelopment $\Psi_\alpha^{(h)}$ by $T^{(h)}$, and it is rigorously defined as:

$$\Phi_\alpha^{(h)}(m) := \begin{cases} \displaystyle \inf_{\{\psi^{(i)}\} \in \mathcal{Q}_\Omega(m)} \liminf_{i \to \infty} \int_\Omega \alpha^{(h)} \left(|\nabla_{\mathrm{P}} \psi^{(i)}|^2 + \frac{1}{h^2} |\partial_3 \psi^{(i)}|^2 \right) d\mathcal{L}^3, \\ \qquad \text{if } |m| = 1, \ \mathcal{L}^3\text{-a.e. in } \Omega, \\ \infty, \text{ otherwise}, \quad \text{for any } m \in L^2(\Omega; \mathbb{R}^3); \end{cases} \tag{6}$$

by using a composition:

$$\alpha^{(h)}(x) := (\alpha \circ T^{(h)})(x) = \alpha(x_1, x_2, hx_3), \quad \text{for all } x = (x_1, x_2, x_3) \in \overline{\Omega};$$

and a class of approximating functions:

$$\mathcal{Q}_\Omega(m) := \left\{ \ \{\psi^{(i)}\} \ \middle| \ \begin{matrix} \psi^{(i)} \in W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2), \ i = 1, 2, 3, \cdots, \\ \text{and } \ \psi^{(i)} \to m \text{ in } L^2(\Omega; \mathbb{R}^3) \text{ as } i \to \infty \end{matrix} \right\}; \tag{7}$$

for any $m \in L^2(\Omega; \mathbb{S}^2)$.

As is easily seen, the inverse transform $(T^{(h)})^{-1}$ provides a bijective correspondence between the minimizers $m^{(h)}$ of (MP)$^{(h)}$ and the minimizers $m_{\mathrm{org}}^{(h)} := m^{(h)} \circ (T^{(h)})^{-1}$ of the original free-energy $\mathcal{E}_\alpha^{(h)}$. Besides, let us note that the domains $\mathrm{Dom}(\mathcal{F}_\alpha^{(h)})$ of free-energies are not uniform, but variable with respect to $0 < h < 1$, and the variation is directly governed by the degenerating part of the coefficient:

$$A_0^{(h)} := (\alpha^{(h)})^{-1}(0) \subset \overline{\Omega}, \quad \text{for } 0 < h < 1.$$

Under very thin situation of the thickness h, it is naturally expected that the minimization problem (MP)$^{(h)}$ can be reduced to a simpler problem, considered in two-dimensional domain S. Such reduction will be realized through the limiting observation for (MP)$^{(h)}$ as $h \searrow 0$, and then, the binary function:

$$\alpha^\circ(x_1, x_2) := \alpha(x_1, x_2, 0) \text{ for any } (x_1, x_2) \in \overline{S}, \\ \text{with the degenerating part } A_0^\circ := (\alpha^\circ)^{-1}(0);$$

will be the material coefficient in the limiting problem. Actually, in the h-independent case of $A_0^{(h)}$, a number of like-minded study results, such as [1, 2, 3, 4, 5, 6, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18], were reported, from various viewpoints, and some of them concluded a definite association between the limiting profile of (MP)$^{(h)}$, as $h \searrow 0$, and the following minimization problem, denoted by (MP)$^\circ$.

(MP)$^\circ$ Find a vectorial function $m^\circ = (m_1^\circ, m_2^\circ, m_3^\circ) \in L^2(S; \mathbb{R}^3)$ of two variables, which minimizes the following functional:

$$\mathcal{F}_\alpha^\circ(m) := \Phi_\alpha^\circ(m) + \int_S \varphi(m) \, d\mathcal{L}^2 + \frac{1}{2} \int_S |m_3|^2 \, d\mathcal{L}^2, \\ \text{for any } m = (m_1, m_2, m_3) \in L^2(S; \mathbb{R}^3); \tag{8}$$

where Φ_α° is a convex function on $L^2(S; \mathbb{R}^3)$, defined as:

$$\Phi_\alpha^\circ(m) := \begin{cases} \displaystyle \inf_{\{\psi^{(i)}\} \in \mathcal{Q}_S(m)} \liminf_{i \to \infty} \int_S \alpha^\circ |\nabla \psi^{(i)}|^2 \, d\mathcal{L}^2, \\ \qquad \text{if } |m| = 1, \ \mathcal{L}^2\text{-a.e. in } S, \\ \infty, \text{ otherwise}, \quad \text{for any } m \in L^2(S; \mathbb{R}^3); \end{cases} \tag{9}$$

by using a class of approximating sequences:

$$\mathcal{Q}_S(m) := \left\{ \ \{\psi^{(i)}\} \ \middle| \ \begin{matrix} \psi^{(i)} \in W^{1,2}(S; \mathbb{R}^3) \cap L^2(S; \mathbb{S}^2), \ i = 1, 2, 3, \cdots, \\ \text{and } \ \psi^{(i)} \to m \text{ in } L^2(S; \mathbb{R}^3) \text{ as } i \to \infty \end{matrix} \right\}.$$

Now, the main theme of this paper is to verify whether analogous observation is available even under h-variable situation of $A_0^{(h)}$ (or energy domains), or not. To this end, we here impose the following two conditions for the material coefficient α:

(a1) $\mathcal{L}^3(A_0^{(h)}) = 0$, for $0 < h < 1$;
(a2) $\mathcal{L}^2(A_0^\circ) = 0$, and $\alpha^\circ(x_{\rm P}) \leq \alpha(x)$, for all $x = (x_1, x_2, x_3) \in \overline{\Omega}$.

Consequently, a certain positive answer for our theme will be demonstrated in the main theorem, stated as follows.

Main Theorem. *(I) Let us assume the condition (a1). Then, for any $0 < h < 1$, the problem (MP)$^{(h)}$ admits at least one solution (minimizer) $m^{(h)}$.*

(II) Under the conditions (a1)-(a2), there exist a sequence $\{h_i \,|\, i = 1, 2, 3, \cdots\} \subset (0, 1)$ and a function $m^\circ \in L^2(S; \mathbb{R}^3)$ of two variables, such that:
(i) $h_i \searrow 0$, $m^{(h_i)} \to m^\circ$ in $L^2(\Omega; \mathbb{R}^3)$, $\mathcal{F}_\alpha^{(h_i)}(m^{(h_i)}) \to \mathcal{F}_\alpha^\circ(m^\circ)$, as $i \to \infty$;
(ii) the limit m° solves the problem (MP)$^\circ$;

where $\{m^{(h)} \,|\, 0 < h < 1\}$ is the sequence of minimizers $m^{(h)}$, $0 < h < 1$, obtained in (I).

The content of this paper is as follows. In the next Section 2, some key-properties for the minimization problems (MP)$^{(h)}$, $0 < h < 1$, and (MP)$^\circ$ are briefly mentioned as preliminaries. In subsequent Section 3, the continuous dependence between the energy sequence $\{\mathcal{F}_\alpha^{(h)} \,|\, 0 < h < 1\}$ and the energy $\mathcal{F}_\alpha^\circ$, as $h \searrow 0$, will be shown by means of the notion of Γ-convergence (cf. [9]). On that basis, the final Section 4 will be devoted to the proof of Main Theorem.

Notation. Throughout this paper, the Lebesgue measure is denoted by $\mathcal{L}^n$, for any observing dimension $n \in \mathbb{N}$.

For any abstract Banach space, the norm of X is denoted by $|\cdot|_X$. However, when X is an Euclidean space, the norm is simply denoted by $|\cdot|$. Besides, for any functional $F : X \longrightarrow (-\infty, \infty]$, we denote by ${\rm Dom}(F)$ the domain of F, and for any $r > 0$, we denote by $L(r; F)$ the sublevel set of F, more precisely:

$$
{\rm Dom}(F) := \left\{ \ \xi \in X \ \middle| \ F(\xi) < \infty \ \right\} \quad \text{and} \quad L(r; F) := \left\{ \ \xi \in X \ \middle| \ F(\xi) \leq r \ \right\}.
$$

For any abstract Hilbert space H, the inner product of H is denoted by $(\cdot, \cdot)_H$. However, when H is an Euclidean space, the inner product between two vectors $\xi, \eta \in H$ is simply denoted by $\xi \cdot \eta$.

2. **Preliminaries.**

Let us start with summarizing the known-facts, concerned with the coupled Maxwell equation (5).

(Fact 1) (Summary of [18, Lemma 3.1]) Let us fix any $0 < h < 1$. Then, for any function $m = (m_1, m_2, m_3) \in L^2(\Omega; \mathbb{R}^3)$, the solution $\zeta^{(h)}$ of the Maxwell equation (5) is prescribed in the scope of a Hilbert space:

$$
V^{(h)} := \left\{ \ v \in H^1_{\rm loc}(\mathbb{R}^3) \ \middle| \ \nabla v \in L^2(\mathbb{R}^3; \mathbb{R}^3) \ \text{ and } \ \int_{B_\Omega} v \, d\mathcal{L}^3 = 0 \ \right\};
$$

endowed with a h-dependent inner product:

$$
(u, v)_{V^{(h)}} := (\nabla_{\rm P} u, \nabla_{\rm P} v)_{L^2(\mathbb{R}^3; \mathbb{R}^{3\times 2})} + \frac{1}{h^2} (\partial_3 u, \partial_3 v)_{L^2(\mathbb{R}^3; \mathbb{R}^3)}, \quad \text{for } u, v \in V^{(h)};
$$

where B_Ω is an (fixed) open ball containing Ω. Then, the solution $\zeta^{(h)} \in V^{(h)}$ is supposed to fulfill a weak formulation by the following variational identity:

$$\int_\Omega \left((\nabla_{\mathrm{P}} \zeta^{(h)} - m_{\mathrm{P}}) \cdot \nabla_{\mathrm{P}} v + \frac{1}{h} \left(\frac{1}{h} \partial_3 \zeta^{(h)} - m_3 \right) \partial_3 v \right) d\mathcal{L}^3 = 0, \quad \text{for any } v \in V^{(h)}; \tag{10}$$

Moreover, taking more one function $\tilde{m} \in L^2(\Omega; \mathbb{R}^3)$, arbitrarily, and taking another solution $\tilde{\zeta}^{(h)}$ of (10) when $m = \tilde{m}$, it follows that:

$$|\zeta^{(h)} - \tilde{\zeta}^{(h)}|_{V^{(h)}} \leq |m - \tilde{m}|_{L^2(\Omega;\mathbb{R}^3)}. \tag{11}$$

Hence, the variational problem (10) is well-posed.

(Fact 2) (Summary of [15, Proposition 4.1]) Let us set:

$$\begin{cases} F^{(h)}_{\mathrm{mag}}(m) := \dfrac{1}{2} \displaystyle\int_\Omega \left(\nabla_{\mathrm{P}} \zeta^{(h)} \cdot m_{\mathrm{P}} + \frac{1}{h} \partial_3 \zeta^{(h)} \, m_3 \right) d\mathcal{L}^3, \\ F^{\circ}_{\mathrm{mag}}(m) := \dfrac{1}{2} \displaystyle\int_\Omega |m_3|^2 \, d\mathcal{L}^3, \end{cases} \quad \text{for any } m = (m_1, m_2, m_3) \in L^2(\Omega; \mathbb{R}^3); \tag{12}$$

by using the solution $\zeta^{(h)}$ of the variational identity (10). On that basis, let us assume that $\{\bar{m}^{(h)} \,|\, 0 < h < 1\} \subset L^2(\Omega; \mathbb{R}^3)$, and $\bar{m}^{(h)} \to \bar{m}$ in $L^2(\Omega; \mathbb{R}^3)$ as $h \searrow 0$, for some $\bar{m} = (\bar{m}_1, \bar{m}_2, \bar{m}_3) \in L^2(\Omega; \mathbb{R}^3)$. Then:

$$F^{(h)}_{\mathrm{mag}}(\bar{m}^{(h)}) \to F^{\circ}_{\mathrm{mag}}(\bar{m}), \quad \text{as } h \searrow 0.$$

Next, let us look toward the key-properties of the lower semi-continuous envelopments $\Phi^{(h)}_\alpha$, $0 < h < 1$, and Φ°_α.

Lemma 2.1. *(I) For any $0 < h < 1$, the functional $\Phi^{(h)}_\alpha$ is a maximal functional in the class of l.s.c. functionals on $L^2(\Omega; \mathbb{R}^3)$, supporting the functional:*

$$\psi \in W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2) \mapsto \int_\Omega \alpha^{(h)} \left(|\nabla_{\mathrm{P}} \psi|^2 + \frac{1}{h^2} |\partial_3 \psi|^2 \right) d\mathcal{L}^3.$$

Moreover:

(i-1) $\displaystyle \Phi^{(h)}_{0,\alpha}(m) := \int_{\Omega \setminus A^{(h)}_0} \alpha^{(h)} \left(|\nabla_{\mathrm{P}} m|^2 + \frac{1}{h^2} |\partial_3 m|^2 \right) d\mathcal{L}^3 \leq \Phi^{(h)}_\alpha(m),$ *for any $m \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus A^{(h)}_0; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2)$;*

(i-2) $W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2) \subset \mathrm{Dom}(\Phi^{(h)}_\alpha) \subset \mathrm{Dom}(\Phi^{(h)}_{0,\alpha}) \subset W^{1,2}_{\mathrm{loc}}(\Omega \setminus A^{(h)}_0; \mathbb{R}^3)$, *and $\Phi^{(h)}_{0,\alpha} = \Phi^{(h)}_\alpha$ on $W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2)$;*

(i-3) for any $m \in \mathrm{Dom}(\Phi^{(h)}_\alpha)$, there exists a sequence $\{\mu^{(i)} \,|\, i = 1, 2, 3, \cdots\} \subset W^{1,2}(\Omega; \mathbb{R}^3) \cap L^2(\Omega; \mathbb{S}^2)$ such that

$$\mu^{(i)} \to m \text{ in } L^2(\Omega; \mathbb{R}^3) \text{ and } \Phi^{(h)}_{0,\alpha}(\mu^{(i)}) \to \Phi^{(h)}_\alpha(m), \text{ as } i \to \infty.$$

(II) The functional Φ°_α is a maximal functional in the class of l.s.c. functionals on $L^2(S; \mathbb{R}^3)$, supporting the functional:

$$\psi \in W^{1,2}(S; \mathbb{R}^3) \cap L^2(S; \mathbb{S}^2) \mapsto \int_S \alpha^\circ |\nabla \psi|^2 \, d\mathcal{L}^2.$$

Moreover:

(ii-1) $\Phi_{0,\alpha}^\circ(m) := \displaystyle\int_{S \setminus A_0^\circ} \alpha^\circ |\nabla m|^2 \, d\mathcal{L}^2 \leq \Phi_\alpha^{(h)}(m)$, *for any* $m \in W^{1,2}_{\rm loc}(S \setminus A_0^\circ; \mathbb{R}^3)$ $\cap L^2(S; \mathbb{S}^2)$*;*

(ii-2) $W^{1,2}(S;\mathbb{R}^3) \cap L^2(S;\mathbb{S}^2) \subset \mathrm{Dom}(\Phi_\alpha^\circ) \subset \mathrm{Dom}(\Phi_{0,\alpha}^\circ) \subset W^{1,2}_{\rm loc}(S \setminus A_0^\circ; \mathbb{R}^3)$, *and* $\Phi_{0,\alpha}^\circ = \Phi_\alpha^\circ$ *on* $W^{1,2}(S;\mathbb{R}^3)$

(ii-3) for any $m \in \mathrm{Dom}(\Phi_\alpha^{(h)})$*, there exists a sequence* $\{\mu^{(i)} \,|\, i = 1,2,3,\cdots\}$ $\subset W^{1,2}(S;\mathbb{R}^3) \cap L^2(S;\mathbb{S}^2)$ *such that*

$$\mu^{(i)} \to m \text{ in } L^2(S;\mathbb{R}^3) \text{ and } \Phi_{0,\alpha}^\circ(\mu^{(i)}) \to \Phi_\alpha^\circ(m), \text{ as } i \to \infty.$$

Proof. This lemma follows, directly, from the definition formulas (6) and (9). □

Remark 1. (Key-properties for free-energies) By virtue of (4), (8) and Lemma 2.1, the functional $\mathcal{F}_\alpha^{(h)}$ (resp. $\mathcal{F}_\alpha^\circ$) turns out to be l.s.c. in $L^2(\Omega;\mathbb{R}^3)$ (resp. in $L^2(S;\mathbb{R}^3)$), and $\mathrm{Dom}(\mathcal{F}_\alpha^{(h)}) = \mathrm{Dom}(\Phi_\alpha^{(h)})$, for $0 < h < 1$ (resp. $\mathrm{Dom}(\mathcal{F}_\alpha^\circ) = \mathrm{Dom}(\Phi_\alpha^\circ)$). Furthermore, under the conditions (a1)-(a2), as in introduction, the variation of energy-domains $\mathrm{Dom}(\mathcal{F}_\alpha^{(h)})$, with respect to h, will be restrictive in the sense that $\mathrm{Dom}(\mathcal{F}_\alpha^{(h)})$ will be included in $W^{1,2}_{\rm loc}(\Omega \setminus (A_0^\circ \times (0,1)); \mathbb{R}^3)$, uniformly, for all $0 < h < 1$.

Taking into account of Lemma 2.1, Remark 1 and [16, Corollary 2], we can derive the following corollary.

Corollary 1. *(Compactness) Let us assume the condition (a1), as in introduction. Then, for any* $0 < h < 1$ *and any* $r > 0$*, the sublevel sets* $L(r;\Phi_\alpha^{(h)})$ *and* $L(r;\mathcal{F}_\alpha^{(h)})$ *are compact in* $L^2(\Omega;\mathbb{R}^3)$*. Additionally, if we assume the conditions (a1)-(a2), as in introduction, then for any* $r > 0$*, the sublevel sets* $L(r;\Phi_\alpha^\circ)$ *and* $L(r;\mathcal{F}_\alpha^\circ)$ *are compact in* $L^2(S;\mathbb{R}^3)$*, and the unions*

$$\mathcal{U}_\Phi(r) := \bigcup_{0<h<1} L(r;\Phi_\alpha^{(h)}) \ \text{ and } \ \mathcal{U}_\mathcal{F}(r) := \bigcup_{0<h<1} L(r;\mathcal{F}_\alpha^{(h)});$$

are relatively compact in $L^2(\Omega;\mathbb{R}^3)$*.*

Proof. Let us assume (a1), and let us fix any $0 < h < 1$ and any $r > 0$. Here, taking the solution $\zeta^{(h)}$ as the test function of (10), we have:

$$F_{\rm mag}^{(h)}(m) = \frac{1}{2}|\zeta^{(h)}|^2_{V^{(h)}} \geq 0, \quad \text{for any } m \in L^2(\Omega;\mathbb{R}^3). \tag{13}$$

Subsequently, we see from (4), (13) and (i-1) of Lemma 2.1 that:

$$L(r;\mathcal{F}_\alpha^{(h)}) \subset L(r;\Phi_\alpha^{(h)}) \subset L(r;\Phi_{0,\alpha}^{(h)}).$$

Since the compactness of $L(r;\Phi_{0,\alpha}^{(h)})$ in $L^2(\Omega;\mathbb{R}^3)$ is already concluded in [16, Corollary 2], we can say that its closed subsets $L(r;\Phi_\alpha^{(h)})$ and $L(r;\mathcal{F}_\alpha^{(h)})$ are also so.

Just as in the above, the assertion for the sublevel sets $L(r;\Phi_\alpha^\circ)$ and $L(r;\mathcal{F}_\alpha^\circ)$ (resp. for the unions $\mathcal{U}_\Phi(r)$ and $\mathcal{U}_\mathcal{F}(r)$) can be concluded, with the helps from the condition (a2) and [16, Corollary 2] (resp. [16, Theorem 3.4]). □

3. **Continuous dependence of energies.** The objective in this section is summarized in the following theorem, concerned with continuous dependence (Γ-convergence) of lower semi-continuous envelopments, as $h \searrow 0$.

Theorem 3.1. *(Γ-convergence from $\Phi_\alpha^{(h)}$ to Φ_α° as $h \searrow 0$) Let us assume the conditions (a1)-(a2), as in introduction. Then, the sequence $\{\Phi_\alpha^{(h)} \,|\, 0 < h < 1\}$ of the lower semi-continuous envelopments converges to Φ_α°, in the sense of Γ-convergence, as $h \searrow 0$. More precisely, referring to [9] (or [1, Lemma 2.3]), the above assertion is equivalent to:*

(γ1) $\liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\check{m}^{(h)}) \geq \Phi_\alpha^\circ(\check{m}^\circ)$, if $\{\check{m}^{(h)} \,|\, 0 < h < 1\} \subset L^2(\Omega;\mathbb{R}^3)$, $\check{m}^\circ \in L^2(\Omega;\mathbb{R}^3)$, and $\check{m}^{(h)} \to \check{m}^\circ$ in $L^2(\Omega;\mathbb{R}^3)$ as $h \searrow 0$;

(γ2) for any $\hat{m}^\circ \in \mathrm{Dom}(\Phi_\alpha^\circ)$, there exists a sequence $\{\hat{m}^{(h)} \,|\, 0 < h < 1\} \subset L^2(\Omega;\mathbb{R}^3)$, such that $\hat{m}^{(h)} \to \hat{m}^\circ$ in $L^2(\Omega;\mathbb{R}^3)$ and $\Phi_\alpha^{(h)}(\hat{m}^{(h)}) \to \Phi_\alpha^\circ(\hat{m}^\circ)$, as $h \searrow 0$.

This theorem is proved by relying on some classes of open sets, mentioned in the following lemma.

Lemma 3.2. *(Open coverings for non-degenerate parts) There exists a sequence $\{\bar{\alpha}_\ell^\circ \,|\, \ell = 1,2,3,\cdots\} \subset (0,1)$ and a covering $\{S_\ell \,|\, \ell = 1,2,3,\cdots\} \subset S \setminus A_0^\circ$ of $S \setminus A_0^\circ$ with smooth boundaries ∂S_ℓ ($\ell = 1,2,3,\cdots$), such that:*

$$\begin{cases} \emptyset \neq S_1 \subset\subset \cdots \subset\subset S_\ell \subset\subset \cdots \subset\subset S \setminus A_0^\circ = \displaystyle\bigcup_{\ell=0}^{\infty} S_\ell, \\ \bar{\alpha}_1^\circ > \cdots > \bar{\alpha}_\ell^\circ > \cdots > 0 = \displaystyle\lim_{\ell \to \infty} \bar{\alpha}_\ell^\circ, \quad \text{and } \alpha^\circ \geq \bar{\alpha}_\ell^\circ \quad \text{on } S_\ell, \text{ for } \ell = 1,2,3,\cdots. \end{cases}$$

Hence, if we assume the conditions (a1)-(a2) as in introduction, then a sequence $\{\Omega_\ell^\dagger\} := \{S_\ell \times (0,1) \,|\, \ell = 1,2,3,\cdots\}$ turns out to be a covering of an open set $\Omega^\dagger := \Omega \setminus (A_0^\circ \times (0,1))$, with Lipschitz boundaries $\partial\Omega_\ell^\dagger$ ($\ell = 1,2,3,\cdots$), such that:

$$\begin{cases} \emptyset \neq \Omega_1^\dagger \subset \cdots \subset \Omega_\ell^\dagger \subset \cdots \subset \Omega^\dagger = \displaystyle\bigcup_{\ell=0}^{\infty} \Omega_\ell^\dagger \subset \Omega \setminus A_0^{(h)}, \quad \mathcal{L}^3(\Omega \setminus \Omega^\dagger) = 0, \\ \alpha^{(h)} \geq \alpha^\circ \geq \bar{\alpha}_\ell^\circ > 0 \text{ on } \Omega_\ell^\dagger, \quad \text{for } \ell = 1,2,3,\cdots \text{ and } 0 < h < 1. \end{cases} \tag{14}$$

Proof of Lemma 3.2. This lemma is a direct consequence of the line of arguments, discussed in [16, Lemma 4.1, Remark 4-5]. □

Proof of Theorem 3.1. First, we verify the assertion (γ1). Then, it is enough to consider only the case when $\liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\check{m}^{(h)}) < \infty$, since another case is trivial.

On account of (i-3) in Lemma 2.1, we find a sequence $\{\check{h}_i \,|\, i = 1,2,3,\cdots\} \subset (0,1)$ and a sequence $\{\check{\mu}^{(i)} \,|\, i = 1,2,3,\cdots\} \subset W^{1,2}(\Omega;\mathbb{R}^3) \cap L^2(\Omega;\mathbb{S}^2)$, such that:

$$\begin{cases} \check{h}_i \searrow 0 \text{ and } \check{\mu}^{(i)} \to \check{m}^\circ \text{ in } L^2(\Omega;\mathbb{R}^3), \text{ as } i \to \infty, \\ \displaystyle\lim_{i \to \infty} \Phi_{0,\alpha}^{(\check{h}_i)}(\check{\mu}^{(i)}) = \liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\check{m}^{(h)}). \end{cases}$$

Here, in the light of (14) and (i-1) in Lemma 2.1,

$$\begin{aligned} |\partial_3 \check{m}^\circ|_{L^2(\Omega_\ell^\dagger;\mathbb{R}^3)}^2 &\leq \liminf_{i \to \infty} |\partial_3 \check{\mu}^{(i)}|_{L^2(\Omega_\ell^\dagger;\mathbb{R}^3)}^2 \\ &\leq \frac{1}{\bar{\alpha}_\ell^\circ} \sup_{i \geq 1} \Phi_{0,\alpha}^{(\check{h}_i)}(\check{\mu}^{(i)}) \lim_{i \to \infty} \check{h}_i^2 = 0, \quad \ell = 1,2,3,\cdots, \end{aligned} \tag{15}$$

which implies $\partial_3 \check{m}^\circ = 0$, $\mathcal{L}^3$-a.e. in $\Omega^\dagger$. Hence, the limit $\check{m}^\circ$ can be regarded to belong to the class $L^2(S \setminus A_0^\circ; \mathbb{R}^3)$ $(= L^2(S; \mathbb{R}^3))$ of binary functions. Subsequently, let us set:

$$\check{\mu}^{(i)}(x_1, x_2) := \check{\mu}^{(i)}(x_1, x_2, c_i), \text{ for } \mathcal{L}^2\text{-a.e. } (x_1, x_2) \in S \text{ and } i = 1, 2, 3, \cdots;$$

by using a collection $\{c_i \,|\, i = 1, 2, 3, \cdots\} \subset (0, 1)$ of constants, such that:

$$\int_S \alpha^{(\check{h}_i)}(x_1, x_2, c_i) |\nabla_{\mathrm{P}} \check{\mu}^{(i)}(x_1, x_2, c_i)|^2 \, d\mathcal{L}^2 \leq \int_\Omega \alpha^{(\check{h}_i)} |\nabla_{\mathrm{P}} \check{\mu}^{(i)}|^2 \, d\mathcal{L}^3, \quad i = 1, 2, 3, \cdots.$$

Then, with the helps from (15) and Fubini's theorem, it is computed that:

$$\begin{aligned}
& \lim_{i \to \infty} |\check{\psi}^{(i)} - \check{\mu}^{(i)}|_{L^2(\Omega; \mathbb{R}^3)}^2 \\
= & \lim_{i \to \infty} \left(|\check{\psi}^{(i)} - \check{\mu}^{(i)}|_{L^2(\Omega_\ell^\dagger; \mathbb{R}^3)}^2 + |\check{\psi}^{(i)} - \check{\mu}^{(i)}|_{L^2(\Omega \setminus \Omega_\ell^\dagger; \mathbb{R}^3)}^2 \right) \\
\leq & \lim_{i \to \infty} \int_{\Omega_\ell^\dagger} \int_0^1 |\partial_3 \check{\mu}^{(i)}|^2 d\mathcal{L}^1 d\mathcal{L}^3 + 4\mathcal{L}^3(\Omega \setminus \Omega_\ell^\dagger) \\
\leq & \lim_{i \to \infty} |\partial_3 \check{\mu}^{(i)}|_{L^2(\Omega_\ell^\dagger)}^2 + 4\mathcal{L}^3(\Omega \setminus \Omega_\ell^\dagger) = 4\mathcal{L}^3(\Omega \setminus \Omega_\ell^\dagger), \quad \text{for } \ell = 1, 2, 3, \cdots.
\end{aligned}$$

It implies that:

$$\check{\psi}^{(i)} \to \check{m}^\circ \text{ in } L^2(\Omega; \mathbb{R}^3), \quad \text{as } i \to \infty; \tag{16}$$

since $\mathcal{L}^3(\Omega \setminus \Omega_\ell^\dagger) \to 0$ as $\ell \to \infty$. Taking into account of (a2), (16), (ii-2) in Lemma 2.1 and the lower semi-continuity of Φ_α°, we deduce that:

$$\begin{aligned}
& \liminf_{h \searrow 0} \Phi_\alpha^{(h)}(\check{m}^{(h)}) = \lim_{i \to \infty} \Phi_\alpha^{(\check{h}_i)}(\check{\mu}^{(i)}) \geq \liminf_{i \to \infty} \int_\Omega \alpha^{(\check{h}_i)} |\nabla_{\mathrm{P}} \check{\mu}^{(i)}|^2 d\mathcal{L}^3 \\
\geq & \liminf_{i \to \infty} \int_S \alpha^{(\check{h}_i)}(x_1, x_2, c_i) |\nabla \check{\psi}^{(i)}(x_1, x_2)|^2 \, d\mathcal{L}^2 \geq \liminf_{i \to \infty} \Phi_\alpha^\circ(\check{\psi}^{(i)}) \geq \Phi_\alpha^\circ(\check{m}^\circ).
\end{aligned}$$

Thus, the assertion (γ1) is concluded.

Next, we verify the assertion (γ2). Let us take any $\hat{m}^\circ \in \mathrm{Dom}(\Phi_\alpha^\circ)$. Then, construction of the required sequence $\{\hat{m}^{(h)} \,|\, 0 < h < 1\}$ will be on the basis of a sequence $\{\hat{\mu}^{(i)} \,|\, i = 1, 2, 3, \cdots\} \subset W^{1,2}(S; \mathbb{R}^3) \cap L^2(S; \mathbb{R}^3)$, which will be obtained as the approximating sequence, as in (ii-3) of Lemma 2.1, when $m = \hat{m}^\circ$. Here, noting that $\alpha^{(h)} \to \alpha^\circ$ in $C(\overline{\Omega})$ as $h \searrow 0$, there exists a sequence $\{\hat{h}_i \,|\, i = 1, 2, 3, \cdots\} \subset (0, 1)$, such that:

$$\begin{cases}
\bullet \ \hat{h}_1 > \cdots > \hat{h}_i > \cdots > 0 = \lim_{i \to \infty} \hat{h}_i, \\
\bullet \ 0 \leq \Phi_{0,\alpha}^{(h)}(\hat{\mu}^{(i)}) - \Phi_{0,\alpha}^\circ(\hat{\mu}^{(i)}) = \displaystyle\int_\Omega (\alpha^{(h)} - \alpha^\circ) |\nabla_{\mathrm{P}} \hat{\mu}^{(i)}|^2 \, d\mathcal{L}^3 < \frac{1}{2^i}, \\
\quad \text{for any } 0 < h < \hat{h}_i, \ \ i = 1, 2, 3, \cdots.
\end{cases}$$

On that basis, the finding sequence $\{\hat{m}^{(h)}\}$ will be constructed by putting:

$$\hat{m}^{(h)} := \hat{\mu}^{(i)} \text{ in } L^2(\Omega; \mathbb{R}^3), \ \text{ if } \hat{h}_{i+1} \leq h < \hat{h}_i, \ i = 1, 2, 3, \cdots;$$

with an optional setting $\hat{m}^{(h)} := \hat{\mu}^{(1)}$ in $L^2(\Omega; \mathbb{R}^3)$, for $\hat{h}_1 \leq h < 1$. □

The above Theorem 3.1 actually implies the Γ-convergence of free-energies, stated as follows.

Corollary 2. *(Γ-convergence from $\mathcal{F}_\alpha^{(h)}$ to $\mathcal{F}_\alpha^\circ$ as $h \searrow 0$) Let us assume the conditions (a1)-(a2). Then, the sequence $\{\mathcal{F}_\alpha^{(h)} \,|\, 0 < h < 1\}$ of free-energies converges to the limiting one $\mathcal{F}_\alpha^\circ$, in the sense of Γ-convergence, as $h \searrow 0$.*

Proof of Corollary 2. This corollary is immediately concluded, by taking into account of Theorem 3.1 and (Fact 2) in Section 2. ☐

Remark 2. On account of (13) and (Fact 2), we will see that the sequence $\{\hat{m}^{(h)} \,|\, 0 < h < 1\}$ as in (γ2) of Theorem 3.1 will realize the convergence:

$$\mathcal{F}_\alpha^{(h)}(\hat{m}^{(h)}) \to \mathcal{F}_\alpha^\circ(\hat{m}^\circ) \ \text{ as } h \searrow 0.$$

4. **Proof of Main Theorem.** We divide this section into two subsections, for the respective assertions (I) and (II) of Main Theorem.

4.1. **Proof of (I) of Main Theorem.** The proof of this assertion will be a slight modification of the argument, discussed in [16, Section 5.1]. In fact, under (a1), and under the fixed setting of $0 < h < 1$, we can take the so-called *minimizing sequence* $\{m_*^{(i)} \,|\, i = 1, 2, 3, \cdots\} \subset \mathrm{Dom}(\mathcal{F}_\alpha^{(h)})$ that is supposed to satisfy:

$$\mathcal{F}_\alpha^{(h)}(m_*^{(i)}) \searrow F_*^{(h)} := \inf_{m \in L^2(\Omega;\mathbb{R}^3)} \mathcal{F}_\alpha^{(h)}(m) \ \text{ as } i \to \infty.$$

Here, on account of (13), (Fact 1) and Corollary 1, a convergence subsequence $\{m_*^{(i_k)} \,|\, k = 1, 2, 3, \cdots\} \subset \{m_*^{(i)}\}$ will be found with the limit $m_* \in L^2(\Omega;\mathbb{R}^3)$, and it will be seen that:

$$\begin{cases} m_*^{(i_k)} \to m_* \text{ in } L^2(\Omega;\mathbb{R}^3), \quad \varphi(m_*^{(i_k)}) \to \varphi(m_*) \text{ in } L^1(\Omega), \\ F_{\mathrm{mag}}^{(h)}(m_*^{(i_k)}) \to F_{\mathrm{mag}}^{(h)}(m_*), \end{cases} \quad \text{as } k \to \infty.$$

In response to the above, we infer from the lower semi-continuity of $\mathcal{F}_\alpha^{(h)}$ that the limit m_* is one of minimizers of $(\mathrm{MP})^{(h)}$. ☐

4.2. **Proof of (II) of Main Theorem.** Let us assume the conditions (a1)-(a2), and let us take a sequence $\{m^{(h)} \,|\, 0 < h < 1\}$ of minimizers of $\mathcal{F}_\alpha^{(h)}$, $0 < h < 1$.

Let us set $\nu^* := [1, 0, 0] \in \mathbb{S}^2$. Then, for the variational identity (10) when $m \equiv \nu^*$, taking the solution itself as the test function v in (10) yields that $F_{\mathrm{mag}}^{(h)}(\nu^*) \leq 1$, for any $0 < h < 1$ (see [16, Section 5.2], for details). In view of this,

$$\begin{aligned} \Phi_\alpha^{(h)}(m^{(h)}) &\leq \mathcal{F}_\alpha^{(h)}(m^{(h)}) \leq \mathcal{F}_\alpha^{(h)}(\nu^*) = \Phi_\alpha^{(h)}(\nu^*) + |\varphi(\nu^*)|_{L^1(\Omega)} + F_{\mathrm{mag}}^{(h)}(\nu^*) \\ &\leq \varphi(\nu^*) + 1, \quad \text{for all } 0 < h < 1. \end{aligned} \tag{17}$$

Since, the above (17) implies that $\{m^{(h)}\} \subset \mathcal{U}_{\mathcal{F}}(\varphi(\nu^*) + 1)$, we can apply Corollary 1, to find a sequence $\{h_i \,|\, i = 1, 2, 3, \cdots\} \subset (0, 1)$ and a limiting function $m^\circ \in L^2(\Omega;\mathbb{R}^3)$, such that:

$$\begin{cases} h_i \searrow 0, \ m^{(h_i)} \to m^\circ \ \text{ in } L^2(\Omega;\mathbb{R}^3), \\ \varphi(m^{(h_i)}) \to \varphi(m^\circ) \ \text{ in } L^1(\Omega), \end{cases} \quad \text{as } i \to \infty.$$

Here, taking into account of Theorem 3.1, Corollary 2 and (17),

$$\mathcal{F}_\alpha^\circ(m^\circ) \leq \liminf_{i \to \infty} \mathcal{F}_\alpha^{(h_i)}(m^{(h_i)}) \leq \varphi(\nu^*) + 1;$$

and hence $m^\circ \in \mathrm{Dom}(\mathcal{F}_\alpha^\circ) \subset L^2(S;\mathbb{R}^3)$.

Next, taking any $\hat{m}^\circ \in \mathrm{Dom}(\mathcal{F}_\alpha^\circ)$ $(= \mathrm{Dom}(\Phi_\alpha^\circ))$, and taking the sequence $\{\hat{m}^{(h)} \,|\, 0 < h < 1\} \subset L^2(\Omega;\mathbb{R}^3)$, obtained in ($\gamma$2) of Theorem 3.1, it will be seen from Theorem 3.1 Corollary 2 and Remark 2 that:

$$\begin{aligned}\mathcal{F}_\alpha^\circ(m^\circ) &\le \liminf_{i\to\infty} \mathcal{F}_\alpha^{(h_i)}(m^{(h_i)}) \le \limsup_{i\to\infty} \mathcal{F}_\alpha^{(h_i)}(m^{(h_i)}) \\ &\le \lim_{i\to\infty} \mathcal{F}_\alpha^{(h_i)}(\hat{m}^{(h_i)}) = \mathcal{F}_\alpha^\circ(\hat{m}^\circ).\end{aligned} \tag{18}$$

It implies that m° solves the limiting problem (MP)$^\circ$. Furthermore, putting $\hat{m}^\circ = m^\circ$ in (18), it is deduced that:

$$\mathcal{F}_\alpha^{(h)}(m^{(h)}) \to \mathcal{F}_\alpha^\circ(m^\circ) \quad \text{as } h \searrow 0.$$

Thus, we conclude the assertion (II). □

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