Large Deviations Principle by viscosity solutions: The case of diffusions with oblique Lipschitz reflections

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LARGE DEVIATIONS PRINCIPLE FOR OBLIQUE LIPSCHITZ REFLECTIONS

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Abstract. We establish a Large Deviations Principle for diffusions with Lipschitz continuous oblique reflections on regular domains. The rate functional is given as the value function of a control problem and is proved to be good. The proof is based on a viscosity solution approach. The idea consists in interpreting the probabilities as the solutions to some PDEs, make the logarithmic transform, pass to the limit, and then identify the action functional as the solution of the limiting equation.

Abstract. Nous établissons un principe de Grandes Déviations pour des diffusions réfléchies obliquement sur le bord d'un domaine régulier lorsque la direction de la réflection est Lipschitz. La fonction de taux s'exprime comme la fonction valeur d'un problème d'arrêt optimal et est compacte. Nous utilisons des techniques de solutions de viscosité. Les probabilités recherchée sont interprétées comme des solutions de certaines EDPs, leur transformées logarithmics donnent lieu à de nouvelles équations dans lesquelles il est aisé de passer à la limite. Enfin les fonctionnelles d'action sont identifiées comme étant les solutions des dites équations limite.

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Large Deviations Principle, diffusions with oblique reflections, viscosity solutions, optimal control, optimal stopping

1. Introduction

According to the terminology of Varadhan [45], a sequence (X^ε) of random variables with values in a metric space $(\mathcal{X}, d)$ satisfies a *Large Deviations Principle* (LDP in short) if

There exists a lower semi-continuous functional $\lambda : \mathcal{X} \to [0,\infty]$ such that for each Borel measurable set $\mathcal{G}$ of $\mathcal{X}$

(I) $\limsup_{\varepsilon\to 0} \left\{-\varepsilon^2 \ln \mathbf{P}[X^\varepsilon \in \mathcal{G}]\right\} \leq \inf_{g\in \overset{\circ}{\mathcal{G}}} \lambda(g)$ *(LDP's upper bound),*

(II) $\inf_{g\in\overline{\mathcal{G}}} \lambda(g) \leq \liminf_{\varepsilon\to 0} \left\{-\varepsilon^2 \ln \mathbf{P}[X^\varepsilon \in \mathcal{G}]\right\}$ *(LDP's lower bound),*

λ is called the rate functional *for the large Deviations principle (LDP). A rate functional is* good *if for any $a \in [0,\infty)$, the set $\{g \in \mathcal{X} : \lambda(g) \leq a\}$ is compact.*

We refer the reder to the books [1], [22], [30], [44],[45], [16], [17] for the general theory, references and different approches to Large Deviations.

Partial Differential Equations (in short PDEs) methods have been applied to establish different types of Large Deviations estimates starting from Fleming [27]. The idea consists in interpreting the probabilities as the solutions to some PDEs, make the logarithmic transform, pass to the limit, and then identify the action functional as the solution of the limiting equation. The notion of viscosity solutions (cf. Crandall-Lions [15], Lions [36]-[37], or Crandall-Ishii-Lions [14]) appeared to be particularly adapted to this problem. Indeed, the half-relaxed semi-limit method (cf. Barles-Perthame [9]) allows to pass to the limit very easily, moreover the notion of strong uniqueness for viscosity solution allows to identify the solution of the limiting equation with the action functional. A number of Large Deviations results have been proved by using this method [23], [28], [9], [10], [6], [3], [40]. One aim of this work is to use this method to provide a general Large Deviations Principle and not just some Large Deviations inequalities. We establish a LDP for small diffusions with oblique Lipischitz continuous direction of reflections which explains the technicity. This result is new to the best of our knowledge. Our method which was first developed in [35], seems very efficient and we hope it gives a new insight.

Recently [25] came to our knowledge. This book shows also, in a very general setting, that viscosity solutions are an adapted tool in order to establish LDPs, and illustrates also how deep are the links between viscosity solutions and Large Deviations.

Let $\mathcal{O}$ be a smooth open bounded subset of $\mathbb{R}^d$. For $(t,x) \in \mathbb{R}^+ \times \overline{\mathcal{O}}$, we consider the oblique reflection problem

$$(1.1) \qquad \begin{cases} dX_s = b(s,X_s)ds - dk_s, \quad X_s \in \overline{\mathcal{O}} \ (\forall s > t), \\ k_s = \displaystyle\int_t^s \mathbf{1}_{\partial\mathcal{O}}(X_\tau)\gamma(X_\tau)d|k|_\tau \ (\forall s > t), \quad X_t = x. \end{cases}$$

where b is a continuous $\mathbb{R}^d$-valued function defined on $\mathbb{R}^+ \times \overline{\mathcal{O}}$ and γ is a $\mathbb{R}^d$-vector field defined on $\partial\mathcal{O}$. The solutions of problem (1.1) are pairs (X,k) of continuous functions from $[t,\infty)$ to $\overline{\mathcal{O}}$ and $\mathbb{R}^d$ respectively such that k has bounded variations, and $|k|$ denotes the total variation of k.

We shall denote by $n(x)$ the unit outward normal to $\partial\mathcal{O}$ at x, and assume that

$$(1.2) \qquad \begin{cases} \gamma : \mathbb{R}^d \to \mathbb{R}^d \text{ is a Lipschitz continuous function} \\ \quad \text{and } \exists c_0 \ \forall x \in \partial\mathcal{O}, \ \gamma(x)\cdot n(x) \geq c_0 > 0. \end{cases}$$

When b is Lipschitz continuous, γ satisfies condition (1.2) and $\mathcal{O}$ is smooth, the existence of the solutions of (1.1) is given as a particular case of the results of Lions and Sznitman [38] and the uniqueness is a corollary of the result of Barles and Lions [7]. For more general domains existence and uniqueness of solutions of (1.1) is given as a particular case of Dupuis and Ishii [21]. The reader can also use the results given in Appendix B.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a standard filtred probability space which satisfies the *usual conditions* and $(W_t)_{t\geq 0}$ be a standard Brownian motion with values in $\mathbb{R}^m$. Consider for each $\varepsilon > 0$, $t \geq 0$, $x \in \overline{\mathcal{O}}$, the following stochastic differential equation

$$(1.3) \qquad \begin{cases} dX_s^\varepsilon = b_\varepsilon(s,X_s^\varepsilon)ds + \varepsilon\sigma_\varepsilon(s,X_s^\varepsilon)dW_s - dk_s^\varepsilon, \quad X_s^\varepsilon \in \overline{\mathcal{O}} \ (\forall s > t), \\ k_s^\varepsilon = \displaystyle\int_t^s \mathbf{1}_{\partial\mathcal{O}}(X_\tau^\varepsilon)\gamma(X_\tau^\varepsilon)d|k^\varepsilon|_\tau \ (\forall s > t), \quad X_t^\varepsilon = x, \end{cases}$$

where σ is continuous $\mathbb{R}^{d\times m}$-valued. A strong solution of (1.3) is a couple $(X_s^\varepsilon, k_s^\varepsilon)_{s\geq t}$ of $(\mathcal{F}_s)_{s\geq t}$-adapted processes which have almost surely continuous paths and such that $(k_s^\varepsilon)_{s\geq t}$ has almost surely bounded variations, and $|k^\varepsilon|$ denotes its total variation.

Let us now make some comments about this reflection problem. Consider equation (1.3) in the case when $\varepsilon = 1$.

This type of stochastic differential equations has been solved by using the Skorokhod map by Lions and Sznitman in [38] in the case when $\mathcal{O}$ belongs to a very large class of admissible open subsets and the direction of reflection is the normal direction n, or when $\mathcal{O}$ is smooth and γ is of class C^2. This problem was also deeply studied by Dupuis and Ishii [19], [20], [21]. When $\mathcal{O}$ is convex these authors proved in [19] that the Skorokhod map is Lipschitz continuous even when trajectories may have jumps. As a corollary, this result gives existence and uniqueness of the solution of the stochastic equation (1.3) and provides the Large Deviations estimates as well. Dupuis and Ishii also proved in [21] the existence of the solution of equation (1.3) in the following cases: either γ is C^2 and $\mathcal{O}$ has only an exterior cone condition, or $\mathcal{O}$ is a finite intersection of C^1 regular bounded domains $\mathcal{O}_i$ and γ is Lipschitz continuous at points $x \in \partial\mathcal{O}$ when x belongs to only one $\partial\mathcal{O}_i$ but when x is a corner point, $\gamma(x)$ can even be multivaluated. A key ingredient is the use of test functions that Dupuis and Ishii build in [18], [31] and [20] in order to study oblique derivative problems for fully nonlinear second-order elliptic PDEs on nonsmooth domains.

Let us point out that these type of diffusions with oblique reflection in domains with corners arise as rescaled queueing networks and related systems with feedback (see [3] and the references within).

We study in the present paper Large Deviations of (1.1) under the simpler condition of a domain without corners. More precisely we suppose that

$$\mathcal{O} \text{ is a } W^{2,\infty} \text{ open bounded set of } \mathbb{R}^d. \tag{1.4}$$

Let us precise now what is the regularity we require on the coefficients b, σ and b_ε, σ_ε and how b_ε and σ_ε are supposed to converge to b and σ.

For all $\varepsilon > 0$, let $b_\varepsilon, b \in \mathcal{C}([0,+\infty) \times \overline{\mathcal{O}}; \mathbb{R}^d)$, $\sigma_\varepsilon, \sigma \in \mathcal{C}([0,+\infty) \times \overline{\mathcal{O}}; \mathbb{R}^{d\times m})$. And assume that for each $T > 0$, there exists a constant C_T such that for all $\varepsilon > 0$, for all $t \in [0,T]$, for all $x, x' \in \overline{\mathcal{O}}$ one has

$$\begin{aligned} |b(t,x) - b(t,x')|,\ |b_\varepsilon(t,x) - b_\varepsilon(t,x')| &\le C_T|x - x'|, \\ \|\sigma(t,x) - \sigma(t,x')\|,\ \|\sigma_\varepsilon(t,x) - \sigma_\varepsilon(t,x')\| &\le C_T|x - x'|. \end{aligned} \tag{1.5}$$

We also assume that

$$(b_\varepsilon), (\sigma_\varepsilon) \text{ converge uniformly to } b \text{ and } \sigma \text{ on } [0,T] \times \overline{\mathcal{O}}. \tag{1.6}$$

By [21] for all $\varepsilon > 0$ and for all $(t,x) \in [0,T] \times \overline{\mathcal{O}}$, there exists a unique solution $(X^{t,x,\varepsilon}, k^{t,x,\varepsilon})$ of (1.3) on $[t,T]$. Morever $X^{t,x,\varepsilon}$ converges in probability to the solution $X^{t,x}$ of (1.1) when ε converges to 0. Obtaining the Large Deviations estimates provides the rate of this convergence.

We now turn to the definition of the rate functional λ. It is defined under conditions (1.2)-(1.4)-(1.5) as the value function of a non standard control problem of a deterministic differential equation with L^2 coefficients and with oblique reflections.

More precisely, let $(t,x) \in [0,T] \times \overline{\mathcal{O}}$ and $\alpha \in L^2(t,T;\mathbb{R}^m)$ and consider equation

$$\begin{cases} dY_s = (b(s,Y_s) - \sigma(s,Y_s)\alpha_s)\,ds - dz_s, \quad Y_s \in \overline{\mathcal{O}}, (\forall s > t), \\ z_s = \displaystyle\int_t^s \mathbf{1}_{\partial\mathcal{O}}(Y_\tau)\gamma(Y_\tau)d|z|_\tau, (\forall s > t), \quad Y_t = x. \end{cases} \tag{1.7}$$

We prove in Appendix B that there exists a unique solution $(Y_s^{t,x,\alpha}, z_s^{t,x,\alpha})_{s\in[t,T]}$ of (1.7), and we study the regularity of Y with respect to t, x, α and s.

In the following we note $\mathcal{X} = \mathcal{C}([0,T];\overline{\mathcal{O}})$ and, for $g \in \mathcal{X}$, $\quad \|g\|_{\mathcal{X}} = \sup_{t\in[0,T]} |g(t)|$.

We make the following abuse of notations. For $\mathcal{G} \subset \mathcal{X}$ and for $g \in \mathcal{C}([t,T];\overline{\mathcal{O}})$ for some $t \in [0,T]$, we write $g \in \mathcal{G}$ if there exists a function in $\mathcal{G}$ whose restriction to $[t,T]$ coincides with g.

For all $g \in \mathcal{X}$, we define $\lambda_{t,x}(g)$ by

$$\lambda_{t,x}(g) = \inf\left\{\frac{1}{2}\int_t^T |\alpha_s|^2 ds;\ \alpha \in L^2(t,T;\mathbb{R}^m),\ Y^{t,x,\alpha} = g\right\}. \tag{1.8}$$

Note that $\lambda_{t,x}(g) \in [0,+\infty]$.

The main result of our paper is the proof of the full Large Deviations type estimates for (1.3), as well as the identification of the rate functional which is proved to be good.

Theorem 1.1. *Assume (1.2)-(1.4)-(1.5)-(1.6). For each $(t,x) \in [0,T]\times\overline{\mathcal{O}}$, and $\varepsilon > 0$, denote by $X^{t,x,\varepsilon}$ the unique solution of (1.3) on $[t,T]$. Consider $\lambda_{t,x}$ defined by (1.8). Then $(X^{t,x,\varepsilon})_\varepsilon$ satisfies a* Large Deviations Principle *with rate functional $\lambda_{t,x}$. Moreover the rate functional is good.*

As far as the partial differential equations are concerned, we use the notion of viscosity solutions. We shall not recall the classical results of the theory of viscosity solutions here and we refer the reader to M.G. Crandall, H. Ishii and P.-L. Lions [14] (Section 7 for viscosity solutions of second order Hamilton-Jacobi equations), to W.H. Fleming and H.M. Soner [29] (Chapter 5 for stochastic controlled processes) and to G. Barles [4] (Chapter 4 for viscosity solutions of first order Hamilton-Jacobi equations with Neumann type boundary conditions and Chapter 5 Section 2 for deterministic controlled processes with reflections).

The paper is organised as follows. In section 2, we prove first that assertion (I) amounts to the proof of this upper bound for a ball $\mathcal{B}$ (assertion (A1)). Second, we prove that if the rate is good, assertion (II) amounts to prove the lower bound for a finite intersection of complementaries of balls (assertion (A2)). Finally, we prove that the fact that the rate is good holds true if a stability result holds true for equation (1.7) (assertion (A3)). In section 3, we give the proof of (A1), and we finish in section 4 by the proof of (A2).

An important Appendix follows. It includes, in Appendix B, the study of equation (1.7) and the proof of (A3). In Appendix C, we study different mixed optimal control-optimal single or multiple stopping times problems and we characterize particular value functions as the minimal (resp. maximal) viscosity supersolution (resp. subsolution) of the related obstacle problems. These caracterizations are important in order to establish (A1) and (A2). Eventually, in Appendix D, we prove a strong comparison result for viscosity solutions to an obstacle problem with Neumann boundary conditions and quadratic growth in the gradient in the case of a continuous obstacle. This result is needed in the proof of the caracterization of the value functions mentioned above. This long and technical Appendix begins in Appendix A, by the construction of an appropriate test function which is useful in order to establish the results concerning equation (1.7) (Appendix B) and the uniqueness result (Appendix D).

2. A preliminary result

We now define the action functional. For each $(t,x) \in [0,T]\times\overline{\mathcal{O}}$, and for each $\mathcal{G} \subset \mathcal{X}$ let us define $\Lambda_{t,x}(\mathcal{G})$ as follows:

$$\Lambda_{t,x}(\mathcal{G}) = \inf\left\{\frac{1}{2}\int_t^T |\alpha_s|^2 ds;\ \alpha \in L^2(t,T;\mathbb{R}^m), Y^{t,x,\alpha} \in \mathcal{G}\right\}, \tag{2.1}$$

where $Y^{t,x,\alpha}$ is defined by (1.7). Clearly $\Lambda_{t,x}$ is non increasing, $\Lambda_{t,x}(\mathcal{G}) = \inf\limits_{g\in\mathcal{G}} \lambda_{t,x}(g)$ and $\lambda_{t,x}(g) = \Lambda_{t,x}(\{g\})$.

We use the following notation: for $g_0 \in \mathcal{X}$ and $r > 0$ we denote by $\mathcal{B}(g_0, r)$ the ball of center g_0 and of radius r that is $\mathcal{B}(g_0, r) = \{g \in \mathcal{X},\ \|g - g_0\|_\infty < r\}$.

We consider the following assertions.

(A1) for all ball $\mathcal{B}$ in $\mathcal{X}$, $\quad \limsup\limits_{\varepsilon\to 0} \left\{-\varepsilon^2 \ln P[X^{t,x,\varepsilon} \in \mathcal{B}]\right\} \leq \Lambda_{t,x}(\mathcal{B})$,

(A2) for all finite collection of balls $(\mathcal{B}_i)_{i\leq N}$ in $\mathcal{X}$, $\quad \liminf\limits_{\varepsilon\to 0} \left\{-\varepsilon^2 \ln P\left[X^{t,x,\varepsilon} \in \bigcap\limits_{i=1}^{N} \mathcal{B}_i^c\right]\right\} \geq \Lambda_{t,x}\left(\bigcap\limits_{i=1}^{N} \mathcal{B}_i^c\right)$,

(A3) for all $\alpha_n, \alpha \in L^2 = L^2(0, T; \mathbb{R}^m)$, if $\alpha_n \rightharpoonup \alpha$ weakly in L^2 then $\|Y^{t,x,\alpha_n} - Y^{t,x,\alpha}\|_{\mathcal{X}} \to 0$.

The following proposition shows that Theorem 1.1 reduces to assertions (A1), (A2) and (A3).

Proposition 2.1. *For all $(t, x) \in [0, T] \times \overline{\mathcal{O}}$ one has,*

(i) (A1) implies (I),

(ii) If the rate is good then (A2) implies (II).

(iii) (A3) implies that the rate functional $\lambda_{t,x}$ defined by (1.8) is good.

Proof: Fix a measurable subset $\mathcal{G}$ in $\mathcal{X}$. To prove (i), take $g \in \overset{\circ}{\mathcal{G}}$ and fix $r > 0$ such that $\mathcal{B} = \mathcal{B}(g, r) \subset \overset{\circ}{\mathcal{G}}$. Then, by (A1), $\limsup_{\varepsilon\to 0} \left\{-\varepsilon^2 \ln P[X^{t,x,\varepsilon} \in \mathcal{G}]\right\} \leq \limsup_{\varepsilon\to 0} \left\{-\varepsilon^2 \ln P[X^{t,x,\varepsilon} \in \mathcal{B}]\right\} \leq \Lambda_{t,x}(\mathcal{B}) \leq \lambda_{t,x}(g)$, and we conclude the proof of point (i) by taking the infimum over all $g \in \overset{\circ}{\mathcal{G}}$.

Let us prove (ii). Fix $a < \Lambda_{t,x}(\overline{\mathcal{G}})$ and put $\mathcal{K} = \{g \in \mathcal{X},\ \lambda_{t,x}(g) \leq a\}$. Note that $\Lambda_{t,x}(\mathcal{K}^c) \geq a$ and that $\mathcal{K} \subset \overline{\mathcal{G}}^c$. Since $\mathcal{K}$ is compact, there exists a finite collection of balls $(\mathcal{B}_i)_{i\leq N}$ in $\mathcal{X}$ such that $\mathcal{K} \subset \bigcup_{i=1}^{N} \mathcal{B}_i \subset \overline{\mathcal{G}}^c$. Passing to the complementaries, $\Lambda_{t,x}\left(\bigcap_{i=1}^{N} \mathcal{B}_i^c\right) \geq \Lambda_{t,x}(\mathcal{K}^c) \geq a$. Note that, by (A2), $\liminf_{\varepsilon\to 0} \left\{-\varepsilon^2 \ln P\left[X^{t,x,\varepsilon} \in \mathcal{G}\right]\right\} \geq \liminf_{\varepsilon\to 0} \left\{-\varepsilon^2 \ln P\left[X^{t,x,\varepsilon} \in \bigcap_{i=1}^{N} \mathcal{B}_i^c\right]\right\} \geq \Lambda_{t,x}\left(\bigcap_{i=1}^{N} \mathcal{B}_i^c\right)$. We have shown that $\liminf\limits_{\varepsilon\to 0} \left\{-\varepsilon^2 \ln P\left[X^{t,x,\varepsilon} \in \mathcal{G}\right]\right\} \geq a$, for all $a < \Lambda_{t,x}(\overline{\mathcal{G}})$, which completes the proof of (ii).

Let us prove (iii). We suppose that the rate functional $\lambda_{t,x}$ defined by (1.8) satisfies (A3). Fix $(t, x) \in [0, T] \times \overline{\mathcal{O}}$, and $a \in \mathbb{R}$. Put $\mathcal{K} = \{g \in \mathcal{X},\ \lambda_{t,x}(g) \leq a\}$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{K}$. Then, for all n, there exists $\alpha_n \in L^2$ such that $Y^{t,x,\alpha_n} = g_n$ and $\frac{1}{2}\int_t^T |\alpha_n(s)|^2 ds \leq a + o(1) \quad [n \to \infty]$. Thus $(\alpha_n)_{n\in\mathbb{N}}$ is bounded in L^2 and extracting a subsequence if necessary, one can suppose that the sequence $(\alpha_n)_{n\in\mathbb{N}}$ converges weakly in L^2 to some $\overline{\alpha}$. By (A3), $(Y^{t,x,\alpha_n})_n$ converges uniformly on $[t, T]$ to $Y^{t,x,\overline{\alpha}}$ and since for all n, $Y^{t,x,\alpha_n} = g_n$ the sequence $(g_n)_{n\in\mathbb{N}}$ converges to some $\overline{g} = Y^{t,x,\overline{\alpha}}$ in $\mathcal{X}$.
Moreover, $\lambda_{t,x}(\overline{g}) \leq \frac{1}{2}\int_t^T |\overline{\alpha}_s|^2 ds \leq \liminf\limits_{n\to\infty} \frac{1}{2}\int_t^T |\alpha_n(s)|^2 ds \leq a$, hence $\overline{g} \in \mathcal{K}$. We have proved that $\mathcal{K}$ is compact. $\diamond$

3. Proof of assertion (A1)

Fix a ball $\mathcal{B} = \mathcal{B}(g_0, r)$. For each $(t,x) \in [0,T] \times \overline{\mathcal{O}}$ consider the probability $u_\varepsilon(t,x)$ defined by $u_\varepsilon(t,x) = P[X^{t,x,\varepsilon} \in \mathcal{B}]$ and define the logarthmic transform $v_\varepsilon(t,x) = -\varepsilon^2 \ln u_\varepsilon(t,x)$ The aim of this section is to prove that $\limsup_{\varepsilon\to 0} \{v_\varepsilon(t,x)\} \le \Lambda_{t,x}(\mathcal{B})$.

Step 1. From a probability to a PDE: We first interpret the probability $u_\varepsilon(t,x)$ as the value function of an optimal stopping problem. This leads naturaly to a PDE, as the value function of an optimal stopping problem with reward ψ is solution to a variational inequality with obstacle ψ.

Let us define the tube $\mathbf{B}$ as the set

$$\mathbf{B} = \{(t,x) \in [0,T] \times \overline{\mathcal{O}},\ |x - g_0(t)| < r\}. \tag{3.1}$$

Proposition 3.1. *$u_\varepsilon(t,x)$ is the value function of the following optimal stopping problem*

$$u_\varepsilon(t,x) = \inf_{\theta \in T_t} E[\mathbf{1_B}(\theta, X_\theta^{t,x,\varepsilon})],$$

where T_t is the set of stopping times θ with value in $[t,T]$.

The proof can be found at the end of this section.

We now recall that the value function of an optimal stopping time problem with reward ψ is a viscosity solution of a variational inequality with obstacle ψ.

More precisely for each bounded Borel function ψ on $[0,T] \times \mathbb{R}^d$ consider

$$U_\varepsilon[\psi](t,x) = \inf_{\theta \in T_t} E[\psi(\theta, X_\theta^{t,x,\varepsilon})], \tag{3.2}$$

where $X^{t,x,\varepsilon}$ is the solution of (1.3), then $U^\varepsilon[\psi]$ is a solution to the following variational inequality with obtacle ψ

$$\begin{cases} \max\left(-\dfrac{\partial u}{\partial t} + \mathcal{L}_\varepsilon u, u - \psi\right) = 0 \text{ in } [0,T) \times \mathcal{O} \\ \dfrac{\partial u}{\partial \gamma} = 0 \text{ in } [0,T) \times \partial\mathcal{O},\ u(T) = \psi(T) \text{ on } \overline{\mathcal{O}}. \end{cases} \tag{3.3}$$

where $\mathcal{L}_\varepsilon u = -\dfrac{\varepsilon^2}{2}\mathrm{Tr}[\sigma_\varepsilon \sigma_\varepsilon^T D^2 u] - b_\varepsilon \cdot Du$.

Proposition 3.2. *Assume (1.2), (1.4) and (1.5). Then the function $U_\varepsilon[\psi]$ defined by (3.2) is a viscosity solution of (3.3).*

This result is a standard consequence of the well-known Dynamic Programming Principle. Under regularity conditions the proof goes back to [7]. For a general proof of the Dynamic Programming Principle see [24] or [11].

This gives that $u_\varepsilon = U_\varepsilon[\mathbf{1_B}]$ is a solution of the variational inequality (3.3) with obstacle $\mathbf{1_B}$.

Step 2. The logarithmic transform: For all non-negative function ψ bounded away from 0, let $V_\varepsilon[\psi]$ be the logarithmic transform of $U_\varepsilon[\psi]$ defined by

$$V_\varepsilon[\psi] = -\varepsilon^2 \ln(U_\varepsilon[\psi]). \tag{3.4}$$

Then $V_\varepsilon[\psi]$ is a viscosity solution of the following the variational inequality with obstacle $\varepsilon^2\ln(\psi)$

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t}+H_\varepsilon(D^2V,DV),V-\varepsilon^2\ln(\psi)\right)=0 \text{ in } [0,T)\times\mathcal{O}, \\ \dfrac{\partial V}{\partial\gamma}=0 \text{ in } [0,T)\times\partial\mathcal{O},\ V(T)=\varepsilon^2\ln(\psi(T)) \text{ on } \overline{\mathcal{O}}, \end{cases} \tag{3.5}$$

where $H_\varepsilon(D^2V,DV)=-\frac{\varepsilon^2}{2}\mathrm{Tr}[\sigma_\varepsilon\sigma_\varepsilon^T D^2V]+\frac{1}{2}|\sigma_\varepsilon^T DV|^2-b_\varepsilon\cdot DV$.

Formaly, $v_\varepsilon=V_\varepsilon[\mathbf{1_B}]$ is a viscosity solution of variational inequality (3.5) with singular obstacle $\chi_{\mathbf{B^c}}=-\varepsilon^2\ln(\mathbf{1_B})$, which takes infinit values on $\mathbf{B}^c$ and is equal to 0 on $\mathbf{B}$.

In order to avoid the singularity, we seek now to approximate the original obstacle $\mathbf{1_B}$ in such a way that after the logarithmic transform, the obstacle becomes $A\mathbf{1_{B^c}}$ with $A>0$. We define for all $A,\varepsilon>0$, the real valued functions ψ_ε^A, u_ε^A and v_ε^A by

$$\psi_\varepsilon^A=\exp(-A\mathbf{1}_{\mathbf{B}^c}/\varepsilon^2),\qquad u_\varepsilon^A=U_\varepsilon[\psi_\varepsilon^A]\quad\text{and}\quad v_\varepsilon^A=V_\varepsilon[\psi_\varepsilon^A]. \tag{3.6}$$

Note that $\psi_\varepsilon^A\geq\mathbf{1_B}$, hence $u_\varepsilon^A\geq u_\varepsilon$ and $v_\varepsilon^A\leq v_\varepsilon$. As our aim is to majorate $\limsup v_\varepsilon$, it seems at first that we have the inequality from the wrong side. However, the following lemma shows that we can reduce ourselves to the study of v_ε^A.

Lemma 3.3. *For all $A>0$, and for all $(t,x)\in[0,T]\times\overline{\mathcal{O}}$, we have* $\limsup_{\varepsilon\to0} v_\varepsilon^A=\limsup_{\varepsilon\to0} v_\varepsilon\wedge A$.

The proof can be found at the end of this section.

Clearly v_ε^A is a viscosity solution of variational inequality (3.5) with obstacle $A\mathbf{1_{B^c}}$.

Step 3. Passing to the limit: when ε goes to 0, equation (3.5) with obstacle $A\mathbf{1_{B^c}}$ converges to the following variational inequality with obstacle $A\mathbf{1_{B^c}}$

$$\begin{cases} \min\left(-\dfrac{\partial v}{\partial t}+\dfrac{1}{2}|\sigma^T Dv|^2-b\cdot Dv, v-A\mathbf{1}_{\mathbf{B^c}}\right)=0 \text{ in } [0,T)\times\mathcal{O}, \\ \dfrac{\partial v}{\partial\gamma}=0 \text{ in } [0,T)\times\partial\mathcal{O},\ v(T)=A\mathbf{1}_{\mathbf{B}^c}(T) \text{ on } \overline{\mathcal{O}}. \end{cases} \tag{3.7}$$

By a general stability result for viscosity solutions (see [4] Theorem 4.1 p.85 or [8]), the half-relaxed upper-limit $\limsup^* v^A$ defined for all (t,x) in $[0,T]\times\overline{\mathcal{O}}$ by

$$\limsup{}^*\ v_\varepsilon^A(t,x)=\limsup_{\substack{(s,y)\to(t,x)\\ \varepsilon\to0}} v_\varepsilon^A(s,y).$$

is a viscosity subsolution of the limit equation (3.7).

Step 4. A first order mixed optimal control-optimal stopping problem: back to the action functional: we now study a value function of a mixed optimal control-optimal stopping problem which appears to be the maximal viscosity subsolution of equation (3.7), and which we compare with $\Lambda_{t,x}(\mathcal{B})$.

For each bounded Borel function ψ, and for all $(t,x)\in[0,T]\times\overline{\mathcal{O}}$, define the following value function

$$v[\psi](t,x)=\inf_{\alpha\in L^2}\sup_{\theta\in[t,T]}\left\{\frac{1}{2}\int_t^\theta|\alpha_s|^2ds+\psi(\theta,Y_\theta^{t,x,\alpha})\right\} \tag{3.8}$$

where $Y^{t,x,\alpha}$ is the unique solution of (1.7).

Proposition 3.4. (1) *For each bounded Borel function ψ the function $v[\psi^*]$ defined by (3.8) is the maximal usc viscosity subsolution of the variational inequality (3.7) with obstacle ψ.*
(2) *For all $A>0$, $(t,x)\in[0,T]\times\overline{\mathcal{O}}$, one has $v[A\mathbf{1}_{\mathbf{B^c}}](t,x)\le A\wedge\Lambda_{t,x}(\mathcal{B})$.*

The proof can be found in Appendix C for point (1.) and at the end of this section for point (2.).

Conclusion: by Lemma 3.3, by using the half-reaxed semi-limit method, and by Proposition 3.4 we have, for each $A>0$, $A\wedge\limsup_{\varepsilon\to0}v_\varepsilon(t,x)\le\limsup{}^*v^A_\varepsilon(t,x)\le v[A\mathbf{1}_{\mathbf{B^c}}]\le A\wedge\Lambda_{t,x}(\mathcal{B})$. The proof of (A1) is now complete. $\diamond$

We now turn to the proofs of Proposition 3.1, of Lemma 3.3 and of Proposition 3.4.

Proof of Proposition 3.1: Obviously, if $X^{t,x,\varepsilon}\in\mathcal{B}$ then $(\theta,X^{t,x,\varepsilon}_\theta)\in\mathbf{B}$ for all $\theta\in T_t$, hence $u_\varepsilon(t,x)\le E[\mathbf{1}_{\mathbf{B}}(\theta,X^{t,x,\varepsilon}_\theta)]$. Taking the infimum over $\theta\in T_t$ we obtain $u_\varepsilon(t,x)\le\inf_{\theta\in T_t}E[\mathbf{1}_{\mathbf{B}}(\theta,X^{t,x,\varepsilon}_\theta)]$. Conversely, define $\tilde\theta:=\inf\{s\ge t,\ |X^{t,x,\varepsilon}_s-g_0(s)|\ge r\}$, the first exit time of $(s,X^{t,x,\varepsilon}_s)_{s\ge t}$ from $\mathbf{B}$. We have $\{(\tilde\theta\wedge T,X^{t,x,\varepsilon}_{\tilde\theta\wedge T})\in\mathbf{B}\}\subset\{X^{t,x,\varepsilon}\in\mathcal{B}\}$. (Indeed, if $(\tilde\theta(\omega)\wedge T,X^{t,x,\varepsilon}_{\tilde\theta(\omega)\wedge T})\in\mathbf{B}$, then $\tilde\theta(\omega)>T$, and for all $s\in[t,T]$ we have $|X^{t,x,\varepsilon}_s(\omega)-g_0(s)|<r$, which means, as both $X^{t,x,\varepsilon}_\cdot(\omega)$ and $g_0(.)$ are continuous on $[t,T]$ that $\|X^{t,x,\varepsilon}(\omega)-g_0\|_{\mathcal{X}}<r$, hence $X^{t,x,\varepsilon}(\omega)\in\mathcal{B}$). We have now $\inf_{\theta\in T_t}E[\mathbf{1}_{\mathbf{B}}(\theta,X^{t,x,\varepsilon}_\theta)]\le E[\mathbf{1}_{\mathbf{B}}(\theta,X^{t,x,\varepsilon}_{\tilde\theta\wedge T}]\le u_\varepsilon(t,x)$ and the proof is complete. $\diamond$

Proof of Lemma 3.3: Fix $(t,x)\in[0,T]\times\overline{\mathcal{O}}$, and $A>0$. Clearly $e^{-A/\varepsilon^2}\vee\mathbf{1}_{\mathbf{B}}(t,x)\le\psi^A_\varepsilon(t,x)\le\mathbf{1}_{\mathbf{B}}(t,x)+e^{-A/\varepsilon^2}$. This gives easily,

$$e^{-A/\varepsilon^2}\vee u_\varepsilon(t,x)\le u^A_\varepsilon(t,x)\le u_\varepsilon(t,x)+e^{-A/\varepsilon^2}.$$

As for any non-negative sequence (u_ε) one has $\limsup_{\varepsilon\to0}\left\{-\varepsilon^2\ln(u_\varepsilon+e^{-A/\varepsilon^2})\right\}=A\wedge\limsup_{\varepsilon\to0}\left\{-\varepsilon^2\ln u_\varepsilon\right\}$, we obtain

$$A\wedge\limsup_{\varepsilon\to0}\left\{-\varepsilon^2\ln u_\varepsilon\right\}\ge\limsup_{\varepsilon\to0}\left\{-\varepsilon^2\ln u^A_\varepsilon\right\}\ge A\wedge\limsup_{\varepsilon\to0}\left\{-\varepsilon^2\ln u_\varepsilon\right\},$$

which completes the proof of the lemma. $\diamond$

Proof of Proposition 3.4: Point (1) is detailed in Appendix C (Proposition 5.8).

Let us prove now the second point. Obviously, $v^A(t,x)\le A$. Now, if $\Lambda_{t,x}(\mathcal{B})<A$, for each $\eta>0$ such that $\Lambda_{t,x}(\mathcal{B})+\eta<A$ there exists $\tilde\alpha\in L^2$ such that $Y^{t,x,\tilde\alpha}\in\mathcal{B}$ and $\Lambda_{t,x}(\mathcal{B})\le\dfrac12\displaystyle\int_t^T|\tilde\alpha_s|^2ds\le\Lambda_{t,x}(\mathcal{B})+\eta$. Thus, for any $\theta\in[t,T]$, one has $\mathbf{1}_{\mathbf{B}^c}(\theta,Y^{t,x,\tilde\alpha}_\theta)=0$ so that $v^A(t,x)\le\displaystyle\sup_{\theta\in[t,T]}\frac12\int_t^\theta|\tilde\alpha_s|^2ds\le\Lambda_{t,x}(\mathcal{B})+\eta$. We have proved that $v^A(t,x)\le A\wedge\Lambda_{t,x}(\mathcal{B})$. $\diamond$

4. Proof of assertion (A2)

Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{X}$ and $(r_n)_{n\in\mathbb{N}}$, a sequence in $]0,\infty[$. For each nonempty finite subset I of $\mathbb{N}$ and for all $(t,x)\in[0,T]\times\overline{\mathcal{O}}$, we define the probability

(4.1)
$$u^I_\varepsilon(t,x)=P[X^{t,x,\varepsilon}\in\mathcal{G}],\ \text{ where }\mathcal{G}=\bigcap_{i\in I}\mathcal{B}(g_i,r_i)^c\quad\text{and its logarithmic transform }v^I_\varepsilon(t,x)=-\varepsilon^2\ln[u^I_\varepsilon(t,x)],$$

and we prove that $\liminf_{\varepsilon\to 0} v_\varepsilon^I(t,x) \geq \Lambda_{t,x}(\mathcal{G})$.

In the following we will denote by θ_I a multiple stopping time $(\theta_i)_{i\in I}$ with $\theta_i \in T_t$ for each $i \in I$, and we write $\theta_I \in T_t^I$. We also define the tube $\mathbf{B}_i$ for $i \in \mathbb{N}$ by $\mathbf{B}_i = \{(t,x) \in [0,T]\times\overline{\mathcal{O}};\ |x-g_i(t)| < r_i\}$.

Step 1. From a probability to a PDE: we first interpret $u_\varepsilon^I(t,x)$ as the value of an optimal multiple stopping times problem and we show that it can also be interpreted as the value of an optimal single stopping time problem with a new reward.

Proposition 4.1. *Let I be a non empty finite subset of $\mathbb{N}$, $\varepsilon > 0$, and $(t,x) \in [0,T] \times \overline{\mathcal{O}}$.*
1. One has $u_\varepsilon^I(t,x) = \sup_{\theta_I\in T_t^I} E\left[\prod_{i\in I} \mathbf{1}_{\mathbf{B}_i^c}(\theta_i, X_{\theta_i}^{t,x,\varepsilon})\right]$
2. One has also $u_\varepsilon^I(t,x) = \sup_{\theta\in T_t} E\left[\psi_\varepsilon^I(\theta, X_\theta^{t,x,\varepsilon})\right]$ *where* $\psi_\varepsilon^I = \begin{cases} \mathbf{1}_{\mathbf{B}_i^c} \text{ if } I=\{i\} \text{ with } i\in\mathbb{N}, \\ \max_{i\in I}(\mathbf{1}_{\mathbf{B}_i^c} u_\varepsilon^{I\setminus\{i\}}) \text{ if } \operatorname{card} I \geq 2 \end{cases}$ *is the new reward.*
3. Consequently, u_ε^I is a viscosity solution of the following variational inequality with obstacle ψ_ε^I

$$\begin{cases} \min\left(-\dfrac{\partial u}{\partial t} + \mathcal{L}_\varepsilon u, u - \psi_\varepsilon^I\right) = 0 \text{ in } [0,T)\times\mathcal{O}, \\ \dfrac{\partial u}{\partial\gamma} = 0 \text{ in } [0,T)\times\partial\mathcal{O},\ u(T) = \psi_\varepsilon^I(T) \text{ on } \overline{\mathcal{O}}. \end{cases} \tag{4.2}$$

Proof: The proof of 1. is similar to the proof of Proposition 3.1. The main difference is in the choice of the optimal stopping time which is here $\tilde\theta_I \in T_t^I$ where for $i \in I$, $\tilde\theta_i$ is the first exit time in $[t,T]$ of $(s, X_s^{t,x,\varepsilon})$ from $\mathbf{B}_i$.
Point 2. is a consequence of the reduction result Theorem 3.1 in [33], and 3. follows by Proposition 3.2. $\diamond$

Note that u_ε^I is not bounded away from 0 and at this point logarithmic transform stays formal. We approximate $\mathbf{1}_{\mathbf{B}_i^c}$ and u_ε^I, in the following way. We define, for each $(t,x) \in [0,T]\times\overline{\mathcal{O}}$,

$$\psi_\varepsilon^{\{i\},A}(t,x) = \exp\left(-\frac{A\mathbf{1}_{\mathbf{B}_i}(t,x)}{\varepsilon^2}\right) \quad \text{and} \quad u_\varepsilon^{I,A}(t,x) = \sup_{\theta_I\in T_t^I} E\left[\prod_{i\in I}\psi_\varepsilon^{\{i\},A}(\theta_i, X_{\theta_i}^{t,x,\varepsilon})\right]. \tag{4.3}$$

Clearly

$$\mathbf{1}_{\mathbf{B}_i^c} \vee e^{-A/\varepsilon^2} \leq \psi_\varepsilon^{\{i\},A} \quad \text{and} \quad u_\varepsilon^I \vee e^{-A/\varepsilon^2} \leq u_\varepsilon^{I,A}. \tag{4.4}$$

The reduction result applies to $u_\varepsilon^{I,A}(t,x)$, hence it can be writen as the following single optimal stopping time problem $u_\varepsilon^{I,A}(t,x) = \sup_{\theta\in T_t} E\left[\psi_\varepsilon^{I,A}(\theta, X_\theta^{t,x,\varepsilon})\right]$ with new reward $\psi_\varepsilon^{I,A}(t,x) = \max_{i\in I}\left(\psi_\varepsilon^{\{i\},A}(t,x)u_\varepsilon^{I\setminus\{i\},A}(t,x)\right)$. By Proposition 3.2 $u_\varepsilon^{I,A}$ is a viscosity subsolution of the variational inequality (4.2) with obstacle $\psi_\varepsilon^{I,A}$.

Step 2. The logarithmic transform: note that by (4.4), $u_\varepsilon^{I,A}$ is bounded away from 0. We define its logarthmic transform $v_\varepsilon^{I,A}$ on $[0,T]\times\overline{\mathcal{O}}$ by $v_\varepsilon^{I,A} = -\varepsilon^2 \ln u_\varepsilon^{I,A}$. Then $v_\varepsilon^{I,A}$ is a viscosity supersolution of the following variational inequality with obstacle $\phi_\varepsilon^{I,A}$

$$\begin{cases} \max\left(-\dfrac{\partial v}{\partial t} + H_\varepsilon(D^2 v, Dv), v - \phi_\varepsilon^{I,A}\right) = 0 \text{ in } [0,T)\times\mathcal{O} \\ \dfrac{\partial v}{\partial \gamma} = 0 \text{ in } [0,T)\times\partial\mathcal{O},\ v(T) = \phi_\varepsilon^{I,A}(T) \text{ on } \overline{\mathcal{O}} \end{cases}$$

where, $\phi_\varepsilon^{I,A} = \begin{cases} A\mathbf{1}_{\mathbf{B}_i} & \text{if } I = \{i\} \text{ with } i \in \mathbb{N}, \\ \min\limits_{i\in I}\left\{A\mathbf{1}_{\mathbf{B}_i} + v_\varepsilon^{I\setminus\{i\},A}\right\} & \text{if } \operatorname{card} I \geq 2. \end{cases}$

Step 3. A mixed optimal control-optimal multiple stopping problem: let us turn now to the study of a mixed optimal control-optimal multiple stopping problem. The value function of this problem will be shown to be smaller than the half-relaxed lower limit $\lim_* v_\varepsilon^{I,A}(t,x)$ and greater than $\Lambda_{t,x}(\mathcal{G}) \wedge A$.

For all finite and nonempty subset I of $\mathbb{N}$ and for all $(t,x) \in [0,T]\times\overline{\mathcal{O}}$, define the following value function

$$v^{I,A}(t,x) = \inf_{\alpha\in L^2} \inf_{\theta_I\in[t,T]^I} \left\{\frac{1}{2}\int_t^{\vee_{i\in I}\theta_i} |\alpha_s|^2 ds + \sum_{i\in I} A\mathbf{1}_{\mathbf{B}_i}(\theta_i, Y_{\theta_i}^{t,x,\alpha})\right\}, \tag{4.5}$$

where $Y^{t,x,\alpha}$ is the unique solution of (1.7).

This mixed optimal *multiple* stopping problem can be reduced to a mixed optimal *single* stopping problem. More precisely, consider for each bounded real valued measurable ϕ on $[0,T]\times\overline{\mathcal{O}}$ and for each $(t,x) \in [0,T]\times\overline{\mathcal{O}}$ the following value function

$$v[\phi](t,x) = \inf_{\alpha\in L^2} \inf_{\theta\in[t,T]} \left\{\frac{1}{2}\int_t^{\theta} |\alpha_s|^2 ds + \phi(\theta, Y_\theta^{t,x,\alpha})\right\}, \tag{4.6}$$

where $Y^{t,x,\alpha}$ is the unique solution of (1.7).
Define also for all nonempty finite subset I of $\mathbb{N}$, for all $A > 0$,

$$\phi^{I,A} = \begin{cases} A\mathbf{1}_{\mathbf{B}_i} & \text{if } I = \{i\} \text{ with } i \in \mathbb{N}, \\ \min\limits_{i\in I}\left\{A\mathbf{1}_{\mathbf{B}_i} + v^{I\setminus\{i\},A}\right\} & \text{if } \operatorname{card} I \geq 2. \end{cases} \tag{4.7}$$

Proposition 4.2. *Let I be a finite subset of $\mathbb{N}$ and $A > 0$, and consider the function $v^{I,A}$ defined by (4.5). Then*

1. *for each $(t,x) \in [0,T]\times\overline{\mathcal{O}}$ one has $v^{I,A}(t,x) = v[\phi^{I,A}](t,x)$ where $\phi^{I,A}$ is defined by (4.7),*
2. *one has for all $(t,x) \in [0,T]\times\overline{\mathcal{O}}$, $v^{I,A}(t,x) \geq A \wedge \Lambda_{t,x}(\mathcal{G})$.*

Proof: The proof of (1) is the concequence of a reduction result for optimal multiple stopping problems. It is detailed in Appendix C (Proposition 5.9).
Let us prove (2). Suppose $v^{I,A}(t,x) < A$, then for each $\eta > 0$ such that $v^{I,A}(t,x) + \eta < A$, there exists $\theta_I \in [0,T]^N$ and $\alpha \in L^2$ such that $\frac{1}{2}\int_t^{\vee_{i\in I}\theta_i} |\alpha_s|^2 ds + \sum_{i\in I} A\mathbf{1}_{\mathbf{B}_i}(\theta_i, Y_{\theta_i}^{t,x,\alpha}) \leq v^{I,A}(t,x) + \eta < A$. This means in particular that $(\theta_i, Y_{\theta_i}^{t,x,\alpha}) \in \mathbf{B}_i^c$ for all $i \in I$ and therefore $Y^{t,x,\alpha} \in \mathcal{G}$. Put $\tilde{\alpha}_s = \alpha_s \mathbf{1}_{\{s \leq \vee_{i\in I}\theta_i\}}$, then $Y^{t,x,\tilde{\alpha}} \in \mathcal{G}$ and $\Lambda_{t,x}(\mathcal{G}) \leq \frac{1}{2}\int_t^T |\tilde{\alpha}_s|^2 ds \leq v^{I,A}(t,x) + \eta$. Letting η to 0 the proof is complete. $\diamond$

We now give some results concerning the mixed optimal single stopping problem (4.6), and its links with the following variational inequality with obstacle ϕ,

$$\begin{cases} \max\left(-\dfrac{\partial v}{\partial t} + \dfrac{1}{2}|\sigma^T Dv|^2 - b\cdot Dv, v-\phi\right) = 0 \text{ in } [0,T)\times\mathcal{O} \\ \dfrac{\partial v}{\partial \gamma} = 0 \text{ in } [0,T)\times\partial\mathcal{O},\ v(T)=\phi(T) \text{ on } \overline{\mathcal{O}} \end{cases} \tag{4.8}$$

Proposition 4.3. *Let $\phi : [0,T]\times\overline{\mathcal{O}}$ be a measurable, bounded, real valued function then the function $v[\phi_*]$ is the minimal lsc viscosity supersolution of the variational inequality (4.8).*

The proof is similar and even simpler than the proof of Proposition 5.8 given in Appendix C. Let us remark that this result is well known for deterministic systems with Lipschitz coefficients in $\mathbb{R}^n$ (see Barles and Perthame [8]). The main difficulty in the present case is to prove the minimality of $v[\phi_*]$. This point is the consequence of a strong comparison result for equation (4.8) when the obstacle is bounded and continuous on $[0,T]\times\overline{\mathcal{O}}$. The proof of this strong comparison result, which is highly technical, is detailed in Appendix D.

Step 4. Passing to the limit: let us now prove the following lemma

Lemma 4.4. *For each finite non-empty subset I of $\mathbb{N}$ and for each $A>0$, one has $\liminf_* v_\varepsilon^{I,A} \geq v[\phi^{I,A}]$.*

Proof: The result is established by induction on the cardinal of I. If $I=\{i\}$ for some i in $\mathbb{N}$, then $\liminf_* v_\varepsilon^{\{i\},A}$ is a viscosity supersolution of (4.8) with $\phi=\phi^{\{i\},A}$. By Proposition 4.3, $v[\phi^{\{i\},A}]$ is the minimal viscosity supersolution of the same equation, hence the proof is complete.

Suppose now that I has N elements with $N\geq 2$ and that the lemma holds for any subset J of $\mathbb{N}^*$ containing $N-1$ elements. Then by using the induction hypothesis on formula (4.7), one has $\liminf_* \phi_\varepsilon^{I,A} = \tilde{\phi}^{I,A} \geq \phi^{I,A}$. By a stability result $\liminf_* v_\varepsilon^{I,A}$ is a viscosity supersolution of (4.8) with obstacle $\tilde{\phi}^{I,A}$. By Proposition 4.3, as $\tilde{\phi}^{I,A}$ is lsc, the minimal viscosity supersolution of this equation is $v[\tilde{\phi}^{I,A}]$. Now as $\tilde{\phi}^{I,A}\geq\phi^{I,A}$ one clearly has $v[\tilde{\phi}^{I,A}]\geq v[\phi^{I,A}]$.

Finally we have $\liminf_* v_\varepsilon^{I,A} \geq v[\tilde{\phi}^{I,A}] \geq v[\phi^{I,A}]$ which completes the proof of the lemma. ⋄

Conclusion: for all finite nonempty $I\subset\mathbb{N}$, $A>0$ and $(t,x)\in[0,T]\times\overline{\mathcal{O}}$ one has, by (4.4), $u_\varepsilon^I(t,x)\leq u_\varepsilon^{I,A}(t,x)$, hence, by Lemma 4.4 and Proposition 4.2, $\liminf_{\varepsilon\to 0} v_\varepsilon^I(t,x) \geq \liminf_* v_\varepsilon^{I,A}(t,x) \geq v[\phi^{I,A}](t,x)\geq \Lambda_{t,x}(\mathcal{G})\wedge A$. The proof of (A3) is complete. ⋄

5. Appendix

Appendix A: the test-function

Lemma 5.1. *We assume that* γ *and* $\mathcal{O}$ *satisfy (1.2) and (1.4). Then, for all* $\varepsilon, \rho > 0$*, there exists* $\psi_{\varepsilon,\rho} \in C^1(\overline{\mathcal{O}} \times \overline{\mathcal{O}}, \mathbb{R})$ *such that,*

$$(\psi i) \qquad \forall x, y \in \overline{\mathcal{O}}, \quad \frac{1}{2}\frac{|x-y|^2}{\varepsilon^2} - K\frac{\rho^2}{\varepsilon^2} \leq \psi_{\varepsilon,\rho}(x,y) \leq K(\frac{|x-y|^2}{\varepsilon^2} + \frac{\rho^2}{\varepsilon^2}),$$

$$(\psi ii) \qquad \begin{cases} \forall x, y \in \overline{\mathcal{O}}, \quad |D_x\psi_{\varepsilon,\rho}(x,y) + D_y\psi_{\varepsilon,\rho}(x,y)| \leq K(\frac{|x-y|^2}{\varepsilon^2} + \frac{\rho^2}{\varepsilon^2}), \\ \forall x, y \in \overline{\mathcal{O}}, \quad |D_x\psi_{\varepsilon,\rho}(x,y)| + |D_y\psi_{\varepsilon,\rho}(x,y)| \leq K(\frac{|x-y|}{\varepsilon^2} + \frac{\rho^2}{\varepsilon^2}), \end{cases}$$

$$(\psi iii) \qquad \begin{cases} \forall x \in \partial\mathcal{O}, y \in \overline{\mathcal{O}}, \quad D_x\psi_{\varepsilon,\rho}(x,y)\cdot\gamma(x) > 0, \\ \forall y \in \partial\mathcal{O}, x \in \overline{\mathcal{O}}, \quad D_y\psi_{\varepsilon,\rho}(x,y)\cdot\gamma(y) > 0. \end{cases}$$

for some constant K *depending only on* $\mathcal{O}$, $||\gamma||_\infty$ $||\gamma||_{Lip}$ *and* c_0.

We use ideas from [5].

Proof: We first define the Lipschitz continuous $\mathbb{R}^d$-valued function μ on $\partial\mathcal{O}$ by $\mu(x) = \dfrac{\gamma(x)}{\gamma(x)\cdot n(x)}$ as well as a smooth approximation $(\mu_\rho)_{\rho>0}$ in $C^1(\mathbb{R}^d, \mathbb{R}^d)$ such that $\|\mu_\rho - \mu\|_\infty \leq \rho$ and $\|\mu_\rho\| + \|D\mu_\rho\| \leq K_1$ for some $K_1 > 0$ independent of ρ.

Then we set,

$$\phi_{\varepsilon,\rho}(x,y) \quad = \quad \frac{|x-y|^2}{\varepsilon^2} + 2\frac{(x-y)}{\varepsilon}\cdot\mu_\rho(\frac{x+y}{2})\frac{(d(x)-d(y))}{\varepsilon} + A\frac{(d(x)-d(y))^2}{\varepsilon^2}.$$

We can choose the constant $A > 0$ large enough in order to get, for some constant $K_2 > 0$ and for all ε, $\rho > 0$,

$$(\phi i) \qquad \forall x, y \in \overline{\mathcal{O}}, \quad \frac{1}{2}\frac{|x-y|^2}{\varepsilon^2} \leq \phi_{\varepsilon,\rho}(x,y) \leq K_2\frac{|x-y|^2}{\varepsilon^2},$$

$$(\phi ii) \qquad \begin{cases} \forall x, y \in \overline{\mathcal{O}}, \quad |D_x\phi_{\varepsilon,\rho}(x,y) + D_y\phi_{\varepsilon,\rho}(x,y)| \leq K_2\frac{|x-y|^2}{\varepsilon^2}, \\ \forall x, y \in \overline{\mathcal{O}}, \quad |D_x\phi_{\varepsilon,\rho}(x,y)| + |D_y\phi_{\varepsilon,\rho}(x,y)| \leq K_2\frac{|x-y|}{\varepsilon^2}, \end{cases}$$

$$(\phi iii) \qquad \begin{cases} \forall x \in \partial\mathcal{O}, y \in \overline{\mathcal{O}}, \quad D_x\phi_{\varepsilon,\rho}(x,y)\cdot\gamma(x) \geq -K_2[\frac{|x-y|^2}{\varepsilon^2} + \frac{\rho^2}{\varepsilon^2}], \\ \forall y \in \partial\mathcal{O}, x \in \overline{\mathcal{O}}, \quad D_y\phi_{\varepsilon,\rho}(x,y)\cdot\gamma(y) \geq -K_2[\frac{|x-y|^2}{\varepsilon^2} + \frac{\rho^2}{\varepsilon^2}]. \end{cases}$$

Indeed (ϕi) comes from a simple application of Cauchy-Schwarz inequality, and from the fact that d is Lipschitz continuous. Now for all $U = (u,v) \in \mathbb{R}^d \times \mathbb{R}^d$, we have

$$\begin{aligned} D\phi_{\varepsilon,\rho}(x,y).U \quad = \quad & \frac{2(x-y)\cdot(u-v)}{\varepsilon^2} + 2\frac{u-v}{\varepsilon^2}\cdot\mu_\rho(\frac{x+y}{2})(d(x)-d(y)) - 2\frac{x-y}{\varepsilon^2}\cdot\mu_\rho(\frac{x+y}{2})(n(x)u - n(y)v) \\ & + \frac{(x-y)}{\varepsilon^2}\cdot D\mu_\rho(\frac{x+y}{2})(u+v)(d(x)-d(y)) - \frac{2A}{\varepsilon^2}(d(x)-d(y))(n(x)u - n(y)v). \end{aligned}$$

Taking $U = (u,u)$, as both d and n are Lipschitz continuous, and using Cauchy-Schwarz inequality we obtain straightforwardly the first inequality in (ϕii). The second inequality in (ϕii) is clear.

Let us now prove (ϕiii). By symmetry, there is only one inequality to prove. Take $x \in \partial\mathcal{O}$ and $y \in \overline{\mathcal{O}}$, and take $U = (\gamma(x), 0)$, and recall that $\gamma(x).n(x) \geq c_0 > 0$. The sum of all the terms that have $(d(x) - d(y)) = -d(y)$ can be made non-negative for A large enough. The remaining term is, taking $\dfrac{2(x-y)}{\varepsilon^2}$ in factor,

$$\gamma(x) - \mu_\rho(\frac{x+y}{2})n(x) \cdot \gamma(x) = \left(\mu(x) - \mu_\rho(\frac{x+y}{2})\right)(n(x) \cdot \gamma(x)).$$

Writing $|\mu(x) - \mu_\rho(\frac{x+y}{2})| \leq |(\mu(x) - \mu(\frac{x+y}{2})| + |\mu(\frac{x+y}{2}) - \mu_\rho(\frac{x+y}{2})| \leq K|\frac{x-y}{2}| + \rho$, we have completed the proof of (ϕiii).

Finally, we set, for $x, y \in \overline{\mathcal{O}}$,

$$\psi_{\varepsilon,\rho}(x,y) = e^{C(2\|d\|_\infty - d(x) - d(y))}\phi_{\varepsilon,\rho}(x,y) - B\frac{\rho^2}{\varepsilon^2}(d(x) + d(y)). \tag{5.1}$$

By choosing B, then C large enough, we obtain the desired result. $\diamond$

Appendix B. A deterministic reflection problem

In this section, we suppose that b and σ satisfy (1.5). We consider for each fixed $\alpha \in L^2 = L^2(0,T;\mathbb{R}^m)$ and for each fixed $(t,x) \in [0,T] \times \overline{\mathcal{O}}$ the deterministic equation with oblique reflection

$$\begin{cases} dY_s = (b(s,Y_s) - \sigma(s,Y_s)\alpha_s)\,ds - dz_s, \quad Y_s \in \overline{\mathcal{O}} \quad (\forall s \in [t,T]), \quad Y_s = x \quad (\forall s \in [0,t]) \\ dz_s = \mathbf{1}_{\partial\mathcal{O}}(Y_s)\gamma(Y_s)d|z|_s \quad (\forall s \in [t,T]), \text{ and } z_s = 0 \quad (\forall s \in [0,t]). \end{cases} \tag{5.2}$$

A solution of equation (5.2) is a couple (Y,z) of continuous functions defined on $[0,T]$ with values in $\mathbb{R}^d$ such that z has bounded variations, and $|z|_s$ denotes the total variation of z on the interval $[0,s]$.

Theorem 5.2. *Assume (1.2)-(1.4)-(1.5) and let $\alpha \in L^2$. Then,*

(1) *there exists a unique solution $(Y^{t,x,\alpha}, z^{t,x,\alpha})$ of (5.2),*
(2) *for each s, the function $(t,x) \mapsto Y_s^{t,x,\alpha}$ is continuous and $|Y_s^{t,x,\alpha} - Y_s^{t',x',\alpha}|^2 \leq K(|x - x'|^2 + |t-t'|^{1/4})$,*
(3) *for each (t,x), $Y_\cdot^{t,x,\alpha}$ is $\mathcal{C}^{0,1/2}$: $|Y_s^{t,x,\alpha} - Y_{s'}^{t,x,\alpha}| \leq K|s-s'|^{1/2}$, $(\forall s,s' \in [0,T])$.*
(4) *Assumption (A3) holds true: $\forall$ $(\alpha^n)_n$, α in L^2, if $\alpha^n \rightharpoonup \alpha$ weakly in L^2, then $\|Y_\cdot^{t,x,\alpha_n} - Y_\cdot^{t,x,\alpha}\|_{\mathcal{X}} \to 0$*

where the constant K in (2) *and* (3) *depends only on $\mathcal{O}$, γ, c_0 the Lipschitz constant K_T of b, σ and $\|\alpha\|_{L^2}$.*

Proof of (1): For the sake of completeness, and as the hypothesis on the coefficient $c = b - \sigma\alpha$ are slightly more general than in [38] or [21], we present a complete proof. To that end, we use the Skorokhod problem. More precisely, fix $(t,x) \in [0,T] \times \overline{\mathcal{O}}$, and $\alpha \in L^2$ and define $c_t(x) = b(t,x) - \sigma(t,x)\alpha_t$ for all $(t,x) \in [0,T] \times \overline{\mathcal{O}}$.

By [38], Theorem 2.2 p.521 for each $X \in \mathcal{C}([t,T];\overline{\mathcal{O}})$, there exists at least one solution (Y,z) of the following Skorokhod problem:

$$\begin{cases} Y_s = x + \displaystyle\int_t^s c_u(X_u)du - z_s, \quad Y \in \mathcal{C}([t,T];\overline{\mathcal{O}}), \\ z_s = \displaystyle\int_t^s \mathbf{1}_{\partial\mathcal{O}}(Y_u)\gamma(Y_u)d|z|_u, \quad z \in \mathcal{C}_{bv}([t,T];\overline{\mathcal{O}}). \end{cases} \tag{5.3}$$

We next show that the solution of (5.3) is unique and then we prove the existence and uniqueness for the solutions to equation (5.2) by a fixed point argument. Note that, as in view of the first equation of (5.3) z is completly defined by X and Y, it is enough to prove the uniqueness for Y only.

We define the binary relation $\mathcal{S}$ on $\mathcal{C}([t,T];\overline{\mathcal{O}})$ in the following way: for all $X, Y \in \mathcal{C}([t,T];\overline{\mathcal{O}})$, $Y \,\mathcal{S}\, X$ if and only if there exists $z \in \mathcal{C}_{bv}([t,T];\overline{\mathcal{O}})$ such that (Y,z) is the solution of (5.3).

Lemma 5.3. *Let $X, X', Y, Y' \in \mathcal{C}([t,T];\overline{\mathcal{O}})$ and suppose $Y \,\mathcal{S}\, X$ and $Y' \,\mathcal{S}\, X'$. Then, there exists $\eta > 0$, depending only on $\overline{\mathcal{O}}$, γ, c_0 and a_T (and independent of t), such that for all $u \in [t, t+\eta] \cap [0,T]$, one has:* $|Y_u - Y'_u| \le \frac{1}{2}|X_u - X'_u|$.

Proof: We use the function $\psi_{\varepsilon,\rho}$ defined in Lemma 5.1 with $\varepsilon = 1$, and fix $s \in [t,T]$ and we put $f_x = D_x\psi_{1,\rho}$, $f_y = D_y\psi_{1,\rho}$. We have

$$\psi_{1,\rho}(Y_s, Y'_s) = \psi_{1,\rho}(x,x) + \int_t^s f_x(Y_u,Y'_u)c_u(X_u)du + \int_t^s f_y(Y_u,Y'_u)c_u(X'_u)du$$

$$- \int_t^s f_x(Y_u,Y'_u)\mathbf{1}_{\partial\mathcal{O}}(Y_u)\gamma(Y_u)d|z|_u - \int_t^s f_y(Y_u,Y'_u)\mathbf{1}_{\partial\mathcal{O}}(Y'_u)\gamma(Y'_u)d|z'|_u.$$

By (ψiii), the two last integrals of the right hand side of the above inequality are non positive. Write the first term of the right hand side of the previous inequality as

$$\int_t^s (f_x+f_y)(Y_u,Y'_u)c_u(X_u)du + \int_t^s f_y(Y_u,Y'_u)(c_u(X'_u) - c_u(X_u))du.$$

Put $a_u = 1 + |\alpha_u|$, by using (ψi), (ψii) and as $|c_u(X_u)| \le K a_u$ and $|c_u(X_u) - c_u(X'_u)| \le K a_u |X_u - X'_u|$,

$$\frac{1}{2}|Y_s - Y'_s|^2 \le K\left(2\rho^2 + \int_t^s (|Y_u - Y'_u|^2 + \rho^2)a_u du + \int_t^s (|Y_u - Y'_u| + \rho^2)|X_u - X'_u| a_u du\right).$$

This equality holds independently of $\rho > 0$ and its right-hand term is nondecreasing with s therefore, by letting ρ to 0 we have for all $s \in [t,T]$, and writing for $g \in \mathcal{X}$, $|g|_{[t,s]} = \sup\{|g(u)|, u \in [t,s]\}$,

$$|Y - Y'|^2_{[t,s]} \le K\left(\int_t^s a_u du\right)\left(|Y - Y'|^2_{[t,s]} + |Y - Y'|_{[t,s]}|X - X'|_{[t,s]}\right).$$

Now, by using Cauchy-Schwarz inequality, we have for $|t-s|$ small enough:
$|Y - Y'|_{[t,s]} \le \dfrac{2K|s-t|^{\frac{1}{2}}\|a\|_{L^2}}{1 - 2K|s-t|^{\frac{1}{2}}\|a\|_{L^2}}|X - X'|_{[t,s]}$, and we can chose $\eta > 0$ independently of t such that

$$\sup_{u \in [t,t+\eta]\cap[0,T]} |Y_u - Y'_u| \le \frac{1}{2} \sup_{u \in [t,t+\eta]\cap[0,T]} |X_u - X'_u|,$$

which completes the proof of the lemma. $\diamond$

Lemma 5.3 shows first that for each X there exists a unique Y such that $X \,\mathcal{S}\, Y$. Changing notation we have proved that $\mathcal{S} : X \mapsto Y$ is a map. Lemma 5.3 shows also that there exists η such that for all $t \in [0,T]$, $\mathcal{S}$ contracts $\mathcal{C}([t,t+\eta] \cap [0,T];\overline{\mathcal{O}})$ onto itself. This gives existence and uniqueness for (5.2) by a fixed point argument hence it proves assertion (1) of the theorem. Stricly speaking, this fixed point argument holds for a fixed $\alpha \in \mathcal{L}([t,T];\mathbb{R}^m)$. Lemma 5.4 below shows in particular that $Y^{t,x,\alpha}$ is uniquely comatible with almost everywhere equivalence of α in L^2.

Proof of (2): Let us first establish the following lemma.

Lemma 5.4. *Fix $\alpha, \alpha' \in L^2$ and $t, t' \in [0,T]$ with $t' \leq t$, x, $x' \in \overline{\mathcal{O}}$ and define $Y = Y^{t,x,\alpha}$ and $Y' = Y^{t,x',\alpha'}$. Then, there exists a constant $K > 0$ (which only depends on $\|\alpha\|_{L^2}$, c_0, $\|\gamma\|_{Lip}$, K_T and $\mathcal{O}$) such that, for all $\rho > 0$ and for all $s \in [t,T]$, we have*

$$|Y_s - Y'_s|^2 \leq K\left(g_\rho(s) + \int_t^s g_\rho(u)\cdot(1+|\alpha_u|+|\alpha'_u|)du\right)$$

where $g_\rho(s) = \rho^2 + |x-x'|^2 + |t-t'|^{1/2} + \left|\int_t^s D_x\psi_{1,\rho}(Y_u, Y'_u)\sigma(u, Y_u)(\alpha_u - \alpha'_u)du\right|.$

Proof: We put for $(s,y) \in [0,T]\times\overline{\mathcal{O}}$, $c_s(y) = b(s,y) - \sigma(s,y)\alpha_s$, $c'_s(y) = b(s,y) - \sigma(s,y)\alpha'_s$, and $a_s = 1+|\alpha_s|$, $a'_s = 1+|\alpha'_s|$. Note that there exists a constant K such that for all $(s,y) \in [0,T]\times\overline{\mathcal{O}}$, one has $|c_s(y)| \leq Ka_s$ and $|c'_s(y)| \leq Ka'_s$. Define as before $f_x = D_x\psi_{1,\rho}$ and $f_y = D_y\psi_{1,\rho}$. Recall also that the function $\psi_{1,\rho}$ is a continuous function from $\mathbb{R}^d\times\mathbb{R}^d$ to $\mathbb{R}^d$ which is bounded on $\overline{\mathcal{O}}\times\overline{\mathcal{O}}$ independently of $\rho \in (0,1)$, and write

$$\psi_{1,\rho}(Y_s, Y'_s) = \psi_{1,\rho}(x,x') + \int_{t'}^t f_y(Y_u,Y'_u)c'_u(Y'_u)du + \int_t^s (f_x+f_y)(Y_u,Y'_u)c_u(X_u)du$$

$$+\int_t^s f_y(Y_u,Y'_u)(c'_u(Y'_u) - c_u(Y_u))du - \int_{t'}^s f_x(Y_u,Y'_u)\mathbf{1}_{\partial\mathcal{O}}(Y_u)\gamma(Y_u)d|z|_u - \int_{t'}^s f_y(Y_u,Y'_u)\mathbf{1}_{\partial\mathcal{O}}(Y'_u)\gamma(Y'_u)d|z'|_u$$

We then follow similar calculations as Lemma 5.3.

By Cauchy-Schwarz inequality, the first integral can be majorated by $K|t'-t|^{1/2}\|a'\|_{L^2}$, the second integral is smaller than $K(\rho^2(t-s) + \int_t^s |Y_u - Y'_u|^2 a_u du)$. For the third integral write $c'_u(Y'_u) - c_u(Y_u) = b(u,Y_u) - b(u,Y'_u) - (\sigma(u,Y_u) - \sigma(u,Y'_u))\alpha'_u + \sigma(u,Y_u)(\alpha_u - \alpha'_u)$ and we use (ψii), and eventually we use (ψiii) in order to estimate the to last integrals. Hence we have

$$\begin{aligned}|Y_s - Y'_s|^2 \quad \leq \quad & K(|x-x'|^2 + (t-t')^{1/2}\|a'\|^2_{L^2} + \rho^2) + K\int_t^s |Y_u - Y'_u|^2 a_u du + K\rho^2(s-t)^{1/2}\|a'\|_{L^2} \\ & + K\int_t^s |Y_u - Y'_u|^2 a'_u du + \left|\int_t^s D_y\psi_{1,\rho}(Y_u,Y'_u)\cdot\sigma(u,Y_u)(\alpha_u - \alpha'_u)du\right|\end{aligned}$$

which implies that

$$|Y_s - Y'_s|^2 \leq Kg_\rho(s) + K\int_t^s |Y_u - Y'_u|^2(1+|\alpha_u|+|\alpha'_u|)du$$

for some positive constants K. By Gronwall's lemma the proof is complete. $\diamond$

Fix $\alpha \in L^2$ and $x, x' \in \overline{\mathcal{O}}$, and apply Lemma 5.4 to $Y = Y^{t,x,\alpha}$ and to $Y' = Y^{t',x',\alpha}$. We have, have $g_\rho(s) = \rho^2 + |x-x'|^2 + |t-t'|^{1/2}$, which gives $|Y_s - Y'_s|^2 \leq K(\rho^2 + |x-x'|^2 + |t-t'|^{1/2})$. Letting ρ to 0, we have obtained the desired result.

Fix $s_0 \in [t,T]$ and $\alpha \in L^2$. Consider $Y : s \mapsto Y^{t,x,\alpha}_s$ and $Y' : s \mapsto Y^{t,x,\alpha}_{s_0}$.

Proof of (3): Let us first prove that for each $(t,x) \in [0,T]\times\overline{\mathcal{O}}$ and for each $s \in [t,T]$ one has

$$\sup_{u\in[s,t]} |Y^{t,x,\alpha}_u - x| \leq K\sqrt{t-s}. \tag{5.4}$$

Indeed, define Y and Y' by $Y_u = Y^{t,x,\alpha}_u$ and $Y'_u = x$ for $u \in [t,T]$. Put $c(u,y) = b(u,y) - \sigma(s,y)\alpha_s$ and $c'(u,y) = 0$ for $(u,y) \in [t,T]\times\overline{\mathcal{O}}$. The same computation as in Lemma 5.4 gives, for all $s' \in [t,s]$,

$$\psi_{1,\rho}(Y_{s'}, x) \leq \int_t^{s'} D_x\psi_{1,\rho}(Y_u, x)c(u,Y_u)du \leq \int_t^s K|Y_u - x|(1+|\alpha_u|)du,$$

which gives, using Cauchy-Schwarz inequality

$$|Y_{s'} - x|^2 \leq K \left(\sup_{u \in [t,s]} |Y_u - x| \right) \sqrt{t-s}\,(1 + \|\alpha\|_{L^2})\,.$$

Passing to the supremum over $s' \in [t,s]$, we obtain $\sup_{u\in[t,s]} |Y_u - x| \leq K\sqrt{t-s}$.

Fix now $(t,x) \in [0,T] \times \overline{\mathcal{O}}$ and $s, s' \in [t,T]$, with $s' \leq s$. Put $x' = Y_{s'}^{t,x,\alpha}$. By the previous result we have $|Y_s^{s',x'} - x'| \leq K\sqrt{s-s'}$, and by the uniqueness result we have the flow property $Y_s^{t,x} = Y_s^{s',x',\alpha}$, hence we have proved that $|Y_s^{t,x,\alpha} - Y_{s'}^{t,x,\alpha}| \leq K\sqrt{s-s'}$.

Proof of (4): We apply Lemma 5.4 to $Y = Y^{t,x,\alpha}$ and $Y^n = Y^{t,x,\alpha^n}$. Then g_ρ^n is given for all $s \in [t,T]$ by

$$g_\rho^n(s) = \rho^2 + \int_t^s f_\rho(u, Y_u, Y_u^n)(\alpha_u - \alpha_u^n)du.$$

where $f_\rho(t,y,y') = D_x\psi_{1,\rho}(y,y')\sigma(t,y)$ is continuous on $[0,T] \times \overline{\mathcal{O}} \times \overline{\mathcal{O}}$.
We first prove that a subsequence of g_ρ^n converges pointwise to ρ^2 as n goes to ∞. We remark, by assertion (3) of Theorem 5.2, that the sequence (Y^n) is bounded in $\mathcal{C}^{0,1/2}([0,T];\overline{\mathcal{O}})$ and therefore is relatively compact. Let $\overline{Y}$ be one of its limit in X. Let us prove that this limit is Y. Extracting a subsequence if necessary, one can suppose that the sequence $(Y^{n_p})_p$ converges to $\overline{Y}$. We write

$$g_\rho^{n_p}(s) = \rho^2 + \int_t^s (f_\rho(u, Y_u, Y_u^{n_p}) - f_\rho(u, Y_u, \overline{Y}_u))(\alpha_u - \alpha_u^{n_p})du + \int_t^s f_\rho(u, Y_u, \overline{Y}_u)\cdot(\alpha_u - \alpha_u^{n_p})du.$$

The first integral converges to 0 as p goes to ∞ by Lebesgue's Theorem and the second integral converges to 0 by definition of the weak convergence of (α^n) to α. Now as (α^n) is bounded in L^2 there exists $K > 0$ (independent of n and ρ) such that for all $n \in \mathbb{N}$ and for all $s \in [0,T]$,

$$g_\rho^n(s) \leq \rho^2 + K.$$

It follows, applying again Lebesgue's Theorem in the inequality given by Lemma 5.4, that for all $\rho \in (0,1)$ and for all $s \in (t,T)$,

$$\lim_{p\to\infty} |Y_s - Y_s^{n_p}|^2 \leq \rho^2(1 + \int_t^s |\alpha_u|du),$$

Letting ρ to 0, we deduce that $(Y^{n_p})_p$ converges pointwise, and even uniformly to Y and by uniqueness of the limit we have $Y = \overline{Y}$. This implies that the whole sequence $(Y^n)_n$ converges uniformly to Y and the proof of (A3) is complete.

The proof of Theorem 5.2 is now complete. $\diamond$

Appendix C: discontinuous mixed single or multiple optimal stopping problems

In this appendix we first study a mixed optimal control-optimal stopping time problem and we prove that a particular value function is the maximal viscosity supersolution of a variational inequality. Then we prove a reduction result: the value function of a mixed optimal control-optimal *multiple* stopping problem can be writen as the value function of a mixed optimal control-optimal *single* stopping problem with a new reward defined recursively.

C.1. A deterministic mixed optimal control-optimal single stopping problem. We first study the following mixed optimal control-optimal single stopping problem. For each bounded borelian real valued function ψ defined on $[0,T]\times\overline{\mathcal{O}}$ and for each (t,x) in $[0,T]\times\overline{\mathcal{O}}$ define the value function $V[\psi](t,x)$ by

$$V[\psi](t,x)=\inf_{\alpha\in L^2}\sup_{\theta\in[t,T]}\left\{\frac{1}{2}\int_t^\theta|\alpha_s|^2ds+\psi(\theta,Y_\theta^{t,x,\alpha})\right\}, \tag{5.5}$$

where $Y^{t,x,\alpha}$ is the unique solution of (1.7).

When ψ is upper-semicontinuous (usc), we show that this value function is characterized as the maximal viscosity subsolution of the following equation

$$\begin{cases}\min\left(-\dfrac{\partial V}{\partial t}+\dfrac{1}{2}|\sigma^T DV|^2-b\cdot DV, V-\psi\right)=0 \text{ in } [0,T)\times\mathcal{O}\\ \dfrac{\partial V}{\partial\gamma}=0 \text{ in } [0,T)\times\partial\mathcal{O},\ V(T)=\psi(T) \text{ on } \overline{\mathcal{O}}\end{cases} \tag{5.6}$$

The proof follows different results of Barles and Perthame [8]. We adapt them here to our context.

Lemma 5.5. *$V[\psi^*]$ is usc and $V[\psi_*]$ is lsc. In particular, if ψ is continuous, $V[\psi]$ is continuous.*

Proof: *Step 1:* Suppose, by contradiction, that $V[\psi^*]$ is not usc. Then there exist $(t,x)\in[0,T]\times\overline{\mathcal{O}}$, a sequence $(t_n,x_n)_n$ that converges to (t,x) and $\varepsilon>0$ such that, $V[\psi^*](t,x)+2\varepsilon\leq\liminf_{n\to\infty}V[\psi^*](t_n,x_n)$. Consider an ε-control $\overline{\alpha}$ for $V[\psi^*]$, for all $\theta\in[t,T]$,

$$\frac{1}{2}\int_t^\theta|\overline{\alpha}_s|^2ds+\psi^*(\theta,Y_\theta^{t,x,\overline{\alpha}})+\varepsilon\leq\liminf_{n\to\infty}V[\psi^*](t_n,x_n). \tag{5.7}$$

Now for each $n\in\mathbb{N}$ there exists $\theta_n\in[t_n,T]$ such that $V[\psi^*](t_n,x_n)\leq\frac{1}{2}\int_{t_n}^{\theta_n}|\overline{\alpha}_s|^2ds+\psi^*(\theta_n,Y_{\theta_n}^{t_n,x_n,\overline{\alpha}})$. Extracting a sequence if necessary, we have that (θ_n) converges to $\overline{\theta}\in[t,T]$. By the regularity of Y given by Appendix B, we obtain that

$$\lim_{n\to\infty}V[\psi^*](t_n,x_n)\leq\frac{1}{2}\int_t^{\overline{\theta}}|\overline{\alpha}_s|^2ds+\psi^*(\overline{\theta},Y_{\overline{\theta}}^{t,x,\overline{\alpha}}). \tag{5.8}$$

Using (5.7) with $\theta=\overline{\theta}$ and (5.8) we obtain $\varepsilon\leq 0$, which is the expected contradiction.

Step 2: Suppose by contradiction that $V[\psi_*]$ is not lsc. Then there exist $(t,x)\in[0,T]\times\overline{\mathcal{O}}$, a sequence $(t_n,x_n)_n$ that converges to (t,x) and $\varepsilon>0$ such that, $V[\psi_*](t,x)\geq\lim_{n\to\infty}V[\psi_*](t_n,x_n)+3\varepsilon$. For each n consider an ε-control $\alpha^n\in L^2$ for $V[\psi_*](t_n,x_n)$

$$V[\psi_*](t,x)\geq\lim_{n\to\infty}\sup_{\theta\in[t_n,T]}\frac{1}{2}\int_{t_n}^\theta|\alpha_s^n|^2ds+\psi_*(\theta,Y_\theta^{t_n,x_n,\alpha^n})+2\varepsilon \tag{5.9}$$

The sequence (α^n) is bounded in L^2 and extracting a subsequence if necessary, it converges to $\overline{\alpha}$ weakly in L^2.

Now, there exists $\overline{\theta}\in[t,T]$ such that

$$\frac{1}{2}\int_t^{\overline{\theta}}|\overline{\alpha}_s|^2ds+\psi_*(\overline{\theta},Y_{\overline{\theta}}^{t,x,\overline{\alpha}})+\varepsilon\geq V[\psi_*](t,x), \tag{5.10}$$

For each $n\in\mathbb{N}$ define $\theta_n=t_n\wedge\overline{\theta}$. One has $\theta_n\in[t_n,T]$ and as $t_n\to t$ one has $\theta^n\to\overline{\theta}$. By (5.9) and (5.10) we obtain $\varepsilon\geq 2\varepsilon$ which provides the expected contradiction. $\diamond$

Lemma 5.6. *$V[\psi^*]$ (resp. $V[\psi_*]$) is a viscosity subsolution (resp. supersolution) of (5.6) with obstacle ψ. In particular, if ψ is continuous, $V[\psi]$ is a continuous solution of (5.6) with obstacle ψ.*

For completeness let us recall the definition of a viscosity subsolution and supersolution of equation (5.6). For simplicity we define $H(D\varphi)(t,x) = \frac{1}{2}|\sigma^T D\varphi(t,x)|^2 - b \cdot D\varphi(t,x)$

Definition 1. *An usc locally bounded function v defined on $[0,T]\times\overline{\mathcal{O}}$ is a viscosity subsolution of equation (5.6) if and only if*

$\forall \varphi \in \mathcal{C}^1([0,T]\times\overline{\mathcal{O}})$, if $(t_0,x_0)\in[0,T]\times\overline{\mathcal{O}}$ is a local minimum of $v-\varphi$, then

1. if $(t_0,x_0)\in[0,T]\times\mathcal{O}$, $\min\left(-\dfrac{\partial\varphi}{\partial t}+H(D\varphi), v-\psi^*\right)(t_0,x_0)\le 0,$

2. if $(t_0,x_0)\in[0,T]\times\partial\mathcal{O}$, $\min\left(-\dfrac{\partial\varphi}{\partial t}+H(D\varphi), v-\psi^*, \dfrac{\partial\varphi}{\partial\gamma}\right)(t_0,x_0)\le 0.$

A lsc locally bounded function u defined on $[0,T]\times\overline{\mathcal{O}}$ is a viscosity supersolution of equation (5.6) if and only if

$\forall \varphi \in \mathcal{C}^1([0,T]\times\overline{\mathcal{O}})$, if $(t_0,x_0)\in[0,T]\times\overline{\mathcal{O}}$ is a local maximum of $u-\varphi$, then

1. if $(t_0,x_0)\in[0,T)\times\mathcal{O}$, $\min\left(-\dfrac{\partial\varphi}{\partial t}+H(D\varphi), u-\psi_*\right)(t_0,x_0)\ge 0,$

2. if $(t_0,x_0)\in\{T\}\times\mathcal{O}$, $(u-\psi_*)(t_0,x_0)\ge 0,$

3. if $(t_0,x_0)\in[0,T)\times\partial\mathcal{O}$, $\max\left(\min\left(-\dfrac{\partial\varphi}{\partial t}+H(D\varphi), u-\psi_*\right), \dfrac{\partial\varphi}{\partial\gamma}\right)(t_0,x_0)\ge 0,$

4. if $(t_0,x_0)\in\{T\}\times\partial\mathcal{O}$, $\max\left(u-\psi_*, \dfrac{\partial\varphi}{\partial\gamma}\right)(t_0,x_0)\ge 0.$

Proof: Let us first recall the Dynamic Programming Principle, which proof is well known in the deterministic case, even for a discontinuous reward. For each $(t,x)\in[0,T]\times\overline{\mathcal{O}}$ and for each $\tau\in[t,T]$ we have

$$V[\psi](t,x) = \inf_{\alpha\in L^2}\sup_{\theta\in[t,T]}\left\{\frac{1}{2}\int_t^{\theta\wedge\tau}|\alpha_s|^2ds + \psi(\theta, Y_\theta^{t,x,\alpha})\mathbf{1}_{\{\theta<\tau\}} + V[\psi](\tau, Y_\tau^{t,x,\alpha})\mathbf{1}_{\{\theta\ge\tau\}}\right\}.$$

Step 1: Let us first prove that $V[\psi^*]$ is a viscosity subsolution.

Let $\varphi\in\mathcal{C}^1([0,T]\times\overline{\mathcal{O}})$ and suppose that (t_0,x_0) is a local maximum of $V[\psi^*]-\varphi$. Without loss of generality we can suppose that $V[\psi^*](t_0,x_0)=\varphi(t_0,x_0)$, and we fix $r>0$ such that for all $(t,x)\in[0,T]\times\overline{\mathcal{O}}$ if $|t-t_0|<r$ and $|x-x_0|<r$ then $V[\psi^*](t,x)\le\varphi(t,x)$.

The only technical point consists in showing that:

if $\Big[V[\psi^*](t_0,x_0) > \psi^*(t_0,x_0)$ and $\left(\frac{\partial\varphi}{\partial\gamma}(t_0,x_0)>0 \text{if } x_0\in\partial\mathcal{O}\right)\Big]$, then $\left(-\frac{\partial\varphi}{\partial t}+H(D\phi)\right)(t_0,x_0)\le 0$.

Consider $\alpha\in L^2$ a constant control whose value will be fixed later, and fix $\overline{\theta}$ in $[t_0,T]$ which maximizes $\frac{1}{2}\int_{t_0}^{\theta}|\alpha|^2du+\psi^*(\theta,Y_\theta^{t_0,x_0,\alpha})$ on $[t_0,T]$. One has $\varphi(t_0,x_0)=V[\psi^*](t_0,x_0)\le\frac{1}{2}(\overline{\theta}-t_0)|\alpha|^2+\psi^*(\overline{\theta},Y_{\overline{\theta}}^{t,x,\alpha})$, and in particular, $\overline{\theta}>t_0$. Consider also $\tilde{\theta}=\inf\{s\in[t_0,T], |Y_s^{t_0,x_0,\alpha}-x_0|\ge r\}$. One has also $\tilde{\theta}>t_0$. For each τ such that $t_0<\tau<\overline{\theta}\wedge\tilde{\theta}\wedge t_0+r$, the Dynamic Programming Principle gives $\varphi(t_0,x_0)=V[\psi^*](t_0,x_0)\le\frac{1}{2}(\tau-t_0)|\alpha|^2+V[\psi^*](\tau,Y_\tau^{t_0,x_0,\alpha})\le\frac{1}{2}(\tau-t_0)|\alpha|^2+\varphi(\tau,Y_\tau^{t_0,x_0,\alpha})$. Substracting $\varphi(t_0,x_0)$, dividing by $-h=t_0-\tau<0$, and letting h to 0 we obtain,

$$-\frac{1}{2}|\alpha|^2-\frac{\partial\varphi}{\partial t}(t_0,x_0)-D\varphi(t_0,x_0)\cdot b(t_0,x_0)+D\varphi(t_0,x_0)\cdot\sigma(t_0,x_0)\alpha\le 0.$$

We chose now $\alpha = (-D\varphi \cdot \sigma)(t_0, x_0)$ and we obtain the desired inequality.

Step 2: Let us prove now that $V[\psi_*]$ is a viscosity supersolution.

Let $\varphi \in \mathcal{C}^1([0,T] \times \overline{\mathcal{O}})$ and suppose that (t_0, x_0) is a local minimum of $V[\psi_*] - \varphi$. Without loss of generality we can suppose that $V[\psi_*](t_0, x_0) = \varphi(t_0, x_0)$, and fix $r > 0$ such that $V[\psi^*](t,x) > \varphi(t,x)$ if $0 < |t - t_0| + |x - x_0| < r$ and $B(x_0, r) \subset \mathcal{O}$ if $x_0 \in \mathcal{O}$.
As clearly $V[\psi*](t_0, x_0) \geq \psi^*(t_0, x_0)$, we only need to show that:

if $\Big[(t_0, x_0) \in [0,T) \times \overline{\mathcal{O}}$ and $\Big(\frac{\partial \varphi}{\partial \gamma}(t_0, x_0) < 0$ if $x_0 \in \partial\mathcal{O}\Big)\Big]$, then $\Big(-\frac{\partial \varphi}{\partial t} + H(D\varphi)\Big)(t_0, x_0) \geq 0$.

By fixing r smaller if necessary, we can suppose that $\frac{\partial \varphi}{\partial \gamma}(t,x) < 0$ if $x_0 \in \partial \mathcal{O}$ and $|t - t_0| + |x - x_0| < r$.
Fix $\tau \in (t_0, T]$. By the Dynamic Programming Principle $V[\psi^*](t_0, x_0) \geq \inf_{\substack{\alpha \in L^2 \\ \|\alpha\|_{L^2} \leq K}} \left[\frac{1}{2}\int_{t_0}^{\tau} |\alpha_s|^2 ds + V[\psi_*](\tau, Y_\tau^{t_0, x_0, \alpha})\right]$,
where $K = V[\psi^*](t_0, x_0)$. By the regularity of Y with respect to α and τ, there exists $h > 0$ such that for all $\alpha \in L^2$ with $\|\alpha\|_{L^2} \leq K$, for all $\tau \in [t_0, t_0 + h]$ one has $|\tau - t_0| + |Y_\tau^{t_0, x_0, \alpha} - X_0| < r$.
Therefore, for each $\tau \in (t_0, t_0 + h]$ we have

$$\varphi(t_0, x_0) = V[\psi^*](t_0, x_0) \geq \inf_{\substack{\alpha \in L^2 \\ \|\alpha\|_{L^2} \leq K}} \left[\frac{1}{2}\int_{t_0}^{\tau} |\alpha_s|^2 ds + \varphi(\tau, Y_\tau^{t_0, x_0, \alpha})\right]. \tag{5.11}$$

Suppose by contradiction that there exists $\varepsilon > 0$ such that $\frac{\partial \varphi}{\partial t}(t_0, x_0) + bD\varphi(t_0, x_0) - \frac{1}{2}|\sigma^T D\varphi|^2(t_0, x_0) > \varepsilon$. Taking r smaller if necessary, we can suppose that this inequality holds for all $(t,x) \in [0,T] \times \overline{\mathcal{O}}$, such that $|t - t_0| + |x - x_0| < r$. Note that if $\tau \in [t_0, t_0 + h]$ then for all $\alpha \in L^2$ with $\|\alpha\|_{L^2} \leq K$ we have $\mathbf{1}_{\partial\mathcal{O}}(Y_\tau^{t_0, x_0, \alpha}) \frac{\partial \varphi}{\partial \gamma}(\tau, Y_\tau^{t_0, x_0, \alpha}) \leq 0$.

Noticing that $\inf_{\alpha \in \mathbb{R}^m} \frac{1}{2}|\alpha|^2 - D\varphi\sigma(t,x)\alpha = -\frac{1}{2}|\sigma^T D\varphi(t,x)|^2$, we have for $\tau \in (t_0, t_0 + h]$, and for all $\alpha \in L^2$ with $\|\alpha\|_{L^2} \leq K$
$\varphi(\tau, Y_\tau^{t_0, x_0, \alpha}) \geq \varphi(t_0, x_0) + \int_{t_0}^{\tau} (\frac{\partial \varphi}{\partial t} + D\varphi b - D\varphi\sigma\alpha)(u, Y_u^{t_0, x_0, \alpha}) du \geq \varphi(t_0, x_0) + \varepsilon(\tau - t_0) - \frac{1}{2}\int_{t_0}^{\tau} |\alpha_u|^2 du$. Now (5.11) gives

$$\varphi(t_0, x_0) \geq \varphi(t_0, x_0) + \varepsilon(\tau - t_0) > \varphi(t_0, x_0).$$

which provides the expected contradiction. $\diamond$

Lemma 5.7. *Let (ψ_n) be a nonincreasing sequence of continuous functions on $[0,T] \times \overline{\mathcal{O}}$ such that $\psi^* = \lim \downarrow \psi_n$. Then $V[\psi^*] = \lim \downarrow V[\psi_n]$.*

Proof: Suppose that $\psi_n \downarrow \psi^*$. Clearly, $V[\psi_n] \geq V[\psi^*]$ and the sequence $V[\psi_n]$ is nonincreasing hence we obtain $V[\psi^*] \leq \lim \downarrow V[\psi_n]$. Let us prove the second inequality. Fix $(t,x) \in [0,T] \times \overline{\mathcal{O}}$, and $\varepsilon > 0$. There exists $\alpha^* \in L^2$ and $\theta^* \in [t,T]$ such that for all $\theta \in [t,T]$ one has

$$V[\psi^*](t,x) + \varepsilon \geq \frac{1}{2}\int_t^{\theta^*} |\alpha_s^*|^2 ds + \psi^*(\theta^*, Y_{\theta^*}^{t,x,\alpha^*}) \geq \frac{1}{2}\int_t^{\theta} |\alpha_s^*|^2 ds + \psi^*(\theta, Y_\theta^{t,x,\alpha^*}). \tag{5.12}$$

Now, for each $n \in \mathbb{N}$, there exists $\theta_n \in [t,T]$ such that $V[\psi_n](t,x) \leq \frac{1}{2}\int_t^{\theta_n} |\alpha_s^*|^2 ds + \psi_n(\theta_n, Y_{\theta_n}^{t,x,\alpha^*})$. Extracting a sequence if necessary, we can suppose that θ_n tends to $\overline{\theta}$. Fix $p \in \mathbb{N}$. For each $n \geq p$, we have

$$V[\psi_n](t,x) \leq \frac{1}{2}\int_t^{\theta_n} |\alpha_s^*|^2 ds + \psi_p(\theta_n, Y_{\theta_n}^{t,x,\alpha^*}).$$

Letting n to ∞, $\lim_{n\to\infty} V[\psi_n](t,x) \leq \frac{1}{2}\int_t^{\overline{\theta}} |\alpha_s^*|^2 ds + \psi_p(\overline{\theta}, Y_{\overline{\theta}}^{t,x,\alpha^*})$. Now passing to the limit in p and using (5.12) we obtain $\lim_{n\to\infty} V[\psi_n](t,x) \leq V[\psi^*](t,x) + \varepsilon$. $\diamond$

Proposition 5.8. *$V[\psi^*]$ is the maximal usc viscosity subsolution of (5.6) with obstacle ψ.*

Proof: In view of Lemmas 5.5 and 5.6, the only point which is left to show is the maximality of the solution. Let v be a usc function which is a viscosity subsolution of (5.6) with obstacle ψ. Let ψ_n be a nonincreasing sequence of continuous functions on $[0,T]\times\overline{\mathcal{O}}$ such that $\psi^* = \lim \downarrow \psi_n$. Since $\psi^n \geq \psi^*$, v is also a viscosity subsolution to the variational inequality (5.6) with obstacle ψ^n. By Lemma 5.6, $V[\psi^n]$ is a continuous viscosity solution to the same equation, and $v \leq V[\psi^n]$ by Theorem 5.10. As by Lemma 5.7, $V[\psi^*] = \lim \downarrow V[\psi_n]$, we obtain $v \leq V[\psi^*]$. $\diamond$

C.2. Reduction of multiple stopping to single stopping problems. Let $(\psi_i)_{i\in\mathbb{N}}$ be a family of real valued bounded measurable functions defined on $[0,T]\times\overline{\mathcal{O}}$ and consider for each nonempty finite subset I of $\mathbb{N}$ and for each $(t,x)\in[0,T]\times\overline{\mathcal{O}}$ the mixed optimal control–optimal *multiple* stopping problem

$$v^I(t,x) = \inf_{\alpha\in L^2}\inf_{\theta_I\in[t,T]^N}\left\{\int_t^{\vee_{i\in I}\theta_i}|\alpha_s|^2 ds + \sum_{i\in I}\psi_i(\theta_i, Y_{\theta_i}^{t,x,\alpha})\right\}. \tag{5.13}$$

the value function of the following mixed optimal control–optimal *single* stopping problem

$$u^I(t,x) = \inf_{\alpha\in L^2}\inf_{\theta\in[t,T]}\left\{\frac{1}{2}\int_t^{\theta}|\alpha_s|^2 ds + \phi(\theta, Y_{\theta}^{t,x,\alpha})\right\}, \tag{5.14}$$

where the *new reward* is defined recursively by

$$\phi = \begin{cases} \psi_i & \text{if } I=\{i\} \text{ with } i\in\mathbb{N},\\ \min_{i\in I}\left\{\psi_i + v^{I\setminus\{i\},A}\right\} & \text{if } I \text{ contains 2 or more elements.}\end{cases} \tag{5.15}$$

Proposition 5.9. *For each finite nonempty subset I of $\mathbb{N}$ let v^I be defined by (5.13) and let u^I be defined by (5.14) and (5.15). Then for each $(t,x)\in[0,T]\times\overline{\mathcal{O}}$ one has $v^I(t,x) = u^I(t,x)$.*

Proof: Fix a nonempty finite subset I of $\mathbb{N}$ of cardinal N. When I contains only one element, there is nothing to prove. Suppose now that I has two or more elments. Let us prove first that for each $(t,x)\in[0,T]\times\overline{\mathcal{O}}$, $v^I(t,x)\leq u^I(t,x)$.
Fix $(t,x)\in[0,T]\times\overline{\mathcal{O}}$. Consider a partition $(A_j)_{j\in I}$ of $[t,T]^N$ such that for each $\theta_I\in[t,T]^N$ if $\theta_I\in A_j$ then one has $\theta_j = \wedge_{i\in I}\theta_i$.

Fix $\alpha\in L^2$ and $\theta_I\in[t,T]^N$, $\displaystyle\int_t^{\vee_{i\in I}\theta_i}|\alpha_s|^2 ds + \sum_{i\in I}\psi_i(\theta_i, Y_{\theta_i}^{t,x,\alpha}) =$

$$\sum_{j\in I}\mathbf{1}_{A_j}(\theta_I)\left\{\int_t^{\theta_j}|\alpha_s|^2 ds + \psi_j(\theta_j, Y_{\theta_j}^{t,x,\alpha}) + \int_{\theta_j}^{\vee_{i\in I\setminus\{j\}}\theta_i}|\alpha_s|^2 ds + \sum_{i\in I\setminus\{j\}}\psi_i(\theta_i, Y_{\theta_i}^{t,x,\alpha})\right\}.$$

Clearly, by uniqueness for equation (1.7), the second terme of the right-hand side can be minorated by $v^{I\setminus\{j\}}(\theta_j, Y_{\theta_j}^{t,x,\alpha})$, hence

$$\begin{aligned}\int_t^{\vee_{i\in I}\theta_i}|\alpha_s|^2 ds + \sum_{i\in I}\psi_i(\theta_i, Y_{\theta_i}^{t,x,\alpha}) &\geq \sum_{j\in J}\mathbf{1}_{A_j}(\theta_I)\left\{\int_t^{\theta_j}|\alpha_s|^2 ds + \psi_j(\theta_j, Y_{\theta_j}^{t,x,\alpha}) + v^{I\setminus\{j\}}(\theta_j, Y_{\theta_j}^{t,x,\alpha})\right\}\\ &\geq \sum_{j\in J}\mathbf{1}_{A_j}(\theta_I)\left\{\int_t^{\theta_j}|\alpha_s|^2 ds + \phi(\theta_j, Y_{\theta_j}^{t,x,\alpha})\right\} \geq u^I(t,x).\end{aligned}$$

Taking the infimum over $\alpha \in L^2$ and $\theta_I \in [t,T]^N$, we obtain $v^I(t,x) \geq u^I(t,x)$.

Let us now prove the reverse inequality. For simplicity, suppose first that there exist an optimal time $\theta^* \in [t,T]$ and an optimal control $\alpha^* \in L^2(t,T;\mathbb{R}^m)$ for $u^I(t,x)$, and for each $i \in I$ there exist $\tilde{\alpha}^{i*} \in L^2(\theta^*,T;\mathbb{R}^m)$ and $\tilde{\theta}^{i*} \in [\theta^*,T]^{N-1}$ that are optimal for $v^{I\backslash\{i\}}(\theta^*, Y_{\theta^*}^{t,x,\alpha^*})$. Define $\alpha^{i*} = \alpha^* \mathbf{1}_{[t,\theta^*)} + \tilde{\alpha}^{i*}\mathbf{1}_{[\theta^*,T)}$. Note that α^{i*} is optimal for $u^I(t,x)$, that $Y_{\theta^*}^{t,x,\alpha^*} = Y_{\theta^*}^{t,x,\alpha^{i*}}$ and that α^{i*} is also optimal for $v^{I\backslash\{i\}}(\theta^*, Y_{\theta^*}^{t,x,\alpha^*})$.

Let $(B_i)_{i\in I}$ be a partition of $[t,T]$ such that for each $s \in [t,T]$, if $s \in B_i$ then $\phi(s, Y_s^{t,x,\alpha^*}) = \psi_i(s, Y_s^{t,x,\alpha^*}) + v^{I\backslash\{i\}}(s, Y_s^{t,x,\alpha^*})$. We have

$$u^I(t,x) = \sum_{i\in I} \mathbf{1}_{B_i}(\theta^*)\left(\int_t^{\theta^*} |\alpha_s^*|^2 ds + \left\{\psi_i + v^{I\backslash\{i\}}\right\}(\theta^*, Y_{\theta^*}^{t,x,\alpha^*})\right)$$

$$= \sum_{i\in I}\left(\mathbf{1}_{B_i}(\theta^*)\left\{\int_t^{\theta^*} |\alpha_s^{i*}|^2 ds + \psi_i(\theta^*, Y_{\theta^*}^{t,x,\alpha^{i*}}) + \int_{\theta^*}^{\vee_{j\in I\backslash\{i\}}\tilde{\theta}_j^{i*}} |\alpha_s^{i*}|^2 ds + \sum_{j\in I\backslash\{i\}} \psi_j(\tilde{\theta}_j^{i*}, Y_{\tilde{\theta}_j^{i*}}^{t,x,\alpha^{i*}})\right\}\right)$$

For each i define $\theta^{i*} \in [t,T]^N$ by $\theta_i^{i*} = \theta_i^*$ and $\theta_j^{i*} = \tilde{\theta}_j^{i*}$ for $j \in I\backslash\{i\}$. One has now

$$u^I(t,x) = \sum_{i\in I}\mathbf{1}_{B_i}(\theta^*)\left\{\int_t^{\vee_{j\in I}\theta_j^{i*}} |\alpha_s^{i*}|^2 ds + \sum_{j\in I}\psi_j(\theta_j^{i*}, Y_{\theta_j^{i*}}^{t,x,\alpha^{i*}})\right\} \geq v^I(t,x).$$

In general, there is no optimal stopping control and stopping times, but for each $\varepsilon > 0$ on can find $\varepsilon/2$ optimal θ^* and α^* for $u^I(t,x)$ and $\varepsilon/2$ optimal $\tilde{\theta}_p^{i*}$ and $\tilde{\alpha}_p^{i*}$ for $v^{I\backslash\{i\}}(\theta^*, Y_{\theta^*}^{t,x,\alpha^*})$. Building α^{i*} and θ^{*i} as above we obtain that for each $\varepsilon > 0$, $u^I(t,x) + \varepsilon \geq v^I(t,x)$. $\diamond$

Appendix D: a strong comparison result

In this appendix we prove a strong comparison result for viscosity solutions to a first order variational inequality with Neumann boundary conditions and with continuous obstacle.

Theorem 5.10. *Assume (1.2) and (1.5)-(1.6) and let $\psi \in \mathcal{C}([0,T]\times\overline{\mathcal{O}};\mathbb{R})$. If u, $v : [0,T]\times\overline{\mathcal{O}} \to \mathbb{R}$ are respectively usc viscosity subsolution and lsc viscosity supersolution to*

$$\begin{cases} \min\left(-\dfrac{\partial w}{\partial t} + \dfrac{1}{2}|\sigma^T Dw|^2 - b\cdot Dw, w - \psi\right) = 0 \ \text{in } [0,T)\times\mathcal{O}, \\ \dfrac{\partial w}{\partial \gamma} = 0 \ \text{in } [0,T)\times\partial\mathcal{O}, \ w(T) = \psi(T) \ \text{on } \overline{\mathcal{O}}, \end{cases} \tag{5.16}$$

then $u \leq v$ on $[0,T]\times\overline{\mathcal{O}}$.

Note that the difficulty of proving this strong comparison result is double. First, we have to handle the Neuman condition, and the test function of Appendix A was built to that aim. Second, even though the equation is of first order and no Ishii lemma is needed, the quadratic term $|\sigma^T Dv|^2$ has to be taken with care.

Proof: In the following we denote for all $\varepsilon > 0$ by Ψ_ε the function $\Psi_\varepsilon := \psi_{\varepsilon,\varepsilon^2}$ where $\psi_{\varepsilon,\rho}$ is the test-function of Lemma 5.1.

Let u, v be respectively a bounded usc viscosity subsolution and a bounded lsc viscosity supersolution of (5.16). We first remark that the terminal condition holds in a stronger sense that the viscosity sense.

Proposition 5.11. *For all $x \in \overline{\mathcal{O}}$, $u(T,x) \le \psi(T,x)$.*

Proof: Fix $x_0 \in \overline{\mathcal{O}}$. For all $\varepsilon > 0$, put $\varphi_\varepsilon(t,x) = \Psi_\varepsilon(x,x_0) + \frac{T-t}{\varepsilon^2}$ and let $(t_\varepsilon, x_\varepsilon)$ be a global maximum of the usc function $u - \varphi_\varepsilon$. Then $u(T,x_0) - \varphi_\varepsilon(x_0,x_0) \le u(t_\varepsilon,x_\varepsilon) - \varphi_\varepsilon(x_\varepsilon,x_0)$ which implies, by Lemma 5.1, $\frac{1}{2}\frac{|x_\varepsilon - x_0|^2}{\varepsilon^2} - K\varepsilon^2 + \frac{T-t_\varepsilon}{\varepsilon^2} \le \varphi(t_\varepsilon,x_\varepsilon) \le u(t_\varepsilon,x_\varepsilon) - u(T,x_0) + \varphi_\varepsilon(T,x_0) \le 2\|u\| + K\varepsilon^2$. We deduce that t_ε and x_ε go respectively to T and x_0 as ε goes to 0 and, by the upper semicontinuity of u, that,

$$\lim_{\varepsilon\to 0} u(t_\varepsilon,x_\varepsilon) = u(T,x_0) \text{ and } \lim_{\varepsilon\to 0}\frac{|x_\varepsilon - x_0|^2}{\varepsilon^2} = 0,\ \lim_{\varepsilon\to 0}\frac{T-t_\varepsilon}{\varepsilon^2} = 0. \tag{5.17}$$

As, by Lemma 5.1, $\frac{\partial\varphi_\varepsilon}{\partial\gamma}(t,x) > 0$ if $x \in \partial\mathcal{O}$, by definition of viscosity subsolution, one has, for all $\varepsilon > 0$ and $x_\varepsilon \in \overline{\mathcal{O}}$,

$$\min\left(\frac{1}{\varepsilon^2} + \frac{1}{2}|\sigma^T D_x\varphi_\varepsilon|^2 - b\cdot D_x\varphi_\varepsilon, u-\psi\right)(t_\varepsilon,x_\varepsilon) \le 0.$$

Now, assume that, for some subsequence, $u(t_\varepsilon,x_\varepsilon) > \psi(t_\varepsilon,x_\varepsilon)$. Then, necessarily, for this subsequence,

$$1 \le \varepsilon^2|D_x\Psi_\varepsilon(x_\varepsilon,x_0)|(\frac{\|\sigma\|^2}{2}|D_x\Psi_\varepsilon(x_\varepsilon,x_0)| + \|b\|). \tag{5.18}$$

But, by lemma 5.1, $\varepsilon|D_x\Psi_\varepsilon(x_\varepsilon - x_0)| \le K(\frac{|x_\varepsilon - x_0|}{\varepsilon} + \varepsilon^3)$ and therefore (5.18) cannot occur because of (5.17) and for all $\varepsilon > 0$, $u(t_\varepsilon,x_\varepsilon) \le \psi(t_\varepsilon,x_\varepsilon)$. We conclude by letting ε go to 0 and using (5.17). $\diamond$

For all $0 < \nu < 1$ and all $\delta > 0$, let

$$M_{\nu,\delta} = \sup_{(t,x)\in[0,T]\times\overline{\mathcal{O}}}(\nu u(t,x) - v(t,x) - \delta(T-t)).$$

Our aim is to prove that $M_{\nu,\delta} \le (1-\nu)\|\psi\|$ which will give the conclusion of the theorem by letting ν and δ tend to 1 and 0 respectively. To do so, we define, for all ε, $\alpha > 0$, $M^{\varepsilon,\alpha}_{\nu,\delta}$ as being the supremum over $[0,T]^2 \times \overline{\mathcal{O}}^2$ of the function

$$(t,s,x,y) \mapsto \nu u(t,x) - v(s,y) - \delta(T-s) - \Psi_\varepsilon(x,y) - \frac{(t-s)^2}{\alpha^2}$$

and denote by $(\hat t, \hat s, \hat x, \hat y)$ an optimal point (recall that u and v are bounded and respectively usc and lsc). It is clear that $M_{\nu,\delta} \le M^{\varepsilon,\alpha}_{\nu,\delta}$.

We first notice that, for all $x \in \partial\mathcal{O}$, since $\Psi_\varepsilon(x,x) = 0$,

$$\nu u(T,x) - v(T,x) \le M^{\varepsilon,\alpha}_{\nu,\delta} = \nu u(\hat t,\hat x) - v(\hat s,\hat y) - \delta(T-\hat s) - \Psi_\varepsilon(\hat x,\hat y) - \frac{(\hat t-\hat s)^2}{\alpha^2},$$

which implies, by Lemma 5.1, that $\dfrac{1}{2}\dfrac{|\hat x - \hat y|^2}{\varepsilon^2} + \dfrac{(\hat t - \hat s)^2}{\alpha^2} \le 2(\|u\| + \|v\|) + K\varepsilon^2$. Therefore, up to some subsequences, $\hat t$, $\hat s$ and $\hat x$, $\hat y$ converge respectively to some $\overline{t}$ and $\overline{x}$ in $[0,T]\times\overline{\mathcal{O}}$ as α and ε go to 0.

Now we use again the maximality of $M^{\varepsilon,\alpha}_{\nu,\delta}$. For all α, $\varepsilon > 0$

$$\nu u(\overline{t},\overline{x}) - v(\overline{t},\overline{x}) - \delta(T-\overline{t}) - \Psi_\varepsilon(\overline{x},\overline{x}) \le M^{\varepsilon,\alpha}_{\nu,\delta} = \nu u(\hat t,\hat x) - v(\hat s,\hat y) - \delta(T-\hat s) - \Psi_\varepsilon(\hat x,\hat y) - \frac{(\hat t-\hat s)^2}{\alpha^2},$$

so that, by Lemma 5.1, the upper semicontinuity of u and the lower semicontinuity of v, we have both

$$\lim_{(\varepsilon,\alpha)\to(0,0)} \frac{|\hat{x}-\hat{y}|^2}{\varepsilon^2} + \frac{(\hat{t}-\hat{s})^2}{\alpha^2} = 0, \quad \text{and} \quad \lim_{(\varepsilon,\alpha)\to(0,0)} M_{\nu,\delta}^{\varepsilon,\alpha} = M_{\nu,\delta}. \tag{5.19}$$

Note that in the above limits the order of the convergence of α and ε to 0 does not matter.

We define, for all (t,x), $(s,y) \in [0,T]\times\overline{\mathcal{O}}$,

$$\varphi_1(t,x) = \frac{1}{\nu}\left(v(\hat{s},\hat{y}) + \delta(T-\hat{s}) + \Psi_\varepsilon(x,\hat{y}) + \frac{|t-\hat{s}|^2}{\alpha^2}\right), \quad \varphi_2(s,y) = u(\hat{t},\hat{x}) - \delta(T-s) - \Psi_\varepsilon(\hat{x},y) - \frac{|\hat{t}-s|^2}{\alpha^2}$$

and we apply the definition of viscosity solutions to u and v: $u-\varphi_1$ reaches its maximum at $(\hat{t},\hat{x})$ and when $\hat{x}\in\partial\mathcal{O}$ we can check easily that $D\varphi_1(\hat{t},\hat{x})\cdot\gamma(\hat{x}) > 0$ by Lemma 5.1 and therefore the Neumann boundary condition never holds. This imply that for all α, ε,

$$\min\left(-\frac{\partial\varphi_1}{\partial t} + \frac{1}{2}|\sigma^T D\varphi_1|^2 - b\cdot\varphi_1, u-\psi\right)(\hat{t},\hat{x}) \le 0. \tag{5.20}$$

For v, the situation is slightly different. As in the former case, we deduce from Lemma 5.1 that the Neumann boundary condition cannot hold when $\hat{y}\in\partial\mathcal{O}$, but if for some subsequence of (α,ε), $\hat{s}=T$ then we can have $v(\hat{s},\hat{y}) \ge \psi(\hat{s},\hat{y})$ and no information on the partial differential inequation. In this case, we remark that $\hat{t}$ goes to T (hence $\overline{t}$=T) and that, by Proposition 5.11 and the upper semicontinuity of u, for all δ_0, $u(\hat{t},\hat{x}) \le u(T,\overline{x}) + \delta_0/2 \le \psi(T,\overline{x}) + \delta_0/2 \le \psi(\hat{t},\hat{x}) + \delta_0$ for α and ε small enough. We deduce, from those two inequalities, by passing to the limit as α and ε go to 0 and using (5.19), that

$$M_{\nu,\delta} = \nu u(T,\overline{x}) - v(T,\overline{x}) \le \nu\psi(T,\overline{x}) - \psi(T,\overline{x}) + \delta_0 \le (1-\nu)\|\psi\| + \delta_0\nu$$

for all $\delta_0 > 0$, so that finally $M_{\nu,\delta} \le (1-\nu)\|\psi\|$.

Now we are left with the case, when $\hat{s} < T$ at least along a subsequence of (ε,α). We have

$$\min\left(-\frac{\partial\varphi_2}{\partial t} + \frac{1}{2}|\sigma^T D\varphi_2|^2 - b\cdot\varphi_2, v-\psi\right)(\hat{s},\hat{y}) \ge 0. \tag{5.21}$$

If, for some subsequence, $u(\hat{t},\hat{x}) > \psi(\hat{t},\hat{x})$ then (5.20) and (5.21) give respectively

$$-\frac{\partial\varphi_1}{\partial t}(\hat{t},\hat{x}) + \frac{1}{2}|\sigma(\hat{t},\hat{x})^T D\varphi_1(\hat{t},\hat{x})|^2 - b(\hat{t},\hat{x})\cdot\varphi_1(\hat{t},\hat{x}) \le 0$$

and

$$-\frac{\partial\varphi_2}{\partial t}(\hat{s},\hat{y}) + \frac{1}{2}|\sigma(\hat{s},\hat{y})^T D\varphi_2(\hat{s},\hat{y})|^2 - b(\hat{s},\hat{y})\cdot\varphi_2(\hat{s},\hat{y}) \ge 0.$$

We multiply the first inequality by ν and substract the second one; we obtain a rather complicated inequality which has three kinds of terms: the time derivative term, the linear term and the quadratic term.

The time derivative term is the simplest one

$$-\nu\frac{\partial\varphi_1}{\partial t}(\hat{t},\hat{x}) + \frac{\partial\varphi_2}{\partial t}(\hat{s},\hat{y}) = \delta.$$

The linear term can be writen

$$(b(\hat{s},\hat{x}) - b(\hat{t},\hat{x}))\cdot D_x\Psi_\varepsilon(\hat{x},\hat{y}) + (b(\hat{s},\hat{y}) - b(\hat{s},\hat{x}))\cdot D_x\Psi_\varepsilon(\hat{x},\hat{y}) - b(\hat{s},\hat{y})\cdot(D_x\Psi_\varepsilon(\hat{x},\hat{y}) + D_y\Psi_\varepsilon(\hat{x},\hat{y}))$$

and can be estimated, if ω_b and K_b denote respectively the modulus of continuity with respect to t and the Lipschitz constant with respect to x of b on $[0,T]\times\overline{\mathcal{O}}$ and by using Lemma 5.1, by

$$K\left((\omega_b(|\hat{t}-\hat{s}|) + K_b|\hat{x}-\hat{y}|)\left(\frac{|\hat{x}-\hat{y}|}{\varepsilon^2} + \varepsilon^2\right) + \|b\|\left(\frac{|\hat{x}-\hat{y}|^2}{\varepsilon^2} + \varepsilon^2\right)\right).$$

We know, by (5.19), that $|\hat{t}-\hat{s}|\leq C\alpha$ for some constant C independent of α and $\varepsilon<1$, therefore, the above term goes to 0 as α and ε go to 0, providing that $1>\varepsilon>\omega_\sigma(C\alpha)$ by (5.19).
As far as *the quadratic term* is concerned, we first remark that for all a, b in $\mathbb{R}^m$ and all $0<\nu<1$,

$$\frac{1}{\nu}|a|^2-|b|^2\geq-\frac{1}{1-\nu}|a+b|^2$$

so that we are reduced to estimate

$$|\sigma(\hat{t},\hat{x})^T D_x\Psi_\varepsilon(\hat{x},\hat{y})+\sigma(\hat{s},\hat{y})^T D_y\Psi_\varepsilon(\hat{x},\hat{y})|^2,$$

which we do as for the linear term, concluding that it goes to 0 as α and ε go to 0, providing that $1>\varepsilon>\omega_\sigma(C\alpha)$.

In conclusion to all those estimates we obtain the contradiction $\delta\leq 0$, and finally we necessarily have, for all α and $\varepsilon>\omega_b(C\alpha)$ small enough, $u(\hat{t},\hat{x})\leq\psi(\hat{t},\hat{x})$. This, combined with (5.21) and (5.19), yields $M_{\nu,\delta}\leq(1-\nu)\|\psi\|$ and the proof is complete. $\diamond$

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References

[1] R. Azencott Grandes déviations et applications, *Ecole d'Eté de Probabilités de Saint-Flour VIII-1978*, in: Lecture Notes in Math. **774**, Springer, Berlin, 1980, 1–176.

[2] R. F. Anderson and S. Orey Small random perturbation of dynamical systems with reflecting boundary, *Nagoya Math. J.* **60** (1976) 189–216.

[3] R. Atar and P. Dupuis Large deviations and queing networks: Methods for rate functional identification, *Stoch. Proc. App.* **84** (1999) 255–296.

[4] G. Barles *Solutions de viscosité des équations de Hamilton-Jacobi*, Mathématiques et applications **17**, Springer, Berlin, 1994.

[5] G. Barles Fully nonlinear Neumann type boundary conditions for second-order elliptic and parabolic equations *J. Diff. Equations* **106** N°1 (1993) 90–106.

[6] G. Barles and A.-P. Blanc Large deviations estimates for the exit probabilities of a diffusion process through some vanishing parts of the boundary *Adv. in Diff. Equations* **2** N°1 (1997) 39–84.

[7] G. Barles and P.-L. Lions Remarques sur les problèmes de réflexion oblique *C.R. Acad. Sci. Paris*, **320** Série I (1995) 69–74.

[8] G. Barles and B. Perthame Discontinuous solutions of deterministic optimal stopping time problems *Math. Modelling Numer. Anal.* **21** N°4 (1987) 557–579.

[9] G. Barles and B. Perthame Exit time problems in optimal control and vanishing viscosity method *SIAM J. in control and Optimization* **26** (1988) 1133–1148.

[10] G. Barles and B. Perthame Comparison Principle for Dirichlet type Hamilton-Jacobi equations and singular perturbations of degenerated elliptic equations *Appl. Math. and Opt.* **21** (1988) 21–44.

[11] B. Bouchard and N. Touzi Weak Dynamic Programming Principle for Viscosity Solutions, *SIAM Journal on Control and Optimization*, to appear.

[12] A. Bensoussan and J.-L. Lions *Applications des Inéquations variationnelles en contrôle stochastique* Dunod, Paris, 1978.

[13] A. Bensoussan and J.-L. Lions *Contrôle Impulsionnel et Inéquations QuasiVariationnelles.* Dunod, Paris, 1982.

[14] M. G. Crandall, H. Ishii and P.-L. Lions User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Soc.***27** (1992) 1–67.

[15] M. G. Crandall and P.-L. Lions Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **83** N°1 (1983) 1–42. MR 0690039

[16] A. Dembo and O. Zeitouni *Large Deviations Techniques and Applications* Jones and Bartlett Publishers, Boston, MA, 1993.

[17] P. Dupuis and R.S. Ellis *A weak convergence approch to the theory of Large Deviations* Wiley Series in Probability and Statistics, John Wiley & Sons, Inc, New-York, 1997.

[18] P. Dupuis and H. Ishii On oblique derivative problems for fully nonlinear second-order equations on nonsmooth domains. *Non Linear Anal.* TMA **12** (1990) 1123–1138.

[19] P. Dupuis and H. Ishii On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications *Stochastics Stochastics Rep.* **35** N°1 (1991) 31–62.

[20] P. Dupuis and H. Ishii On oblique derivative problems for fully nonlinear second-order elliptic PDE's on domains with corners.*Hokkaido Math. J.* **20** (1991) 135–164.

[21] P. Dupuis and H. Ishii SDEs with oblique reflection on nonsmooth domains. *The Annals of Prob.* **21** (1993) 554-580.

[22] H. Doss and P. Priouret Petites perturbations de systèmes dynamiques avec réflexion*Séminaire de Probabilités XVII*, in: Lecture Notes in Math.**986** 353–370 Springer, Berlin, 1983.

[23] Evans and Ishii A PDE approach to some aymptotic problems concerning random differential equations with small noise intensities. *Ann. Inst. H. Poncaré Anal. Non Linéaire* **2** N°1 (1985) 1–20.

[24] N. El Karoui, J.-P. Lepeltier and A. Millet A probabilistic approach to the reduite in optimal stopping *Prob. and Math. Stat.* **13** N°1 (1992) 97–121.

[25] J. Feng and T. Kurtz *Large Deviations for Stochastic Processes* Mathematical Surveys and Monographs **131** American Mathematical Society, Providence, RI, 2006.

[26] W. H. Fleming A stochastic control approach to some large deviations problems *Proceedings of Rome Conference 1984* in: Lecture Note in Math. **1119**, Springer-Verlag, Berlin, 1985, 52–66. MR790689

[27] W. H. Fleming Exit probabilities and optimal stochastic control. *Applied Math. and Opt.* **4** (1978) 329–346.

[28] W. H. Fleming and P.E. Souganidis A PDE-viscosity solution approach to some problems of large deviations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **4** (13) N°2 (1986) 171–192. MR 0876121

[29] W. H. Fleming and H. M. Soner *Controlled Markov processes and viscosity solutions*, Applications of Math. **25**, Springer-Verlag, New-York,1993.

[30] M. I. Freidlin and A. D. Wentzell *Random perturbations of dynamical systems*, Comp. Studies in Math. **260**, Springer-Verlag, New York, 1984.

[31] H. Ishii Fully nonlinear oblique derivative problems for nonlinear second-order elliptic PDE's. *Duke Math. J.* **62** (1991) 663–691.

[32] I. Karatzas and S. E. Shreve *Brownian Motion and Stochastic Calculus* Springer-Verlag, 2^{nd} edition, 1994.

[33] M. Kobylanski, M.-C. Quenez and E. Rouy-Mironescu Optimal multiple stopping time problem *Annals of applied prob* , to appear.

[34] M. Kobylanski and M.-C. Quenez Discontinuous optimal stopping: the Markovian case. Working paper.

[35] M. Kobylanski PhD dissertation. Thèse de l'université de Tours, 1998.

[36] P.-L. Lions Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations, Part I: The dynamic programming principle and applications; Part II: Viscosity solutions and uniqueness *Comm. P.D.E.* **8** (1983) 1101–1174, 1229–1276.

[37] P.-L. Lions Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations, Part III, in Nonlinear PDE and Appl., Séminaire du Collège de France, vol V, Pitman, 1985 .

[38] P.-L. Lions and A. S. Sznitman Stochastic differential equations with reflecting boundary conditions *Comm. Pure and Appl. Math.* **37** (1984) 511–537.

[39] B. Perthame Perturbed dynamical systems with attracting singularities and weak viscosity limits in Hamilton-Jacobi Equations. *Trans. Amer. Math. Soc.* **317** (1990) 723–747.

[40] H. Pham A large deviations approach to optimal long term investment *Finance and Stochastics* **7** (2003) 169–195.

[41] P. Priouret Remarques sur les petites perturbations de systèmes dynamiques, *Séminaire de Strasbourg XVI* in: Lecture Notes in Math. **920**, Springer-Verlag Berlin Heidelberg New-York, 1982, 159–183.

[42] Y. Saisho Stochastic Differential Equations for multi-dimensional domain with reflecting boundary *P.T.R.F.* **74** (1987) 455–477.

[43] A. V. Skorokhod Stochastic processes for diffusion processes in a bounded region *1, 2 Theor. Veroyatnost. i Primenen* **6** (1962) 264–274, **7** (1963) 5–25.

[44] D. W. Stroock *An introduction to the theory of Large Deviations.* Springer-Verlag, New-York, 1984 .

[45] S.R.S. Varadhan Large Deviations and Applications, in: CBMS-NSF Regional Conference Series in Applied Mathematics **46** SIAM, Philadelphia, PA, 1984.