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Microscopic renormalized energy for a pinned Ginzburg-Landau functional. The circular and degree one case

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Abstract

We study the vortex location for minimizers of a Ginzburg-Landau energy with a discontinuous constraint. The discontinuous constraint appears in the potential $(a^2 - |u|^2)^2$. The function a is piecewise constant: it takes the value $0 < b < 1$ in small disjoint domains (called *inclusions*) and 1 otherwise. It is proved, under some assumptions on the smallness of the inclusions and on their inter-distances, that the vortices of minimizers are trapped inside the inclusions. Moreover the asymptotic location of the vortices inside an inclusion depends only on three parameters: the value b, the geometry of the inclusion and the number of vortices inside the inclusion.

It is expected that, if an inclusion containing a unique vortex is a disk, then the asymptotic location of the vortex is the center of the inclusion. This article is dedicated to the proof of this expectation.

Keywords. Ginzburg-Landau type energy, pinning, renormalized energy
Mathematics Subject Classification (2000). 49K20 35J66 35J50

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1 The problem

The aim of this article is to improve a result of a previous work [Dos] by answering a problem left open: Perspective (4) in [Dos] (Section 5 page 31).

We study a variant of the *simplified Ginzburg-Landau energy*

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2. \tag{1}$$

Here Ω is a smooth bounded two-dimensional domain, u is a complex-valued function and ε is a positive parameter. [The variant studied is presented in Formula (3)]

This Ginzburg-Landau type energy is a simplification of the *full Ginzburg-Landau energy* with magnetic field. The full Ginzburg-Landau energy was introduced by Ginzburg and Landau [GL50] in the 1950s as a phenomenological model to describe superconductivity. (see [Tin96])

In the energy $\mathcal{E}_\varepsilon$, the induced magnetic field is ignored and the application of an exterior field is replaced by the use of a boundary condition for the functions u.

The simplified Ginzburg-Landau energy to model the state of a superconductor.

The mathematical study of Ginzburg-Landau type energies was the subject of a huge number of publications since 1990. One of the main ingredients in the popularity of such models is probably the celebrated monograph [BBH94].

In [BBH94], Bethuel, Brezis and Hélein considered the minimization of the energy $\mathcal{E}_\varepsilon$ in the asymptotic $\varepsilon \to 0$. They assumed that Ω is simply connected [1] and they minimized $\mathcal{E}_\varepsilon$ under a smooth $\mathbb{S}^1$-valued Dirichlet boundary condition $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$ with a non zero degree $d \in \mathbb{N}^*$.[2] There, the Dirichlet boundary condition is independent of ε. In the following we let $H^1_g = \{u \in H^1(\Omega,\mathbb{C}) \,|\, \mathrm{tr}_{\partial\Omega} u = g\}$ where $H^1(\Omega,\mathbb{C}) = W^{1,2}(\Omega,\mathbb{C})$ is the usual Sobolev space of order 1 modeled on $L^2(\Omega,\mathbb{C})$ and $\mathrm{tr}_{\partial\Omega}$ stands for the Trace Operator on $\partial\Omega$.

Note that, by considering a minimizing sequence, we get easily that minimizers of $\mathcal{E}_\varepsilon$ with Dirichlet boundary condition always exist. Moreover, such minimizer u_ε is smooth and satisfies $|u_\varepsilon| \leq 1$

Physical and relevant informations that may be understood *via* this model concern the vorticity defects in the superconductor. Vorticity defects are areas in a superconductor where the superconductivity phenomenon is destroyed. [SS07]

Namely, if the superconductor is an infinitely long cylinder $\mathcal{S} = \Omega \times \mathbb{R}$ and if we apply a magnetic field (with sufficiently large magnitude) parallel to $\mathcal{S}$, then vorticity defects appear. The vorticity defects take the form of small wires parallel to $\mathcal{S}$ [SS07]. A minimizer u_ε of $\mathcal{E}_\varepsilon$ (with a boundary condition) allows to describe the vorticity defects. Their cross sections are modeled in Ω by small neighborhoods of zeros of u_ε. These cross sections may be seen as $\{|u_\varepsilon| \leq \ell\}$ for $\ell \in]0,1[$, say $\ell = 1/2$.

By increasing the magnitude of the applied field above a first critical field, a first vorticity defect appears. While the magnitude increases a second vorticity defect may be observed in the material, and so on. We thus may obtain an arbitrary large number of vorticity defects. The simplified Ginzburg-Landau energy (1) is essentially used to treat the case of a bounded number of vorticity defects (which essentially correspond to small neighborhoods of the vortices of a minimizer, see Definition 1 & Remark 2.2 below). Namely, the asymptotic analysis is done (when $\varepsilon \to 0$) with a bounded number of vorticity defects. For studying unbounded number of vortices, the full Ginzburg-Landau is required.

In the context of the energy $\mathcal{E}_\varepsilon$ defined by (1), the boundary condition acts as a high magnetic field by creating vorticity defects for test functions and *a fortiori* for minimizers. Since $\deg_{\partial\Omega}(g) \neq 0$, for $u \in H^1(\Omega,\mathbb{C})$ s.t. $\mathrm{tr}_{\partial\Omega} u = g$, the set $\{x \in \Omega \,|\, |u(x)| < 1/2\}$ has a non zero two dimensional Hausdorff measure.

A part of the main results of [BBH94] concerns *quantization* and *location* of the *vorticity defects*.

[1] For technical reasons, they assumed that Ω is star-shaped, this hypothesis was replaced by Ω is simply connected by Struwe in [Str94].

[2] The degree (or winding number) of a map $g \in C^\infty(\partial\Omega,\mathbb{S}^1)$, denoted by $\deg_{\partial\Omega}(g)$, may be computed by the classical formula $\deg_{\partial\Omega}(g) := \frac{1}{2\pi}\int_{\partial\Omega} g \wedge \partial_\tau g \,\mathrm{d}\tau$ where "$\wedge$" stands for the vectorial product in $\mathbb{C}$, *i.e.* $z_1 \wedge z_2 = \mathrm{Im}(\overline{z_1} z_2)$, $z_1, z_2 \in \mathbb{C}$ and ∂_τ is the tangential derivative. This degree coincides with the winding number when g is continuous [Bre06].
This formula may be extended by duality for a map $g \in H^{1/2}(\partial\Omega,\mathbb{S}^1)$ ([BGP91], Appendix).

Theorem 1. *[Bethuel-Brezis-Hélein] Let $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$ with degree $d > 0$ and let u_ε be a minimizer of $\mathcal{E}_\varepsilon$ in H^1_g. Then we have*

- *There are $\varepsilon_0 > 0$ and $C > 0$ s.t. for $0 < \varepsilon < \varepsilon_0$ u_ε has exactly d zeros $x_1^\varepsilon, ..., x_d^\varepsilon$ and $\{x \in \Omega \,|\, |u_\varepsilon(x)| \leq 1/2\} \subset \cup_i B(x_i^\varepsilon, C\varepsilon)$;*
  [Here $B(z,r) \subset \mathbb{R}^2$ is the open ball of center z and radius r]
- *Each zero is of degree 1: $\deg(u_\varepsilon, x_i^\varepsilon) = 1$ for all $i = 1, ..., d$* [3] *;*
- *As $\varepsilon \to 0$, up to extraction of a subsequence, there exist d distinct points $a_1, ..., a_d \in \Omega$ s.t.*
  i) (up to relabeling of the points x_i^ε) we have $x_i^\varepsilon \to a_i$,
  ii) the set $\{a_1, ..., a_d\}$ minimizes a renormalized energy W_g.

Definition 1. *We say that the zeros of u_ε are vortices (for $\varepsilon < \varepsilon_0$) by using the ad-hoc definition of a vortex: a vortex of a continuous map $u : \overline{\Omega} \to \mathbb{C}$ is an isolated zero of u with a non zero degree.*

Remark 2.

1. The function u_ε represents (in this simplified model) the state of a superconductor whose Ginzburg-Landau material parameter is $\kappa = \varepsilon^{-1}$. In the asymptotic $\varepsilon \to 0$ we are considering extreme type II superconductor.

2. The vorticity defects [*i.e.* the connected components of $\{x \in \Omega \,|\, |u_\varepsilon(x)| \leq 1/2\}$] may be seen as ball centered at a vortex and with radius of order ε. The two first bullets in Theorem 1 correspond to quantization results and the last bullet deals with the location of vorticity defects.

3. Although the Dirichlet boundary condition is not physically relevant, this mathematical artifact creates the same quantized vortices as a magnetic field in type-II superconductors [BBH94]. In particular, the degrees of the vortices correspond to the circulation of supercurrents around the vorticity defects ([SS07]). Since all the degrees are equal to 1 we have vorticity defects with a similar circulation of supercurrents.

   Moreover for $\varepsilon < \varepsilon_0$, the vortices $x_1^\varepsilon, ..., x_d^\varepsilon$ are subject of two repulsive effects ($\eta > 0$ is independent of ε):

   - $|x_i^\varepsilon - x_j^\varepsilon| \geq \eta$ for $i, j = 1, ..., d$, $i \neq j$ [Coulomb repulsion],
   - $\mathrm{dist}(x_i^\varepsilon, \partial\Omega) \geq \eta$ for $i = 1, ..., d$ [confinement].

4. The renormalized energy $W_g(a_1, ..., a_d) \in \mathbb{R}$ is defined for $a_1, ..., a_d \in \Omega$ with $a_i \neq a_j$ for $i \neq j$ by the following formula:

$$W_g(a_1, ..., a_d) \quad := \quad \lim_{\rho \to 0} \left\{ \inf_{\substack{w \in H^1(\Omega_\rho, \mathbb{S}^1) \\ \mathrm{tr}_{\partial\Omega} w = g \\ w(a_i + \rho e^{\imath\theta}) = \mathrm{Cst}_i e^{\imath\theta}}} \frac{1}{2} \int_{\Omega_\rho} |\nabla w|^2 - \pi d |\ln \rho| \right\} \tag{2}$$

   where $\Omega_\rho = \Omega \setminus \cup \overline{B(a_i, \rho)}$. Here we used the abuse of notation $w(a_i + \rho e^{\imath\theta}) = \mathrm{tr}_{\partial B(a_i,\rho)} w(a_i + \rho e^{\imath\theta})$.

   It is clear that W_g depends on g. Note that we may replace the condition $w(a_i + \rho e^{\imath\theta}) = \mathrm{Cst}_i e^{\imath\theta}$ by $\deg_{\partial B(a_i,\rho)}(w) = 1$ in Definition (2) of W_g.

The Pinned energy to model the state of a superconductor with impurities.
Some variants of this model has been treated in the mathematic literature [this list is not exhaustive]:

- Minimization of $\mathcal{E}_\varepsilon$ with a Dirichlet boundary condition of modulus not equal to one and even possibly vanishing [AS98b][AS04].
- Minimization of $\mathcal{E}_\varepsilon$ with a degree type boundary conditions [BM06], [BMRS12] (The main issue is the existence of *minimizer/ critical points*).

[3] Since u_ε is continuous and because it has exactly d zeros, we may define $\deg(u_\varepsilon, x_i^\varepsilon) := \deg_{C_\rho(x_i^\varepsilon)}(u_\varepsilon/|u_\varepsilon|)$ independently of small ρ with $C_\rho(x_i^\varepsilon) = \{x \in \mathbb{C} \,|\, |x - x_i^\varepsilon| = \rho\}$.

- Weighted energy ([BH95], [AS98a]) or pinned energy with a Dirichlet boundary condition ([LM99], [DMM11], [Dos]).
- Extension to the dimension $N \geq 3$ [San01] [BOS05].
- Non local version of the energy with a Dirichlet boundary condition [MS].

The main result of this article (Theorem 3 below) is connected with pinned energy in a 2D domain in the asymptotic $\varepsilon \to 0$:

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon^2 - |u|^2)^2. \tag{3}$$

Here, the function $a_\varepsilon : \Omega \to \mathbb{R}_+^*$ is called *pinning term*.

Various versions of pinning terms are considered in the literature. Note that if a_ε is in $L^\infty(\Omega, [0,1])$, then minimizers of E_ε w.r.t. a Dirichlet boundary condition $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$ always exist. Moreover, there are in $H^2(\Omega, \mathbb{C})$ (thus continuous) and if u_ε is a such minimizer, then it satisfies $|u_\varepsilon| \leq 1$.

If $a_\varepsilon \in L^\infty(\Omega, [b,1])$ for some $b \in]0,1[$, then, the use of a pinning term in the modification of the potential part of $\mathcal{E}_\varepsilon$ to get the energy E_ε, may be easily interpreted in term of heterogeneity in a superconductor [[Dos10]-Introduction]. In this context vorticity defects can be modeled by $\{x \in \Omega \,|\, |u_\varepsilon(x)| \leq b/2\}$.

It is clear that if the Dirichlet boundary condition has a non zero degree, then, letting u_ε be a minimizer of E_ε, the set $\{x \in \Omega \,|\, |u_\varepsilon(x)| \leq b/2\}$ is not empty. Moreover, for small ε, the potential part $\frac{1}{2\varepsilon^2}\int_\Omega (a_\varepsilon^2 - |u|^2)^2$ of the energy E_ε creates a strong penalty on $|a_\varepsilon - |u||$. The vorticity defects should be located close to the minimal points of a_ε. This fact is called the *pinning phenomenon*: the vorticity defects are attracted by the *minima* of a_ε . We want to study this phenomenon. In order to understand the mechanism of this pinning, we focus in this article on pinning terms which are simple functions.

The function a_ε is a step function which takes only the value 1 and the value $b \in]0,1[$ (the main result of this article is still true for pinning terms taking more values). More precisely:

$$\begin{array}{rccc} a_\varepsilon : & \Omega & \to & \{1, b\} \\ & x & \mapsto & \begin{cases} b & \text{if } x \in \cup_{I_\varepsilon} B(z_i^\varepsilon, r_i(\varepsilon)) \\ 1 & \text{otherwise} \end{cases} \end{array}, \tag{4}$$

where I_ε is a finite subset of $\mathbb{N}^*$, $\overline{B(z_i^\varepsilon, r_i(\varepsilon))} \subset \Omega$, $\varepsilon \ll r_i(\varepsilon) \ll 1$ and

$$\max_{i \neq j} \frac{r_i(\varepsilon) + r_j(\varepsilon)}{|z_i^\varepsilon - z_j^\varepsilon|} \underset{\varepsilon \to 0}{\to} 0. \tag{5}$$

From now on we omit the dependence on ε for r_i: we write r_i instead of $r_i(\varepsilon)$.

This kind of pinning terms are the (simplified) dilute ones. They are characterized by two properties:

1. The connected components of $\omega_\varepsilon := \{x \in \Omega \,|\, a_\varepsilon = b\}$, called *inclusions*, have a small diameter;
2. The inter distance between two inclusions is much larger than the diameter of each inclusion.

This pinned energy is used to model impurities in a superconductor (see [Kac09]). During the use of a superconductor, an energy dissipation may be observed. It is a consequence of the motion of the vorticity defects (see [LD97], [BS65]). This dissipation can be limited by confining the vorticity defects in some portions of a superconductor (pinning of the vorticity defects). These portions are included in the impurities. Hence, impurities in a superconductor allow to decrease the energy dissipation by pinning the vorticity defects (see [Dev02]). Impurities act as traps for vorticity defects. Since we aim at limiting the motion of the vorticity defects, the smaller the traps are, the better their efficiency should be.

The balls $B(z_i^\varepsilon, r_i)$ represents the (small) impurities. The main questions related with vorticity defects in the minimization of the pinned energy E_ε are, as for the simplified energy $\mathcal{E}_\varepsilon$, about *quantization* and *location* of the *vorticity defects*.

The quantization part takes the standard form: prove that (for small ε) a minimizer u_ε of E_ε has exactly d zeros and all with degree 1 (d vortices). Prove also that $\{x \in \Omega \,|\, |u_\varepsilon(x)| \leq b/2\}$ is contained in a small neighborhood of the vortices.

The location part concerning pinned energy maybe be done in three steps:

Step 1. Prove that vorticity defects are trapped by the inclusions. *[pinning]*

Step 2. Explain how the repartition of the vorticity defects is done between the connected components of ω_ε. *[macroscopic location]*

Step 3. Explain the location of the vorticity defects inside a connected component of ω_ε ; this is the microscopic location of the vorticity defects. *[microscopic location]*

The goal of this article is to provide a new information about Step 3. To illustrate our main result, without loss of generality, we restrict our study to the case of a Dirichlet boundary condition of degree one.

From Theorems 3&5 in [Dos] we have the following theorem.

Theorem 2. *[MDS] Let Ω be a simply connected domain, let $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$ and let $b \in]0,1[$. Consider a pinning term $a_\varepsilon : \Omega \to \{b; 1\}$ as in (4)&(5).*

Assume that

- $\deg_{\partial\Omega}(g) = 1$
- *There exists $\eta > 0$ s.t. for small ε there is z_i^ε satisfying $r_i = \max_j r_j$ and $\operatorname{dist}(z_i^\varepsilon, \partial\Omega) \geq \eta$.*

Then there is $\varepsilon_0 > 0$ s.t. for $\varepsilon < \varepsilon_0$, a minimizer u_ε of E_ε has a unique zero x_ε.

Moreover:

1. *There exists an index i_ε s.t. the corresponding radius satisfies*

$$r_{i_\varepsilon} = \max_j r_j =: r_0$$

   and s.t. $x_\varepsilon \in B(z_{i_\varepsilon}^\varepsilon, r_0)$. We write $z_\varepsilon = z_{i_\varepsilon}^\varepsilon$.

2. *There is $C > 0$ s.t. we have $\{x \in \Omega \,|\, |u_\varepsilon(x)| \leq b/2\} \subset B(x_\varepsilon, C\varepsilon)$.*

3. *For a sequence $\varepsilon_n \downarrow 0$, up to extraction of a subsequence, we have*

$$\frac{x_{\varepsilon_n} - z_{\varepsilon_n}}{r_0(\varepsilon_n)} \underset{n\to\infty}{\to} x_0 \in \mathbb{D}.$$

4. *The point x_0 minimizes a renormalized energy $W_b : \mathbb{D} \to \mathbb{R}$ which depends only on b (and is in particular independent of g).*

Remark 3. In [Dos] this result is proved under the hypotheses:

(H.1) $r_i \in \{\lambda(\varepsilon)\delta(\varepsilon), ..., \lambda(\varepsilon)\delta(\varepsilon)^P\}$ with $P \in \mathbb{N}^*$ independent of ε,

(H.2) $\lambda(\varepsilon), \delta(\varepsilon) \underset{\varepsilon\to 0}{\to} 0$ and with the technical condition: $\lim_\varepsilon \dfrac{|\ln(\lambda\delta)|^3}{|\ln\varepsilon|} = 0$,

(H.3) $|z_i^\varepsilon - z_j^\varepsilon| \geq \dfrac{r_i + r_j}{\lambda}$.

Theses hypotheses may be slightly relaxed. Namely:

- In [Dos], the inclusions are not necessarily discs. The pattern is given by a smooth simply connected open set $\omega \subset \mathbb{D}$. The main results are obtained under the assumption that the pattern is unique: all the inclusions are (small) copies of an unique $\omega \subset \mathbb{D}$.

  We may easily extend the main results of [Dos] for a finite (and independent of ε) collection $\{\omega_1, ..., \omega_N\}$ of smooth and simply connected open set included in $\mathbb{D}$.

  Moreover we may consider $b_1, ...b_N \in]0,1[$ (independent of ε) and, for example, we may let $a_\varepsilon \equiv b_i$ in the inclusions obtained from ω_i.

- Hypothesis (H.1) may be replaced by $r_i = \lambda\delta_i > 0$ and $\dfrac{\delta_i + \delta_j}{|z_i^\varepsilon - z_j^\varepsilon|} \leq C$ (with C independent of ε) as long as $\lambda \underset{\varepsilon\to 0}{\to} 0$, $\sup_i \delta_i \underset{\varepsilon\to 0}{\to} 0$ and $\dfrac{\max_i |\ln(\lambda\delta_i)|^3}{|\ln\varepsilon|} \underset{\varepsilon\to 0}{\to} 0$.
- Hypothesis (H.2) seems to be more difficult to relax.
- Hypothesis (H.3) corresponds to the dilution of the inclusions:

$$\text{if } \max_{i\neq j} \frac{r_i + r_j}{|z_i^\varepsilon - z_j^\varepsilon|} \underset{\varepsilon\to 0}{\to} 0, \text{ then we may take } r_i = \lambda\delta_i \text{ with } \lambda = \sqrt{\max_{i\neq j} \frac{r_i + r_j}{|z_i^\varepsilon - z_j^\varepsilon|}}.$$

The goal of this article is to prove that the limiting microscopic location of the vortices is 0, *i.e.*, $x_0 = 0$. In order to do this we prove that 0 is the unique minimizer of $W_b : \mathbb{D} \to \mathbb{R}$.

More precisely the main result of this article is the following

Theorem 3. *For $x \in \mathbb{D} \setminus \{0\}$ we have $W_b(x) > W_b(0)$.*

This article answers positively to Perspective (4) in [Dos] (Section 5 page 31).

2 Expression of the renormalized energy $W_b : \mathbb{D} \to \mathbb{R}$

In this section we obtain a nice form for W_b in the spirit of (2) for W_g. This expression is given in (11) and (12). In this section the inclusions are disks but, without any modification, we may consider inclusions which are small copies of a domain $\omega \subset \mathbb{D}$.

Once this is done, we then formulate a lemma [specific to circular inclusions] which directly implies Theorem 3 (see Lemma 6). The plan of its proof is given at the end of this section.

In order to get an expression of W_b we estimate $E_\varepsilon(u_\varepsilon)$. This is essentially done by following the strategy of Bethuel-Brezis-Hélein: u_ε as a modulus "close" to a_ε outside a small ball centered at the zero x_ε & standard energy core estimates around x_ε.

The discontinuity of the pinning term creates an *energetic noise* in the problem because $|u_\varepsilon|$ has to be close to a_ε. This noise is essentially carried by $|u_\varepsilon|$ and localized in a small layer around $\partial\omega_\varepsilon$. It is studied in [DMM11] when a_ε is periodic w.r.t. a $\delta \times \delta$-grid ($\delta \to 0$).

In order to denoise the problem, a nice trick was introduced by Lassoued-Mironescu [LM99]. The strategy consists first in letting U_ε be **the** unique global minimizer of E_ε with the Dirichlet boundary condition identically equal to 1. Note that U_ε depends only on Ω, a_ε and ε.

This special solution may be seen as a regularization of a_ε. For example, one may prove that U_ε takes its values in $[b, 1]$ and that it is exponentially close to a_ε far away from $\partial\omega_\varepsilon$ (see Proposition 3 in [Dos]). The energetic noise previously mentioned corresponds to the energetic cost of the abrupt transition of U_ε between values b and 1.

With the help of U_ε we have the following decoupling: for all $u \in H^1_g$

$$E_\varepsilon(u) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(u/U_\varepsilon)$$

where

$$F_\varepsilon(v) = \frac{1}{2}\int_\Omega \left\{ U_\varepsilon^2 |\nabla v|^2 + \frac{1}{2\varepsilon^2} U_\varepsilon^4 (1 - |v|^2)^2 \right\}. \tag{6}$$

Since $\mathrm{tr}_{\partial\Omega} U_\varepsilon = 1$ and $0 < b \leq U_\varepsilon \leq 1$, it is clear that u_ε is a minimizer of E_ε in H^1_g if and only if $v_\varepsilon = \dfrac{u_\varepsilon}{U_\varepsilon}$ is a minimizer of F_ε in H^1_g. The zeros (and their degrees) of a minimizer u_ε of E_ε in H^1_g are the same than those of $v_\varepsilon = u_\varepsilon/U_\varepsilon$. Therefore the study of the vorticity defects may be done *via* the minimization problem of F_ε.

Remark 4.
- Here $E_\varepsilon(U_\varepsilon) \sim \dfrac{\sum_i r_i}{\varepsilon}$ and $\inf_{H^1_g} F_\varepsilon \sim |\ln\varepsilon|$ with $\dfrac{\sum_i r_i}{\varepsilon} \gg |\ln\varepsilon|$.

- The minimization of F_ε in H^1_g may be done by following the strategy developed in [BBH94]: a minimizer v_ε is almost $\mathbb{S}^1$-valued "not too close" its zero x_ε & standard energy core estimates around x_ε.

In order to obtain an expression of the renormalized energy W_b we divide Ω in three regions. For this purpose we need the asymptotic (microscopic) location of x_ε inside the inclusion of ω_ε which contains x_ε.

We consider a sequence $\varepsilon_n \downarrow 0$. From Theorem 2, up to pass to a subsequence, we have $\frac{x_{\varepsilon_n} - z_{\varepsilon_n}}{r_0(\varepsilon_n)} \underset{n\to\infty}{\to} x_0 \in B(0,1)$. For sake of the presentation, from now on we write ε instead of ε_n.

Let $R(\varepsilon) > 1$ be s.t. [recall that $r_0 = \max_j r_j$]

$$R(\varepsilon) = \sqrt{\frac{\text{dist}\left[z_\varepsilon\,;\,\omega_\varepsilon \setminus B(z_\varepsilon, r_0)\right]}{r_0}} \tag{7}$$

From (5), we know that $R(\varepsilon) \to \infty$ as $\varepsilon \to 0$. We drop the dependence on ε by writing R instead of $R(\varepsilon)$.

For $A \subset \Omega$ an open set, we write

$$F_\varepsilon(v_\varepsilon, A) := \frac{1}{2}\int_A \left\{ U_\varepsilon^2 |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2} U_\varepsilon^4 (1 - |v_\varepsilon|^2)^2 \right\}.$$

We obtain easily

$$\begin{aligned} F_\varepsilon(v_\varepsilon) \;\; = \;\; & F_\varepsilon\left[v_\varepsilon, \Omega \setminus \overline{B(z_\varepsilon, Rr_0)}\right] + F_\varepsilon\left[v_\varepsilon, B(z_\varepsilon, Rr_0) \setminus \overline{B(x_\varepsilon, \frac{r_0}{R})}\right] + \\ & + F_\varepsilon\left[v_\varepsilon, B(x_\varepsilon, \frac{r_0}{R})\right]. \end{aligned}$$

A more complete presentation of this energetic study is done Appendix A.1.

With the help of [Dos] [by matching upper and lower bounds and Equation (4.30) in [Dos]], we get

$$F_\varepsilon\left[v_\varepsilon, B(x_\varepsilon, \frac{r_0}{R})\right] = b^2(\pi \ln \frac{br_0}{R\varepsilon} + \gamma) + o_\varepsilon(1)$$

where γ is a universal constant defined in [BBH94] Lemma IX.1.

This estimate is the standard energy core estimate around a vortex (of degree one). It is not specific to dilute pinning terms.

Once again, by matching upper and lower bounds, Proposition 18 in [Dos] and the estimate $|v_\varepsilon| \simeq 1$ in $\Omega \setminus \overline{B(z_i^\varepsilon, Rr_0)}$ (see Theorem 3.(3) in [Dos]), we have

$$F_\varepsilon\left[v_\varepsilon, \Omega \setminus \overline{B(z_i^\varepsilon, Rr_0)}\right] = W_g(z_i^\varepsilon) + \pi |\ln Rr_0| + o_\varepsilon(1).$$

This estimate is specific to the dilute case. A sharp energetic estimate of $F_\varepsilon\left[v_\varepsilon, \Omega \setminus \overline{B(z_i^\varepsilon, Rr_0)}\right]$ in non-dilute cases is a problem still open (see Perspective (2) page 31 in [Dos]). Such energetic estimate deals with *homogenized (macroscopic) renormalized energy*.

We now treat the remaining term. In order to get a nice form for the part $F_\varepsilon\left[v_\varepsilon, B(z_\varepsilon, Rr_0) \setminus \overline{B(x_\varepsilon, r_0/R)}\right]$ we rescall the inclusion $B(z_\varepsilon, Rr_0)$. Let $\Theta : \mathbb{C} \to \mathbb{C}$ be the conformal mapping s.t. $\Theta(x) = (x - z_\varepsilon)/r_0$. We denote $\mathcal{D} := B(z_\varepsilon, r_0 R) \setminus \overline{B(x_\varepsilon, r_0/R)}$ and $\hat{\mathcal{D}} := \Theta(\mathcal{D}) = B(0,R) \setminus \overline{B(\hat{x}_\varepsilon, 1/R)}$ where $\hat{x}_\varepsilon := \Theta(x_\varepsilon)$.

For the remaining term, by matching upper a lower bounds in [Dos], we have

$$\begin{aligned} F_\varepsilon\left[v_\varepsilon,\mathcal{D}\right] &= \inf_{\substack{w\in H^1(\mathcal{D},\mathbb{S}^1)\\ w(z_\varepsilon+r_0Re^{\imath\theta})=e^{\imath\theta}\\ w(x_\varepsilon+\frac{r_0}{R}e^{\imath\theta})=\mathrm{Cst}\,e^{\imath\theta}}} \frac{1}{2}\int_{\mathcal{D}} U_\varepsilon^2|\nabla w|^2+o_\varepsilon(1) \qquad (8)\\ &= \inf_{\substack{\hat{w}\in H^1(\hat{\mathcal{D}},\mathbb{S}^1)\\ \hat{w}(Re^{\imath\theta})=e^{\imath\theta}\\ \hat{w}(\hat{x}_\varepsilon+R^{-1}e^{\imath\theta})=\mathrm{Cst}\,e^{\imath\theta}}} \frac{1}{2}\int_{\hat{\mathcal{D}}} a^2|\nabla\hat{w}|^2+o_\varepsilon(1) \qquad (9)\\ &= \inf_{\substack{\hat{w}\in H^1[B(0,R)\setminus\overline{B(x_0,R^{-1})},\mathbb{S}^1]\\ \hat{w}(Re^{\imath\theta})=e^{\imath\theta}\\ \hat{w}(x_0+R^{-1}e^{\imath\theta})=\mathrm{Cst}\,e^{\imath\theta}}} \frac{1}{2}\int_{B(0,R)\setminus\overline{B(x_0,R^{-1})}} a^2|\nabla\hat{w}|^2+o_\varepsilon(1) \qquad (10)\end{aligned}$$

where $a(x)=\begin{cases} b & \text{if } x\in\mathbb{D}\\ 1 & \text{otherwise}\end{cases}$.

For the convenience of the reader, the proofs of (8), (9) and (10) are postponed to Appendix.

Thus, for $x_0\in\mathbb{D}$ and $R>1$ (sufficiently large to have $B(x_0,R^{-1})\subset\mathbb{D}$) we consider in $\mathcal{D}(x_0):=B(0,R)\setminus\overline{B(x_0,R^{-1})}$ the auxiliary minimization problem (with only $x_0\in\mathbb{D}$, $b\in]0,1[$ and $R>1$ as parameters)

$$I_R(x_0) = \inf\left\{\frac{1}{2}\int_{\mathcal{D}(x_0)} a^2|\nabla w|^2 \;\middle|\; \begin{array}{c} w\in H^1\left[\mathcal{D}(x_0),\mathbb{S}^1\right]\\ w(Re^{\imath\theta})=e^{\imath\theta}\\ w(x_0+R^{-1}e^{\imath\theta})=\mathrm{Cst}\,e^{\imath\theta},\mathrm{Cst}\in\mathbb{S}^1\end{array}\right\}. \qquad (11)$$

It is easy to check that the *infimum* in (11) is achieved (see *e.g.* Proposition 10 [Dos]).

Moreover, it is direct to prove that $I_R(x_0)-\pi(1+b^2)\ln R$ is uniformly bounded when $R\to\infty$. The quantity $I_R(x_0)-\pi(1+b^2)\ln R$ corresponds to $W_b(x_0)$ plus a small error term. Here $W_b:\mathbb{D}\to\mathbb{R}$ is the renormalized energy introduced in Theorem 2.

More precisely, when $R\to\infty$ we have ([DM11])

$$I_R(x_0)=\pi(1+b^2)\ln R+W_b(x_0)+o_R(1). \qquad (12)$$

Namely, with the previous estimates we obtain:

$$\begin{aligned} F_\varepsilon(v_\varepsilon) &= \pi\ln\frac{1}{Rr_0}+W_g(z_\varepsilon)+b^2(\pi\ln\frac{br_0}{R\varepsilon}+\gamma)+\pi(1+b^2)\ln R+\\ &\qquad\qquad +W_b(x_0)+o_\varepsilon(1)\\ &= \pi\ln\frac{1}{r_0}+W_g(z_\varepsilon)+b^2(\pi\ln\frac{br_0}{\varepsilon}+\gamma)+W_b(x_0)+o_\varepsilon(1). \qquad (13)\end{aligned}$$

Remark 5.

- The renormalized energy (of the unit disc) $W_b:\mathbb{D}\to\mathbb{R}$ depends only on b. It provides informations about the asymptotic microscopic location of the vortex (inside an inclusion). From the expression of W_b given in [DM11] [Formula (91)] we may get obviously Estimate (12).
- Formulas (12)&(13) are still valid for inclusions whose geometries are given by a domain $\omega\subset\mathbb{D}$. In this case, the microscopic renormalized energy depends also on ω.
- In (13), $W_g(z_\varepsilon)$ gives information about the macroscopic location of the vortex x_ε: the position of x_ε in Ω. Indeed, by Theorem 2, we know that the vortex of v_ε is in one of the largest inclusions (those of size r_0). The choice among the largest inclusions is done *via* the minimization of W_g (for sufficiently small ε).

In order to prove Theorem 3, we first notice that $W_b(0)=0$.

Our aim is to prove that for $x\in\mathbb{D}\setminus\{0\}$ we have $W_b(x)>0$.

With (11), it is obvious that $W_b(|x|)=W_b(x)$. Then, in order to compute $W_b(x)$, we may restrict our argument to $|x|\in[0,1[$.

Theorem 3 is a direct consequence of the following lemma.

Lemma 6. *For $t \in [0,1)$ we have*

$$W_b(t) \geq \pi \frac{\min\{b^2\,;\,1-b^2\}}{16} \ln \frac{1}{1-t^2}. \tag{14}$$

It is clear that if Lemma 6 holds, then for $x \in \mathbb{D}$ we have

$$W_b(x) = W_b(|x|) \geq \pi \frac{\min\{b^2\,;\,1-b^2\}}{16} \ln \frac{1}{1-|x|^2} > 0 = W_b(0).$$

Therefore Theorem 3 holds also.

From now on, we fix $x = t \in \mathbb{D}$, $t > 0$. The rest of this article is dedicated to the proof of Estimate (14).

We adopt the following plan:

- In Section 3 we formulate a stronger version of Lemma 6 [see Lemma 8 below]. This lemma is obtained *via* two successive minimization problems. We first fix the trace on $\mathbb{S}^1$, $h \in H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)$, of the test functions. We consider two minimization problems where h is used in the boundary conditions. Second, we minimize among all the possible traces $h \in H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)$. [See Equality (15)]
- In Sections 4 and 5, for fixed $h \in H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)$, we establish lower bounds related with Problem (11) in $B(0,R)\setminus\overline{\mathbb{D}}$ and $\mathbb{D}\setminus\overline{B(t,R^{-1})}$ respectively. [See Lower bounds (18) and (25) respectively]
- In Section 6 we couple both previous lower bounds to get an estimate independent of $h \in H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)$. [See Estimates (45), (46) and (47)]
- We compute explicitly in term of t and b this new lower bound in Section 7 [see (49)] and finally we conclude.

3 Reformulation of Problem (11)

Because the *infimum* in (11) is attained by a map $w \in H^1[B(0,R)\setminus\overline{B(t,R^{-1})},\mathbb{S}^1]$, it is clear that we have

$$\begin{aligned} I_R(t) \;=\; \inf_{\substack{h\in H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)\\ \deg_{\mathbb{S}^1}(h)=1}} \Bigg\{ & \inf_{\substack{w\in H^1[B(0,R)\setminus\overline{B(0,1)},\mathbb{S}^1]\\ w(R\mathrm{e}^{\imath\theta})=\mathrm{e}^{\imath\theta}\\ w(\mathrm{e}^{\imath\theta})=h(\mathrm{e}^{\imath\theta})}} \frac{1}{2}\int_{B(0,R)\setminus B(0,1)} |\nabla w|^2 \;+ \\ & + \inf_{\substack{w\in H^1[B(0,1)\setminus\overline{B(t,R^{-1})},\mathbb{S}^1]\\ w(\mathrm{e}^{\imath\theta})=h(\mathrm{e}^{\imath\theta})\\ w(t+R^{-1}\mathrm{e}^{\imath\theta})=\mathrm{Cst}\,\mathrm{e}^{\imath\theta},\mathrm{Cst}\in\mathbb{S}^1}} \frac{b^2}{2}\int_{B(0,1)\setminus\overline{B(t,R^{-1})}} |\nabla w|^2 \Bigg\}. \end{aligned} \tag{15}$$

Remark 7. We may easily prove that each *infimum* in the right hand side (RHS in short) of (15) is attained.

Thus, for $h \in H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)$, it is natural to introduce

$$\begin{aligned} \mathrm{W}_{R,t}(h) \;:=\; & \inf_{\substack{w\in H^1[B(0,R)\setminus\overline{B(0,1)},\mathbb{S}^1]\\ w(R\mathrm{e}^{\imath\theta})=\mathrm{e}^{\imath\theta}\\ w(\mathrm{e}^{\imath\theta})=h(\mathrm{e}^{\imath\theta})}} \frac{1}{2}\int_{B(0,R)\setminus\overline{B(0,1)}} |\nabla w|^2 \;+ \\ & + \inf_{\substack{w\in H^1[\mathcal{D}_{t,R},\mathbb{S}^1]\\ w(\mathrm{e}^{\imath\theta})=h(\mathrm{e}^{\imath\theta})\\ w(t+R^{-1}\mathrm{e}^{\imath\theta})=\mathrm{Cst}\,\mathrm{e}^{\imath\theta},\mathrm{Cst}\in\mathbb{S}^1}} \frac{b^2}{2}\int_{\mathcal{D}_{t,R}} |\nabla w|^2 - \pi(1+b^2)|\ln R|, \end{aligned} \tag{16}$$

where $\mathcal{D}_{t,R} := B(0,1) \setminus \overline{B(t,R^{-1})}$.

From (15) and (16) we immediately deduce

$$\begin{aligned} W_b(t) + o_R(1) = I_R(t) - I_R(0) &= I_R(t) - \pi(1+b^2)|\ln R| \\ &= \inf_{\substack{h \in H^{1/2}(\mathbb{S}^1,\mathbb{S}^1) \\ \deg_{\mathbb{S}^1}(h)=1}} \mathrm{W}_{R,t}(h). \end{aligned} \tag{17}$$

From Equality (17), Lemma 6 is a direct consequence of:

Lemma 8. *There is $R_0 > 0$ (depending only on t and b) s.t. for $R > R_0$ and $h \in H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)$ we have*

$$\mathrm{W}_{R,t}(h) \geq \pi \frac{\min\{b^2\,;\,1-b^2\}}{16} \ln \frac{1}{1-t^2}.$$

The rest of this article is dedicated to the proof of this lemma.

Remark 9. Lemma 8 implies Lemma 6. Which in turn implies Theorem 3.

4 Lower bound for the first term of the RHS of (15)

We fix $t > 0$, and consider a function $h \in H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)$ s.t. $\deg_{\mathbb{S}^1}(h) = 1$ and a real $R > 1$ which will go to infinity. We denote $r = R^{-1}$.

We estimate the first term in the RHS of (16).

Notations.

- The notation $o_R(1)$ means a quantity depending on R which tends to 0 when R tends to $+\infty$.
- The notation $\mathcal{O}[f(R)]$ means a quantity $g(R)$ depending on R s.t. $\dfrac{g(R)}{f(R)}$ is bounded (independently of R) when $R > 1$.
- For $\rho > \rho' > 0$, we let $\mathscr{R}_{\rho,\rho'} = B(0,\rho) \setminus \overline{B(0,\rho')}$.
- We fix a *dephasing* of h, *i.e.*, we fix $\varphi_0 \in H^{1/2}(\mathbb{S}^1,\mathbb{R})$ s.t. $h(\mathrm{e}^{\imath\theta}) = \mathrm{e}^{\imath[\theta+\varphi_0(\mathrm{e}^{\imath\theta})]}$.
- Let Ψ be the conformal mapping

$$\begin{array}{rccc} \Psi: & \mathscr{R}_{R,1} & \to & \mathscr{R}_{1,r} \\ & x & \mapsto & \dfrac{1}{x} \end{array}.$$

This section is devoted to the proof of the following inequality:

$$\inf_{\substack{w \in H^1(\mathscr{R}_{R,1},\mathbb{S}^1) \\ w(R\mathrm{e}^{\imath\theta})=\mathrm{e}^{\imath\theta} \\ w(\mathrm{e}^{\imath\theta})=h(\mathrm{e}^{\imath\theta})}} \frac{1}{2}\int_{\mathscr{R}_{R,1}} |\nabla w|^2 \geq \pi \ln R + |\varphi_0|^2_{H^{1/2}}. \tag{18}$$

Here $|\varphi_0|^2_{H^{1/2}} = \dfrac{1}{2}\displaystyle\int_{B(0,1)} |\nabla \varphi_0^{\mathrm{harm}}|^2$ with $\varphi_0^{\mathrm{harm}}$ which is the harmonic extension of φ_0 in $B(0,1)$.

Note that a map $w \in H^1(\mathscr{R}_{R,1},\mathbb{S}^1)$ may be written $w(\rho\mathrm{e}^{\imath\theta}) = \mathrm{e}^{\imath[\theta+\varphi(\rho\mathrm{e}^{\imath\theta})]}$ with $\varphi \in H^1(\mathscr{R}_{R,1},\mathbb{R})$ and $1 < \rho < R$. Note that φ is not unique but we may freeze this non uniqueness by using the dephasing φ_0:

- The boundary condition $\mathrm{tr}_{\mathbb{S}^1} w = h = \mathrm{e}^{\imath(\theta+\varphi_0)}$ implies that we may choose $\varphi \in H^1(\mathscr{R}_{R,1},\mathbb{R})$ s.t.

$$\mathrm{tr}_{\mathbb{S}^1}\varphi = \varphi_0. \tag{19}$$

- From the condition $\mathrm{tr}_{\partial B(0,R)} w = \mathrm{e}^{\imath\theta}$ we get that

$$\mathrm{tr}_{\partial B(0,R)}\varphi = 2k\pi \text{ with } k \in \mathbb{Z}. \tag{20}$$

With an integration by parts [using that $\partial_\nu\theta = 0$ on $\partial(\mathscr{R}_{R,1})$ & $\Delta\theta = 0$] we obtain for $w = \mathrm{e}^{\imath(\theta+\varphi)}$ satisfying (19) and (20):

$$\begin{aligned}\frac{1}{2}\int_{\mathscr{R}_{R,1}}|\nabla w|^2 &= \frac{1}{2}\int_{\mathscr{R}_{R,1}}|\nabla\theta|^2 + 2\nabla\theta\cdot\nabla\varphi + |\nabla\varphi|^2\\ &= \frac{1}{2}\int_{\mathscr{R}_{R,1}}|\nabla\theta|^2 + \frac{1}{2}\int_{\mathscr{R}_{R,1}}|\nabla\varphi|^2\\ &= \pi\ln R + \frac{1}{2}\int_{\mathscr{R}_{R,1}}|\nabla\varphi|^2.\end{aligned}$$

Therefore, in order to get (18), it suffices to prove that for $\varphi \in H^1(\mathscr{R}_{R,1},\mathbb{R})$ satisfying (19) and (20) we have

$$\frac{1}{2}\int_{\mathscr{R}_{R,1}}|\nabla\varphi|^2 \geq |\varphi_0|^2_{H^{1/2}}. \tag{21}$$

In order to prove (21), we let $\varphi_0^- \in H^{1/2}(\mathbb{S}^1,\mathbb{R})$ be defined by $\varphi_0^-(\mathrm{e}^{\imath\theta}) = \varphi_0(\mathrm{e}^{-\imath\theta})$. It is straight forward to prove that $|\varphi_0^-|^2_{H^{1/2}} = |\varphi_0|^2_{H^{1/2}}$.

It follows from the definition of φ_0^- that [note that $r = R^{-1}$ and Ψ is defined above]

$$\begin{aligned}&\tilde{\varphi} \in \left\{ \psi \in H^1(\mathscr{R}_{1,r},\mathbb{R}) \;\middle|\; \begin{array}{c} \psi = \varphi_0^- \text{ sur } \mathbb{S}^1 \\ \exists k\in\mathbb{Z} \text{ tq} \\ \psi = 2k\pi \text{ sur } \partial B(0,r)\end{array} \right\}\\ \iff\quad &\tilde{\varphi}\circ\Psi \in \left\{\varphi\in H^1(\mathscr{R}_{R,1},\mathbb{R})\,|(19)\text{ and }(20)\text{ hold}\right\}.\end{aligned}$$

For $\tilde{\varphi} \in H^1(\mathscr{R}_{1,r},\mathbb{R})$ s.t. $\mathrm{tr}_{\partial B(0,r)}\tilde{\varphi} = 2k\pi$ ($k\in\mathbb{Z}$), we fill the hole $B(0,r)$ by letting $\underline{\tilde{\varphi}} \in H^1(B(0,1),\mathbb{R})$ be s.t. $\underline{\tilde{\varphi}} = \begin{cases}\tilde{\varphi} & \text{in } \mathscr{R}_{1,r}\\ 2k\pi & \text{in } B(0,r)\end{cases}$.

Finally, for $\varphi \in H^1(\mathscr{R}_{R,1},\mathbb{R})$ satisfying (19) and (20), letting $\tilde{\varphi} = \varphi\circ\Psi^{-1}$, we have [by conformal invariance]

$$\frac{1}{2}\int_{\mathscr{R}_{R,1}}|\nabla\varphi|^2 = \frac{1}{2}\int_{\mathscr{R}_{1,r}}|\nabla\tilde{\varphi}|^2. \tag{22}$$

We deduce from $|\nabla\underline{\tilde{\varphi}}| = 0$ in $B(0,r)$, that

$$\frac{1}{2}\int_{\mathscr{R}_{1,r}}|\nabla\tilde{\varphi}|^2 = \frac{1}{2}\int_{B(0,1)}|\nabla\underline{\tilde{\varphi}}|^2. \tag{23}$$

We let $u \in H^1(B(0,1),\mathbb{R})$ be the harmonic extension of φ_0^- in $B(0,1)$. It suffices to see that $\underline{\tilde{\varphi}} \in H^1(B(0,1),\mathbb{R})$ with $\mathrm{tr}_{\mathbb{S}^1}\underline{\tilde{\varphi}} = \varphi_0^-$ in order to obtain, by the use of the Dirichlet Principle, that

$$\begin{aligned}\frac{1}{2}\int_{B(0,1)}|\nabla\underline{\tilde{\varphi}}|^2 &\geq \frac{1}{2}\int_{B(0,1)}|\nabla u|^2\\ &=: |\varphi_0^-|^2_{H^{1/2}} = |\varphi_0|^2_{H^{1/2}}.\end{aligned} \tag{24}$$

Therefore, by combining (22), (23) and (24) we deduce that (21) holds and thus (18) is valid.

5 Lower bound for the second term of the RHS of (15)

The aim of this section is to establish that

$$\inf_{\substack{w\in H^1[B(0,1)\setminus\overline{B(t,r)},\mathbb{S}^1]\\ w(t+r\mathrm{e}^{\imath\theta})=\mathrm{Cst}\,\mathrm{e}^{\imath\theta}\\ w(\mathrm{e}^{\imath\theta})=h(\mathrm{e}^{\imath\theta})}} \frac{1}{2}\int_{B(0,1)\setminus\overline{B(t,r)}}|\nabla w|^2 \geq \pi|\ln\tilde{r}| + |\tilde{\varphi}_0|^2_{H^{1/2}} - Cr \tag{25}$$

where $\tilde{\varphi}_0 \in H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ is defined below [see (38)] and C is a constant depending only on t.

We first replace the annular type domain $B(0,1) \setminus \overline{B(t,r)}$ by a ring $\mathscr{R}_{1,\tilde{r}}$.

Let $\Phi : B(0,1) \setminus \overline{B(t,r)} \to \mathscr{R}_{1,\tilde{r}}$ be a conformal representation s.t. $\Phi(\mathbb{S}^1) = \mathbb{S}^1$.

It is obvious that Φ and $\tilde{r}$ depend (only) on R and t. The following result is standard.

Lemma 10. *1. The map Φ is a Möbius and we may take Φ to be the following function*

$$\Phi(z) = \frac{z-\alpha}{\alpha z - 1} \text{ with } \alpha \in [0,1[.$$

We immediately obtain that for $Z = \Phi(z)$, where $z \in B(0,1) \setminus \overline{B(t,r)}$ and $Z \in B(0,1) \setminus \overline{B(0,\tilde{r})}$, we have $z = \Phi(Z)$.

2. We have

$$\alpha = \frac{1 + t^2 - r^2 - \sqrt{(1+t^2-r^2)^2 - 4t^2}}{2t} = t + \mathcal{O}(r^2) \tag{26}$$

and

$$\tilde{r} = \Phi(t-r) = \frac{r}{1-t^2}[1 + \mathcal{O}(r)]. \tag{27}$$

For $x \in \overline{B(0,1) \setminus \overline{B(t,r)}}$ we have $\Phi(x) \neq 0$ and we then may define

$$\tilde{\theta}(x) := \text{argument}(\Phi(x)).$$

Here $\tilde{\theta}(x)$ is the main determination of the argument of $\Phi(x)$: $\tilde{\theta}(x) \in]-\pi, \pi]$.

In this section we work first on $\partial B(0,\tilde{r})$ and we use $\tilde{\theta}$ as a variable. Namely, we first compute estimates for $\theta - \tilde{\theta}$ (see (29)) and then we prove (25).

We are going to study the dephasing function (w.r.t the variable $\tilde{\theta}$) $\theta - \tilde{\theta}$ where $\theta \in]-\pi, \pi]$ is s.t. $t + re^{\imath\theta} = \Phi\left[\tilde{r}e^{\imath\tilde{\theta}}\right]$.

It is not difficult to see that, since $t + re^{\imath\theta} = \Phi(\tilde{r}e^{\imath\tilde{\theta}})$, we have on $\partial B(0,\tilde{r})$.

$$\begin{aligned}
e^{\imath\theta} &= \frac{1}{r}\left[\frac{\tilde{r}e^{\imath\tilde{\theta}} - \alpha}{\alpha\tilde{r}e^{\imath\tilde{\theta}} - 1} - t\right] \\
&= e^{\imath\tilde{\theta}}\frac{\tilde{r}}{r}\frac{1 - t\alpha}{\alpha\tilde{r}e^{\imath\tilde{\theta}} - 1} + \frac{t-\alpha}{r(\alpha\tilde{r}e^{\imath\tilde{\theta}} - 1)} \\
&= -e^{\imath\tilde{\theta}}\frac{\tilde{r}}{r}(1 - t\alpha) + \frac{\alpha\tilde{r}e^{\imath\tilde{\theta}}}{\alpha\tilde{r}e^{\imath\tilde{\theta}} - 1}\frac{\tilde{r}}{r}(1 - t\alpha) + \frac{t-\alpha}{r(\alpha\tilde{r}e^{\imath\tilde{\theta}} - 1)} \\
\text{[Estimates (26)\&(27)]} &= -e^{\imath\tilde{\theta}} + \frac{1}{\alpha\tilde{r}e^{\imath\tilde{\theta}} - 1}\left[\mathcal{O}(r)e^{2\imath\tilde{\theta}} + \mathcal{O}(r)e^{\imath\tilde{\theta}} + \mathcal{O}(r)\right].
\end{aligned}$$

Remark 11. In the last estimate, the notations "$\mathcal{O}(r)$" stand for complex numbers whose moduli are bounded by Cr where C is a constant depending only on t.

Therefore we get for the dephasing $\theta - \tilde{\theta}$ on $\partial B(0,\tilde{r})$:

$$e^{\imath(\theta-\tilde{\theta})} = -1 + \frac{1}{\alpha\tilde{r}e^{\imath\tilde{\theta}} - 1}\left[\mathcal{O}(r)e^{\imath\tilde{\theta}} + \mathcal{O}(r) + \mathcal{O}(r)e^{-\imath\tilde{\theta}}\right]. \tag{28}$$

It follows that, since $\theta, \tilde{\theta} \in]-\pi, \pi]$, there exists $k \in \{-1; 1\}$ s.t.

$$\begin{cases} |\theta - \tilde{\theta} - k\pi| \leq Cr \\ |\frac{\mathrm{d}}{\mathrm{d}\tilde{\theta}}(\theta - \tilde{\theta})| \leq Cr \end{cases} \quad \text{with } C \text{ depending only on } t. \tag{29}$$

By letting

$$\begin{array}{rccc} \phi_R : & B(0,\tilde{r}) & \to & \mathbb{R} \\ & \rho e^{\imath\tilde{\theta}} & \mapsto & \frac{\rho}{\tilde{r}}[\theta(\tilde{r}e^{\imath\tilde{\theta}}) - \tilde{\theta}] - \frac{\rho - \tilde{r}}{\tilde{r}}k\pi \end{array}, \tag{30}$$

we get with the help of (29) that

$$\inf_{\substack{\phi\in H^1(B(0,\tilde r),\mathbb{R})\\ \phi=\theta-\tilde\theta \text{ on } \partial B(0,\tilde r)}} \frac{1}{2}\int_{B(0,\tilde r)}|\nabla\phi|^2 \le \frac{1}{2}\int_{B(0,\tilde r)}|\nabla\phi_R|^2 \le Cr \tag{31}$$

with C depending only on t.

For a map $w\in H^1[B(0,1)\setminus\overline{B(t,r)},\mathbb{S}^1]$ we consider the following boundary conditions

$$\mathrm{tr}_{\mathbb{S}^1} w = h, \tag{32}$$

$$w(t+r\mathrm{e}^{\imath\theta}) = \mathrm{Cst}\,\mathrm{e}^{\imath\theta}. \tag{33}$$

And, letting $\tilde h := h\circ\Phi^{-1}$, for a map $\tilde w\in H^1(\mathscr{R}_{1,\tilde r},\mathbb{S}^1)$ we focus on the boundary conditions

$$\mathrm{tr}_{\mathbb{S}^1}\tilde w = \tilde h, \tag{34}$$

$$\tilde w\circ\Phi(t+r\mathrm{e}^{\imath\theta}) = \mathrm{Cst}\,\mathrm{e}^{\imath\theta}. \tag{35}$$

It is easy to check that

$$\begin{array}{rl} & w\in H^1[B(0,1)\setminus\overline{B(t,r)},\mathbb{S}^1] \text{ satisfies (32)\&(33)}\\ \iff & \tilde w = w\circ\Phi^{-1}\in H^1(\mathscr{R}_{1,\tilde r},\mathbb{S}^1) \text{ satisfies (34)\&(35).}\end{array}$$

We may rewrite (35) as

$$\tilde w(\tilde r\mathrm{e}^{\imath\tilde\theta}) = \mathrm{Cst}\,\mathrm{e}^{\imath\tilde\theta}\mathrm{e}^{\imath(\theta-\tilde\theta)}. \tag{36}$$

Consider $w\in H^1[B(0,1)\setminus\overline{B(t,r)},\mathbb{S}^1]$ s.t. (32)&(33) hold. Let $\tilde w = w\circ\Phi^{-1}$ [then (34)&(36) hold for $\tilde w$].

By conformal invariance,

$$\frac{1}{2}\int_{B(0,1)\setminus\overline{B(t,r)}}|\nabla w|^2 \;=\; \frac{1}{2}\int_{\mathscr{R}_{1,\tilde r}}|\nabla\tilde w|^2. \tag{37}$$

Let us fix a dephasing of $\tilde h$, *i.e.* fix $\tilde\varphi_0\in H^{1/2}(\mathbb{S}^1,\mathbb{R})$ s.t.

$$\tilde h(\mathrm{e}^{\imath\tilde\theta}) = \mathrm{e}^{\imath\left[\tilde\theta+\tilde\varphi_0(\mathrm{e}^{\imath\tilde\theta})\right]}. \tag{38}$$

We write $\tilde w = \mathrm{e}^{\imath(\tilde\theta+\tilde\varphi)}$ with $\tilde\varphi\in H^1(\mathscr{R}_{1,\tilde r},\mathbb{R})$ s.t.

$$\begin{cases}\mathrm{tr}_{\mathbb{S}^1}\tilde\varphi = \tilde\varphi_0\\ \tilde\varphi(\tilde r\mathrm{e}^{\imath\tilde\theta}) = \mathrm{Cst}+(\theta-\tilde\theta)\end{cases}. \tag{39}$$

The second equality comes from (36).

An integration by part [and also $\partial_\nu\tilde\theta = 0$ on $\partial(\mathscr{R}_{1,\tilde r})$ & $\Delta\tilde\theta = 0$ in $\mathscr{R}_{1,\tilde r}$] yields

$$\begin{array}{rcl}\frac{1}{2}\int_{\mathscr{R}_{1,\tilde r}}|\nabla\tilde w|^2 &=& \frac{1}{2}\int_{\mathscr{R}_{1,\tilde r}}|\nabla\tilde\theta|^2+|\nabla\tilde\varphi|^2\\ &=& \pi|\ln\tilde r| + \frac{1}{2}\int_{\mathscr{R}_{1,\tilde r}}|\nabla\tilde\varphi|^2.\end{array} \tag{40}$$

Thus, from (37) and (40), in order to get (25), it suffices to prove that if $\tilde\varphi\in H^1(\mathscr{R}_{1,\tilde r},\mathbb{R})$ satisfies (39), then we have

$$\frac{1}{2}\int_{B(0,1)\setminus\overline{B(t,r)}}|\nabla\tilde\varphi|^2 \ge |\tilde\varphi_0|^2_{H^{1/2}} - Cr \tag{41}$$

where C is a constant depending only on t.

Let us focus on the proof of (41) and we let

$$\underline{\tilde{\varphi}} = \begin{cases} \tilde{\varphi} & \text{in } \mathscr{R}_{1,\tilde{r}} \\ \phi_R + \text{Cst} & \text{in } B(0,\tilde{r}) \end{cases}.$$

Here "Cst" is s.t. $\underline{\tilde{\varphi}} \in H^1(B(0,1),\mathbb{R})$ and ϕ_R is defined in (30).

On the one hand we have [with (31)]

$$\frac{1}{2}\int_{\mathscr{R}_{1,\tilde{r}}} |\nabla\tilde{\varphi}|^2 \geq \frac{1}{2}\int_{B(0,1)} |\nabla\underline{\tilde{\varphi}}|^2 - Cr \tag{42}$$

where C is a constant depending only on t.

On the other hand, the equality $\mathrm{tr}_{\mathbb{S}^1}\underline{\tilde{\varphi}} = \tilde{\varphi}_0$ and the Dirichlet Principle ensure that

$$\frac{1}{2}\int_{B(0,1)} |\nabla\underline{\tilde{\varphi}}|^2 \geq |\tilde{\varphi}_0|^2_{H^{1/2}}. \tag{43}$$

Combining (42) with (43) we finally get (41). Thus (25) holds.

6 Coupling of both previous lower bounds

We recall that

- $\Phi : B(0,1)\setminus\overline{B(t,r)} \to B(0,1)\setminus\overline{B(0,\tilde{r})} =: \mathscr{R}_{1,\tilde{r}}$ is a conformal representation.
- $\tilde{\theta}(x) = \text{argument}[\Phi(x)]$ for $x \in B(0,1)\setminus\overline{B(t,r)}$ with $\tilde{\theta} \in]-\pi,\pi]$.
- We fixed $h \in H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)$ and we denoted $\tilde{h} = h\circ\Phi^{-1}$.
- We fixed two dephasings $\varphi_0, \tilde{\varphi}_0 \in H^{1/2}(\mathbb{S}^1,\mathbb{R})$ s.t.

$$h(\mathrm{e}^{\imath\theta}) = \mathrm{e}^{\imath[\theta+\varphi_0(\mathrm{e}^{\imath\theta})]} \text{ and } \tilde{h}(\mathrm{e}^{\imath\tilde{\theta}}) = \mathrm{e}^{\imath[\tilde{\theta}+\tilde{\varphi}_0(\mathrm{e}^{\imath\tilde{\theta}})]}.$$

With the help of (16), (18) and (25) we have

$$\mathrm{W}_{R,t}(h) \geq -\pi b^2\ln(\tilde{r}R) + |\varphi_0|^2_{H^{1/2}} + b^2|\tilde{\varphi}_0|^2_{H^{1/2}} - Cr \tag{44}$$

where C is a constant depending only on t.

Note that from (27) we have $\tilde{r}R = \frac{1}{1-t^2} + o_R(1)$. Therefore, from (44), we have [with, in the following the "$o_R(1)$" which depend only on t]

$$\begin{aligned} \mathrm{W}_{R,t}(h) &\geq -\pi b^2\ln(\tilde{r}R) + |\varphi_0|^2_{H^{1/2}} + b^2|\tilde{\varphi}_0|^2_{H^{1/2}} - Cr \\ &\geq -\pi b^2\ln\frac{1}{1-t^2} + |\varphi_0|^2_{H^{1/2}} + b^2|\tilde{\varphi}_0|^2_{H^{1/2}} - o_R(1) \\ &= -\pi b^2\ln\frac{1}{1-t^2} + b^2(|\varphi_0|^2_{H^{1/2}} + |\tilde{\varphi}_0|^2_{H^{1/2}}) \\ &\qquad + (1-b^2)|\varphi_0|^2_{H^{1/2}} - o_R(1). \end{aligned} \tag{45}$$

The goal of this section [see (47) below] is to get a lower bound independent of h for

$$H := \begin{cases} |\varphi_0|^2_{H^{1/2}} + |\tilde{\varphi}_0|^2_{H^{1/2}} & \text{if } |\varphi_0|_{H^{1/2}} > 3^{-1}|\tilde{\varphi}_0|_{H^{1/2}} \\ \frac{8}{9}|\tilde{\varphi}_0|^2_{H^{1/2}} & \text{if } |\varphi_0|_{H^{1/2}} \leq 3^{-1}|\tilde{\varphi}_0|_{H^{1/2}} \end{cases}.$$

Note that we have:

$$\begin{aligned} b^2(|\varphi_0|^2_{H^{1/2}} + |\tilde{\varphi}_0|^2_{H^{1/2}}) + (1-b^2)|\varphi_0|^2_{H^{1/2}} &\geq b^2H + \min\left\{\frac{(1-b^2)H}{10}\,;\,\frac{b^2H}{8}\right\} \\ &\geq b^2H + \frac{H\min\left\{1-b^2\,;\,b^2\right\}}{10}. \end{aligned} \tag{46}$$

In order to couple the previous lower bounds we claim that, since for $\mathrm{e}^{\imath\theta} \in \mathbb{S}^1$ we have $h(\mathrm{e}^{\imath\theta}) = \tilde{h} \circ \Phi(\mathrm{e}^{\imath\theta})$, the following equality holds on $\mathbb{S}^1$

$$\mathrm{e}^{\imath(\theta+\varphi_0)} = \mathrm{e}^{\imath(\tilde{\theta}+\tilde{\varphi}_0\circ\Phi)}.$$

This infers that on $\mathbb{S}^1$ we have

$$\mathrm{e}^{\imath\left[(\theta-\tilde{\theta})-(\tilde{\varphi}_0\circ\Phi-\varphi_0)\right]} = 1.$$

We write $(\theta-\tilde{\theta})_{|\mathbb{S}^1} := \mathrm{tr}_{\mathbb{S}^1}(\theta-\tilde{\theta})$. Since $(\theta-\tilde{\theta})_{|\mathbb{S}^1} - (\tilde{\varphi}_0 \circ \Phi - \varphi_0) \in H^{1/2}(\mathbb{S}^1, 2\pi\mathbb{Z})$, there is $k \in \mathbb{Z}$ s.t.

$$\tilde{\varphi}_0 \circ \Phi - \varphi_0 = (\theta-\tilde{\theta})_{|\mathbb{S}^1} + 2k\pi.$$

It follows that

$$|\tilde{\varphi}_0 \circ \Phi - \varphi_0|^2_{H^{1/2}} = |(\theta-\tilde{\theta})_{|\mathbb{S}^1}|^2_{H^{1/2}}.$$

Thus, if $|\varphi_0|_{H^{1/2}} > 3^{-1}|\tilde{\varphi}_0|_{H^{1/2}}$, then we have

$$|\varphi_0|^2_{H^{1/2}} + |\tilde{\varphi}_0|^2_{H^{1/2}} \geq \frac{1}{2}|(\theta-\tilde{\theta})_{|\mathbb{S}^1}|^2_{H^{1/2}}.$$

Note that the previous estimate always holds, but we use it only when $|\varphi_0|_{H^{1/2}} > 3^{-1}|\tilde{\varphi}_0|_{H^{1/2}}$.

Otherwise, if $|\varphi_0|_{H^{1/2}} \leq 3^{-1}|\tilde{\varphi}_0|_{H^{1/2}}$, then we have

$$\frac{8}{9}|\tilde{\varphi}_0|^2_{H^{1/2}} \geq \frac{1}{2}|(\theta-\tilde{\theta})_{|\mathbb{S}^1}|^2_{H^{1/2}}.$$

Therefore, by combining both previous estimates with the definition of H, we get

$$H \geq \frac{1}{2}|(\theta-\tilde{\theta})_{|\mathbb{S}^1}|^2_{H^{1/2}}. \tag{47}$$

7 Computation of $\left|(\theta-\tilde{\theta})_{|\mathbb{S}^1}\right|^2_{H^{1/2}}$ and epilog

Let us compute $|(\theta-\tilde{\theta})_{|\mathbb{S}^1}|^2_{H^{1/2}}$ in order to deduce from (47) a nice lower bound for H (see (49)).

More precisely we prove that [recall that from Equality (26) we have $\alpha \in [0,1[$]

$$|(\theta-\tilde{\theta})_{|\mathbb{S}^1}|^2_{H^{1/2}} := \frac{1}{2}\int_{B(0,1)} |\nabla(\theta-\tilde{\theta})^{\mathrm{harm}}_{|\mathbb{S}^1}|^2 = 2\pi \ln\frac{1}{1-\alpha^2}. \tag{48}$$

Here, $(\theta-\tilde{\theta})^{\mathrm{harm}}_{|\mathbb{S}^1}$ is the harmonic extension of $(\theta-\tilde{\theta})_{|\mathbb{S}^1}$ in $B(0,1)$.

Estimate (48), with the help of (47), implies

$$H \geq \pi \ln\frac{1}{1-t^2} + o_R(1) \tag{49}$$

where $o_R(1)$ depends only on t (and consequently it is independent of h).

In order to establish (48), we use the following formula based on the Fourier decomposition of a map $f \in H^{1/2}(\mathbb{S}^1,\mathbb{R})$: if $f \in H^{1/2}(\mathbb{S}^1,\mathbb{R})$ is s.t. $f(\mathrm{e}^{\imath\theta}) = \sum_{\mathbb{Z}} c_n \mathrm{e}^{\imath n\theta}$ then

$$|f|^2_{H^{1/2}} = \pi\sum_{n\in\mathbb{Z}} |n||c_n|^2.$$

In order to establish (48), we find the Fourier expansion series of $(\theta-\tilde{\theta})_{|\mathbb{S}^1}(\mathrm{e}^{\imath\theta})$. To this aim, we observe that $\frac{\mathrm{d}}{\mathrm{d}\theta}(\mathrm{e}^{\imath\tilde{\theta}})$ is the Poisson Kernel for the unit disk.

It is not difficult to see that, since $\Phi(\mathrm{e}^{\imath\theta}) = \dfrac{\mathrm{e}^{\imath\theta}-\alpha}{\alpha\mathrm{e}^{\imath\theta}-1} = \mathrm{e}^{\imath\tilde{\theta}}$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}(\mathrm{e}^{\imath\tilde{\theta}(\mathrm{e}^{\imath\theta})}) = \imath\mathrm{e}^{\imath\tilde{\theta}(\mathrm{e}^{\imath\theta})}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\tilde{\theta}(\mathrm{e}^{\imath\theta})\right] &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{\mathrm{e}^{\imath\theta}-\alpha}{\alpha\mathrm{e}^{\imath\theta}-1}\right] \\
&= \imath\mathrm{e}^{\imath\theta}\frac{\alpha^2-1}{(\alpha\mathrm{e}^{\imath\theta}-1)^2} \\
&= \imath\mathrm{e}^{\imath\tilde{\theta}(\mathrm{e}^{\imath\theta})}\frac{1-\alpha^2}{|\alpha\mathrm{e}^{\imath\theta}-1|^2} \\
&= \imath\mathrm{e}^{\imath\tilde{\theta}(\mathrm{e}^{\imath\theta})}\frac{1-\alpha^2}{1-2\alpha\cos(\theta)+\alpha^2}.
\end{aligned}$$

From these estimates we get:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\tilde{\theta}(\mathrm{e}^{\imath\theta})\right] = \frac{1-\alpha^2}{1-2\alpha\cos(\theta)+\alpha^2} = \sum_{n\in\mathbb{Z}} \alpha^{|n|}\mathrm{e}^{\imath n\theta}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\tilde{\theta}(\mathrm{e}^{\imath\theta})-\theta\right] = \sum_{n\in\mathbb{Z}^*} \alpha^{|n|}\mathrm{e}^{\imath n\theta}.$$

Thus

$$\tilde{\theta}(\mathrm{e}^{\imath\theta})-\theta = \sum_{n\in\mathbb{Z}^*} \frac{\alpha^{|n|}}{\imath n}\mathrm{e}^{\imath n\theta} + \mathrm{Cst}.$$

Consequently

$$|(\theta-\tilde{\theta})_{|\mathbb{S}^1}|^2_{H^{1/2}} = 2\pi\sum_{n\geq 1}\frac{\alpha^{2n}}{n} = 2\pi\ln\frac{1}{1-\alpha^2}.$$

Which proves (48).

We are now able to prove Lemma 8.

From (45) (with, in the following the "$o_R(1)$" which depend only on t)

$$\begin{aligned}
\mathrm{W}_{R,t}(h) &\geq -\pi b^2\ln\frac{1}{1-t^2} + b^2(|\varphi_0|^2_{H^{1/2}} + |\tilde{\varphi}_0|^2_{H^{1/2}}) + \\
&\qquad + (1-b^2)|\varphi_0|^2_{H^{1/2}} + o_R(1) \\
[\text{Est. (46)}] &\geq -\pi b^2\ln\frac{1}{1-t^2} + b^2 H + \frac{H\min\left\{1-b^2\,;\,b^2\right\}}{10} + o_R(1) \\
[\text{Est. (49)}] &\geq \pi\frac{\min\left\{1-b^2\,;\,b^2\right\}}{10}\ln\frac{1}{1-t^2} + o_R(1).
\end{aligned}$$

Since $\mathrm{W}_{R,t}(h) \geq \pi\frac{\min\left\{1-b^2\,;\,b^2\right\}}{10}\ln\frac{1}{1-t^2} + o_R(1)$ (with $o_R(1)$ which depends only on t), there is $R_0 > 1$ depending only on t and b s.t. for $R > R_0$ we have

$$\mathrm{W}_{R,t}(h) \geq \pi\frac{\min\left\{1-b^2\,;\,b^2\right\}}{16}\ln\frac{1}{1-t^2}.$$

This last estimate concludes the proof of Lemma 8.

A Proofs of (8), (9) and (10)

Since the arguments used in this appendix do not use the circular geometry of the inclusions, we assume, in this appendix, that the inclusions are small copies of a smooth simply connected open set s.t. $\overline{\omega} \subset \mathbb{D}$ and s.t. $0 \in \omega$.

Therefore the pinning term a_ε has the following form:

$$\begin{array}{rccc} a_\varepsilon: & \Omega & \to & \{1,b\} \\ & x & \mapsto & \begin{cases} b & \text{if } x \in \cup_{I_\varepsilon}\left\{z_i^\varepsilon + r_i\cdot\omega\right\} \\ 1 & \text{otherwise} \end{cases}. \end{array}$$

Here $\varepsilon \ll r_i \ll 1$ and the dilute condition (5) is satisfied. We assume that letting $r_0 = \max r_i$ we have

$$\lim_{\varepsilon\to 0}\frac{|\ln r_0|^3}{|\ln\varepsilon|} = 0. \tag{50}$$

Recall that, for sufficiently small ε, a minimizer v_ε of F_ε in H^1_g has a unique zero x_ε. Moreover there is $i_\varepsilon \in I_\varepsilon$ s.t. letting z_ε be s.t. $z_\varepsilon := z^\varepsilon_{i_\varepsilon}$, we have $r_{i_\varepsilon} = r_0$ and $x_\varepsilon \in z_\varepsilon + r_0\cdot\omega$.

We denote $\mathcal{D} := B(z_\varepsilon, r_0 R) \setminus \overline{B(x_\varepsilon, r_0/R)}$ where R is defined in (7).
We are going to prove in this appendix the validity of (8), (9) and (10).
The main ingredient in the proof is the estimate [See Theorem 3.(3) in [Dos]]

$$|v_\varepsilon| \geq 1 - C\sqrt{\frac{|\ln r_0|}{\ln \varepsilon}} \text{ in } \Omega \setminus \overline{B(x_\varepsilon, r_0/R)}. \tag{51}$$

A.1 Preliminary energetic analysis

A.1.1 Global upper bound

We have the following upper bound:

$$\begin{aligned} F_\varepsilon(v_\varepsilon) \quad \leq \quad & \pi \ln Rr_0 + W_g(z_\varepsilon) + \inf_{\substack{w \in H^1(\mathcal{D}, \mathbb{S}^1) \\ w(z_\varepsilon + r_0 R \mathrm{e}^{\imath\theta}) = \mathrm{e}^{\imath\theta} \\ w(x_\varepsilon + r_0 R^{-1} \mathrm{e}^{\imath\theta}) = \mathrm{Cst}\, \mathrm{e}^{\imath\theta}}} \frac{1}{2} \int_{\mathcal{D}} U_\varepsilon^2 |\nabla w|^2 \\ & + b^2(\pi \ln \frac{br_0}{R\varepsilon} + \gamma) + o_\varepsilon(1). \end{aligned} \tag{52}$$

This estimate is obtained with the test function

$$v_0 = \begin{cases} w_{Rr_0} & \text{in } \Omega \setminus \overline{B(z_\varepsilon, Rr_0)} \\ \mathrm{Cst}_1\, w_{\mathcal{D}} & \text{in } \mathcal{D} \\ \mathrm{Cst}_2\, u_0 & \text{in } B(x_\varepsilon, \frac{r_0}{R}) \end{cases}$$

where

- The function w_{Rr_0} is a minimizer of $\frac{1}{2} \int_{\Omega \setminus \overline{B(z_\varepsilon, Rr_0)}} |\nabla w|^2$ among the maps $w \in H^1(\Omega \setminus \overline{B(z_\varepsilon, Rr_0)}, \mathbb{S}^1)$ satisfying $\mathrm{tr}_{\partial\Omega} w = g$ and $w(z_\varepsilon + Rr_0 \mathrm{e}^{\imath\theta}) = \mathrm{Cst}\, \mathrm{e}^{\imath\theta}$.
  From Formula (15) in [CM96] we have

$$\begin{aligned} F_\varepsilon \left[w_{Rr_0}, \Omega \setminus \overline{B(z_\varepsilon, Rr_0)} \right] \quad &= \quad \frac{1}{2} \int_{\Omega \setminus \overline{B(z_\varepsilon, Rr_0)}} U_\varepsilon^2 |\nabla w_{Rr_0}|^2 \\ &\leq \quad \pi |\ln Rr_0| + W_g(z_\varepsilon) + o_\varepsilon(1). \end{aligned}$$

- The function $w_{\mathcal{D}}$ is a minimizer for $\frac{1}{2} \int_{\mathcal{D}} U_\varepsilon^2 |\nabla w|^2$ among the maps $w \in H^1(\mathcal{D}, \mathbb{S}^1)$ satisfying the conditions $w(z_\varepsilon + r_0 R \mathrm{e}^{\imath\theta}) = \mathrm{e}^{\imath\theta}$ and $w(x_\varepsilon + r_0 R^{-1} \mathrm{e}^{\imath\theta}) = \mathrm{Cst}\, \mathrm{e}^{\imath\theta}$.
- The function u_0 is a minimizer for $\mathcal{E}_{\varepsilon/b}[u, B(x_\varepsilon, R^{-1} r_0)]$ [where $\mathcal{E}_\varepsilon$ is defined in (1)] among maps $u \in H^1(B(x_\varepsilon, R^{-1} r_0), \mathbb{C})$ with the boundary condition $u(x_\varepsilon + R^{-1} r_0 \mathrm{e}^{\imath\theta}) = \mathrm{e}^{\imath\theta}$.
  From Lemma IX.1 in [BBH94] and since $U_\varepsilon = b^2 + V_\varepsilon$ with $\|V_\varepsilon\|_{L^\infty} = \mathcal{O}(\varepsilon^4)$ [see Proposition 3 in [Dos]], we have

$$F_\varepsilon(u_0, B(x_\varepsilon, R^{-1} r_0)) = b^2(\pi \ln \frac{br_0}{R\varepsilon} + \gamma) + o_\varepsilon(1).$$

- The constants Cst_1 and Cst_2 are s.t. $v_0 \in H^1(\Omega, \mathbb{C})$.

A.1.2 Some lower bounds

Arguing as in the proof of Proposition 18 in [Dos] we get

$$\begin{aligned} F_\varepsilon \left[v_\varepsilon, \Omega \setminus \overline{B(z_\varepsilon, Rr_0)} \right] \quad &\overset{(51)}{\geq} \quad \frac{1}{2} \int_{\Omega \setminus \overline{B(z_\varepsilon, Rr_0)}} U_\varepsilon^2 \left| \nabla \frac{v_\varepsilon}{|v_\varepsilon|} \right|^2 - o_\varepsilon(1) \\ \left[\begin{array}{c} \text{Same argument than} \\ \text{in the proof of} \\ \text{Prop. 18 in [Dos]} \end{array} \right] \quad &\geq \quad \inf_{\substack{w \in H^1[\Omega \setminus \overline{B(z_i^\varepsilon, Rr_0)}, \mathbb{S}^1)] \\ \mathrm{tr}_{\partial\Omega} w = g}} \frac{1}{2} \int_{\Omega \setminus \overline{B(z_\varepsilon, Rr_0)}} |\nabla w|^2 - o_\varepsilon(1) \\ \text{[Lemma 2 in [CM96]]} \quad &= \quad W_g(z_i^\varepsilon) + \pi |\ln Rr_0| - o_\varepsilon(1). \end{aligned} \tag{53}$$

From Estimate (4.30) in [Dos] we get easily

$$F_\varepsilon\left[v_\varepsilon, B(x_\varepsilon, \frac{r_0}{R})\right] = b^2(\pi \ln \frac{br_0}{R\varepsilon} + \gamma) + o_\varepsilon(1). \tag{54}$$

By combining (52), (53) and (54) we have

$$F_\varepsilon\left[v_\varepsilon, \mathcal{D}\right] \leq \inf_{\substack{w\in H^1(\mathcal{D},\mathbb{S}^1)\\ w(z_\varepsilon+r_0Re^{\imath\theta})=e^{\imath\theta}\\ w(x_\varepsilon+r_0R^{-1}e^{\imath\theta})=\mathrm{Cst}\, e^{\imath\theta}}} \frac{1}{2}\int_{\mathcal{D}} U_\varepsilon^2|\nabla w|^2 + o_\varepsilon(1).$$

On the other hand, since $|v_\varepsilon| \geq 1 - C\sqrt{\frac{|\ln r_0|}{\ln \varepsilon}}$ in $\mathcal{D}$, we get easily that

$$\begin{aligned} F_\varepsilon\left[v_\varepsilon, \mathcal{D}\right] &\geq \frac{1}{2}\int_{\mathcal{D}} U_\varepsilon^2 \left|\nabla \frac{v_\varepsilon}{|v_\varepsilon|}\right|^2 + o_\varepsilon(1) \\ &\geq \inf_{\substack{w\in H^1(\mathcal{D},\mathbb{S}^1)\\ \deg_{\partial B(z_\varepsilon, r_0R)}(w)=1}} \frac{1}{2}\int_{\mathcal{D}} U_\varepsilon^2|\nabla w|^2 + o_\varepsilon(1). \end{aligned}$$

We let $\hat{U}_\varepsilon(\hat{x}) = U_\varepsilon(z_\varepsilon + r_0\hat{x})$ for $\hat{x} \in B(0,R)$. For $w \in H^1(\mathcal{D},\mathbb{C})$, by conformal invariance we have

$$\frac{1}{2}\int_{\mathcal{D}} U_\varepsilon^2|\nabla w|^2 = \frac{1}{2}\int_{\hat{\mathcal{D}}} \hat{U}_\varepsilon^2|\nabla \hat{w}|^2$$

with $\hat{w}(\hat{x}) = w(z_\varepsilon + r_0\hat{x})$ and $\hat{\mathcal{D}} := B(0,R) \setminus B(\hat{x}_\varepsilon, r)$ where $\hat{x}_\varepsilon = \dfrac{x_\varepsilon - z_\varepsilon}{r_0}$.

Therefore, in order to get (8), it suffices to prove:

$$\inf_{\substack{\hat{w}\in H^1(\hat{\mathcal{D}},\mathbb{S}^1)\\ \hat{w}(Re^{\imath\theta})=e^{\imath\theta}\\ \hat{w}(\hat{x}_\varepsilon+R^{-1}e^{\imath\theta})=\mathrm{Cst}\, e^{\imath\theta}}} \frac{1}{2}\int_{\hat{\mathcal{D}}} \hat{U}_\varepsilon^2|\nabla \hat{w}|^2 = \inf_{\substack{\hat{w}\in H^1(\hat{\mathcal{D}},\mathbb{S}^1)\\ \deg_{\partial B(0,R)}(\hat{w})=1}} \frac{1}{2}\int_{\hat{\mathcal{D}}} \hat{U}_\varepsilon^2|\nabla \hat{w}|^2 + o_\varepsilon(1). \tag{55}$$

A.2 Auxiliary problems and the proof of (10)

Estimate (10) is a direct consequence of Proposition 1 below. Moreover with the help of this proposition we may prove (8)&(9) (see in the next section).

In the following we drop the *"hat"* for the maps: write w instead of $\hat{w}$.

Notations.

- We fix a sequence $(x_n)_n \subset \omega$ s.t. $x_n \to x_0 \in \omega$;
- We fix two sequences $R_n = R \uparrow \infty$ and $r_n = r \downarrow 0$ s.t. $r < 1 < R$ (we are not necessarily in the case $r = R^{-1}$).
- For $\rho_1 > \rho_2 > 0$ and $x \in \mathbb{C}$ we write

$$\mathcal{D}_{\rho_1,\rho_2}(x) = B(0,\rho_1) \setminus \overline{B(x,\rho_2)} \text{ and } \mathscr{R}_{\rho_1,\rho_2}(x) = B(x,\rho_1) \setminus \overline{B(x,\rho_2)}.$$

For $\alpha \in L^\infty(\mathbb{C}, [b^2, 1])$ and for $x_0 \in \omega$ (with sufficiently large n s.t. $B(x_0, r) \subset \omega$), consider the following minimization problems

$$\mu_\alpha(\mathcal{D}_{R,r}(x_0)) = \inf_{\substack{w\in H^1(\mathcal{D}_{R,r}(x_0),\mathbb{S}^1)\\ \deg_{\partial B(0,R)}(w)=1}} \frac{1}{2}\int_{\mathcal{D}_{R,r}(x_0)} \alpha|\nabla w|^2$$

and

$$\mu_\alpha^{\mathrm{Dir}}(\mathcal{D}_{R,r}(x_0)) = \inf_{\substack{w\in H^1(\mathcal{D}_{R,r}(x_0),\mathbb{S}^1)\\ w(R\mathrm{e}^{\imath\theta})=\mathrm{e}^{\imath\theta}\\ w(x_0+r\mathrm{e}^{\imath\theta})=\mathrm{Cst}\,\mathrm{e}^{\imath\theta}}} \frac{1}{2}\int_{\mathcal{D}_{R,r}(x_0)} \alpha|\nabla w|^2.$$

In this appendix, α is either the function identically equal to one (denoted by $\mathbb{1}$), or the function $\hat{U}_\varepsilon^2$ or $\alpha : \mathbb{C} \to \{1, b^2\}$, $x \mapsto \alpha(x) = \begin{cases} b^2 & \text{if } x \in \omega_\alpha \\ 1 & \text{otherwise} \end{cases}$ with ω_α "close to" ω.

Note that if $B(x_0, c) \subset \omega$ for some $x_0 \in \omega$ and $0 < c < 1$, then there exists $r_c > 0$ s.t. for $r \le r_c$ we have

$$\mu_{a^2}^{\mathrm{Dir}}(\mathcal{D}_{R,r}(x_0)) \le \pi \ln R + b^2\pi|\ln r| + C_0(c). \tag{56}$$

Here $C_0(c)$ is a constant depending on c; it may be considered decreasing w.r.t. $c \in]0, 1[$.

In the context of a sequence $(x_n)_n \subset \omega$ s.t. $x_n \to x_0 \in \omega$, from (56), there is $\tilde{C}_0$ s.t. for sufficiently large n we have the uniform estimate

$$\mu_{a^2}^{\mathrm{Dir}}(\mathcal{D}_{R,r}(x_n)) \le \pi \ln R + b^2\pi|\ln r| + \tilde{C}_0. \tag{57}$$

To get Estimate (56), we fix a smooth map $\tilde{g} \in C^\infty(\partial\omega, \mathbb{S}^1)$ s.t. $\deg_{\partial\omega}(\tilde{g}) = 1$ and we write $W_{\tilde{g}}^\omega(\cdot)$ the renormalized energy of Bethuel-Brezis-Hélein defined Formula (2) for $\Omega := \omega$ and $g := \tilde{g}$.

We first note that from Theorem 6 in [CM96], we have that $W_{\tilde{g}}^\omega : \omega \to \mathbb{R}$ is a smooth function. Therefore for $c > 0$, there is $C_1(c) > 0$ s.t. $W_{\tilde{g}}^\omega(x) \le C_1(c)$ if $B(x, c) \subset \omega$.

Moreover if we have $\mathrm{dist}(x_0, \partial\omega) \ge c$, then there exists $r_c > 0$ (depending only on c, ω and $\tilde{g}$) s.t. for $0 < r < r_c$ we have the existence of a map $w_r \in H^1(\omega\setminus\overline{B(x_0, r)}, \mathbb{S}^1)$ s.t. $\mathrm{tr}_{\partial\omega} w_r = \tilde{g}$, $w_r(x_0+r\mathrm{e}^{\imath\theta}) = \mathrm{Cst}\,\mathrm{e}^{\imath\theta}$ and s.t.

$$\frac{1}{2}\int_{\omega\setminus\overline{B(x_0,r)}} |\nabla w_r|^2 \le \pi \ln r + W_{\tilde{g}}^\omega(x_0) + 1. \tag{58}$$

This estimate is obtained from Estimate (15) in [CM96].

The upper bound (56) is obtained with the following test function

$$w(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } x \in \mathscr{R}_{R,1}(0) \\ w^0(x) & \text{if } x \in \mathbb{D}\setminus\overline{\omega} \\ w_r(x) & \text{if } x \in \omega\setminus\overline{B(x_0, r)} \end{cases}$$

where

- $w^0 \in H^1(\mathbb{D}\setminus\overline{\omega}, \mathbb{S}^1)$ is a map independent of r s.t. $\mathrm{tr}_{\partial(\mathbb{D}\setminus\overline{\omega})}(w^0) = \begin{cases} \mathrm{e}^{\imath\theta} & \text{on } \mathbb{S}^1 \\ \tilde{g} & \text{on } \partial\omega \end{cases}$;
- w_r is given by (58).

It is standard to get that $\frac{1}{2}\int_{\mathscr{R}_{R,1}(0)} \left|\nabla \frac{x}{|x|}\right|^2 = \pi \ln R$.

Thus, for $r < r_c$, Estimate (56) holds.

The main result of this section is the following proposition.

Proposition 1. *1. For sufficiently large n, letting $y = x_0$ or $y = x_n$, we have*

$$0 \le \mu_{a^2}^{\mathrm{Dir}}(\mathcal{D}_{R,r}(y)) - \mu_{a^2}(\mathcal{D}_{R,r}(y)) \le \frac{C}{\min\{|\ln r|; \ln R\}}$$

where the constant C is independent of n.

2. When $n \to \infty$ we have $\mu_{a^2}(\mathcal{D}_{R,r}(x_n)) - \mu_{a^2}(\mathcal{D}_{R,r}(x_0)) = o_n(1)$

Proof. We prove the first assertion. Let $y = x_0$ or $y = x_n$ and let w be a minimal map for $\mu_{a^2}(\mathcal{D}_{R,r}(y))$ [it is standard to prove that such map exists]. We are going to construct a map $\underline{w}$ [see (70)] s.t., for a suitable constant Cst, Cst $\underline{w}$ is a test function for the problem $\mu_{a^2}^{\mathrm{Dir}}(\mathcal{D}_{R,r}(y))$ and s.t.

$$\frac{1}{2}\int_{\mathcal{D}_{R,r}(y)} \alpha|\nabla \underline{w}|^2 \leq \mu_{a^2}(\mathcal{D}_{R,r}(y)) + \frac{C}{\min\{|\ln r|; \ln R\}} \tag{59}$$

where the constant C is independent of n.

Since $\mu_{a^2}(\mathcal{D}_{R,r}(y)) \leq \mu_{a^2}^{\mathrm{Dir}}(\mathcal{D}_{R,r}(y))$, if (59) holds, then the first assertion of Proposition 1 holds also.

In order to construct the map $\underline{w}$, we first choose two good circles. This is done by getting two upper bounds [see (63)&(64) below].

We let $c < 10^{-2}\mathrm{dist}(x_0, \partial\omega)$ and n sufficiently large s.t. $|x_n - x_0| < c$ and s.t. $r < c$. In particular we have $B(y, c) \subset \omega$ with $y = x_0$ or $y = x_n$.

Recall that for $\rho_1 > \rho_2 > 0$ and $x \in \mathbb{C}$, we denote $\mathscr{R}_{\rho_1,\rho_2}(x) := B(x, \rho_1) \setminus \overline{B(x, \rho_2)}$.

We switch to polar coordinates in $\mathscr{R}_{R,1}(0)$ and in $\mathscr{R}_{c,r}(y)$ by writing

- in $\mathscr{R}_{R,1}(0)$: $w(\rho, \theta) := w(\rho e^{\imath\theta})$, $\rho \in]1, R[$, $\theta \in [0, 2\pi]$;
- in $\mathscr{R}_{c,r}(y)$: $w(\rho, \theta) := w(y + \rho e^{\imath\theta})$, $\rho \in]r, c[$, $\theta \in [0, 2\pi]$.

Let us define for $\rho \in]r, c[\cup]1, R[$ the circle $\mathcal{C}_\rho := \begin{cases} \partial B(0, \rho) & \text{if } \rho > 1 \\ \partial B(y, \rho) & \text{if } \rho < c \end{cases}$.

Our first observation is that, since the map w is of degree 1 on $\mathcal{C}_\rho$ for $\rho \in]r, c[\cup]1, R[$, then we have $\frac{1}{2\pi}\int_0^{2\pi} w \wedge \partial_\theta w = 1$ (here we used the formula of the degree given in Footnote 2 page 2). Thus we obtain

$$1 \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |\partial_\theta w(\rho, \theta)|\right)^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |\partial_\theta w(\rho, \theta)|^2. \tag{60}$$

Consequently, by using polar coordinates we get easily

$$\left| \begin{array}{l} \frac{1}{2}\int_{\mathscr{R}_{R,R^{2/3}}(0)\cup\mathscr{R}_{R^{1/3},1}(0)} |\nabla w|^2 \geq 2\pi \ln R^{1/3} \\ \frac{1}{2}\int_{\mathscr{R}_{R^{2/3},R^{1/3}}(0)} |\nabla w|^2 \geq \pi \ln R^{1/3} \end{array} \right. \tag{61}$$

and

$$\left| \begin{array}{l} \frac{1}{2}\int_{\mathscr{R}_{c,r^{1/3}}(y)\cup\mathscr{R}_{r^{2/3},r}(y)} |\nabla w|^2 \geq \pi\left(\ln c + 2|\ln r^{1/3}|\right) \\ \frac{1}{2}\int_{\mathscr{R}_{r^{1/3},r^{2/3}}(y)} |\nabla w|^2 \geq \pi|\ln r^{1/3}| \end{array} \right. . \tag{62}$$

Therefore, from (57), (61) and (62), we get

$$\frac{1}{2}\int_{\mathscr{R}_{R^{2/3},R^{1/3}}(0)} |\nabla w|^2 \leq \pi \ln R^{1/3} + C_2 \tag{63}$$

and

$$\frac{1}{2}\int_{\mathscr{R}_{r^{1/3},r^{2/3}}(y)} |\nabla w|^2 \leq \pi|\ln r^{1/3}| + C_2 \tag{64}$$

where $C_2 > 0$ is independent of n.

From (63) and (64), and arguing by contradiction, it is easy to get the existence of $\tilde{R} \in]R^{1/3}, R^{2/3}[$ and $\tilde{r} \in]r^{2/3}, r^{1/3}[$ s.t.

$$\frac{1}{2}\int_0^{2\pi} |\partial_\theta w(\tilde{R}, \theta)|^2 \leq \pi + \frac{C_2}{\ln R^{1/3}} \tag{65}$$

and

$$\frac{1}{2}\int_0^{2\pi} |\partial_\theta w(\tilde{r},\theta)|^2 \leq \pi + \frac{C_2}{|\ln r^{1/3}|}. \tag{66}$$

We write $w(\tilde{R},\theta) = \mathrm{e}^{\imath[\theta+\varphi(\tilde{R},\theta)]}$ and $w(\tilde{r},\theta) = \mathrm{e}^{\imath[\theta+\varphi(\tilde{r},\theta)]}$.

Since for $\rho \in \{\tilde{r},\tilde{R}\}$ we have $\displaystyle\int_0^{2\pi} \partial_\theta\varphi(\rho,\theta)\,\mathrm{d}\theta = 0$ and $|\partial_\theta w| = |1+\partial_\theta\varphi(\rho,\theta)|$, Estimates (65)&(66) may be rewritten in term of φ:

$$\frac{1}{2}\int_0^{2\pi} |\partial_\theta\varphi(\tilde{R},\theta)|^2 \leq \frac{C_2}{\ln R^{1/3}} \tag{67}$$

and

$$\frac{1}{2}\int_0^{2\pi} |\partial_\theta\varphi(\tilde{r},\theta)|^2 \leq \frac{C_2}{|\ln r^{1/3}|}. \tag{68}$$

Note that:

- from Estimates (67)&(68), we get that the maps $\varphi(\tilde{R},\cdot)$,$\varphi(\tilde{r},\cdot) : [0,2\pi] \to \mathbb{R}$ are continuous;
- for $\rho \in \{\tilde{R},\tilde{r}\}$ and $\theta_0 \in [0,2\pi]$ we have

$$|\varphi(\rho,\theta_0) - \varphi(\rho,0)|^2 = \left|\int_0^{\theta_0} \partial_\theta\varphi(\rho,\theta)\,\mathrm{d}\theta\right|^2 \leq 2\pi\int_0^{2\pi} |\partial_\theta\varphi(\rho,\theta)|^2\,\mathrm{d}\theta. \tag{69}$$

We now construct the map $\underline{w}$ by letting

$$\underline{w}(x) = \begin{cases} \mathrm{e}^{\imath(\theta+\phi_1(x))} & \text{if } x = \rho\mathrm{e}^{\imath\theta} \in \mathscr{R}_{R,\tilde{R}}(0) \\ w(x) & \text{if } x \in \mathcal{D}_{\tilde{R},\tilde{r}}(y) \\ \mathrm{e}^{\imath(\theta+\phi_2(x))} & \text{if } x = y+\rho\mathrm{e}^{\imath\theta} \in \mathscr{R}_{\tilde{r},r}(y) \end{cases}; \tag{70}$$

where

$$\phi_1(\rho,\theta) = \varphi(\tilde{R},0) + \frac{\tilde{R}}{\rho}\frac{R-\rho}{R-\tilde{R}}\left[\varphi(\tilde{R},\theta) - \varphi(\tilde{R},0)\right] \text{ for } \tilde{R} < \rho < R$$

and

$$\phi_2(\rho,\theta) = \varphi(\tilde{r},0) + \frac{\rho-r}{\tilde{r}-r}\left[\varphi(\tilde{r},\theta) - \varphi(\tilde{r},0)\right] \text{ for } r < \rho < \tilde{r}.$$

From direct computations and with the help of (67),(68) and (69) we get easily that

$$\begin{aligned} \frac{1}{2}\int_{\mathscr{R}_{R,\tilde{R}}(0)} |\nabla\phi_1|^2 + \frac{1}{2}\int_{\mathscr{R}_{\tilde{r},r}(y)} |\nabla\phi_2|^2 &\leq C_3 \max_{\rho\in\{\tilde{r},\tilde{R}\}} \frac{1}{2}\int_0^{2\pi} |\partial_\theta\varphi(\rho,\theta)|^2 \\ &\leq \frac{C}{\min\{|\ln r|;\ln R\}} \end{aligned}$$

where $C_3, C > 0$ are independent of n.

Now, it is straightforward to see that

$$\begin{aligned} \mu_{a^2}^{\mathrm{Dir}}(\mathcal{D}_{R,r}(y)) &\leq \frac{1}{2}\int_{\mathcal{D}_{R,r}(y)} \alpha|\nabla\underline{w}|^2 \\ \text{[Definition of } \underline{w}] &\leq \frac{1}{2}\int_{\mathcal{D}_{\tilde{R},\tilde{r}}(y)} \alpha|\nabla w|^2 + \pi\ln\frac{R}{\tilde{R}} + b^2\pi\ln\frac{\tilde{r}}{r} + \frac{C}{\min\{|\ln r|;\ln R\}} \\ \text{[Est. (60)]} &\leq \frac{1}{2}\int_{\mathcal{D}_{R,r}(y)} \alpha|\nabla w|^2 + \frac{C}{\min\{|\ln r|;\ln R\}} \\ \text{[by minimality of } w] &\leq \mu_{a^2}(\mathcal{D}_{R,r}(y)) + \frac{C}{\min\{|\ln r|;\ln R\}}, \end{aligned}$$

which is exactly (59).

We now prove the second assertion. Let $\eta := |x_n - x_0|$ and assume that $\eta > 0$ (if $\eta = 0$, then there is nothing to prove).

Let us consider:

- w_1 a minimizing map for $\mu^{\mathrm{Dir}}_{a^2}(\mathcal{D}_{R,\max\{r,\sqrt{\eta}\}}(x_n))$;
- w_2 a minimizer for $\mu^{\mathrm{Dir}}_{a^2}\left(\tilde{\mathcal{D}}\right)$ with $\tilde{\mathcal{D}} := B(x_n,\max\{r,\sqrt{\eta}\}) \setminus \overline{B(x_0,\min\{r,\eta^2\})}$.

  We have by conformal invariance that

$$\begin{aligned}
\mu^{\mathrm{Dir}}_{a^2}\left(\tilde{\mathcal{D}}\right) &= b^2\mu^{\mathrm{Dir}}_{\mathbb{I}}\left[\mathbb{D}\setminus\overline{B\left(\frac{x_0-x_n}{\max\{r,\sqrt{\eta}\}},\frac{\min\{r,\eta^2\}}{\max\{r,\sqrt{\eta}\}}\right)}\right]\\
[(15)\text{ in [CM96]}] &= b^2\left[\pi\ln\frac{\max\{r,\sqrt{\eta}\}}{\min\{r,\eta^2\}} + W^{\mathbb{D}}_{\mathrm{e}^{\imath\theta}}\left(\frac{x_0-x_n}{\max\{r,\sqrt{\eta}\}}\right)+o_n(1)\right]\\
&= b^2\pi\ln\frac{\max\{r,\sqrt{\eta}\}}{\min\{r,\eta^2\}}+o_n(1). \qquad (71)
\end{aligned}$$

  Here we used the continuity of $W^{\mathbb{D}}_{\mathrm{e}^{\imath\theta}}(\cdot)$, the renormalized energy of Bethuel-Brezis-Hélein defined Formula (2) for $\Omega := \mathbb{D}$ and $g(\mathrm{e}^{\imath\theta}) := \mathrm{e}^{\imath\theta}$, combined with $W^{\mathbb{D}}_{\mathrm{e}^{\imath\theta}}(0) = 0$ & $\dfrac{x_0-x_n}{\max\{r,\sqrt{\eta}\}} \to 0$.

Let $w : \mathcal{D}_{R,\min\{r,\eta^2\}}(x_0) \to \mathbb{S}^1$ be s.t.

$$w(x) = \begin{cases} w_1(x) & \text{if } x \in \mathcal{D}_{R,\max\{r,\sqrt{\eta}\}}(x_n)\\ \mathrm{Cst}\, w_2(x) & \text{if } x \in B(x_n,\max\{r,\sqrt{\eta}\})\setminus\overline{B(x_0,\min\{r,\eta^2\})}\end{cases}.$$

On the one hand, since the map w is of degree one, we have

$$\mu_{a^2}(\mathcal{D}_{R,r}(x_0)) + b^2\pi\ln\frac{r}{\min\{r,\eta^2\}} \le \int_{\mathcal{D}_{R,\min\{r,\eta^2\}}(x_0)} a^2|\nabla w|^2.$$

Therefore:

$$\begin{aligned}
\mu_{a^2}(\mathcal{D}_{R,r}(x_0)) &\le \int_{\mathcal{D}_{R,\min\{r,\eta^2\}}(x_0)} a^2|\nabla w|^2 - b^2\pi\ln\frac{r}{\min\{r,\eta^2\}}\\
\left[\begin{array}{c}\text{by the definitions of}\\ w_1\&w_2\text{ and (71)}\end{array}\right] &\le \mu^{\mathrm{Dir}}_{a^2}(\mathcal{D}_{R,\max\{r,\sqrt{\eta}\}}(x_n)) + b^2\pi\ln\frac{\max\{r,\sqrt{\eta}\}}{r}+o_n(1)\\
[\text{Proposition 1.1}] &\le \mu_{a^2}(\mathcal{D}_{R,\max\{r,\sqrt{\eta}\}}(x_n)) + b^2\pi\ln\frac{\max\{r,\sqrt{\eta}\}}{r}+o_n(1)\\
[\text{Estimate (60)}] &\le \mu_{a^2}(\mathcal{D}_{R,r}(x_n))+o_n(1).
\end{aligned}$$

Following the same strategy, we may reverse the role of x_0 an x_n. Therefore, the second assertion holds. □

A.3 Proofs of (8) and (9)

In this subsection, we turn to the case where R is defined in (7) and $r = R^{-1}$.

We are now able to prove that Estimates (9) and (55) hold. Recall that we consider a sequence $\varepsilon_n \downarrow 0$ s.t. $\hat{x}_{\varepsilon_n} \to x_0$. We drop the extra-subscript n by writing ε instead ε_n.

Let two functions α^+ and α^- be s.t. $\alpha^\pm = \begin{cases} b^2 & \text{in } (1\pm\sqrt{\varepsilon})\cdot\omega\\ 1 & \text{otherwise}\end{cases}$. It is easy to see that $\alpha^+ \le a^2 \le \alpha^-$.

Moreover from Proposition 3 in [Dos] we have the existence of $V_\varepsilon \in L^\infty(\mathbb{C},\mathbb{R})$ s.t.

$$\alpha^+ - V_\varepsilon \le \hat{U}^2_\varepsilon \le \alpha^- + V_\varepsilon \text{ and } \|V_\varepsilon\|_{L^\infty} \le C\varepsilon^4.$$

Therefore we get easily that

$$\mu^{\mathrm{Dir}}_{\alpha^+}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) - o_\varepsilon(1) \le \mu^{\mathrm{Dir}}_{\hat{U}^2_\varepsilon}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) \le \mu^{\mathrm{Dir}}_{\alpha^-}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) + o_\varepsilon(1) \qquad (72)$$

and

$$\mu_{\alpha^+}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) - o_\varepsilon(1) \le \mu_{\hat{U}^2_\varepsilon}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) \le \mu_{\alpha^-}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) + o_\varepsilon(1). \qquad (73)$$

Therefore, it suffices to prove that

$$\begin{cases} \mu^{\rm Dir}_{\alpha^\pm}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) - \mu^{\rm Dir}_{a^2}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) = o_\varepsilon(1) \\ \mu_{\alpha^\pm}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) - \mu_{a^2}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) = o_\varepsilon(1) \end{cases} \tag{74}$$

to get Estimate (55) as a direct consequence of Proposition 1.1.

We prove that $\mu_{\alpha^+}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) - \mu_{a^2}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) = o_\varepsilon(1)$, the same estimate with α^- instead of α^+ or with Dirichlet boundary condition is obtained in a similar way.

Let w be a minimizer for $\mu_{\alpha^+}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon))$.

Let us define $\tilde{w} \in H^1\left[\mathcal{D}_{R/(1+\sqrt{\varepsilon}),r/(1+\sqrt{\varepsilon})}\left(\frac{\hat{x}_\varepsilon}{1+\sqrt{\varepsilon}}\right), \mathbb{S}^1\right]$ be s.t. $\tilde{w}(\tilde{x}) = w[(1+\sqrt{\varepsilon})\tilde{x}]$.

It is easy to see that

$$\begin{aligned}
\mu_{a^2}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) &\geq \mu_{\alpha^+}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) \\
&= \frac{1}{2}\int_{\mathcal{D}_{R,r}(\hat{x}_\varepsilon)} \alpha^+ |\nabla w|^2 \\
&= \frac{1}{2}\int_{\mathcal{D}_{R/(1+\sqrt{\varepsilon}),r/(1+\sqrt{\varepsilon})}(\frac{\hat{x}_\varepsilon}{1+\sqrt{\varepsilon}})} a^2 |\nabla \tilde{w}|^2 \\
&\geq \mu_{a^2}\left[\mathcal{D}_{R/(1+\sqrt{\varepsilon}),r}\left(\frac{\hat{x}_\varepsilon}{1+\sqrt{\varepsilon}}\right)\right] \\
\text{[Proposition 1.1\&2]} \quad &\geq \mu^{\rm Dir}_{a^2}(\mathcal{D}_{R/(1+\sqrt{\varepsilon}),r}(x_0)) + o_\varepsilon(1) \\
\left[\begin{array}{c}\text{by extending with } \mathrm{e}^{\imath\theta} \\ \text{the maps in } \mathscr{R}_{R,R/(1+\sqrt{\varepsilon})}(0)\end{array}\right] \quad &\geq \mu^{\rm Dir}_{a^2}(\mathcal{D}_{R,r}(x_0)) + o_\varepsilon(1) \\
\text{[Proposition 1.1\&2]} \quad &\geq \mu_{a^2}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) + o_\varepsilon(1).
\end{aligned}$$

Therefore $\mu_{a^2}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) = \mu_{\alpha^+}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) + o_\varepsilon(1)$. The same argument leads to $\mu_{a^2}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) = \mu_{\alpha^-}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) + o_\varepsilon(1)$ and $\mu^{\rm Dir}_{a^2}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) = \mu^{\rm Dir}_{\alpha^\pm}(\mathcal{D}_{R,r}(\hat{x}_\varepsilon)) + o_\varepsilon(1)$. Therefore (74) holds.

And, by using (72)&(73), we get that the same inequality holds with $\hat{U}^2_\varepsilon$ instead of $\alpha^\pm$. Thus (9) &(55) hold. Consequently (8) is valid.

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