Divergence

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DIVERGENCE

Abstract. This note is dedicated to a few questions related to the divergence equation which have been motivated by recent studies concerning the Neumann problem for the Laplace equation or the (evolutionary) Stokes system in domains of $\mathbb{R}^n$. For simplicity, we focus on the classical Sobolev spaces framework in bounded domains, but our results have natural and simple extensions to the Besov spaces framework in more general domains.

1. Introduction.

We would like to present some recent advances related to the divergence operator, in connection with our new result in [4]. We are interested in functionals generated by the divergence of a vector-field and in the solvability of the divergence equation

$$\operatorname{div} u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{at} \quad \partial\Omega, \tag{1}$$

where the function f satisfies the compatibility condition

$$\int_\Omega f\,dx = 0. \tag{2}$$

For simplicity, we here assume that Ω is a C^2 bounded domain of $\mathbb{R}^n$.

Solving (1) in Sobolev spaces $W^k_p(\Omega)$ with $k \geq 0$ is a classical issue whenever f (together with enough derivatives) vanishes at the boundary $\partial\Omega$ of the domain (see e.g. [8] and the references therein). Motivated by recent works in incompressible fluid mechanics, we address two natural questions related to (1) which, to our knowledge, have been overlooked in the literature.

First, we want to investigate the problem of the solvability of (1) in high regularity if f *does not* vanish at $\partial\Omega$, and in low regularity, namely if the right-hand side of (1) belongs only to $(W^1_{p'}(\Omega))^*$.

Second, we want to consider the case where the divergence operator is replaced with a "twisted" or "curvilinear" divergence operator such as

$$\operatorname{div}_A = A : D \tag{3}$$

where A stands for some given matrix-valued function.

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Note that as $A = \mathrm{Id}$ then $\mathrm{div}_A = \mathrm{div}$. Basically, we shall show that if A is measure preserving and close enough to the identity then the results for the standard divergence equation (1) remain true for the twisted divergence equation

$$\mathrm{div}_A\, u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{at} \quad \partial\Omega. \tag{4}$$

Our investigations are motivated by the study of the inhomogeneous Navier-Stokes equations

$$\begin{array}{lcr} \rho_t + v\cdot\nabla\rho = 0 & \text{in} & \Omega\times(0,T), \\ \rho v_t + \rho v\cdot\nabla v - \nu\Delta v + \nabla Q = 0 & \text{in} & \Omega\times(0,T), \\ \mathrm{div}\, v = 0 & \text{in} & \Omega\times(0,T), \\ v = 0 & \text{on} & \partial\Omega\times(0,T), \\ v|_{t=0} = v_0, \qquad \rho|_{t=0} = \rho_0 & \text{in} & \Omega, \end{array} \tag{5}$$

in various domains of $\mathbb{R}^n$ (see [3, 6, 7]). It turns out that solving this problem for rough data involves the divergence equation *in a setting where the trace of the right-hand side of* (1) *or* (4) *need not be defined.* Furthermore, in recent works (see [5, 6]), we noticed that considering System (5) *in the Lagrangian coordinates* may help to handle very general initial densities. However, the Lagrangian velocity is no longer divergence-free for positive times: it satisfies (4) where $A = (DX)^{-1}$ and X stands for the Lagrangian flow. Note in particular that incompressibility is equivalent to $\det A \equiv 1$ so that we will restrict ourselves to this case in what follows.

2. **The divergence operator in low regularity.** In order to investigate (1) in low regularity, one has to find a proper meaning of $\mathrm{div}\, k$ as a distribution acting on smooth functions up to the boundary. To be more specific, let us consider the following Neumann problem:

$$\begin{array}{lcr} \Delta u = \mathrm{div}\, k & \text{in} & \Omega, \\ \frac{\partial u}{\partial \vec{n}} = 0 & \text{at} & \partial\Omega, \end{array} \qquad \int_\Omega u\, dx = 0. \tag{6}$$

Suppose we are able to solve such a system in the case where $k \in L_2(\Omega)$. Then we expect ∇u to be in $L_2(\Omega)$. This implies that

$$\mathrm{div}\,(\nabla u - k) = 0 \qquad \text{and} \qquad \nabla u - k \in L_2(\Omega).$$

The trace theorem entails that the normal part of the vector-field $\nabla u - k$ at the boundary is well defined as a functional over the trace space $W^{1/2}_2(\partial\Omega)$. In other words, denoting by $W^{-1/2}_2(\partial\Omega)$ the space of those functionals, and by $\vec{n}$ the outer unitary normal vector to $\partial\Omega$, we may write

$$\vec{n}\cdot(\nabla u - k)|_{\partial\Omega} \in W^{-1/2}_2(\partial\Omega).$$

On the other hand, if Equation (6) is solvable for $k \in L_2(\Omega)$ then we must have $\vec{n}\cdot\nabla u|_{\partial\Omega} = 0$, so

$$\vec{n}\cdot k|_{\partial\Omega} \in W^{-1/2}_2(\partial\Omega).$$

However, the above condition does not make sense for k an arbitrary vector-field with coefficients in $L_2(\Omega)$. In short, there is no chance to solve system (6) in this context.

This obstacle is classical and may be overcome in different ways (see e.g. [16] and the references therein). In this section, we present the approach that has been proposed in our recent work [4], in connection with the divergence equation. The key idea is to “prescribe” the normal part of the vector-field in a case where it should not make sense. This may be done by introducing a functional acting on smooth functions up to the boundary of Ω, which contains both the information over the divergence of k and over some distribution at the boundary. Here is the corresponding definition:

Definition 2.1. For any couple (k,ζ) of smooth functions with $k \in C^\infty(\overline{\Omega};\mathbb{R}^n)$ and $\zeta \in C^\infty(\partial\Omega;\mathbb{R})$ we define $\mathcal{DIV}[k,\zeta]$ as the functional over $C^\infty(\overline{\Omega};\mathbb{R})$ such that[1]

$$\mathcal{DIV}[k,\zeta](\phi) := -\int_\Omega k\cdot\nabla\phi\,dx + \int_{\partial\Omega}\zeta\phi\,d\sigma \qquad \text{for } \phi\in C^\infty(\overline{\Omega}).$$

We aim at generalizing the above definition to Lebesgue spaces $L_p(\Omega)$, that is the set of measurable functions over Ω with integrable p-th power. In passing, let us also introduce the Sobolev space W^m_p $(m\in\mathbb{N})$ which is the closure of smooth functions (up to boundary) for the following norm:

$$\|u\|_{W^m_p(\Omega)} = \|u\|_{L_p(\Omega)} + \sum_{0<|\alpha|\le m}\|D^\alpha u\|_{L_p(\Omega)}. \tag{7}$$

The space $W^{-1/p}_p(\partial\Omega)$ is the dual space to $W^{1/p}_q(\partial\Omega)$ with $1/p = 1-1/q$. The latter space can be viewed as the trace space of functions from $W^1_q(\Omega)$, where the norm is determined as follows:

$$\|u\|_{W^{1-1/q}_q(\partial\Omega)} = \inf\bigl\{\|w\|_{W^1_q(\Omega)} : w\in W^1_q(\Omega) \text{ and } w|_{\partial\Omega} = u\bigr\}. \tag{8}$$

Thanks to the trace theorem and to the definition given in (8), it is obvious that for all smooth functions ϕ, k and ζ, one has

$$\left|\int_\Omega \mathcal{DIV}[k,\zeta]\phi\,dx\right| \le C(\|k\|_{L_p(\Omega)} + \|\zeta\|_{W^{-1/p}_p(\partial\Omega)})\|\phi\|_{W^1_q(\Omega)}. \tag{9}$$

Therefore, arguing by density, Definition 2.1 naturally extends as follows:

Proposition 1. *Let $1<p,q<\infty$ with $1/p+1/q=1$, $k\in L_p(\Omega;\mathbb{R}^n)$ and $\zeta\in W^{-1/p}_p(\partial\Omega;\mathbb{R})$. Then $\mathcal{DIV}[k,\zeta]$ extends as the functional over $W^1_q(\Omega;\mathbb{R})$ such that*

$$\int_\Omega \mathcal{DIV}[k,\zeta]\phi\,dx = -\int_\Omega k\cdot\nabla\phi\,dx + \int_{\partial\Omega}\zeta\phi\,d\sigma \qquad \textit{for } \phi\in W^1_q(\Omega).$$

Remark 1. Note that in the above equality, ζ need not be integrable. It is understood that

$$\int_{\partial\Omega}\zeta\,d\sigma := \langle\zeta,1\rangle_{(W^{-1/p}_p(\partial\Omega),W^{1/p}_q(\partial\Omega))}. \tag{10}$$

We shall keep this notation throughout the paper.

Definition 2.2. In all that follows, we denote by $\mathcal{W}^{-1}_p(\Omega)$ the set of functionals $\mathcal{DIV}[k,\zeta]$ with $k\in L_p(\Omega;\mathbb{R}^n)$ and $\zeta\in W^{-1/p}_p(\partial\Omega;\mathbb{R})$ satisfying the compatibility condition

$$\int_{\partial\Omega}\zeta\,d\sigma = 0. \tag{11}$$

Granted with the above formalism and definitions, the above Neumann problem (6) recasts in

$$\mathcal{DIV}[\nabla u,0] = \mathcal{DIV}[k,0] \qquad \text{in } \mathcal{D}'(\overline{\Omega}). \tag{12}$$

This setting yields the following relations on the traces

$$\vec{n}\cdot(\nabla u - k) = 0 \ \text{ at } \ \partial\Omega.$$

[1] We will rather adopt the notation $\int_\Omega \mathcal{DIV}[k,\zeta]\phi\,dx := \mathcal{DIV}[k,\zeta](\phi)$ in the rest of the paper.

In this context, it is natural to generalize (6) as so:

$$\mathcal{DIV}\,[\nabla u,\zeta_2]=\mathcal{DIV}\,[k,\zeta_1]\qquad\text{in }\ \mathcal{D}'(\overline{\Omega})\tag{13}$$

where ζ_1 and ζ_2 are given distributions over $\partial\Omega$ satisfying the compatibility condition (with the convention (10)):

$$\int_{\partial\Omega}(\zeta_1-\zeta_2)\,d\sigma=0.\tag{14}$$

Under assumptions such as $k\in L_2(\Omega)$ and $\zeta_1,\zeta_2\in W_2^{-1/2}(\partial\Omega)$, solving (13) amounts to finding $u\in W^1_2(\Omega)$ such that

$$\int_\Omega\nabla u\cdot\nabla\phi\,dx-\int_{\partial\Omega}\zeta_2\phi\,d\sigma=\int_\Omega k\cdot\nabla\phi\,dx-\int_{\partial\Omega}\zeta_1\phi\,d\sigma\quad\text{for all }\ \phi\in W^1_2(\Omega).$$

This is of course equivalent to

$$\int_\Omega\nabla u\cdot\nabla\phi\,dx=\int_\Omega k\cdot\nabla\phi\,dx-\int_{\partial\Omega}(\zeta_1-\zeta_2)\phi\,d\sigma,\tag{15}$$

or to

$$\int_\Omega\nabla u\cdot\nabla\phi\,dx-\int_{\partial\Omega}(\zeta_2-\zeta_1)\phi\,d\sigma=\int_\Omega k\cdot\nabla\phi\,dx.\tag{16}$$

Those latter two problems rewrite respectively

$$\mathcal{DIV}\,[\nabla u,0]=\mathcal{DIV}\,[k,\zeta_1-\zeta_2]\qquad\text{in }\ \mathcal{D}'(\overline{\Omega})\tag{17}$$

and

$$\mathcal{DIV}\,[\nabla u,\zeta_2-\zeta_1]=\mathcal{DIV}\,[k,0]\qquad\text{in }\ \mathcal{D}'(\overline{\Omega}).\tag{18}$$

In any case, the distributional interpretation is that

$$\Delta u=\operatorname{div}k\ \text{ in }\ \Omega\quad\text{ and }\quad(\nabla u-k)\cdot\vec{n}=\zeta_2-\zeta_1\ \text{ on }\ \partial\Omega.$$

In the case of regular data (that is if k is smooth), then the normal trace of k at the boundary is defined. Therefore decorrelating k and its "formal" normal trace ζ_1 is not relevant any longer: keeping (6) in mind, the only equation that has to be considered is (13) with $\zeta_2=0$ and $\zeta_1=k\cdot\vec{n}$.

The below result ends our considerations for systems (6) and (13).

Theorem 2.3. *Let Ω be a C^2 bounded domain. Let $k\in L_2(\Omega)$ and $\zeta_1,\zeta_2\in W_2^{-1/2}(\partial\Omega)$ satisfying the compatibility condition (14).*

Then there exists a unique solution to (13) such that $u\in W^1_2(\Omega)$ and $\int_\Omega u\,dx=0$. Moreover, the following estimate is valid:

$$\|u\|_{W^1_2(\Omega)}\leq C(\|k\|_{L_2(\Omega)}+\|\zeta_1-\zeta_2\|_{W_2^{-1/2}(\partial\Omega)}).\tag{19}$$

The proof of Theorem 2.3 follows directly from classical arguments based on Definition 2.1 and Inequality (9).

3. **The divergence equation in high regularity.** The objective of this part is to analyze the construction of a solution to the divergence equation

$$\operatorname{div}u=f\ \text{ in }\ \Omega,\qquad u=0\ \text{ at }\ \partial\Omega.\tag{20}$$

We are interested in the case where f is smooth as well as in the case where it is merely a functional (that is $f=\mathcal{DIV}\,[k,\zeta]$). In particular, we want to show that the corresponding solution operators coincide, a property that is of importance for the analysis of (5). Our main result reads:

Theorem 3.1. *Let $m \in \{-1\} \cup \mathbb{N}$, $p \in (1,\infty)$ and Ω a bounded domain with $\partial\Omega \in C^{\max\{2,m\}}$. There exists a linear operator $B_m : \mathcal{W}_p^{-1}(\Omega) \to L_p(\Omega)$ satisfying the following properties:*

- *if $f = \mathcal{DIV}\,[k,\zeta] \in \mathcal{W}_p^{-1}(\Omega)$ then $u := B_m(f)$ fulfills*
$$\mathcal{DIV}\,[u,0] = \mathcal{DIV}\,[k,\zeta] \qquad \text{in } \mathcal{D}'(\overline{\Omega}) \tag{21}$$
and, in addition,
$$\|u\|_{L_p(\Omega)} \le C(\|k\|_{L_p(\Omega)} + \|\zeta\|_{W_p^{-1/p}(\partial\Omega)}). \tag{22}$$
- *For all $j \in \{0,\cdots,m\}$ and $f \in W_p^j(\Omega)$ satisfying (2), the vector-field $u := B_m(f)$ is in $W_p^{j+1}(\Omega)$, fulfills (1) and*
$$\|u\|_{W_p^{j+1}(\Omega)} \le C\|f\|_{W_p^j(\Omega)}. \tag{23}$$

The above result is a natural extension of the following well-known theorem [1, 8, 14]:

Theorem 3.2. *Let $\partial\Omega \in C^2$, $m \in \mathbb{N}$ and $p \in (1,\infty)$. Then there exists a linear map $\mathcal{B} : \bar{W}_p^m(\Omega) \to \bar{W}_p^{m+1}(\Omega)$, where, for $j \in \mathbb{N}$,*
$$\bar{W}_p^j(\Omega) := \left\{ f \in W_p^j(\Omega) \ \text{ such that } \ \nabla^k f|_{\partial\Omega} = 0 \ \text{ for } \ k = 0,\cdots,j-1 \right\},$$
and the vector-field $u := \mathcal{B}[f]$ fulfills (20) and
$$\|u\|_{W_p^{m+1}(\Omega)} \le C\|f\|_{W_p^m(\Omega)}. \tag{24}$$

The main improvement in Theorem 3.1 lies in the case of low regularity data and on the fact that the trace of f at the boundary need not be zero. A similar result has been proved in [10] (see Cor. 1.4) for the high-regularity case and by the authors in [4] in the Besov spaces $B^s_{p,q}(\Omega)$ $(-1+1/p < s < 1/p)$ setting. However, in those two works it does not appear clearly that high and low regularity estimates hold true for *the same* solution map: we have to keep in mind that solutions to (1) are not unique.

Proving Theorem 3.2 is based on an accurate analysis of the following integral formula :
$$u(x) = \mathcal{B}(f) := \int_\Omega f(y)\frac{x-y}{|x-y|^n}\int_0^\infty \omega\Big(x + r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{n-1}drdy, \tag{25}$$
where ω is a fixed smooth function with average 1 supported in a ball $B(x_0,R)$. Equality (25) is known in the literature as the Bogovskiĭ formula [1]. The result of Bogovskiĭ has been inspired by an idea of Sobolev [15]. It is well-known (see e.g. [8]) that the above vector-field u fulfills the divergence equation (1) in Ω, provided the domain is star-shaped with respect to the ball $B(x_0,R)$. The expression of (25) guarantees that for smooth f supported in Ω the solution vanishes away from Ω. Hence the boundary condition for u is satisfied immediately. Theorem 3.2 then follows by density once suitable a priori estimates have been established. Let us emphasize that arguing as so requires the function f to vanish at the boundary.

In the case of low regularity the formula should be modified. Formally, if one performs an integration by parts in (25) in the case where $f = \operatorname{div} k$, and set $\zeta = k\cdot\vec{n}$, we obtain
$$u = Ik + J\zeta, \tag{26}$$

where

$$Ik(x) = -\int_\Omega k(y)\cdot\nabla_y\Big[\frac{x-y}{|x-y|^n}\int_0^\infty \omega\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{n-1}dr\Big]dy, \quad (27)$$

$$J\zeta(x) = \int_{\partial\Omega}\zeta(y)\frac{x-y}{|x-y|^n}\int_0^\infty \omega\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{n-1}\,dr\,d\sigma_y. \quad (28)$$

Even if there is no correlation between k and ζ, these two integrals make sense independently from one another if seen in the principal value meaning. Of course, in the special case where the data are regular and $\zeta = k\cdot\vec{n}$, we may write

$$Ik + J(k\cdot\vec{n}) = \mathcal{B}(\operatorname{div} k) \quad (29)$$

so that we do get a solution to the initial divergence equation (1).

The main achievement of Formula (26) is that it keeps some marginal information about the trace, not neglecting influence of data located in neighborhood of the boundary and provides a solution to (21) in the star-shaped case. The case of more general domains may be treated by a suitable decomposition as in [8]. This enabled to get the following result (see [4]):

Theorem 3.3. *Let $f = \mathcal{DIV}[k,\zeta]$ be in $\mathcal{W}_p^{-1}(\Omega)$. There exists a linear operator B_{-1} so that $u := B_{-1}(f)$ fulfills the (generalized) divergence equation (21) with the estimate*

$$\|u\|_{L_p(\Omega)} \le C(\|k\|_{L_p(\Omega)} + \|\zeta\|_{W_p^{-1/p}(\partial\Omega)}).$$

In addition if $\operatorname{div} k \in L_p(\Omega)$ and $\zeta = \vec{n}\cdot k$ at the boundary, then $u = \mathcal{B}[\operatorname{div} k]$.

Remark 2. In the literature we can find some results concerning the divergence equation in spaces with very low regularity (see [2, 9] and [10], Th. 4.1). However, to our knowledge, questions concerning the meaning of boundary conditions have not been discussed: the spaces (of negative regularity order) that have been used in the aforementioned works have the property that any element can be extended by zero onto $\mathbb{R}^n$. In other words, we are allowed to put $\mathbb{R}^n$ in (25) instead of Ω.

Proof of Theorem 3.1. We omit the cases $m \in \{-1, 0\}$ that have been treated in Theorem 3.3 and go directly to the case $m \ge 1$.

Let us first spend some time on the most natural approach (which, unfortunately, seems to fail). We want to reduce the study to the case which is treated in Theorem 3.2. So it is enough to find a vector-field $E_m u$ in $W_p^{m+1}(\Omega)$ such that

$$\nabla^k \operatorname{div} E_m u = \nabla^k f \;\text{ at }\; \partial\Omega, \quad \text{for }\; k = 0,\cdots,m-1.$$

Of course $E_m u$ will depend on both m and f.

The issue becomes solvable whenever we are able to determine derivatives at the boundary in terms of $\operatorname{div} u$ and its derivatives at $\partial\Omega$. The conditions $\operatorname{div} u = f$ in Ω and $u = 0$ at $\partial\Omega$ imply that the tangential derivatives of u vanish at $\partial\Omega$ and that

$$\frac{\partial(u\cdot\vec{n})}{\partial\vec{n}} = f \quad \text{at } \partial\Omega. \quad (30)$$

However higher derivatives obtained that way depend on the regularity of the boundary itself– see [18]. To get the general form of (30) corresponding to the case where

$$\operatorname{div} u = f \;\text{ in }\; \Omega \quad \text{and} \quad u = g \;\text{ on }\; \partial\Omega,$$

we look at a neighborhood of a point $x_0 \in \partial\Omega$. Taking a local coordinate system given by an extension of boundary frame $(\vec{n}, \vec{\tau}_1, ..., \vec{\tau}_{n-1})$ with coordinates $(s, t_1, ..., t_{n-1})$ we obtain the following boundary relations

$$M \cdot \nabla_{(s,t_1,\dots,t_{n-1})} \tilde{u} = \tilde{f} \quad \text{and} \quad \tilde{u} = \tilde{g} \qquad \text{at } s = 0. \tag{31}$$

Here $M \cdot \nabla_{(s,t_1,\dots,t_{n-1})}$, $\tilde{u}, \tilde{f}, \tilde{g}$ correspond to div_x, u, f, g in the new coordinates system. Of course the coefficients of the matrix M depend on the geometry of $\partial\Omega$. Furthermore, choosing the change of coordinates in such a way that the normal vector is preserved, we are ensured that (31) yields

$$\frac{\partial(u \cdot \vec{n})}{\partial \vec{n}} \quad \text{is a linear combination of } [f, g, g_{\tau_1}, ..., g_{\tau_{n-1}}]. \tag{32}$$

The coefficients of the linear combination depend only on those of the matrix M – see [13, 18]. The rest of the components of $\nabla^k u|_{\partial\Omega}$, which are not involved in the constraint $\operatorname{div} u = f$, are put to be zero.

So applying the existence theorem [17], Chap. 3, we are able to find $E_m u \in W_p^{m+1}(\Omega)$ such that

$$\nabla^k \operatorname{div} E_m u|_{\partial\Omega} = \nabla^k f|_{\partial\Omega} \qquad \text{for } k = 0, \cdots, m-1$$

and

$$\|E_m u\|_{W_p^{m+1}(\Omega)} \leq C\|f\|_{W_p^m(\Omega)}. \tag{33}$$

In conclusion, subtracting E_m from f reduces the problem to that when f vanishes at the boundary. Therefore, applying Theorem 3.2 provides a solution in $W_p^{m+1}(\Omega)$ to (1). Unfortunately, this construction solves the problem only for the fixed highest regularity: whether (22) and (23) (for $j < m$) are satisfied too, is unclear.

For the sake of simplicity, we focus now on the case $m = 1$ (which is the most important for the applications that we have in mind) and assume in addition that Ω is star-shaped with respect to some ball $B(x_0, R)$. The case of a general C^2 bounded domain may be achieved by means of classical decomposition techniques as in [4, 8]. To get W_p^2 regularity we differentiate (25) once. We obtain for $i = 1, \cdots, n$,

$$\begin{aligned} \partial_{x_i} u(x) &= \int_\Omega f(y)\partial_{x_i}\left[\frac{x-y}{|x-y|^n}\int_0^\infty \omega\left(x + r\frac{x-y}{|x-y|}\right)(|x-y|+r)^{n-1}dr\right]dy \\ &= [K(\partial_{y_i} f)](x) + L_i f(x) \end{aligned}$$

with, for some suitable modification ω^* of ω,

$$\begin{aligned} Kg(x) &:= -\int_\Omega g(y)\frac{x-y}{|x-y|^n}\int_0^\infty \omega^*\left(x, r\frac{x-y}{|x-y|}\right)(|x-y|+r)^{n-1}drdy, \\ L_i f(x) &:= \int_{\partial\Omega} f(y) n_{y_i}\frac{x-y}{|x-y|^n}\int_0^\infty \omega^*\left(x, r\frac{x-y}{|x-y|}\right)(|x-y|+r)^{n-1}drd\sigma_y. \end{aligned}$$

It is clear that $K : L_p(\Omega) \to W_p^1(\Omega)$ is a bounded map (see e.g. [4]). That L_i maps $W_p^{1-1/p}(\partial\Omega)$ to $W_p^1(\Omega)$ is a consequence of Theorem 1.1 of [11]. In our case, the assumption that $L_i(1)$ is a constant is not satisfied. However, by looking more closely at the proof therein, page 192, in the case $k = 0$ and $s = 1 - 1/p$, we see that it suffices to have the property that $\nabla L_i(1) \in L_p(\Omega)$. In our case, as the domain is C^2, we even have that $L_i(1) \in C^1$.

The general case $m > 1$ which is more technical, is not presented here. Basically, we still have to start from formula (25) and to use the full C^m regularity of the domain. Note that the higher regularity of the boundary is important to control operators L_i. At the same time, as regards operator K, the C^2 regularity (and probably even less) suffices. □

Remark 3. If we look at the regularity of $\partial\Omega$ then our approach is not optimal (see [10]). However our main goal in the present note is to justify that *the same* operator B_m provides estimates in any space $W^{j+1}_p(\Omega)$ with $j \in \{-1, \cdots, m\}$. In the applications that we have in mind (see the introduction), it is important to have a solution operator providing both low and high regularity estimates.

4. **The twisted divergence equation.** The last part of this note concerns the resolution of the twisted divergence equation (4). As indicated above, our motivation comes mainly from the study of incompressible flows in the Lagrangian coordinates [5, 12], so that we restrict ourselves to the case where the matrix-valued function A is associated with a measure preserving map, that is $\det A \equiv 1$. Then the following formula (see e.g. the appendix of [5]):

$$\operatorname{div}_A \xi = A : D\xi = \operatorname{div}(A\xi) \tag{34}$$

holds true for sufficiently smooth vector-fields ξ.

Let us now state our main result:

Theorem 4.1. *Let $A \in L_\infty(\Omega; \mathbb{R}^n \times \mathbb{R}^n)$ satisfy $\det A \equiv 1$. Let $m \geq -1$ be an integer and $p > 1$, a real number. There exist two positive constants ε and C depending only on m, p and Ω, and a linear map $B_A : \mathcal{W}^{-1}_p(\Omega) \to L_p(\Omega)$ such that if*

$$\|A - \mathrm{Id}\|_{L_\infty(\Omega)} \leq \varepsilon, \tag{35}$$

then the following results are true:

- *for any functional $\mathcal{DIV}[k,\zeta]$ in $\mathcal{W}^{-1}_p(\Omega)$, the vector-field $u := B_A(\mathcal{DIV}[k,\zeta])$ belongs to $L_p(\Omega)$, satisfies*

$$\mathcal{DIV}[Au, 0] = \mathcal{DIV}[k,\zeta] \qquad \text{in } \Omega \tag{36}$$

$$\|u\|_{L_p(\Omega)} \leq C\big(\|k\|_{L_p(\Omega)} + \|\zeta\|_{W^{-1/p}_p(\partial\Omega)}\big). \tag{37}$$

- *If moreover $m \geq 0$ then for any $f \in L_p(\Omega; \mathbb{R}^n)$ satisfying (2) the vector-field $u := B_A(f)$ is a solution in $W^1_p(\Omega)$ to the twisted divergence equation (4), and*

$$\|u\|_{W^1_p(\Omega)} \leq C\|f\|_{L_p(\Omega)}. \tag{38}$$

- *If in addition $m \geq 1$, $p > n/m$ and $\nabla A \in W^{m-1}_p(\Omega)$ then $u := B_A(f)$ with $f \in W^m_p(\Omega)$ satisfying (2), fulfills*

$$\|u\|_{W^{m+1}_p(\Omega)} \leq C\Big(\|f\|_{W^m_p(\Omega)} + \|\nabla A\|^{\frac{1}{1-n/(pm)}}_{W^{m-1}_p(\Omega)}\|f\|_{L_p(\Omega)}\Big). \tag{39}$$

Proof. Granted with Theorem 3.1, this is a mere application of the Banach fixed point theorem. Let us introduce the linear map

$$T : W^{m+1}_p(\Omega) \to W^{m+1}_p(\Omega) \tag{40}$$

such that

$$T\bar\xi := B_m((\mathrm{Id} - A) : D\bar\xi + f) \qquad \text{if} \quad m \geq 0,$$

$$T\bar\xi := B_{-1}(\mathcal{DIV}[((\mathrm{Id} - A)\bar\xi) + k), \zeta]) \qquad \text{if} \quad m = -1.$$

Let us first assume that $m = -1$. Then Theorem 3.1 ensures that

$$\|T\bar\xi\|_{L_p(\Omega)} \leq C\big(\|\mathrm{Id} - A\|_{L_\infty(\Omega)}\|\bar\xi\|_{L_p(\Omega)} + \|k\|_{L_p(\Omega)} + \|\zeta\|_{W^{-1/p}_p(\Omega)}\big),$$

whence, by virtue of (35),

$$\|T\bar\xi\|_{L_p(\Omega)} \leq C\varepsilon\|\bar\xi\|_{L_p(\Omega)} + C\big(\|k\|_{L_p(\Omega)} + \|\zeta\|_{W_p^{-1/p}(\Omega)}\big). \tag{41}$$

Next, let us look at the case $m=0$. Then Theorem 3.1 yields

$$\begin{aligned}\|T\bar\xi\|_{W^1_p(\Omega)} &\leq C\big(\|(\mathrm{Id}-A):\nabla\bar\xi\|_{L_p(\Omega)} + \|f\|_{L_p(\Omega)}\big)\\ &\leq C\big(\|\mathrm{Id}-A\|_{L_\infty(\Omega)}\|\nabla\bar\xi\|_{L_p(\Omega)} + \|f\|_{L_p(\Omega)}\big),\end{aligned}$$

whence, thanks to (35)

$$\|T\bar\xi\|_{W^1_p(\Omega)} \leq C\big(\varepsilon\|\bar\xi\|_{W^1_p(\Omega)} + \|f\|_{L_p(\Omega)}\big). \tag{42}$$

So putting (41) and (42) together, and choosing ε small enough, we get

$$\|T\bar\xi\|_{W_p^{m+1}(\Omega)} \leq \frac12\|\bar\xi\|_{W_p^{m+1}(\Omega)} + \begin{cases} C\|f\|_{W_p^m(\Omega)} & \text{if} \quad m=0,\\ C(\|k\|_{L_p(\Omega)} + \|\zeta\|_{W_p^{-1/p}(\partial\Omega)}) & \text{if} \quad m=-1.\end{cases}$$

As the map T is linear, this implies that

$$\|T\bar\xi_1 - T\bar\xi_2\|_{W_p^{m+1}(\Omega)} \leq \frac12\|\bar\xi_1-\bar\xi_2\|_{W_p^{m+1}(\Omega)}.$$

So we proved that T is a contraction, hence it has a unique fixed point ξ. This fixed point obviously satisfies (4) (or (36)) and the desired estimates. The case $m\in\{-1,0\}$ is done.

The case $m\geq 1$ is slightly more involved. Firstly, according to Theorem 3.1,

$$\|T\bar\xi\|_{W_p^{m+1}(\Omega)} \leq C(\|(\mathrm{Id}-A):\nabla\bar\xi\|_{W_p^m(\Omega)} + \|f\|_{W_p^m(\Omega)}). \tag{43}$$

For bounding the first term of the right-hand side, we may use the following classical tame estimate:

$$\|(\mathrm{Id}-A):\nabla\bar\xi\|_{W_p^m(\Omega)} \leq \|\mathrm{Id}-A\|_{L_\infty(\Omega)}\|\nabla\bar\xi\|_{W_p^m(\Omega)} + C\|\nabla\bar\xi\|_{L_\infty(\Omega)}\|\nabla A\|_{W_p^{m-1}(\Omega)},$$

and the following interpolation inequality (recall that $p>n/m$):

$$\|\nabla\bar\xi\|_{L_\infty(\Omega)} \leq \eta\|\nabla\bar\xi\|_{W_p^m(\Omega)} + C\eta^{-\frac{\theta}{1-\theta}}\|\nabla\bar\xi\|_{L_p(\Omega)} \quad \text{with} \quad \theta := \frac{n}{pm}\cdot$$

We end up with

$$\begin{aligned}\|T\bar\xi\|_{W_p^{m+1}(\Omega)} \leq C\Big(&\varepsilon\|\bar\xi\|_{W_p^{m+1}(\Omega)}\\ &+\|\nabla A\|_{W_p^{m-1}(\Omega)}\big(\eta\|\nabla\bar\xi\|_{W_p^m(\Omega)} + \eta^{-\frac{\theta}{1-\theta}}\|\nabla\bar\xi\|_{L_p(\Omega)}\big) + \|f\|_{W_p^m(\Omega)}\Big).\end{aligned}$$

Now, combining (42) with the above inequality yields for any $M>0$:

$$\begin{aligned}\|T\bar\xi\|_{W_p^{m+1}(\Omega)} + M\|T\bar\xi\|_{W_p^1(\Omega)} \leq C\Big(&\varepsilon(\|\bar\xi\|_{W_p^{m+1}(\Omega)} + M\|\bar\xi\|_{W_p^1(\Omega)})\\ +\|\nabla A\|_{W_p^{m-1}(\Omega)}\big(\eta\|\nabla\bar\xi\|_{W_p^m(\Omega)} &+ \eta^{-\frac{\theta}{1-\theta}}\|\nabla\bar\xi\|_{L_p(\Omega)}\big) + \|f\|_{W_p^m(\Omega)} + M\|f\|_{L_p(\Omega)}\Big).\end{aligned}$$

Let us take $\eta = \varepsilon\|\nabla A\|^{-1}_{W_p^{m-1}(\Omega)}$ and $M=\varepsilon^{-1}\eta^{-\frac{\theta}{1-\theta}}\|\nabla A\|_{W_p^{m-1}(\Omega)}$. Then the above inequality becomes (up to an irrelevant change of C):

$$\begin{aligned}&\|T\bar\xi\|_{W_p^{m+1}(\Omega)} + M\|T\bar\xi\|_{W_p^1(\Omega)}\\ &\qquad\leq C\Big(\varepsilon(\|\bar\xi\|_{W_p^{m+1}(\Omega)} + M\|\bar\xi\|_{W_p^1(\Omega)}) + \|f\|_{W_p^m(\Omega)} + M\|f\|_{L_p(\Omega)}\Big).\end{aligned} \tag{44}$$

So it is now clear that if one takes ε small enough then the linear map T is strictly contractive in the sense of the norm that is defined by the l.h.s. of (44). So one may end the proof as in the case $m \in \{-1, 0\}$. Note that Inequality (44) yields (39). $\square$

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REFERENCES

[1] M. Bogovskiĭ: Solution of the first boundary value problem for an equation of continuity of an incompressible medium. (Russian) *Dokl. Akad. Nauk SSSR*, **248**(5), pages 1037–1040 (1979).

[2] M. Costabel and A. McIntosch, *On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains,* Math. Zeitschrift, **265**, 2010, pages 297–320.

[3] R. Danchin and P.B. Mucha, A critical functional framework for the inhomogeneous Navier-Stokes equations in the half-space, *J. Funct. Anal.*, **256**(3), pages 881–927 (2009).

[4] R. Danchin and P.B. Mucha, The divergence equation in rough spaces, *Journal of Mathematical Analysis and Applications*, **386**, pages 10–31 (2012).

[5] R. Danchin and P.B. Mucha, A Lagrangian approach for solving the incompressible Navier-Stokes equations with variable density, *Communications on Pure and Applied Mathematics*, in press.

[6] R. Danchin and P.B. Mucha, Incompressible flows with piecewise constant density, submitted.

[7] R. Danchin and P.B. Mucha, Critical functional framework and maximal regularity in action on systems of incompressible flows, in progress.

[8] G. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I. Linearized steady problems,* Springer, Berlin, 1994.

[9] M. Geissert, H. Heck and M. Hieber, On the equation $\operatorname{div} u = g$ and Bogovskiĭ's operator in Sobolev spaces of negative order, Partial differential equations and functional analysis, Oper. Theory Adv. Appl., **168**, Birkhäuser, Basel, pages 113–121, 2006.

[10] D. Mitrea, M. Mitrea and S. Monniaux, The Poisson problem for the exterior derivative operator with Dirichlet boundary condition on nonsmooth domains, *Commun. Pure Appl. Anal.*, **7**(6), pages 1295–1333 (2008).

[11] D. Mitrea, M. Mitrea, S. Monniaux, Weighted Sobolev space estimates for a class of singular integral operators. Around the research of Vladimir Maz'ya. III, 179-200, Int. Math. Ser. (N. Y.), 13, Springer, New York, 2010.

[12] P.B. Mucha, W.M. Zajączkowski, On local existence of solutions of the free boundary problem for an incompressible viscous self-gravitating fluid motion, *Appl. Math.* , **27**(3), pages 319–333 (2000).

[13] P.B. Mucha, On the Stefan problem with surface tension in the L_p framework. *Adv. Differential Equations*, **10**(8), pages 861–900 (2005).

[14] A. Novotný and I. Straškraba, *Introduction to the Mathematical Theory of Compressible Flow*, Oxford Univ. Press, 2004

[15] S. Sobolev, Applications of functional analysis to mathematical physics, *American Mathematical Society.* Translation of Monographs, Vol. 7, 1964.

[16] R. Temam: *Navier-Stokes equations. Theory and numerical analysis.* Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., 1977.

[17] H. Triebel, *Theory of function spaces. Monographs in Mathematics,* 78. Birkhäuser Verlag, Basel, 1983.

[18] W.M. Zajączkowski, Existence and regularity of solutions of some elliptic system in domains with edges, *Dissertationes Math.*, **274** (1988), 95 pp.

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