

MIXED HODGE STRUCTURES

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MIXED HODGE STRUCTURES

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Abstract. With a basic knowledge of cohomology theory, the background necessary to understand Hodge theory and polarization, Deligne's Mixed Hodge Structure on cohomology of complex algebraic varieties is described.

Introduction

We assume that the reader is familiar with the basic theory of manifolds, basic algebraic geometry as well as cohomology theory. For instance, the reader should be familiar with the contents of [39], [2] or [34], the beginning of [25], the notes [13] in this book.

According to Deligne [6] and [7], the cohomology space $H^n(X,\mathbb{C})$ of a complex algebraic variety X carries two finite filtrations by complex sub-vector spaces, the weight filtration W and the Hodge filtration F. In addition, the subspaces W_j are rationally defined, i.e., W_j is generated by rational elements in $H^n(X,\mathbb{Q})$.

For a smooth compact variety, the filtration W is trivial; however the filtration F and its conjugate $\overline{F}$, with respect to the rational cohomology, define the Hodge decomposition. The structure in linear algebra defined on a complex vector space by such decomposition is called a Hodge Structure (HS).

For any complex algebraic variety X, the filtration W is defined on $H^n(X,\mathbb{Q})$. We can define the homogeneous part $Gr_j^W H^n(X,\mathbb{Q}) = W_j/W_{j-1}$ of degree j in the graded vector space $Gr_*^W H^n(X,\mathbb{Q}) := \oplus_{j\in\mathbb{Z}} W_j/W_{j-1}$ associated to the filtration W of $H^n(X,\mathbb{Q})$. For each integer j, there exists an isomorphism:

$$Gr_j^W H^n(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \simeq Gr_j^W H^n(X,\mathbb{C}),$$

such that conjugation on $Gr_j^W H^n(X,\mathbb{C})$ with respect to $Gr_j^W H^n(X,\mathbb{Q})$ is well-defined. Deligne showed that the filtration F with its conjugate induce a Hodge decomposition on $Gr_j^W H^n(X,\mathbb{C})$ of weight j. These data define a Mixed Hodge Structure on the cohomology of the complex algebraic variety X.

On a smooth variety X, the weight filtration W and the Hodge filtration F reflect properties of the Normal Crossing Divisor (NCD) at infinity of a completion of the variety, but are independent of the choice of the Normal Crossing Divisor.

Inspired by the properties of étale cohomology of varieties over fields with positive characteristic, constructed by A. Grothendieck and M. Artin, Deligne established

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the existence of a Mixed Hodge Structure on the cohomology of complex algebraic varieties, depending naturally on algebraic morphisms but not on continuous maps. The theory has been fundamental in the study of topological properties of complex algebraic varieties.

After this introduction, we recall the analytic background of Hodge decomposition on the cohomology of Kähler manifolds. We mention [18] and [37], which are references more adapted to algebraic geometers, and where one can also find other sources as classical books by A. Weil [40] and R.O. Wells [41] for which we refer for full proofs in the first two sections. In the second section, we recall the Hard Lefschetz Theorem and Riemann bilinear relations in order to define a polarization on the cohomology of projective smooth varieties.

We give an abstract definition of Mixed Hodge Structures in the third section as an object of interest in linear algebra, following closely Deligne [6]. The algebraic properties of Mixed Hodge Structures are developed on objects with three opposite filtrations in any abelian category. In section four, we need to develop algebraic homology techniques on filtered complexes up to filtered quasi-isomorphisms of complexes. The main contribution by Deligne was to fix the axioms of a Mixed Hodge Complex (MHC) giving rise to a Mixed Hodge Structure on its cohomology. A delicate lemma of Deligne on two filtrations needed to prove that the weight spectral sequence of a Mixed Hodge Complex is in the category of Hodge Structure is explained. This central result is stated and applied in section five, where in particular, the extension of Hodge Structures to all smooth compact complex algebraic varieties is described by Hodge filtrations on the analytic de Rham complex and not harmonic forms, although the proof is by reduction to Kähler manifolds.

For a non-compact smooth algebraic variety X, we need to take into account the properties at infinity, that is the properties of the Normal Crossing Divisor, the complement of the variety in a proper compactification, and introduce Deligne's logarithmic de Rham complex in section 6.

If X is singular, we introduce a smooth simplicial covering to construct the Mixed Hodge Complex in section 7. We mention also an alternative construction.

Each section starts with the statement of the results. The proofs are given in later subsections because they can be technical. Since a Mixed Hodge Structure involves rational cohomology constructed via techniques different than de Rham cohomology, when we define a morphism of Mixed Hodge Structures, we ask for compatibility with the filtrations as well as the identification of cohomology via the various constructions. For this reason it is convenient to work constantly at the level of complexes in the language of filtered and bi-filtered derived categories to ensure that the Mixed Hodge Structures constructed on cohomology are canonical and do not depend on the particular definition of cohomology used. Hence, we add a lengthy introduction on hypercohomology in section 4.

As applications let us mention Deformations of smooth proper analytic families which define a linear invariant called Variation of Hodge Structure (VHS) introduced by P. Griffiths, and limit Mixed Hodge Structures adapted to study the degeneration of Variation of Hodge Structures. Variation of Mixed Hodge Structure (VMHS) are the topics of other lectures.

Finally we stress that we only introduce the necessary language to express the statements and their implications in Hodge theory, but we encourage mathematicians to look into the foundational work of Deligne in references [6] and [7] to

discover his dense and unique style, since here intricate proofs in Hodge theory and spectral sequences are only surveyed. We remark that further developments in the theory occurred for cohomology with coefficients in Variations of Hodge Structure and the theory of differential Hodge modules ([1],[32]).

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1. Hodge decomposition

In this section we explain the importance of the Hodge decomposition and its relation to various concepts in geometry. Let $\mathcal{E}^*(X)$ denotes the de Rham complex of differential forms with complex values on a complex manifold X and $\mathcal{E}^{p,q}(X)$ the differentiable forms with complex coefficients of type (p,q) (see 1.2.3 below). The cohomology subspaces of type (p,q) are defined as:

$$H^{p,q}(X) = \frac{Z_d^{p,q}(X)}{d\mathcal{E}^*(X) \cap Z_d^{p,q}(X)} \quad \text{where} \quad Z_d^{p,q}(X) = Ker\, d \cap \mathcal{E}^{p,q}(X)$$

Theorem 1.1 (Hodge decomposition)**.** *Let X be a compact Kähler manifold (see 1.5). There exists a decomposition, called the Hodge decomposition, of the complex cohomology spaces into a direct sum of complex subspaces:*

$$H^i(X,\mathbb{C}) = \oplus_{p+q=i} H^{p,q}(X), \quad \text{satisfying} \quad H^{p,q}(X) = \overline{H^{q,p}(X)}.$$

Since a smooth complex projective variety is Kähler (see 1.24), we deduce:

Corollary 1.2. There exists a Hodge decomposition on the cohomology of a smooth complex projective variety.

The above Hodge decomposition theorem uses successive fundamental concepts from Riemannian geometry such as harmonic forms, complex analytic manifolds such as Dolbeault cohomology and from Hermitian geometry. It will be extended later to algebraic geometry.

The manifold X will denote successively in this section a differentiable, complex analytic, and finally a compact Kähler manifold. We give in this section a summary of the theory for the reader who is not familiar with the above theorem, which includes the definition of the *Laplacian* in Riemannian geometry and the space of harmonic forms isomorphic to the cohomology of X. On complex manifolds, we start with a *Hermitian structure*, its underlying *Riemannian structure* and its associated *fundamental* $(1,1)$ *form* used to construct the Lefschetz decomposition. It is only on *Kähler manifolds* when the form is closed that the components of type (p,q) of an *harmonic form* are harmonics. For an extended exposition of the theory with full proofs, including the subtle linear algebra of Hermitian metrics needed here, we refer to [41], see also [18], [37] (for original articles see [27] and even [29]).

1.1. **Harmonic forms on a differentiable manifold.** A Riemannian manifold X is endowed with a *scalar product on its tangent bundle* defining a metric. Another basic result in analysis states that the *cohomology of a compact smooth Riemannian manifold is represented by real harmonic global differential forms.* To define harmonic forms, we need to introduce Hodge $\mathbb{S}$*tar-operator and the Laplacian.*

1.1.1. *Riemannian metric.* A bilinear form g on X is defined at each point x as a product on the tangent space T_x to X at x

$$g_x(,) : T_x \otimes_{\mathbb{R}} T_x \to \mathbb{R}$$

where g_x vary smoothly with x, that is $h_{ij}(x) := g_x(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ is differentiable in x, then the product is given as $g_x = \sum_{i,j} h_{ij}(x) dx_i \otimes dx_j$. It is the local expression of a global product on fiber bundles $g : T_X \otimes T_X \to \mathbb{R}_X$. It is called a metric if moreover the matrix of the the product defined by $h_{ij}(x)$ is positive definite, that is: $g_x(u,u) > 0$ for all $u \neq 0 \in T_x$.

A globally induced metric on the cotangent bundle, is defined locally on the fiber $\mathcal{E}^1_x := T^*_x := Hom(T_x, \mathbb{R})$ and more generally on the differential forms $\mathcal{E}^p_x$ as follows.

Let $e_1, \ldots, e_n$ be an orthonormal basis of T_x, and $(e^*_j)_{j\in[1,n]}$ its dual basis of T^*_x. The vectors $e^*_{i_1} \wedge \ldots \wedge e^*_{i_p}$, for all $i_1 < \ldots < i_p$, is a basis of $\mathcal{E}^p_x$, for all $x \in X$.

Lemma and Definition 1.3. *There exists a unique metric globally defined on the bundle $\mathcal{E}^p_X$ such that, for all $x \in X$, the vectors $e^*_{i_1} \wedge \ldots \wedge e^*_{i_p}$, $i_1 < \ldots < i_p$, is an orthonormal basis of $\mathcal{E}^p_x$ whenever $e_1, \ldots, e_n$ is an orthonormal basis of T_x.*

Indeed, $e_{i_1}^* \wedge \ldots, \wedge e_{i_p}^*$ defines a local differentiable section of $\mathcal{E}_X^p$ due to the fact that the Gram-Schmidt process of normalization construct an orthonormal basis varying differentiably in terms of the point $x \in X$.

Definition 1.4. We suppose that the manifold X is oriented. For each point $x \in X$ let $e_j, j \in [1, n]$ be an orthonormal positively oriented basis of $T_{X,x}$. The volume form, denoted *vol*, is defined by the metric as the unique nowhere vanishing section of Ω_X^n, satisfying $vol_x := e_1^* \wedge \ldots \wedge e_n^*$ for each point $x \in X$.

Exercise 1.5. The volume form may be defined directly without orthonormalization process as follows. Let $x_1, \ldots, x_n$ denote a local ordered set of coordinates on an open subset U of a covering of X, compatible with the orientation. Prove that

$$\sqrt{det(h_{ij})} dx_1 \wedge \cdots \wedge dx_n$$

defines the global volume form.

1.1.2. *L^2 metric.* For X compact, we deduce from the volume form a global pairing called the L^2 metric

$$\forall \psi, \eta \in \mathcal{E}^i(X), (\psi, \eta)_{L^2} = \int_X g_x(\psi(x), \eta(x)) vol(x)$$

1.1.3. *Laplacian.* We prove the existence of an operator $d^* : \mathcal{E}_X^{i+1} \to \mathcal{E}_X^i$ called formal adjoint to the differential $d : \mathcal{E}_X^i \to \mathcal{E}_X^{i+1}$, satisfying:

$$(d\psi, \eta)_{L^2} = (\psi, d^*\eta)_{L^2}, \quad \forall \psi \in \mathcal{E}^i(X), \forall \eta \in \mathcal{E}^{i+1}(X).$$

The adjoint operator is defined by constructing first an operator $*$

$$\mathcal{E}_X^i \stackrel{*}{\to} \mathcal{E}_X^{n-i}$$

by requiring at each point $x \in X$:

$$g_x(\psi(x), \eta(x)) vol_x = \psi_x \wedge *\eta_x, \quad \forall \psi_x, \eta_x \in \mathcal{E}_{X,x}^i.$$

The section *vol* defines an isomorphism of bundles $\mathbb{R}_X \simeq \mathcal{E}_X^n$, inducing the isomorphisms: $Hom(\mathcal{E}_X^i, \mathbb{R}_X) \simeq Hom(\mathcal{E}_X^i, \mathcal{E}_X^n) \simeq \mathcal{E}_X^{n-i}$. Finally, we deduce $*$ by composition with the isomorphism $\mathcal{E}_X^i \to Hom(\mathcal{E}_X^i, \mathbb{R}_X)$ defined by the scalar product on $\mathcal{E}_X^i$. Locally, in terms of an orthonormal basis, the formula apply for $\psi(x) = e_{i_1}^* \wedge \ldots \wedge e_{i_p}^* = \eta(x)$; it shows

$$*(\psi(x)) = \epsilon e_{j_1}^* \wedge \ldots \wedge e_{j_{n-p}}^*$$

where $\{j_1, \ldots, j_{n-p}\}$ is the complement of $\{i_1, \ldots, i_p\}$ in $[1, n]$ and ϵ is the sign of the permutation of $[1, n]$ defined by $i_1, \ldots, i_p, j_1, \ldots, j_{n-p}$. We have on $\mathcal{E}_X^i$:

$$*^2 = (-1)^{i(n-i)} Id, \quad d^* = (-1)^{n+in+1} * \circ d \circ * = (-1)^i *^{-1} \circ d \circ *$$

Lemma 1.6. *We have for all $\alpha, \beta \in \mathcal{E}^i(X)$:*

$$(\alpha, \beta)_{L^2} = \int_X \alpha \wedge *\beta.$$

Definition 1.7. The formal adjoint d^* of d is a differential operator and the operator Δ defined as:

$$\Delta = d^* \circ d + d \circ d^*$$

is called the Laplacian. Harmonic forms are defined as the solutions of the Laplacian and denoted by:

$$\mathcal{H}^i(X) = \{\psi \in \mathcal{E}^i(X) : \Delta(\psi) = 0\}$$

The Harmonic forms depend on the choice of the metric. A fundamental result due to Hodge and stated here without proof is that harmonic forms represents the cohomology spaces:

Theorem 1.8. *On a compact smooth oriented Riemannian manifold each cohomology class is represented by a unique real harmonic global differential form:*

$$\mathcal{H}^i(X) \simeq H^i(X, \mathbb{R}).$$

1.2. **Complex manifolds and decomposition into types.** An underlying real differentiable structure is attached to a complex manifold X, then a real and a complex tangent bundles are defined and denoted by $T_{X,\mathbb{R}}$ and T_X. The comparison is possible by embedding T_X into $T_{X,\mathbb{R}} \otimes \mathbb{C}$, then a delicate point is to describe holomorphic functions among differentiable functions, and more generally the complex tangent space and holomorphic differential forms in terms of the real structure of the manifold. Such description is a prelude to Hodge theory and is fundamental in Geometry, whether we mention Cauchy-Riemann equations satisfied by the real components of a holomorphic function, or the position of the holomorphic tangent bundle inside the complexified real tangent bundle. A weaker feature of a complex manifold is the existence of an almost complex structure J on the real tangent bundle which is enough to define the type of differential forms.

1.2.1. *Almost-complex structure.* Let V be a real vector space. An almost-complex structure on V is defined by a linear map $J : V \to V$ satisfying $J^2 = -Id$; then $\dim_{\mathbb{R}} V$ is even and the eigenvalues are i and $-i$ with associated eigenspaces V^+ and V^- in $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$:

$$V^+ = \{x - iJx : x \in V\} \subset V_{\mathbb{C}}, \quad V^- = \{x + iJx : x \in V\} \subset V_{\mathbb{C}}.$$

where we write v for $v \otimes 1$ and iv for the element $v \otimes i$. The conjugation σ on $\mathbb{C}$ extends to a real linear map σ on $V_{\mathbb{C}}$ such that $\sigma(a+ib) = a-ib$; we write $\sigma(\alpha) = \overline{\alpha}$ for $\alpha \in V_{\mathbb{C}}$. We note that the subspaces V^+ and V^- are conjugate: $\overline{V^+} = V^-$ such that σ induces an anti-linear isomorphism $\sigma : V^+ \simeq V^-$.

Example 1.9. Let V be a complex vector space, then the identity bijection with the underlying real space $V_{\mathbb{R}}$ of V, $\varphi : V \to V_{\mathbb{R}}$ transports the action of i to a real linear action J satisfying $J^2 = -Id$, defining an almost complex structure on $V_{\mathbb{R}}$. A complex basis $(e_j)_{j\in[1,n]}$ defines a real basis $(\varphi(e_j), i(\varphi(e_j))$ for $j \in [1, n]$, such that: $J(\varphi(e_j) := i\varphi(e_j)$. For $V = \mathbb{C}^n$, $\varphi(z_1, \ldots, z_n) = (x_1, y_1, \ldots, x_n, y_n)$ and

$$J(x_1, y_1, \ldots, x_n, y_n) = (-y_1, x_1, \ldots, -y_n, x_n).$$

Reciprocally, an almost-complex structure J on a real vector space W corresponds to a structure of complex vector space on W as follows. There exists a family of vectors $(e_j)_{j\in[1,n]}$ in W such that $\{e_j, J(e_j)\}_{j\in[1,n]}$ form a basis of W. First, one chooses a non-zero vector e_1 in W and proves that the space W_1 generated by

$e_1, J(e_1)$ is of dimension two, then we continue with $e_2 \notin W_1$ and so on; then, there is a complex structure on W for which $e_j, j \in [1,n]$ is a complex basis with the action of $i \in \mathbb{C}$ defined by $i.e_j := J(e_j)$.

Remark 1.10. An almost-complex structure on a real space V is equivalent to a complex structure. In what follows we work essentially with $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$, then the product with a complex number $\lambda.(v \otimes 1) = v \otimes \lambda$ will be written λv. This product is not to confuse with the product $\lambda * v$ for the complex structure on V; in particular, $v \otimes i := iv$ while $i * v := Jv$. We note that the map $\varphi : V \to V^+ \subset V_{\mathbb{C}}$ defined by $x \mapsto x - iJx$ is an isomorphism of complex vector spaces.

1.2.2. *Decomposition into types.* Let V be a complex vector space and $(V_{\mathbb{R}}, J)$ the underlying real vector space with its involution J defined by multiplication by i. The following decomposition of $V_{\mathbb{C}}$ is called a decomposition into types

$$V_{\mathbb{C}} \simeq V^{1,0} \oplus V^{0,1} \text{ where } V^{1,0} = V^+, V^{0,1} = V^-, \overline{V}^{1,0} = V^{0,1}$$

the conjugation is with respect to the real structure of V, that is inherited from the conjugation on $\mathbb{C}$. Let $W := Hom_{\mathbb{R}}(V_{\mathbb{R}}, \mathbb{R})$ and $W_{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C}$; a morphism in $Hom_{\mathbb{C}}(V, \mathbb{C})$ embeds, by extension to $V_{\mathbb{C}}$, into $W_{\mathbb{C}}$ and its image is denoted by $W^{1,0}$, then $W^{1,0}$ vanishes on $V^{0,1}$. Let $W^{0,1}$ be the subspace vanishing on $V^{1,0}$, thus we deduce from the almost complex structure a decomposition $W_{\mathbb{C}} \simeq W^{1,0} \oplus W^{0,1}$ orthogonal to the decomposition of $V_{\mathbb{C}}$. Such decomposition extends to the exterior power:

$$\wedge^n W_{\mathbb{C}} = \oplus_{p+q=n} W^{pq}, \quad \text{where } W^{pq} = \wedge^p W^{1,0} \otimes \wedge^q W^{0,1}, \overline{W}^{p,q} = W^{q,p}.$$

Remark 1.11. On a fixed vector space, the notion of almost-complex structure is equivalent to complex structure. The interest in this notion will become clear for a differentiably varying complex structure on the tangent bundle of a manifold, since not all such structures determine an analytic complex structure on M. In order to lift an almost-complex structure on T_M to a complex structure on M, necessary and sufficient conditions on are stated in Newlander-Nirenberg theorem (see [37] Theorem 2.24).

1.2.3. *Decomposition into types on the complexified tangent bundle.* The existence of a complex structure on the manifold gives an almost-complex structure on the differentiable tangent bundle $T_{X,\mathbb{R}}$ and a decomposition into types of the complexified tangent bundle $T_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ which splits as a direct sum $T_X^{1,0} \oplus T_X^{0,1}$ where $T_{X,z}^{1,0} = \{u - iJu : u \in T_{X,\mathbb{R},z}\}$ and $\overline{T_X^{1,0}} = T_X^{0,1}$.
For a complex manifold X with local complex coordinates $z_j = x_j + iy_j, j \in [1,n]$, we define $J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$ on the real tangent space $T_{X,\mathbb{R},z}$ of the underlying differentiable manifold. Since the change of coordinates is analytic, the definition of J is compatible with the corresponding change of differentiable coordinates, hence J is globally defined on the tangent bundle $T_{X,\mathbb{R}}$.

The holomorphic tangent space $T_{X,z}$ embeds isomorphically onto $T_X^{1,0}$ generated by $\frac{\partial}{\partial z_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$ of the form $u_j - iJ(u_j)$ such that the basis $(\frac{\partial}{\partial z_j}, j \in [1,n])$ in $T_{X,\mathbb{R},z} \otimes \mathbb{C}$ is complex dual to $(dz_j = dx_j + idy_j, j \in [1,n])$. From now on we identify the two complex bundles T_X and $T_X^{1,0}$ via the isomorphism:

$$T_{X,\mathbb{R},z} \xrightarrow{\sim} T_{X,z}^{1,0} : \frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}), \frac{\partial}{\partial y_j} \mapsto i\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial y_j} + i\frac{\partial}{\partial x_j})$$

which transforms the action of J into the product with i since $J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$. The inverse is defined by twice the real part: $u + iv \to 2u$.

The dual of the above decomposition of the tangent space is written as:

$$\mathcal{E}^1_X \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathcal{E}^{1,0}_X \oplus \mathcal{E}^{0,1}_X$$

which induces a decomposition of the sheaves of differential forms into types:

$$\mathcal{E}^k_X \otimes_{\mathbb{R}} \mathbb{C} \simeq \oplus_{p+q=k} \mathcal{E}^{p,q}_X$$

In terms of complex local coordinates on an open set U, $\phi \in \mathcal{E}^{p,q}(U)$ is written as a linear combination with differentiable functions as coefficients of:

$dz_I \wedge d\overline{z}_J := dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$ where $I = \{i_1, \cdots, i_p\}, J = \{j_1, \cdots, j_q\}$.

1.2.4. *The double complex.* The differential d decomposes as well into $d = \partial + \overline{\partial}$, where:

$$\partial : \mathcal{E}^{p,q}_X \to \mathcal{E}^{p+1,q}_X : f dz_I \wedge d\overline{z}_J \mapsto \sum_i \frac{\partial}{\partial z_i}(f) dz_i \wedge dz_I \wedge d\overline{z}_J,$$

$$\overline{\partial} : \mathcal{E}^{p,q}_X \to \mathcal{E}^{p,q+1}_X : f dz_I \wedge d\overline{z}_J \mapsto \sum_i (\frac{\partial}{\partial \overline{z}_i})(f) d\overline{z}_i \wedge dz_I \wedge d\overline{z}_J$$

are compatible with the decomposition up to a shift on the bidegree, we deduce from $d^2 = \partial^2 + \overline{\partial}^2 + \partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0$ that $\partial^2 = 0$, $\overline{\partial}^2 = 0$ and $\partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0$. It follows that $(\mathcal{E}^{*,*}_X, \partial, \overline{\partial})$ is a double complex (see [18] p 442). Its associated simple complex, denoted $s(\mathcal{E}^{*,*}_X, \partial, \overline{\partial})$, is isomorphic to the de Rham complex:

$$(\mathcal{E}^*_X \otimes_{\mathbb{R}} \mathbb{C}, d) \simeq s(\mathcal{E}^{*,*}_X, \partial, \overline{\partial}).$$

In particular we recover the notion of the subspace of cohomology of type (p, q):

Definition 1.12. The subspace of de Rham cohomology represented by a global closed form of type (p, q) on X is denoted as:

$H^{p,q}(X) := \{\text{cohomology classes representable by a global closed form of type } (p, q)\}$

We have: $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Remark 1.13. The definition shows that $\sum_{p+q=i} H^{p,q}(X) \subset H^i_{DR}(X)$. Although the differential forms decompose into types, it does not follow that the cohomology also decomposes into a direct sum of subspaces $H^{p,q}(X)$, since the differentials are not compatible with types. It is striking that there is a condition, easy to state, on Kähler manifolds, that imply the full decomposition. Hodge theory on such manifolds shows that:

$$H^i_{DR}(X) = \oplus_{p+q=i} H^{p,q}(X).$$

Example 1.14. Let $\Lambda \subset \mathbb{C}^r$ be a rank-$2r$ lattice

$$\Lambda = \{m_1\omega_1 + \ldots + m_{2r}\omega_{2r}; \quad m_1, \ldots, m_{2r} \in \mathbb{Z}\}$$

where ω_i are complex vectors in $\mathbb{C}^r$, linearly independent over $\mathbb{R}$. The quotient group $T_\Lambda = \mathbb{C}^r/\Lambda$ inherits a complex structure locally isomorphic to $\mathbb{C}^r$, with change of coordinates induced by translations by elements of Λ. Moreover T_Λ is compact. The quotient $T_\Lambda = \mathbb{C}^r/\Lambda$ is a compact complex analytic manifold called a complex torus. The projection $\pi : \mathbb{C}^r \to T_\Lambda$ which is holomorphic and locally isomorphic, is in fact a universal covering.

Family of tori. The above torus depends on the choice of the lattice. Given two lattices Λ_1 and Λ_2, there exists an $\mathbb{R}-$linear isomorphism $g \in GL(2r, \mathbb{R})$ such that $g(\Lambda_1) = \Lambda_2$, hence it induces a diffeomorphism of the underlying differentiable structure of the tori $g : T_{\Lambda_1} \simeq T_{\Lambda_2}$. However, it is not true that all tori are isomorphic as complex manifolds.

Lemma 1.15. *Two complex tori T_{Λ_1} and T_{Λ_2} are isomorphic if and only if there exists a complex linear map $g \in GL(r, \mathbb{C})$ inducing an isomorphism of lattices $g : \Lambda_1 \simeq \Lambda_2$.*

Proof. Given two complex torus T_{Λ_1} and T_{Λ_2}, an analytic morphism $f : T_{\Lambda_1} \to T_{\Lambda_2}$ will induce a morphism on the fundamental groups and there exists an holomorphic lifting to the covering $F : \mathbb{C}^r \to \mathbb{C}^r$ compatible with f via the projections:

$$\pi_2 \circ F = f \circ \pi_1.$$

Moreover, given a point $z_1 \in \mathbb{C}^r$ and $\lambda_1 \in \Lambda_1$, there exists $\lambda_2 \in \Lambda_2$ such that $F(z_1 + \lambda_1) = F(z_1) + \lambda_2$.

By continuity, this relation will remain true by varying $z \in \mathbb{C}^r$; then F and its derivative are periodic. This derivative depends holomorphically on parameters of T_{Λ_1}, hence it is necessarily constant; then, F is complex affine on $\mathbb{C}^r$, since its derivative is constant. We may suppose that F is complex linear after composition with a translation. □

Moduli space. We may parameterize all lattices as follows:
- the group $GL(2r, \mathbb{R})$ acts transitively on the set of all lattices of $\mathbb{C}^r$.
We choose a basis $\tau = (\tau_1, \dots, \tau_{2i-1}, \tau_{2i} \dots \tau_{2r})$, $i \in [1, r]$, of a lattice L, then it defines a basis of $\mathbb{R}^{2r}$ over $\mathbb{R}$. An element φ of $GL(2r, \mathbb{R})$ is given by the linear transformation which sends τ into the basis $\varphi(\tau) = \tau'$ of $\mathbb{R}^{2r}$ over $\mathbb{R}$. The element φ of $GL(2r, \mathbb{R})$ is carrying the lattice L onto the lattice L' defined by the τ'.
- The isotopy group of this action is $GL(2r, \mathbb{Z})$, since τ and τ' define the same lattice if and only if $\varphi \in GL(2r, \mathbb{Z})$.
Hence the space of lattices is the quotient group $GL(2r, \mathbb{R})/GL(2r, \mathbb{Z})$.
- Two tori defined by the lattice L and L' are analytically isomorphic if and only if there is an element of $GL(r, \mathbb{C})$ which transform the lattice L into the lattice L', as the preceding lemma states.

It follows that the parameter space of complex tori is the quotient:

$$GL(2r, \mathbb{Z}) \backslash GL(2r, \mathbb{R}) / GL(r, \mathbb{C})$$

where $GL(r, \mathbb{C})$ is embedded naturally in $GL(2r, \mathbb{R})$ as a complex linear map is $\mathbb{R}-$linear.

For $r = 1$, the quotient $GL(2, \mathbb{R})/GL(1, \mathbb{C})$ is isomorphic to $\mathbb{C} - \mathbb{R}$, since, up to complex isomorphisms, a lattice is generated by $1, \tau \in \mathbb{C}$ independent over $\mathbb{R}$, hence completely determined by $\tau \in \mathbb{C} - \mathbb{R}$. The moduli space is the orbit space of the action of $GL(2, \mathbb{Z})$ on the space $GL(2, \mathbb{R})/GL(1, \mathbb{C}) = \mathbb{C} - \mathbb{R}$. Since $GL(2, \mathbb{Z})$ is the disjoint union of $SL(2, \mathbb{Z})$ and the integral 2×2-matrices of determinant equal to -1, that orbit space is the one of the action of $SL(2, \mathbb{Z})$ on the upper half plane:

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \mapsto \frac{az + b}{cz + d}.$$

Hodge decomposition on complex tori. The cohomology of a complex torus T_Λ is easy to compute by Künneth formula, since it is diffeomorphic to a product of circles: $T_\Lambda \simeq (S^1)^{2r}$. Hence $H^1(T_\Lambda, \mathbb{Z}) \simeq \mathbb{Z}^{2r}$ and $H^j(T_\Lambda, \mathbb{Z}) \simeq \wedge^j H^1(T_\Lambda, \mathbb{Z})$.

The cohomology with complex coefficients may be computed by de Rham cohomology. In this case, since the complex tangent space is trivial, we have natural cohomology elements given by the translation-invariant forms, hence with constant coefficients. The finite complex vector space T_0^* of constant holomorphic $1-$forms is isomorphic to $\mathbb{C}^r$ and generated by $dz_j, j \in [1, r]$ and the Hodge decomposition reduces to prove $H^j(X, \mathbb{C}) \simeq \oplus_{p+q=j} \wedge^p T_0^* \otimes \wedge^q \overline{T_0^*}, p \geq 0, q \geq 0$ which is here a consequence of the above computation of the cohomology.

1.2.5. *Dolbeault cohomology.*
For $r \geq 0$, the complex $(\mathcal{E}_X^{r,*}, \overline{\partial})$, is called the r-Dolbeault complex.

Definition 1.16. The i-th cohomology of the global sections of the r-Dolbeault complex is called the $\overline{\partial}$ cohomology of X of type (r, i)

$$H_{\overline{\partial}}^{r,i}(X) := H^i(\mathcal{E}_X^{r,*}(X), \overline{\partial}) = \frac{Ker(\overline{\partial}^i : \mathcal{E}^{r,i}(X) \to \mathcal{E}^{r,i+1}(X))}{Im(\overline{\partial}^{i-1} : \mathcal{E}^{r,i-1}(X) \to \mathcal{E}^{r,i}(X))}$$

On complex manifolds, we consider the sheaves of holomorphic forms $\Omega_X^r := \wedge \Omega 1_X$.

Lemma 1.17. *The Dolbeault complex $(\mathcal{E}_X^{r,*}, \overline{\partial})$ for $r \geq 0$, is a fine resolution of Ω_X^r, hence*

$$H^i(X, \Omega_X^r) \simeq H^i(\mathcal{E}_X^{r,*}(X), \overline{\partial}) := H_{\overline{\partial}}^{r,i}(X)$$

For the proof, see the $\overline{\partial}$-Poincaré lemma in [18] p. 25 (continued p. 45).
A cohomology class of X of type (r, i) defines a $\overline{\partial}-$cohomology class of the same type, since $(d\varphi = 0) \Rightarrow (\overline{\partial}\varphi = 0)$.

On a Kähler manifold (see below in 1.5) we can find a particular representative ω in each class of $\overline{\partial}$-cohomology, called harmonic form for which:

$$(\overline{\partial}\omega = 0) \Rightarrow (d\omega = 0).$$

1.2.6. *Holomorphic Poincaré lemma.* The holomorphic version of Poincaré lemma shows that the de Rham complex of holomorphic forms Ω_X^* is a resolution of the constant sheaf $\underline{\mathbb{C}}_X$. However, since the resolution is not acyclic, cohomology spaces are computed only as hypercohomology of the global sections functor, that is after acyclic resolution.

$$H^i(X, \mathbb{C}) \simeq \mathbb{H}^i(X, \Omega_X^*) := R^i\Gamma(X, \Omega_X^*) \simeq H^i(\mathcal{E}^*(X) \otimes_{\mathbb{R}} \mathbb{C})$$

1.3. **Hermitian metric, its associated Riemannian metric and $(1,1)$ form.** The Hermitian product h on the complex manifold X is defined at each point z as a product on the holomorphic tangent space T_z to X at z

$$h_z : T_z \times T_z \to \mathbb{C}$$

satisfying $h(u, v) = \overline{h(v, u)}$, linear in u and anti-linear in v, moreover h_z vary smoothly with z. In terms of the basis $(\frac{\partial}{\partial z_j}, j \in [1, n])$ dual to the basis $(dz_j, j \in [1, n])$ of T_z^*, the matrix $h_{ij}(z) := h_z(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$ is differentiable in z and $h_{ij}(z) =$

$\overline{h_{ji}(z)}$. In terms of *the above isomorphism* $T_z \xrightarrow{\sim} T_z^{1,0}$ defined by $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$, it is equivalent to define the Hermitian product as

$$h_z : T_z^{1,0} \times T_z^{1,0} \to \mathbb{C},\ h_z = \sum_{i,j} h_{ij}(z) dz_i \otimes d\overline{z_j} \in \mathcal{E}_z^{1,0} \otimes_{\mathbb{C}} \mathcal{E}_z^{0,1}$$

where the Hermitian product on $T_z^{1,0}$ is viewed as a linear form on $T_z^{1,0} \otimes \overline{T_z^{1,0}}$. The product is called a Hermitian metric if moreover the real number $h_z(u, \overline{u}) > 0$ is positive definite for all $u \neq 0 \in T_z$; equivalently the matrix h of the product defined by $h_{ij}(z)$ (equal to its conjugate transpose: $h = {}^t\overline{h}$ is positive definite: ${}^t\overline{u}hu > 0$.

1.3.1. *Associated Riemannian structure.* The Hermitian metric may be defined on $T_{X,\mathbb{R},z}$ identified with $T_z^{1,0}$ via $\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial z_i}$ and extended via the complex structure s.t. $\frac{\partial}{\partial y_i} = J(\frac{\partial}{\partial x_i}) := i\frac{\partial}{\partial x_i} \to i\frac{\partial}{\partial z_i}$, then using the decomposition as the sum of its real and imaginary parts

$$h_{z|T_{X,\mathbb{R},z}} = Re\,(h_z) + iIm\,(h_z)$$

we associate to the metric a Riemannian structure on X defined by $g_z := Re\,(h_z)$.

In this way, the complex manifold X of dimension n is viewed as a Riemann manifold of dimension $2n$ with metric g_z. Note that g is defined over the real numbers, since it is represented by a real matrix over a basis of the real tangent space.

Since h_z is hermitian, the metric satisfy: $g_z(Ju, Jv) = g_z(u, v)$. In fact we can recover the Hermitian product from a Riemannian structure satisfying this property. From now on, the Hodge operator star will be defined on $T_{X,\mathbb{R},z}$ with respect to g and extended complex linearly to $T_{X,\mathbb{R},z} \otimes \mathbb{C}$.

Lemma and Definition 1.18 (The (1,1)-form ω)**.** *The hermitian metric defines a real exterior* $2-$*form* $\omega := -\frac{1}{2}Im\,(h)$ *of type* $(1,1)$ *on* $T_{X,\mathbb{R},z}$*, satisfying:* $\omega(Ju, v) = -\omega(u, Jv)$ *and* $\omega(Ju, Jv) = \omega(u, v)$*; moreover:* $2\omega(u, Jv) = g(u, v)$ *and*

$$\omega = \frac{i}{2}\sum_{i,j} h_{ij}(z) dz_i \wedge d\overline{z_j} \quad \textit{if} \quad h_z = \sum_{i,j} h_{ij}(z) dz_i \otimes d\overline{z_j}$$

.

Proof. We have an exterior 2-form since $2\omega(v, u) = -Im\,h(v, u) = Im\,h(u, v) = -2\omega(u, v)$. We check: $2\omega(u, Jv) = -Im\,h(u, iv) = -Im\,(-ih(u, v)) = Re\,(h(u, v) = g(u, v)$, and $2\omega(Ju, v) = -Im\,h(iu, v) = -Im\,(ih(u, v)) = -Re\,h(u, v) = -g(u, v)$, while $Im\,h = \frac{1}{2i}\sum_{i,j} h_{ij}(z)(dz_i \otimes d\overline{z_j} - d\overline{z_j} \otimes dz_i) = -i\sum_{i,j} h_{ij}(z) dz_i \wedge d\overline{z_j}$ □

Example 1.19. 1) The Hermitian metric on $\mathbb{C}^n$ is defined by $h := \sum_{i=1}^n dz_i \otimes d\overline{z_i}$:

$$h(z, w) = \sum_{i=1}^n z_i \overline{w}_i$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$. Considering the underlying real structure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ defined by $(z_k = x_k + iy_k) \mapsto (x_k, y_k)$, $Re\,h$ is the Euclidean inner product on $\mathbb{R}^{2n}$ and $Im\,h$ is a non-degenerate alternating bilinear form, i.e., a symplectic form: $\Omega = \frac{i}{2}\sum_{i=1}^n dz_i \wedge d\overline{z_i} = \sum_{i=1}^n dx_i \wedge dy_i$.

Explicitly, on $\mathbb{C}^2$, the standard Hermitian form is expressed on $\mathbb{R}^4$ as:

$$\begin{aligned} h((z_{1,1}, z_{1,2}), (z_{2,1}, z_{2,2})) =& x_{1,1}x_{2,1} + x_{1,2}x_{2,2} + y_{1,1}y_{2,1} + y_{1,2}y_{2,2} \\ &+ i(x_{2,1}y_{1,1} - x_{1,1}y_{2,1} + x_{2,2}y_{1,2} - x_{1,2}y_{2,2}). \end{aligned}$$

2) If $\Lambda \subset \mathbb{C}^n$ is a full lattice, then the above metric induces a Hermitian metric on the torus $\mathbb{C}^n/\Lambda$.
3) The *Fubini -Study metric on the projective space* $\mathbb{P}^n$. Let $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ be the projection and consider a section of π on an open subset $Z : U \to \mathbb{C}^{n+1} - \{0\}$, then the form $\omega = \frac{i}{2\pi}\partial\overline{\partial} log||Z||^2$ is a fundamental $(1,1)$ form, globally defined on $\mathbb{P}^n$, since the forms glue together on the intersection of two open subsets (this formula is related to the construction of the first Chern class of the canonical line bundle on the projective space, in particular the coefficient $\frac{i}{2}$ leads later to a positive definite form, while π leads to an integral cohomology class). For the section $Z = (1, w_1, \ldots, w_n)$,

$$\omega = \frac{i}{2\pi}\frac{(1+|w|^2)\sum_{i=1}^n dw_i \wedge d\overline{w_j} - \sum_{i,j=1}^n \overline{w_j} w_i dw_i \wedge d\overline{w_j}}{(1+|w|^2)^2}$$

which reduces at the point $O = (1, 0, \ldots, 0)$, to $\omega = \frac{i}{2\pi}\sum_{i=1}^n dw_i \wedge d\overline{w_i} > 0$, hence a positive real form , i.e., it takes real values on real tangent vectors, and is associated to a positive Hermitian form. Since ω is invariant under the transitive holomorphic action of $SU(n+1)$ on $\mathbb{P}^n$, we deduce that the hermitian form

$$h = (\sum_{i,j=1}^n h_{i,j}(w) dw_i \otimes d\overline{w_i})(1+|w|^2)^{-2},\ \pi h_{i,j}(w) = (1+|w|^2)\delta_{ij} - \overline{w_j} w_i$$

is a positive definite Hermitian form. Hence the projective space has a Hermitian metric.

1.4. **Harmonic forms on compact complex manifolds.** In this section, a study of the $\overline{\partial}$ operator, similar to the previous study of d, leads to the representation of Dolbeault cohomology by $\overline{\partial}$-harmonic forms (Hodge theorem).

1.4.1. *L^2-inner product.* We choose a Hermitian metric h with associated $(1,1)$-form ω and volume form $vol = \frac{1}{n!}\omega^n$. We extend the L^2-inner product $(\psi, \eta)_{L^2}$ (resp. the Hodge star $*$ operator) defined on the real forms by the underlying Riemannian metric, first linearly into a complex bilinear form $< \psi, \eta >_{L^2}$ on $\mathcal{E}_X^* := \mathcal{E}_{X,\mathbb{R}}^* \otimes \mathbb{C}$ (resp. $\tilde{*} : \mathcal{E}_{X,\mathbb{R}}^p \otimes \mathbb{C} \to \mathcal{E}_{X,\mathbb{R}}^{2n-p} \otimes \mathbb{C}$). However, it is more appropriate to introduce the conjugate operator:

$$* : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{n-p,n-q}, \quad *(\varphi) := \tilde{*}(\overline{\varphi})$$

and then we consider the Hermitian product : $(\psi, \eta)_{L^2} :=< \psi, \overline{\eta} >_{L^2}$, hence:

$$(\psi, \eta)_{L^2} = \int_X \psi \wedge *\eta,\ \psi, \eta \in \mathcal{E}^p(X), \text{ and } (\psi, \eta)_{L^2} = 0 \text{ if } \psi \in \mathcal{E}^p(X), \eta \in \mathcal{E}^q(X), p \neq q.$$

This definition enables us to obtain a Hermitian metric on the complex vector space of global forms $\mathcal{E}^*(X)$ on X, out of the real metric since we have $v \wedge *v = \sum_{|I|=p} |v_I|^2 * (1)$ where $*(1) = vol, v = \sum_{|I|=p} v_I e_I$ for ordered multi-indices $I = i_1 < \ldots < i_p$ of an orthonormal basis e_i , hence the integral vanish iff all v_I vanish everywhere ([41] p.174). With respect to such Hermitian metric, the adjoint operator to $\overline{\partial}$ is defined:

$$\overline{\partial}^* : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p,q-1},\ \overline{\partial}^* = - * \circ \overline{\partial} \circ * : (\overline{\partial}^* \psi, \eta)_{L^2} = (\psi, \overline{\partial}\eta)_{L^2}.$$

Then, we introduce the $\overline{\partial}$-Laplacian:

Definition 1.20. The $\overline{\partial}$-Laplacian is defined as:

$$\Delta_{\overline{\partial}} = \overline{\partial} \circ \overline{\partial}^* + \overline{\partial}^* \circ \overline{\partial}$$

Harmonic forms of type (p, q) are defined as the solutions of the $\overline{\partial}$-Laplacian

$$\mathcal{H}^{p,q}_{\overline{\partial}}(X) = \{\psi \in \mathcal{E}^{p,q}(X) \otimes \mathbb{C} : \Delta_{\overline{\partial}}(\psi) = 0\}.$$

A basic result, which proof involves deep analysis in Elliptic operator theory and it is well documented in various books, for instance ([41] chap. IV, sec. 4) or [18], p. 82 - 84, is stated here without proof:

Theorem 1.21 (Hodge theorem)**.** *On a compact complex Hermitian manifold each $\overline{\partial}$-cohomology class of type (p, q) is represented by a unique $\overline{\partial}$-harmonic global complex differential form of type (p, q):*

$$H^{p,q}_{\overline{\partial}}(X) \simeq \mathcal{H}^{p,q}_{\overline{\partial}}(X)$$

moreover the space of $\overline{\partial}$-harmonic forms is finite dimensional, hence the space of $\overline{\partial}$-cohomology also.

Corollary 1.22. For a compact complex Hermitian manifold:

$$H^q(X, \Omega^p_X) \simeq \mathcal{H}^{p,q}_{\overline{\partial}}(X)$$

In general there is a spectral sequence relating Dolbeault and de Rham cohomology groups [14] and [9] . It will follow from Hodge theory that this spectral sequence degenerates at rank 1.

1.5. **Kähler manifolds.** The exterior product of harmonic forms is not in general harmonic neither the restriction of harmonic forms to a submanifold is harmonic for the induced metric. In general, the $\overline{\partial}$-Laplacian $\Delta_{\overline{\partial}}$ and the Laplacian Δ_d are not related. The theory becomes natural when we add the Kähler condition on the metric, when the fundamental form is closed with respect to the differential d; we refer then to ([41], ch. V, section 4) to establish the relation between $\Delta_{\overline{\partial}}$ and Δ_d and for full proofs in general, most importantly the type components of harmonic forms become harmonic. The proofs in Hodge theory involve the theory of elliptic differential operators on a manifold and will not be given here as we aim just to give the statements.

Definition 1.23. The Hermitian metric is Kähler if its associated $(1, 1)$-form ω is closed: $d\omega = 0$. In this case, the manifold X is called a Kähler manifold.

Example 1.24. 1) A Hermitian metric on a compact Riemann surface (of complex dimension 1) is Kähler since $d\omega$ of degree 3 must vanish.
2) The Hermitian metric on a compact complex torus is Kähler. However, in general a complex tori is not a projective variety. Indeed, by the projective embedding theorem of Kodaira (see [37] Theorem 7.11 p.164), the cohomology must be polarized, or the Kähler form is integral in order to get an algebraic torus, called also an abelian variety since it is a commutative group. This will be the case for $r = 1$, indeed a complex tori of dimension 1 can be always embedded as a cubic curve in the projective plane via Weierstrass function and its derivative.
3) The projective space with the Fubini-Study metric is Kähler.
4) The restriction of a Kähler metric on a submanifold is Kähler with associated $(1, 1)$-form induced by the associated $(1, 1)$-form on the ambient manifold.
5) The product of two Kähler manifolds is Kähler.

6) On the opposite, a Hopf surface is an example of a compact complex smooth surface which is not Kähler. Indeed such surface is the orbit space of the action of a group isomorphic to $\mathbb{Z}$ on $\mathbb{C}^2 - \{0\}$ and generated by the automorphism: $(z_1, z_2) \to (\lambda_1 z_1, \lambda_2 z_2)$ where λ_1, λ_2 are complex of module < 1 (or > 1).

For example, consider $S^3 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ and consider

$$f : S^3 \times \mathbb{R} \simeq \mathbb{C}^2 - \{0\} : f(z_1, z_2, t) = (e^t z_1, e^t z_2)$$

then the action of $\mathbb{Z}$ on $\mathbb{R}$ defined for $m \in \mathbb{Z}$ by $t \to t + m$ is transformed into the action by $(\lambda_1^m, \lambda_2^m) = (e^m, e^m)$. We deduce that the action has no fixed point and the quotient surface is homeomorhic to $S^3 \times S^1$, hence compact. Its fundamental group is isomorphic to $\mathbb{Z}$ as the surface is a quotient of a simply connected surface and its cohomology in degree 1 is also isomorphic to $\mathbb{Z}$, while the cohomology of a Kähler variety in odd degree must have an even dimension as a consequence of the conjugation property in the Hodge decomposition.

On Kähler manifolds, the following key relation between Laplacians is just stated here (see [41] for a proof, or [18] p. 115):

Lemma 1.25.

$$\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_{\partial}$$

We denote by $\mathcal{H}^{p,q}(X)$ the space of harmonic forms of type (p, q), that is the global sections $\varphi \in \mathcal{E}_X^{p,q}(X)$ such that $\Delta_d(\varphi) = 0$, which is equivalent to $\Delta_{\overline{\partial}}\varphi = 0$ as a consequence of the lemma.
Let $\varphi := \sum_{p+q=i} \varphi^{p,q} \in \oplus_{p+q=i}\mathcal{E}^{p,q}(X)$ be harmonic,

$$\Delta_d\varphi = 2 \sum_{p+q=r} (\Delta_{\overline{\partial}}\varphi^{p,q}) = 0 \in \oplus_{p+q=r}\mathcal{E}^{p,q}(X) \Rightarrow \forall p, q,\ 2\Delta_{\overline{\partial}}\varphi^{p,q} = \Delta_d\varphi^{p,q} = 0$$

since $\Delta_{\overline{\partial}}$ is compatible with type. Hence, we deduce:

Corollary 1.26. i) The $\overline{\partial}$-harmonic forms of type (p, q) coincide with the harmonic forms of same type: $\mathcal{H}_{\overline{\partial}}^{p,q}(X) = \mathcal{H}^{p,q}(X)$.
ii) The projection on the (p, q) component of an harmonic form is harmonic and we have a natural decomposition:

$$\mathcal{H}^r(X) \otimes \mathbb{C} = \oplus_{p+q=r}\mathcal{H}^{p,q}(X),\ \mathcal{H}^{p,q}(X) = \overline{\mathcal{H}^{q,p}(X)}.$$

Since Δ_d is real, we deduce the conjugation property.

1.5.1. *Hodge decomposition.* Let $H^{p,q}(X)$ denotes the Dolbeault groups represented by $\overline{\partial}-$closed $(p, q)-$ forms. In general, such forms need not to be $d-$closed, and reciprocally the (p, q) components of a $d-$ closed form are not necessarily $\overline{\partial}-$closed nor $d-$ closed, however this cannot happens on compact Kähler manifolds.

Theorem 1.27. *Let X be a compact Kähler manifold. There is an isomorphism of cohomology classes of type (p, q) with harmonic forms of the same type*

$$H^{p,q}(X) \simeq \mathcal{H}^{p,q}(X).$$

Indeed, let $\varphi = \varphi^{r,0} + \cdots + \varphi^{0,r}$, then $\Delta_{\overline{\partial}}\varphi = \Delta_{\overline{\partial}}\varphi^{r,0} + \cdots + \Delta_{\overline{\partial}}\varphi^{0,r} = 0$ is equivalent to $\Delta_d\varphi^{p,q} = 0$ for each $p + q = r$, since $\Delta_d = 2\Delta_{\overline{\partial}}$.

1.5.2. *Applications to Hodge theory.* We remark first the isomorphisms:

$$H^{p,q}(X) \simeq H^{p,q}_{\overline{\partial}}(X) \simeq H^q(X, \Omega^p_X)$$

which shows in particular for $q = 0$ that the space of global holomorphic p-forms $H^0(X, \Omega^p_X)$ injects into $H^p_{DR}(X)$ with image $H^{p,0}(X)$.
- Global holomorphic p-forms are closed and harmonic for any Kähler metric. They vanish if and only if they are exact.
- There are no non-zero global holomorphic forms on $\mathbb{P}^n$, besides locally constant functions, and more generally:

$$H^{p,q}_{\overline{\partial}}(\mathbb{P}^n) \simeq H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}) = \begin{cases} 0, & \text{if } p \neq q \\ \mathbb{C} & \text{if } p = q \end{cases}$$

Remark 1.28. i) The holomorphic form $z\,dw$ on $\mathbb{C}^2$ is not closed.
ii) The spaces $H^{p,q}$ in H^{p+q}_{DR} are isomorphic to the holomorphic invariant $H^q(X, \Omega^p)$, but the behavior relative to the fiber of a proper holomorphic family $X \to T$ for $t \in T$ is different for each space: $H^q(X_t, \Omega^p)$ is holomorphic in t, de Rham cohomology $H^{p+q}_{DR}(X_t)$ is locally constant but the embedding of $H^{p,q}(X_t)$ into de Rham cohomology $H^{p+q}_{DR}(X_t)$ is not holomorphic.

Exercise 1.29. Let X be a compact Kähler manifold and $\omega := -Im\, h$ its real 2-form of type $(1,1)$ ($\omega \in \mathcal{E}^{1,1}_X \cap \mathcal{E}^2_X$). The volume form $vol \in \mathcal{E}^{2n}(X)$ defined by g on X, can be defined by ω and is equal to $\frac{1}{n!}\omega^n$.
Indeed, if we consider an orthonormal complex basis $e_1, \ldots, e_n$ of the tangent space $T_{X,\mathbb{R},x}$, then $e_1, Ie_1, \ldots, e_n, Ie_n$ is a real oriented orthonormal basis for g and we need to prove $\frac{1}{n!}\omega^n(e_1 \wedge Ie_1 \wedge \cdots \wedge Ie_n) = 1$.
Let $\varphi_i, i \in [1,n]$ denote a local basis of the complex cotangent bundle $T^{*(1,0)}$, unitary for the Hermitian metric, then it is orthogonal and $\|\varphi_j\|^2 = 2$ with respect to the Riemannian metric (for example on $\mathbb{C}^n$, the complex unitary basis dz_j is written as $dx_j + i dy_j$ with dx_j and dy_j elements of an orthonormal basis for the Riemannian structure). We deduce from the decomposition $\varphi_j = \psi'_j + i\psi''j$ into real components, the following formula of the volume:

$$\omega = \frac{i}{2}\sum_j \varphi_j \wedge \overline{\varphi}_j, \quad vol = \frac{\omega^n}{n!} = \psi'_1 \wedge \psi''_1 \wedge \cdots \wedge \psi'_n \wedge \psi''_n.$$

since $\varphi_j \wedge \overline{\varphi}_j = (\frac{2}{i})\psi'_j \wedge \psi''_j$. In particular, ω^n never vanish.

Lemma 1.30. *For a compact Kähler manifold X:*

$$H^{p,p}(X, \mathbb{C}) \neq 0 \quad \textit{for } 0 \leq p \leq \dim X.$$

In fact, the integral of the volume form $\int_X \omega^n > 0$. It follows that the cohomology class $\omega^n \neq 0 \in H^{2n}(X, \mathbb{C})$, hence the cohomology class $\omega^p \neq 0 \in H^{p,p}(X, \mathbb{C})$ since its cup product with ω^{n-p} is not zero.

1.6. **Cohomology class of a subvariety and Hodge conjecture.** To state the Hodge conjecture, we construct the class in de Rham cohomology of a closed complex algebraic subvariety (resp. complex analytic subspace) of codimension p in a smooth complex projective variety (resp. compact Kähler manifold); it is of Hodge type (p,p).

Lemma 1.31. *Let X be a complex manifold and Z a compact complex subanalytic space of dim m in X. The integral of the restriction of a form ω on X to the smooth open subset of Z is convergent and defines a linear function on forms of degree $2m$. It induces a linear map on cohomology*

$$cl(Z) : H^{2m}(X, \mathbb{C}) \to \mathbb{C}, \quad [\omega] \mapsto \int_{Z_{smooth}} \omega_{|Z}$$

Moreover, if X is compact Kähler, $cl(Z)$ vanish on all components of the Hodge decomposition which are distinct from $H^{m,m}$.

If Z is compact and smooth, the integral is well defined, because, by Stokes theorem, if $\omega = d\eta$, the integral vanish since it is equal to the integral of η on the boundary $\partial Z = \emptyset$ of Z.

If Z is not smooth, the easiest proof is to use a desingularisation (see [26]) $\pi : Z' \to Z$ inducing an isomorphism $Z'_{smooth} \simeq Z_{smooth}$ which implies an equality of the integrals of $\pi^*(\omega_{|Z})$ and $\omega_{|Z}$, which is independent of the choice of Z'. The restriction of ω of degree $2m$ vanish unless it is of type m, m since Z_{smooth} is an analytic manifold of dimension m.

1.6.1. *Poincaré duality.* On compact oriented differentiable manifolds, we use the wedge product of differential forms to define the cup-product on de Rham cohomology and integration to define the trace, so that we can state Poincaré duality [18].

Cup-product. For a manifold X, the cup product is a natural bilinear operation on de Rham cohomology

$$H^i(X, \mathbb{C}) \otimes H^j(X, \mathbb{C}) \xrightarrow{\smile} H^{i+j}(X, \mathbb{C})$$

defined by the wedge product on the level of differential forms.

It is a topological product defined on cohomology with coefficients in $\mathbb{Z}$, but its definition on cohomology groups with integer coefficients is less straightforward.

The Trace map. On a compact oriented manifold X of dimension n, the integral over X of a differential ω of highest degree n depends only on its class modulo boundary by Stokes theorem, hence it defines a map called the trace:

$$Tr : H^n(X, \mathbb{C}) \to \mathbb{C} \quad [\omega] \mapsto \int_X \omega.$$

The following theorem is stated without proof:

Theorem 1.32 (Poincaré duality)**.** *Let X be a compact oriented manifold of dimension n. The cup-product and the trace map:*

$$H^j(X, \mathbb{C}) \otimes H^{n-j}(X, \mathbb{C}) \xrightarrow{\smile} H^n(X, \mathbb{C}) \xrightarrow{Tr} \mathbb{C}$$

define an isomorphism:

$$H^j(X, \mathbb{C}) \xrightarrow{\sim} Hom(H^{n-j}(X, \mathbb{C}), \mathbb{C}).$$

If the compact complex analytic space Z is of codimension p in the smooth compact complex manifold X its class $cl(Z) \in H^{2n-2p}(X, \mathbb{C})^*$ corresponds, by Poincaré duality on X, to a fundamental cohomology class $[\eta_Z] \in H^{2p}(X, \mathbb{C})$. Then we have by definition:

Lemma and Definition 1.33. *For a complex compact manifold X, the fundamental cohomology class $[\eta_Z] \in H^{p,p}(X,\mathbb{C})$ of a closed complex submanifold Z of codimension p satisfies the following relation:*

$$\int_X \varphi \wedge \eta_Z = \int_Z \varphi_{|Z}, \quad \forall \varphi \in \mathcal{E}^{n-p,n-p}(X).$$

Lemma 1.34. *For a compact Kähler manifold X, the cohomology class of a compact complex analytic closed submanifold Z of codim p is a non-zero element $[\eta_Z] \neq 0 \in H^{p,p}(X,\mathbb{C})$, for $0 \leq p \leq$ dim X.*

Proof. For a compact Kähler manifold X, let ω be a Kähler form, then the integral on Z of the restriction $\omega_{|Z}$ is positive since it is a Kähler form on Z, hence $[\eta_Z] \neq 0$:

$$\int_X (\wedge^{n-p}\omega) \wedge \eta_Z = \int_Z \wedge^{n-p}(\omega_{|Z}) > 0.$$

□

1.6.2. *Topological construction.* The dual vector space $Hom_{\mathbb{C}}(H^{2m}(X,\mathbb{C}),\mathbb{C})$ is naturally isomorphic to the homology vector space which suggests that the fundamental class is defined naturally in homology. Indeed, Hodge conjecture has gained so much attention that it is of fundamental importance to discover what conditions can be made on classes of algebraic subvarieties, including the definition of the classes in cohomology with coefficients in $\mathbb{Z}$.

The theory of homology and cohomology traces its origin to the work of Poincaré in the late nineteenth century. There are actually many different theories, for example, simplicial and singular. In 1931, Georges de Rham proved a conjecture of Poincaré on a relationship between cycles and differential forms that establishes for a compact orientable manifold an isomorphism between singular cohomology with real coefficients and what is known now as de Rham cohomology. In fact the direct homological construction of the class of Z is natural and well known.

Homological class. The idea that homology should represent the classes of topological subspaces has been probably at the origin of homology theory, although the effective construction of homology groups is different and more elaborate. The simplest way to construct Homology groups is to use singular simplices. But the definition of homology groups $H_j(X,\mathbb{Z})$ of a triangulated space is quite natural. Indeed, the sum of oriented triangles of highest dimensions of an oriented triangulated topological space X of real dimension n, defines an homology class $[X] \in H_n(X,\mathbb{Z})$ ([18], Ch 0, paragraph 4).

We take as granted here that a closed subvariety Z of dimension m in a compact complex algebraic variety X can be triangulated such that the sum of its oriented triangles of highest dimensions defines its class in the homology group $H_{2m}(X,\mathbb{Z})$.

1.6.3. *Cap product.* Cohomology groups of a topological space $H^i(X,\mathbb{Z})$ are dually defined , and there exists a topological operation on homology and cohomology, called the cap product:

$$\frown : H^q(X,\mathbb{Z}) \otimes H_p(X,\mathbb{Z}) \to H_{p-q}(X,\mathbb{Z})$$

We can state now a duality theorem of Poincaré, frequently used in geometry.

Theorem 1.35 (Poincaré duality isomorphism)**.** *Let X be a compact oriented topological manifold of dimension n. The cap product with the fundamental class $[X] \in H_n(X, \mathbb{Z})$ defines an isomorphism, for all j, $0 \leq j \leq n$:*

$$D_X : H^j(X, \mathbb{Z}) \xrightarrow{\frown [X]} H_{n-j}(X, \mathbb{Z})$$

The homology class $i_*[Z]$ of a compact complex analytic subspace $i : Z \to X$ of codimension p in the smooth compact complex manifold X of dimension n corresponds by the inverse of Poincaré duality isomorphism D_X, to a fundamental cohomology class:

$$[\eta_Z]^{top} \in H^{2p}(X, \mathbb{Z}).$$

Lemma 1.36. *The canonical morphism $H_{2n-2p}(X, \mathbb{Z}) \to H^{2n-2p}(X, \mathbb{C})^*$ carry the topological class $[Z]$ of an analytic subspace Z of codimension p in X into the fundamental class $cl(Z)$.*
Respectively. the morphism $H^{2p}(X, \mathbb{Z}) \to H^{2p}(X, \mathbb{C})$ carry the topological class $[\eta_Z]^{top}$ to $[\eta_Z]$.

1.6.4. *Intersection product in topology and geometry.* On a triangulated space, cycles are defined as a sum of triangles with boundary zero, and homology is defined by classes of cycles modulo boundaries. It is always possible to represent two homology classes of degree p and q by two cycles of codim. p and q in 'transversal position' , so that their intersection is defined as a cycle of codim.$p + q$. Moreover, for two representations by transversal cycles, the intersections cycles are homologous [16], ([12] 2.8). Then a theory of intersection product on homology can be deduced:

$$H_{n-p}(X, \mathbb{Z}) \otimes H_{n-q}(X, \mathbb{Z}) \xrightarrow{\cap} H_{n-p-q}(X, \mathbb{Z})$$

In geometry, two closed submanifolds V_1 and V_2 of a compact oriented manifold M can be deformed into a transversal position so that their intersection can be defined as a submanifold W with a sign depending on the orientation ([23], ch 2), then W is defined up to a deformation such that the homology classes satisfy the relation $[V_1] \cap [V_2] = \pm[W]$. The deformation class of W with sign is called $V_1 \cap V_2$.

Poincaré duality in Homology ([18], p 53) .
The intersection pairing:

$$H_j(X, \mathbb{Z}) \otimes H_{n-j}(X, \mathbb{Z}) \xrightarrow{\cap} H_0(X, \mathbb{Z}) \xrightarrow{degree} \mathbb{Z}$$

for $0 \leq j \leq n$ is unimodular: the induced morphism

$$H_j(X, \mathbb{Z}) \to Hom(H_{n-j}(X, \mathbb{Z}), \mathbb{Z})$$

is surjective and its kernel is the torsion subgroup of $H_j(X, \mathbb{Z})$.

1.6.5. *Relation between Intersection and cup products.* The cup product previously defined in de Rham cohomology is a topological product defined with coefficients in $\mathbb{Z}$, but its definition on cohomology groups is less straightforward. The trace map can also be defined with coefficients in $\mathbb{Z}$, and we also have the corresponding duality statement,

Poincaré duality in cohomology.
The cup-product and the trace map:

$$H^j(X, \mathbb{Z}) \otimes H^{n-j}(X, \mathbb{Z}) \xrightarrow{\smile} H^n(X, \mathbb{Z}) \xrightarrow{Tr} \mathbb{Z}$$

define a unimodular pairing inducing an isomorphism:

$$H^j(X,\mathbb{Q}) \xrightarrow{\sim} Hom(H^{n-j}(X,\mathbb{Q}),\mathbb{Q})$$

compatible with the pairing in de Rham cohomology.

Poincaré duality isomorphism transforms the intersection pairing into the cup product:
The following result is proved in [18] (p. 59) in the case $k' = n - k$:
Let σ be a $k-$cycle on an oriented manifold X of real dimension n and τ an $k'-$cycle on X with Poincaré duals $\eta_\sigma \in H^{n-k}(X)$ and $\eta_\tau \in H^{n-k'}(X)$, then:

$$\eta_\sigma \smile \eta_\tau = \eta_{\sigma\frown\tau} \in H^{n-k-k'}(X)$$

1.6.6. The Hodge type (p,p) of the fundamental class of analytic compact submanifold of codimension p is an analytic condition. The search for properties characterizing classes of cycles has been motivated by a question of Hodge.

Definition 1.37. Let $A = \mathbb{Z}$ or $\mathbb{Q}$, $p \in \mathbb{N}$ and let $\varphi : H^{2p}(X,A) \to H^{2p}(X,\mathbb{C})$ denotes the canonical map. The group of cycles

$$H^{p,p}(X,A) := \{x \in H^{2p}(X,A) : \varphi(x) \in H^{p,p}(X,\mathbb{C})\}$$

is called the group of Hodge classes of type (p,p).

Definition 1.38. An $r-$cycle of an algebraic variety X is a formal finite linear combination $\sum_{i\in[1,h]} m_i Z_i$ of closed irreducible subvarieties Z_i of dimension r with integer coefficients m_i. The group of $r-$cycles is denoted by $\mathcal{Z}_r(X)$.

For a compact complex algebraic manifold, the class of closed irreducible subvarieties of codimension p extends into a linear morphism:

$$cl_A : \mathcal{Z}_p(X) \otimes A \to H^{p,p}(X,A) : \sum_{i\in[1,h]} m_i Z_i \mapsto \sum_{i\in[1,h]} m_i \eta_{Z_i}, \forall m_i \in A$$

The elements of the image of $cl_{\mathbb{Q}}$ are called rational algebraic Hodge classes of type (p,p).

1.6.7. *Hodge conjecture.* Is the map $cl_{\mathbb{Q}}$ surjective when X is a projective manifold? In other terms, is any rational Hodge class algebraic?

The question is to construct a cycle of codimension p out of a rational cohomology element of type (p,p) knowing that cohomology is constructed via topological technique.

Originally, the Hodge conjecture was stated with $\mathbb{Z}$-coefficients, but there are torsion elements which cannot be represented by algebraic cycles. There exists compact Kähler complex manifolds not containing enough analytic subspaces to represent all Hodge cycles [38].

Remark 1.39 (Absolute Hodge cycle). In fact Deligne added another property for algebraic cycles by introducing the notion of absolute Hodge cycle (see [8]). An algebraic cycle Z is necessarily defined over a field extension K of finite type over $\mathbb{Q}$. Then its cohomology class in the de Rham cohomology of X over the field K defines, for each embedding $\sigma : K \to \mathbb{C}$, a cohomology class $[Z]_\sigma$ of type (p,p) in the cohomology of the complex manifold X^{an}_σ.

Remark 1.40 (Grothendieck fundamental class). For an algebraic subvariety Z of codimension p in a variety X of dimension n, the fundamental class can be defined as an element of the group $Ext^p(\mathcal{O}_Z, \Omega^p_X)$ (see [20], [24]). Let U be an affine subset and suppose that $Z \cap U$ is defined as a complete intersection by p regular functions f_i, if we use the Koszul resolution of $\mathcal{O}_{Z\cap U}$ defined by the elements f_i to compute the extension groups, then the cohomology class is defined by a symbol:

$$\left[\begin{array}{c} df_1 \wedge \cdots \wedge df_p \\ f_1 \cdots f_p \end{array}\right] \in Ext^p(\mathcal{O}_{Z\cap U}, \Omega^p_U).$$

This symbol is independent of the choice of generators of $\mathcal{O}_{Z\cap U}$, and it is the restriction of a unique class defined over Z which defines the cohomology class of Z in the de Rham cohomology group $\mathbb{H}^{2p}_Z(X, \Omega^*_X)$ with support in Z ([10]). The extension groups and cohomology groups with support in closed algebraic subvarieties form the basic idea to construct the dualizing complex K^*_X of X as part of Grothendieck duality theory (see [24]).

2. Lefschetz decomposition and Polarized Hodge structure

In this chapter, we give more specific results for smooth complex *projective* varieties, namely the Lefschetz decomposition and primitive cohomology subspaces, Riemann bilinear relations and their abstract formulation into polarization of Hodge Structures. We start with the statement of the results. Proofs and complements follow.

2.1. **Lefschetz decomposition and primitive cohomology.** Let (X, ω) be a compact Kähler manifold of class $[\omega] \in H^2(X, \mathbb{R})$ of Hodge type $(1, 1)$. The cup-product with $[\omega]$ defines morphisms:

$$L : H^q(X, \mathbb{R}) \to H^{q+2}(X, \mathbb{R}), \quad L : H^q(X, \mathbb{C}) \to H^{q+2}(X, \mathbb{C})$$

Referring to de Rham cohomology, the action of L is represented on the level of forms as $\varphi \mapsto \varphi \wedge \omega$. Since ω is closed, the image of a closed form (resp. a boundary) is closed (resp. a boundary). Let $n = \dim X$.

Definition 2.1. The primitive cohomology subspaces are defined as:

$$H^{n-i}_{prim}(\mathbb{R}) := Ker(L^{i+1} : H^{n-i}(X, \mathbb{R}) \to H^{n+i+2}(X, \mathbb{R}))$$

and similarly for complex coefficients $H^{n-i}_{\text{prim}}(\mathbb{C}) \simeq H^{n-i}_{\text{prim}}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

The operator L is compatible with Hodge decomposition since it sends the subspace $H^{p,q}$ to $H^{p+1,q+1}$. We shall say that the morphism L is of Hodge type $(1, 1)$; hence the action $L^{i+1} : H^{n-i}(X, \mathbb{C}) \to H^{n+i+2}(X, \mathbb{C})$ is a morphism of Hodge type $(i+1, i+1)$, and the kernel is endowed with an induced Hodge decomposition. This is a strong condition on the primitive subspace, since if we let:

$$H^{p,q}_{\text{prim}} := H^{p+q}_{\text{prim}} \cap H^{p,q}(X, \mathbb{C}),$$

then:

$$H^i_{\text{prim}}(X, \mathbb{C}) = \oplus_{p+q=i} H^{p,q}_{prim}.$$

The following isomorphism, referred to as Hard Lefschetz Theorem, puts a strong condition on the cohomology of projective, and more generally compact Kähler, manifolds and gives rise to a decomposition of the cohomology in terms of primitive subspaces:

Theorem 2.2. *Let X be a compact Kähler manifold.*
i) Hard Lefschetz Theorem. *The iterated linear operator L induces isomorphisms for each i:*

$$L^i : H^{n-i}(X,\mathbb{R}) \xrightarrow{\sim} H^{n+i}(X,\mathbb{R}),\ L^i : H^{n-i}(X,\mathbb{C}) \xrightarrow{\sim} H^{n+i}(X,\mathbb{C})$$

ii) Lefschetz Decomposition. *The cohomology decomposes into a direct sum of image of primitive subspaces by L^r, $r \geq 0$:*

$$H^q(X,\mathbb{R}) = \oplus_{r\geq 0} L^r H^{q-2r}_{\text{prim}}(\mathbb{R}), \quad H^q(X,\mathbb{C}) = \oplus_{r\geq 0} L^r H^{q-2r}_{\text{prim}}(\mathbb{C})$$

The Lefschetz decomposition is compatible with Hodge decomposition.
iii) If X is moreover projective, then the action of L is defined with rational coefficients and the decomposition applies to rational cohomology.

2.1.1. *Hermitian product on cohomology.* From the isomorphism in the Hard Lefschetz theorem and Poincaré duality, we deduce a scalar product on cohomology of smooth complex projective varieties compatible with Hodge Structures and satisfying relations known as Hodge Riemann relations leading to a polarization of the primitive cohomology which is an additional highly rich structure characteristic of such varieties.
Representing cohomology classes by differential forms, we define a bilinear form:

$$Q(\alpha,\beta) = (-1)^{\frac{j(j-1)}{2}} \int_X \alpha \wedge \beta \wedge \omega^{n-j}, \quad \forall [\alpha],[\beta] \in H^j(X,\mathbb{C})$$

where ω is the Kähler class, the product of α with ω^{n-j} represents the action of L^{n-j} and the integral of the product with β represents Poincaré duality.

Properties of the product. The above product $\mathrm{Q}(\alpha,\beta)$ depends only on the class of α and β. The following properties are satisfied:
i) the product Q is real (it takes real values on real forms) since ω is real, in other terms the matrix of Q is real, skew-symmetric if j is odd and symmetric if j is even;
ii) It is non degenerate, by Lefschetz isomorphism and Poincaré duality;
iii) By consideration of type, the Hodge and Lefschetz decompositions are orthogonal relative to Q:

$$\mathrm{Q}(H^{p,q}, H^{p',q'}) = 0, \quad \text{unless } p = p', q = q'.$$

On projective varieties the Kähler class is in the integral lattice defined by cohomology with coefficients in $\mathbb{Z}$, hence the product is defined on rational cohomology and preserves the integral lattice. In this case we have more precise positivity relations in terms of the primitive component $H^{p,q}_{prim}(\mathbb{C})$ of the cohomology $H^{p+q}(X,\mathbb{C})$.

Proposition 2.3 (Hodge-Riemann bilinear relations)**.** The product $i^{p-q}Q(\alpha,\overline{\alpha})$ is positive definite on the primitive component $H^{p,q}_{prim}$:

$$i^{p-q}Q(\alpha,\overline{\alpha}) > 0, \quad \forall \alpha \in H^{p,q}_{prim},\ \alpha \neq 0$$

2.1.2. *Summary of the proof of Lefschetz decomposition.*
1) First, we consider the action of L on sheaves, $L = \wedge\omega : \mathcal{E}^r_X \to \mathcal{E}^{r+2}_X$, then we introduce its formal adjoint operator with respect to the Hermitian form: $\Lambda = L^* : \mathcal{E}^r_X \to \mathcal{E}^{r-2}_X$ which can be defined, using the Hodge star operator, by $\Lambda = *^{-1} \circ L \circ *$.

Note that the operator:

$$h = \sum_{p=0}^{2n}(n-p)\Pi^p, \quad h(\sum_p \omega_p) = \sum_p (n-p)\omega_p, \quad \text{for } \omega_p \in \mathcal{E}_X^p,$$

where Π^p is the projection in degree p on $\oplus_{p\in[0,2n]}\mathcal{E}_X^p$ and $n = \dim X$, satisfies the relation:

$$[\Lambda, L] = h.$$

from which we can deduce the injectivity of the morphism:

$$L^i : \mathcal{E}_X^{n-i} \to \mathcal{E}_X^{n+i}.$$

For this we use the following formula, for $\alpha \in \mathcal{E}_X^k$:

$$[L^r, \Lambda](\alpha) = (r(k-n) + r(r-1))L^{r-1}(\alpha),$$

which is proved by induction on r.
The morphism L^j *commutes with the Laplacian* and, since cohomology classes can be represented by global harmonic sections, L^j induces an isomorphism on cohomology vector spaces which are of finite equal dimension by Poincaré duality:

$$L^j : H^{n-j}(X, \mathbb{R}) \xrightarrow{\sim} H^{n+j}(X, \mathbb{R})$$

Moreover the extension of the operator L to complex coefficients is compatible with the bigrading (p, q) since ω is of type $(1, 1)$. The decomposition of the cohomology into direct sum of image of primitive subspaces by L^r, $r \geq 0$ follows from the previous isomorphisms.
2) Another proof is based on the representation theory of the Lie algebra sl_2. We represent L by the action on global sections of the operator:

$$L = \wedge\omega : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+1,q+1}(X)$$

and its adjoint $\Lambda = L^* : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p-1,q-1}(X)$ is defined by $\Lambda = *^{-1} \circ L \circ *$. We have the relations:

$$[\Lambda, L] = h, \quad [h, L] = -2L, \quad [h, \Lambda] = 2\Lambda$$

Hence, the operators L, Λ, h generate a Lie algebra isomorphic to sl_2. We deduce from the action of the operators L, Λ, h on the space of harmonic forms $\oplus\mathcal{H}_d^*(X) \simeq \oplus H^*(X)$, a representation of the Lie algebra sl_2 by identifying the matrices:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ resp. with } \Lambda,\ L,\ h.$$

Then the theorem follows from the general structure of such representation (see [18] p. 122-124).

2.1.3. *Proof of Hodge-Riemann bilinear relations.* We recall the notion of a primitive form. A differential form $\alpha \in \Omega^k_{X,x,\mathbb{R}}, k \leq n$ is said to be primitive if $L^{n-k+1}\alpha = 0$; then every form $\beta \in \Omega^k_{X,x,\mathbb{R}}$ decomposes in a unique way into a direct sum $\beta = \sum_r L^r\beta_r$ where each β_r is primitive of degree $k-2r$ with the condition $k-2r \leq \inf(k, 2n-k)$. Moreover, the decomposition applies to the complexified forms and it is compatible with bidegrees (p, q) of forms.

The following lemma is admitted and is useful in the proof ([37] prop.6.29)

Lemma 2.4. *Let $\gamma \in \Omega^{q,p}_{X,x} \subset \mathcal{E}^j_{X,x}$ be a primitive element, then*

$$*\gamma = (-1)^{\frac{j(j+1)}{2}} i^{q-p} \frac{L^{n-j}}{(n-j)!}\gamma$$

It is applied to $\overline{\alpha}$ in the definition of the product Q as follows. We represent a primitive cohomolgy class by a primitive harmonic form $\tilde{\alpha}$ of degree $j = p+q$, then we write the product in terms of the L^2-norm, as follows

$$\mathrm{F}(\alpha,\alpha) = i^{p-q}Q(\alpha,\overline{\alpha}) = (n-j)!\int_X \tilde{\alpha}\wedge *\overline{\tilde{\alpha}} = (n-j)!||\tilde{\alpha}||^2_{L^2}$$

Indeed, since $(-1)^j i^{q-p} = (-1)^{q-p} i^{q-p} = i^{p-q}$ we deduce from the lemma $L^{n-j}\overline{\tilde{\alpha}} = i^{q-p}(n-j)!(-1)^{\frac{j(j-1)}{2}}(*\overline{\tilde{\alpha}})$.

This result suggest to introduce the Weil operator C on cohomology:

$$C(\alpha) = i^{p-q}\alpha, \quad \forall \alpha \in H^{p,q}$$

Notice that C is a real operator since for a real vector $v = \sum_{p+q=j} v^{p,q}$, $v^{q,p} = \overline{v^{p,q}}$, hence $\overline{Cv} = \sum \overline{i^{p-q}v^{p,q}} = \sum \overline{i}^{p-q} v^{q,p} = i^{q-p}v^{q,p}$, as $\overline{i}^{p-q} = i^{q-p}$. It depends on the decomposition, in particular the action of C in a variation of Hodge structure is differentiable in the parameter space. We deduce from Q a new non degenerate Hermitian product:

$$\mathrm{F}(\alpha,\beta) = Q(C(\alpha),\overline{\beta}),\ \mathrm{F}(\beta,\alpha) = \overline{\mathrm{F}(\alpha,\beta)} \quad \forall [\alpha],[\beta] \in H^j(X,\mathbb{C})$$

We use $\overline{Q(\alpha,\beta)} = Q(\overline{\alpha},\overline{\beta})$ since Q is real, to check for $\alpha,\beta \in H^{p,q}$:
$\overline{\mathrm{F}(\alpha,\beta)} = \overline{Q(i^{p-q}\alpha,\overline{\beta})} = \overline{i}^{p-q}Q(\overline{\alpha},\beta) = (-1)^j i^{p-q}Q(\overline{\alpha},\beta) = (-1)^{2j}i^{p-q}Q(\beta,\overline{\alpha}) = \mathrm{F}(\beta,\alpha)$.

Remark 2.5. i) When the class $[\omega] \in H^j(X,\mathbb{Z})$ is integral, which is the case for algebraic varieties, the product Q is integral, i.e., with integral value on integral classes.
ii) The integral bilinear form Q extends by linearity to the complex space $H^k_{\mathbb{Z}} \otimes \mathbb{C}$. Its associated form $\mathrm{F}(\alpha,\beta) := Q(\alpha,\overline{\beta})$ is not Hermitian if k is odd. One way to overcome this sign problem is to define F as $\mathrm{F}(\alpha,\beta) := i^k Q(\alpha,\overline{\beta})$, still this form will not be positive definite in general.

2.1.4. *Projective case.* In the special projective case we can choose the fundamental class of an hyperplane section to represent the Kähler class $[\omega]$, hence we have an integral representative of the class $[\omega]$ in the image of $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$, which has an important consequence since the operator $L : H^q(X,\mathbb{Q}) \to H^{q+2}(X,\mathbb{Q})$ acts on rational cohomology. This fact characterizes projective varieties among compact Kähler manifolds since a theorem of Kodaira ([41] chap. VI) states that a Kähler variety with an integral class $[\omega]$ is projective, i.e., it can be embedded as a closed analytic subvariety in a projective space, hence by Chow Lemma it is necessarily an algebraic subvariety.

Remark 2.6 (Topological interpretation). In the projective case, the class $[\omega]$ corresponds to the homology class of an hyperplane section $[H] \in H_{2n-2}(X,\mathbb{Z})$, so that the operator L corresponds to the intersection with $[H]$ in X and the result is an isomorphism:

$$H_{n+k}(X) \xrightarrow{(\cap[H])^k} H_{n-k}(X)$$

The primitive cohomology $H^{n-k}_{prim}(X)$ corresponds to the image of:

$$H_{n-k}(X - H, \mathbb{Q}) \to H_{n-k}(X, \mathbb{Q}).$$

In his original work, Lefschetz used the hyperplane section of a projective variety in homology to investigate the decomposition theorem proved later in the above analysis setting.

Definition 2.7. The cohomology of a projective complex smooth variety with its Hodge decomposition and the above positive definite Hermitian product is called a polarized Hodge Structure.

2.2. **The category of Hodge Structures.** It is rewarding to introduce the Hodge decomposition as a formal structure in linear algebra without any reference to its construction.

Definition 2.8. A Hodge structure (HS) of weight n is defined by the data:
i) A finitely generated abelian group $H_{\mathbb{Z}}$;
ii) A decomposition by complex subspaces:

$$H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{p+q=n} H^{p,q} \quad \text{satisfying} \quad H^{p,q} = \overline{H^{q,p}}.$$

The conjugation on $H_{\mathbb{C}}$ makes sense with respect to $H_{\mathbb{Z}}$. A subspace $V \subset H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ satisfying $\overline{V} = V$ has a real structure, that is $V = (V \cap H_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$. In particular $H^{p,p} = (H^{p,p} \cap H_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$. We may suppose that $H_{\mathbb{Z}}$ is a free abelian group (the lattice), if we are interested only in its image in $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$.

With such an abstract definition we can perform linear algebra operations on a Hodge Structure and define morphisms. We remark that *Poincaré duality* is compatible with the Hodge Structure defined by Hodge decomposition, after an adequate shift in the weight (see below). We deduce from the previous results the notion of polarization:

Definition 2.9 (Polarization of HS)**.** A Hodge structure $(H_{\mathbb{Z}}, (H_{\mathbb{C}} \simeq \oplus_{p+q=n} H^{p,q}))$ of weight n is polarized if a non-degenerate scalar product Q is defined on $H_{\mathbb{Z}}$, alternating if n is odd and symmetric if n is even, such that the terms $H^{p,q}$ of the Hodge decomposition are orthogonal to each other relative to the Hermitian form F on $H_{\mathbb{C}}$ defined as $\mathrm{F}(\alpha, \beta) := i^{p-q} Q(\alpha, \overline{\beta})$ for $\alpha, \beta \in H^{p,q}$ and $\mathrm{F}(\alpha, \alpha)$ is positive definite on the component of type (p, q), i.e., it satisfies the Hodge-Riemann bilinear relations.

2.2.1. *Equivalent definition of HS.* (see 2.2.5 below) Let $\mathbb{S}(\mathbb{R})$ denotes the subgroup:

$$\mathbb{S}(\mathbb{R}) = \left\{ M(u, v) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \in GL(2, \mathbb{R}), \quad u, v \in \mathbb{R} \right\}.$$

It is isomorphic to $\mathbb{C}^*$ via the group homomorphism $M(u, v) \mapsto z = u + iv$.
The interest in this isomorphism is to put the structure of a real algebraic group on $\mathbb{C}^*$; indeed the set of matrices $M(u, v)$ is a real algebraic subgroup of $GL(2, \mathbb{R})$.

Definition 2.10. A rational Hodge Structure of weight $m \in \mathbb{Z}$, is defined by a $\mathbb{Q}-$vector space H and a representation of real algebraic groups $\varphi : \mathbb{S}(\mathbb{R}) \to GL(H_{\mathbb{R}})$ such that for $t \in \mathbb{R}^*$, $\varphi(t)(v) = t^m v$ for all $v \in H_{\mathbb{R}}$.

2.2.2. *The Hodge filtration.* To study variations of Hodge Structure, Griffiths introduced an equivalent structure defined by the Hodge filtration which varies holomorphically with parameters.

Definition 2.11. Given a Hodge Structure $(H_{\mathbb{Z}}, H^{p,q})$ of weight n, we define a decreasing filtration F by subspaces of $H_{\mathbb{C}}$:

$$F^p H_{\mathbb{C}} := \oplus_{r \geq p} H^{r,n-r}.$$

Then, the following decomposition is satisfied:

$$H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{n-p+1} H_{\mathbb{C}}}.$$

The Hodge decomposition may be recovered from the filtration by the formula:

$$H^{p,q} = F^p H_{\mathbb{C}} \cap \overline{F^q H_{\mathbb{C}}}, \quad \text{for } p+q=n.$$

Hence, we obtain an equivalent definition of Hodge Structures, which played an important role in the development of Hodge theory, since we will see in chapter 5 that the Hodge filtration exists naturally on the cohomology of any complex proper smooth algebraic variety X and it is induced by the trivial filtration on the de Rham complex which may be defined algebraically. However, the Hodge decomposition itself is still deduced from the existence of a birational projective variety over X and the decomposition on the cohomology of this projective variety viewed as a Kähler manifold.

2.2.3. *Linear algebra operations on Hodge structures.* The interest of the above abstract definition of Hodge Structures is to apply classical linear algebra operations.

Definition 2.12. A morphism $f : H = (H_{\mathbb{Z}}, H^{p,q}) \to H' = (H'_{\mathbb{Z}}, H'^{p,q})$ of Hodge Structures, is an homomorphism $f\colon H_{\mathbb{Z}} \to {H'}_{\mathbb{Z}}$ such that $f_{\mathbb{C}}\colon H_{\mathbb{C}} \to {H'}_{\mathbb{C}}$ is compatible with the decompositions, i.e, for any p, q, $f_{\mathbb{C}}$ induces a $\mathbb{C}$-linear map from $H^{p,q}$ into $H'^{p,q}$.

Therefore Hodge Structures form a category whose objects are Hodge Structures and morphisms are morphisms of Hodge Structures that we have just defined. We have the following important result:

Proposition 2.13. *The Hodge Structures of same weight n form an abelian category.*

In particular, the decomposition on the kernel (resp. cokernel) of a morphism $\varphi : H \to H'$ is induced by the decomposition of H (resp. H'). For a general subvector space V of H, the induced subspaces $H^{p,q} \cap V$ do not define a decomposition on V. If H and H' are of distinct weights, f is necessarily 0.

2.2.4. *Tensor product and Hom.* Let H and H' be two HS of weight n and n'.
1) We define their Hodge Structure tensor product $H \otimes H'$ of weight $n + n'$ as follows:
i) $(H \otimes H')_{\mathbb{Z}} = H_{\mathbb{Z}} \otimes H'_{\mathbb{Z}}$
ii) the bigrading of $(H \otimes H')_{\mathbb{C}} = H_{\mathbb{C}} \otimes H'_{\mathbb{C}}$ is the tensor product of the bigradings of $H_{\mathbb{C}}$ and $H'_{\mathbb{C}}$: $(H \otimes H')^{a,b} := \oplus_{p+p'=a, q+q'=b} H^{p,q} \otimes H'^{p',q'}$.

Example 2.14. Tate Hodge structure $\mathbb{Z}(1)$ is a Hodge Structure of weight -2 defined by:

$$H_{\mathbb{Z}} = 2i\pi\mathbb{Z} \subset \mathbb{C}, \quad H_{\mathbb{C}} = H^{-1,-1}$$

It is purely bigraded of type $(-1,-1)$ of rank 1. The $m-$tensor product $\mathbb{Z}(1) \otimes \cdots \otimes \mathbb{Z}(1)$ of $\mathbb{Z}(1)$ is a Hodge Structure of weight $-2m$ denoted by $\mathbb{Z}(m)$:

$$H_{\mathbb{Z}} = (2i\pi)^m\mathbb{Z} \subset \mathbb{C}, \quad H_{\mathbb{C}} = H^{-m,-m}$$

Let $H = (H_{\mathbb{Z}}, \oplus_{p+q=n} H^{p,q})$ be a Hodge Structure of weight n, its $m-$twist is a Hodge Structure of weight $n-2m$ denoted $H(m)$ and defined by

$$H(m)_{\mathbb{Z}} := H_{\mathbb{Z}} \otimes (2i\pi)^m\mathbb{Z}, \quad H(m)^{p,q} := H^{p+m,q+m}$$

2) Similarly, $\wedge^p H$ is a Hodge Structure of weight pn.

3) We define a Hodge Structure on $Hom(H, H')$ of weight $n' - n$ as follows:
i) $Hom(H, H')_{\mathbb{Z}} := Hom_{\mathbb{Z}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})$;
ii) the decomposition of $Hom(H, H')_{\mathbb{C}} := Hom_{\mathbb{Z}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}}) \otimes \mathbb{C} \simeq Hom_{\mathbb{C}}(H_{\mathbb{C}}, H'_{\mathbb{C}})$ is given by $Hom(H, H')^{a,b} := \oplus_{p'-p=a, q'-q=b} Hom_{\mathbb{C}}(H^{p,q}, H^{p',q'})$.
In particular the dual H^* to H is a Hodge Structure of weight $-n$.

Remark 2.15. The group of morphisms of Hodge Structures is called the internal morphism group of the category of Hodge Structures and denoted by $Hom_{HS}(H, H')$; it is the sub-group of $Hom_{\mathbb{Z}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})$ of elements of type $(0,0)$ in the Hodge Structure on $Hom(H, H')$. A homomorphism of type (r, r) is a morphism of the Hodge Structures: $H \to H'(-r)$.

Poincaré duality. If we consider the following trace map:

$$H^{2n}(X, \mathbb{C}) \xrightarrow{\sim} \mathbb{C}(-n), \quad \omega \to \frac{1}{(2i\pi)^n} \int_X \omega.$$

then Poincaré duality will be compatible with Hodge Structures:

$$H^{n-i}(X, \mathbb{C}) \simeq Hom(H^{n+i}(X, \mathbb{C}), \mathbb{C}(-n)).$$

The Hodge Structure on homology is defined by duality:

$$(H_i(X, \mathbb{C}), F) \simeq Hom((H^i(X, \mathbb{C}), F), \mathbb{C})$$

with $\mathbb{C}$ pure of weight 0, hence $H_i(X, \mathbb{Z})$ is pure of weight $-i$. Then, Poincaré duality becomes an isomorphism of Hodge Structures: $H^{n+i}(X, \mathbb{C}) \simeq H_{n-i}(X, \mathbb{C})(-n)$.

Gysin morphism. Let $f : X \to Y$ be an algebraic morphism with $\dim X = n$ and $\dim Y = m$, since $f^* : H^i(Y, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$ is compatible with Hodge Structures, its Poincaré dual $Gysin(f) : H^j(Y, \mathbb{Q}) \to H^{j+2(m-n)}(Y, \mathbb{Q})(m-n)$ is compatible with Hodge Structures after a shift by $-2(m-n)$ on the right term.

2.2.5. *Proof of the equivalence with the action of the group* $\mathbb{S}$.

Lemma 2.16. *Let $(H_{\mathbb{R}}, F)$ be a real Hodge Structure of weight m, defined by the decomposition $H_{\mathbb{C}} = \oplus_{p+q=m} H^{p,q}$, then the action of $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ on $H_{\mathbb{C}}$, defined by:*

$$(z, v) \mapsto \sum_{p+q=m} z^p \overline{z}^q v_{p,q},$$

for $v = \sum_{p+q=m} v_{p,q}$, corresponds to a real representation $\varphi : \mathbb{S}(\mathbb{R}) \to GL(H_{\mathbb{R}})$ satisfying $\varphi(t)(v) = t^m v$ for $t \in \mathbb{R}^*$.

We prove that $\overline{\varphi(z)} = \varphi(z), z \in \mathbb{C}^*$, hence it is defined by a real matrix. We deduce from $\overline{v} = \sum_{p+q=m} \overline{v_{p,q}} \in \oplus H^{q,p}$ that $(\overline{v})_{q,p} = \overline{v_{p,q}} \in H^{q,p}$, then:
$\overline{\varphi(z)}(v) := \overline{\varphi(z)(\overline{v})} = \overline{\sum_{p+q=m} z^q \overline{z}^p \overline{v_{p,q}}} = \sum_{p+q=m} z^p \overline{z}^q v_{p,q} = \varphi(z)(v)$.
In particular, for $t \in \mathbb{R}$, $\varphi(t)(v) = t^m v$ and $\varphi(i)(v) = \sum_{p+q=m} i^{p-q} v_{p,q}$ is the Weil operator C on H.

Reciprocally, a representation of the multiplicative commutative torus:

$$\mathbb{S}(\mathbb{C}) \to GL(H_{\mathbb{C}})$$

splits into a direct sum of representations $\mathbb{S}(\mathbb{C}) \to GL(H^{p,q})$ acting via $v \to z^p \overline{z}^q v$, moreover if the representation is real, then $\overline{H^{p,q}} = H^{q,p}$. The real group $\mathbb{R}^*$ acts on $H^{p,q}$ by multiplication by t^{p+q}, so that the sum $\oplus_{p+q=m} H^{p,q}$ defines a sub-Hodge Structure of weight m.

Remark 2.17. i) The complex points of:

$$\mathbb{S}(\mathbb{C}) = \left\{ M(u,v) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \in GL(2,\mathbb{C}), \quad u, v \in \mathbb{C} \right\}$$

are the matrices with complex coefficients $u, v \in \mathbb{C}$ with determinant $u^2 + v^2 \neq 0$. Let $z = u + iv, z' = u - iv$, then $zz' = u^2 + v^2 \neq 0$ such that $S(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^* \times \mathbb{C}^* : (u,v) \mapsto (z, z')$ is an isomorphism satisfying $z' = \overline{z}$, if $u, v \in \mathbb{R}$; in particular $\mathbb{R}^* \hookrightarrow \mathbb{C}^* \times \mathbb{C}^* : t \mapsto (t,t)$.
ii) We can write $\mathbb{C}^* \simeq S^1 \times \mathbb{R}^*$ as the product of the real points of the unitary subgroup $U(1)$ of $\mathbb{S}$ defined by $u^2 + v^2 = 1$ and of the multiplicative subgroup $\mathbb{G}_m(\mathbb{R})$ defined by $v = 0$ in $\mathbb{S}(\mathbb{R})$, and $\mathbb{S}(\mathbb{R})$ is the semi-product of S^1 and $\mathbb{R}^*$. Then the representation gives rise to a representation of $\mathbb{R}^*$ called the scaling since it fixes the weight and of $U(1)$ which fixes the decomposition into $H^{p,q}$ and on which $\varphi(z)$ acts as multiplication by z^{p-q}.

2.3. **Examples.** *Cohomology of projective spaces.* The polarization is defined by the first Chern class of the canonical line bundle $H = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$, dual to the homology class of a hyperplane.

Proposition 2.18. $H^i(\mathbb{P}^n, \mathbb{Z}) = 0$ for i odd and $H^i(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}$ for i even with generator $[H]^i$, the cup product to the power i of the cohomology class of an hyperplane, hence: $H^{2r}(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}(-r)$ as Hodge Structure.

2.3.1. *Hodge Structures of dimension 2 and weight 1.* Let H be a real vector space of dimension 2 endowed with a skew symmetric quadratic form, (e_1, e_2) a basis in which the matrix of Q is

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(u,v) = {}^t v Q u, \quad u, v \in \mathbb{Q}^2$$

then Hodge decomposition $H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$ is defined by the one dimensional subspace $H^{1,0}$ with generator v of coordinates $(v_1, v_2) \in \mathbb{C}^2$. While $Q(v,v) = 0$, the Hodge-Riemann positivity condition is written as $iQ(v, \overline{v}) = -i(v_1 \overline{v}_2 - \overline{v}_1 v_2) > 0$, hence $v_2 \neq 0$, so we divide by v_2 to get a unique representative of $H^{1,0}$ by a vector of the form $v = (\tau, 1)$ with $\mathrm{Im}(\tau) > 0$. Hence the Poincaré half plane $\{ z \in \mathbb{C} : \mathrm{Im} z > 0\}$ is a classifying space for polarized Hodge Structures of dimension 2. This will apply to the cohomology $H := H^1(T, \mathbb{R})$ of a complex torus of dim.1. Note that the torus is a projective variety. Indeed, a non degenerate lattice in $\mathbb{C}$ defines a Weirstrass function $\mathcal{P}(z)$ that defines with its derivative an embedding of the torus onto a smooth elliptic curve $(\mathcal{P}(z), \mathcal{P}'(z)) : T \to \mathbb{P}^2_C$.

2.3.2. *Moduli space of Hodge structures of weight 1* (see [17]). The first cohomology group $H_{\mathbb{Z}}$ of a smooth compact algebraic curve C of genus g is of rank $2g$. The cohomology with coefficients in $\mathbb{C}$ decomposes into the direct sum:

$$H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$$

of two conjugate subspaces of dimension g. We view here the differentiable structure as fixed, on which the algebraic structure vary, hence the cohomology is a fixed vector space on which the decomposition vary. The problem here, is the classification of the different Hodge Structures on the cohomology. Recall that $H_{\mathbb{C}}$ is endowed via the cup-product with a non degenerate alternating form:

$$Q : (\alpha, \beta) \to \int \alpha \wedge \beta.$$

The subspace $H^{1,0}$ is an element of the Grassmann variety $G(g, H_{\mathbb{C}})$ parametrizing complex vector subspaces of dimension g.

The first Riemann bilinear relation states that $Q(\alpha, \beta) = 0$, $\forall \alpha, \beta \in H^{1,0}$, which translates into a set of algebraic equations satisfied by the point representing $H^{1,0}$ in $G(g, H_{\mathbb{C}})$, hence the family of such subspaces is parameterized by an algebraic subvariety denoted $\check{\mathrm{D}} \subset G(g, H_{\mathbb{C}})$. Let $Sp(H_{\mathbb{C}}, Q)$ denotes the symplectic subgroup of complex linear automorphisms of $H_{\mathbb{C}}$ commuting with Q; it acts transitively on $\check{\mathrm{D}}$ (see [3]), hence such variety is smooth.

The second positivity relation is an open condition, hence it defines an open subset D of elements representing $H^{1,0}$ in $\check{\mathrm{D}}$. The subspace D is called the classifying space of polarized Hodge Structures of type 1 (the cohomology here is primitive, hence polarized).

To describe D with coordinates, we choose a basis $\{b_1, \ldots, b_g, b_{g+1}, \ldots, b_{2g}\}$ of $H_{\mathbb{C}}$ in which the matrix of Q is written as a matrix J below, then $H^{1,0}$ is represented by a choice of free g generators which may be written as a matrix M with g columns and $2g$ rows satisfying the condition:

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

$$\left({}^tM_1, {}^tM_2\right) \cdot \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \cdot \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = {}^tM_1M_2 - {}^tM_2M_1 = 0$$

The positivity condition $iQ(\alpha, \overline{\alpha}) > 0, \forall \alpha \neq 0 \in H^{1,0}$ translates into

$$i \cdot \left({}^t\overline{M}_1\, {}^t\overline{M}_2\right) \cdot \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \cdot \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = i({}^t\overline{M}_1M_2 - {}^t\overline{M}_2M_1)$$

is positive definite. We deduce in particular that the determinant $\det(M_2) \neq 0$ and we may choose $M_2 = I_g$, $M_1 = Z$ such that

$$D \simeq \{Z : {}^tZ = Z, \mathrm{Im(Z)} > 0\}.$$

that is the imaginary part of the matrix Z defines a positive definite form. Then D is called the generalized Siegel upper half-plane. If $g = 1$, then D is the upper half plane. Let $\mathrm{Sp}(H_{\mathbb{C}}, Q) \simeq \mathrm{Sp}(g, \mathbb{C}) = \{X \in \mathrm{GL}(n, \mathbb{C}) : {}^tX \cdot J = J \cdot X\}$, then $\mathrm{Sp}(V, Q) := \mathrm{Sp}(H_{\mathbb{C}}, Q) \cap \mathrm{GL}(H_{\mathbb{R}})$ acts transitively on D as follows:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (A \cdot Z + B) \cdot (C \cdot Z + D)^{-1}.$$

It is a classical but not a trivial result that smooth compact curves are homeomorphic as real surfaces to the "connected" sum of g tori with a natural basis of paths $\gamma_1, \ldots, \gamma_g, \gamma_{g+1}, \ldots, \gamma_{2g}$, whose classes generates the homology and such that the matrix of the Intersection product is J.

The classes obtained by Poincaré duality isomorphism $\eta_1, \ldots, \eta_{2g}$ form a basis of $H_{\mathbb{C}}$ on which the matrix of the form Q defined by the product on cohomology is still J. Let $[\gamma_i^*]$ denotes the ordinary dual basis of cohomology. It is not difficult to check on the definitions that $[\gamma_i^*] = -\eta_{i+g}$ for $i \in [1, g]$ and $[\gamma_{i+g}^*] = \eta_i$ for $i \in [g, 2g]$, from which we deduce that the dual basis satisfy also the matrix J. Let ω_i be a set of differential forms defining a basis of $H^{1,0}$, then the period matrix of the basis ω_i with respect to the basis $[\gamma_i^*]$ is defined by the coordinates on that basis given by the entries $\int_{\gamma_i} \omega_j$ for $i \in [1, 2g], j \in [1, g]$. Hence such matrices represent elements in D.

2.3.3. *Hodge structures of weight 1 and abelian varieties.* Given a Hodge Structure:

$$(H_{\mathbb{Z}}, H^{1,0} \oplus H^{0,1}),$$

the projection on $H^{0,1}$ induces an isomorphism of real vector spaces:

$$H_{\mathbb{R}} \to H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1} \to H^{0,1}$$

since $\overline{H^{0,1}} = H^{1,0}$. Then we deduce that $H_{\mathbb{Z}}$ is a lattice in the complex space $H^{0,1}$, and the quotient $T := H^{0,1}/H_{\mathbb{Z}}$ is a complex torus . When $H_{\mathbb{Z}}$ is identified with the image of cohomology spaces $Im(H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{R}))$ of a complex manifold X, resp. $H^{0,1}$ with $H^1(X, \mathcal{O})$, the torus T will be identified with the Picard variety $Pic^0(X)$ parameterizing the holomorphic line bundles on X with first Chern class equal to zero as follows. We consider the exact sequence of sheaves defined by $f \mapsto e^{2i\pi f}$:

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1$$

where 1 is the neutral element of the sheaf of multiplicative groups $\mathcal{O}_X^*$, and its associated long exact sequence of derived functors of the global sections functor:

$$\to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$$

where the morphisms can be interpreted geometrically; when the space $H^1(X, \mathcal{O}_X^*)$ is identified with isomorphisms classes of line bundles on X, the last morphism defines the Chern class of the line bundle. Hence the torus T is identified with the isomorphism classes $\mathcal{L}$ with $c_1(\mathcal{L}) = 0$:

$$T := \frac{H^1(X, \mathcal{O}_X)}{ImH^1(X, \mathbb{Z})} \simeq Pic^0(X) := Ker(H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}))$$

It is possible to show that the Picard variety of a smooth projective variety is an abelian variety. (Define a Kähler form with integral class on $Pic^0(X)$).

2.3.4. *Hodge structures of weight 2.*

$$(H_{\mathbb{Z}},\ H^{2,0} \oplus H^{1,1} \oplus H^{0,2};\ H^{0,2} = \overline{H^{2,0}},\ H^{1,1} = \overline{H^{1,1}};\ Q)$$

the intersection form Q is symmetric and $\mathrm{F}(\alpha, \beta) = Q(\alpha, \overline{\beta})$ is Hermitian. The decomposition is orthogonal for F with F positive definite on $H^{1,1}$, negative definite on $H^{2,0}$ and $H^{0,2}$.

Lemma 2.19. *The HS is completely determined by the subspace $H^{2,0} \subset H_{\mathbb{C}}$ such that $H^{2,0}$ is totally isotropic for Q and the associated Hermitian product H is negative definite on $H^{2,0}$. The signature of Q is $(h^{1,1}, 2h^{2,0})$.*

In the lemma $H^{0,2}$ is determined by conjugation and $H^{1,1} = (H^{2,0} \oplus H^{0,2})^{\perp}$.

3. Mixed Hodge Structures (MHS)

The theory of Mixed Hodge Structures (MHS) introduced by Deligne in [6] is a generalization of Hodge Structures that can be defined on the cohomology of all algebraic varieties.

Since we are essentially concerned by filtrations of vector spaces, it is not more difficult to describe this notion in the terminology of abelian categories. We start with a formal study of Mixed Hodge Structures.

Let $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$, and define $A \otimes \mathbb{Q}$ as $\mathbb{Q}$ if $A = \mathbb{Z}$ or $\mathbb{Q}$ and $\mathbb{R}$ if $A = \mathbb{R}$. For an A-module H_A of finite type, we write $H_{A\otimes\mathbb{Q}}$ for the $(A \otimes \mathbb{Q})$-vector space $(H_A) \otimes_A (A \otimes \mathbb{Q})$. It is a rational space if $A = \mathbb{Z}$ or $\mathbb{Q}$ and a real space if $A = \mathbb{R}$.

Definition 3.1 (Mixed Hodge structure (MHS))**.** An A-Mixed Hodge Structure H consists of:
1) an A-module of finite type H_A;
2) a finite increasing filtration W of the $A \otimes \mathbb{Q}$-vector space $H_A \otimes \mathbb{Q}$ called the weight filtration;
3) a finite decreasing filtration F of the $\mathbb{C}$-vector space $H_{\mathbb{C}} = H_A \otimes_A \mathbb{C}$, called the Hodge filtration, such that the systems:

$$Gr_n^W H = (Gr_n^W(H_{A\otimes\mathbb{Q}}), (Gr_n^W H_{\mathbb{C}}, F))$$

consist of $A \otimes \mathbb{Q}$-Hodge Structure of weight n.

The Mixed Hodge Structure is called real, if $A = \mathbb{R}$, rational, if $A = \mathbb{Q}$, and integral, if $A = \mathbb{Z}$. We need to define clearly, how the filtrations are induced, for example:
$F^p Gr_n^W H_{\mathbb{C}} := ((F^p \cap W_n \otimes \mathbb{C}) + W_{n-1} \otimes \mathbb{C})/W_{n-1} \otimes \mathbb{C} \subset (W_n \otimes \mathbb{C})/W_{n-1} \otimes \mathbb{C}$.

Morphism of MHS. A morphism $f : H \to H'$ of Mixed Hodge Structures is a morphism $f : H_A \to H'_A$ whose extension to $H_{\mathbb{Q}}$ (resp. $H_{\mathbb{C}}$) is compatible with the filtration W, i.e., $f(W_j H_A) \subset W_j H'_A$ (resp. F, i.e., $f(F^j H_A) \subset F^j H'_A$), which implies that it is also compatible with $\overline{F}$.

These definitions allow us to speak of the category of Mixed Hodge Structures.

The main result of this chapter is

Theorem 3.2 (Deligne)**.** *The category of Mixed Hodge Structures is abelian.*

The proof relies on the following:

Canonical decomposition of the weight filtration. While there is an equivalence between the Hodge filtration and the Hodge decomposition, there is no such result for the weight filtration of a Mixed Hodge Structure. In the category of Mixed Hodge Structures, the short exact sequence $0 \to Gr_{n-1}^W \to W_n/W_{n-2} \to Gr_n^W \to 0$ is a non-split extension of the two pure Hodge structures Gr_n^W and Gr_{n-1}^W. To

construct the Hodge decomposition, we introduce for each pair of integers (p,q), the subspaces of $H = H_{\mathbb{C}}$:

$$I^{p,q} = (F^p \cap W_{p+q}) \cap (\overline{F^q} \cap W_{p+q} + \overline{F^{q-1}} \cap W_{p+q-2} + \overline{F^{q-2}} \cap W_{p+q-3} + \cdots)$$

By construction they are related for $p+q=n$ to the components $H^{p,q}$ of the Hodge decomposition: $Gr_n^W H \simeq \oplus_{n=p+q} H^{p,q}$;

Proposition 3.3. The projection $\varphi : W_{p+q} \to Gr_{p+q}^W H \simeq \oplus_{p'+q'=p+q} H^{p',q'}$ induces an isomorphism $I^{p,q} \xrightarrow{\sim} H^{p,q}$. Moreover,

$$W_n = \oplus_{p+q\leq n} I^{p,q}, \quad F^p = \oplus_{p'\geq p} I^{p',q'}$$

Remark 3.4. i) For three decreasing opposed filtrations in an abelian category, Deligne uses in the proof an inductive argument based on the formula for $i > 0$

$$F^{p_i} \oplus \overline{F^{q_i}} \simeq Gr_{n-i}^W H, \quad p_i + q_i = n - i + 1,$$

to construct for $i > 0$ a decreasing family (p_i, q_i) starting with $(p_0 = p, q_0 = q) : p + q = n$ and deduces the existence of a subspace $A^{p,q} \subset W_m$ projecting isomorphically onto the subspace of type (p,q) of the Hodge Structure $Gr_n^W H$. To simplify, we choose here, for W increasing, a sequence of the following type $(p_0, q_0) = (p,q)$, and for $i > 0$, $p_i = p, q_i = q - i + 1$ which explains theasymmetry in the formula (see also [31] Lemma-Definition 3.4), including the fact that for $i = 1$, $\overline{F^q} \cap W_{p+q-1} \subset \overline{F^q} \cap W_{p+q}$.
ii) In general $I^{p,q} \neq \overline{I^{q,p}}$, we have only $I^{p,q} \equiv \overline{I^{q,p}}$ modulo W_{p+q-2}.
iii) A morphism of Mixed Hodge Structures is necessarily compatible with the decomposition, which will be the main ingredient to prove the strictness with respect to W and F (see 3.1.3).

The proof of this proposition is given below. This proposition is used to prove Theorem 3.2.

The knowledge of the linear algebra underlying Mixed Hodge Structures is supposed to help the reader before the reader is confronted with their construction. The striking result to remember is that morphisms of Mixed Hodge Structures are *strict* (see 3.1.3) for both filtrations, the weight W and Hodge F. The complete proofs are purely algebraic [6]; the corresponding theory in an abelian category is developed for objects with opposite filtrations.

3.1. **Filtrations.** Given a morphism in an additive category, the isomorphism between the image and co-image is one of the conditions to define an abelian category. In the abelian category of vector spaces endowed with finite filtrations by subspaces, if we consider a morphism compatible with filtrations $f : (H,F) \to (H',F')$ such that $f(F^j) \subset F'^j$, the induced filtration on the image by F' does not coincide with the induced filtration by F on the co-image. In this case we say that the morphism is strict if they coincide (see 3.1.3). This kind of problem will occur for various repeated restrictions of filtrations and we need here to define with precision the properties of induced filtrations, since this is at the heart of the subject of Mixed Hodge Structures.

On a sub-quotient of a filtered vector space and in general of an object of an abelian category, there are two ways to restrict the filtration (first on the sub-object, then on the quotient and the reciprocal way). On an object A with two filtrations

W and F, we can repeat the restriction for each filtration and we get different objects if we start with W then F or we inverse the order.

We need to know precise relations between the induced filtrations in these various ways, and to know the precise behavior of a linear morphism with respect to such filtrations. For example, we will indicate in section 5, three different ways to induce a filtration on the terms of a spectral sequence. A central result for Mixed Hodge Structures is to give conditions on the complex such that the three induced filtrations coincide. Hence, we recall here preliminaries on filtrations after Deligne.

Let $\mathbb{A}$ denote an abelian category.

Definition 3.5. A decreasing (resp. increasing) filtration F of an object A of $\mathbb{A}$ is a family of sub-objects of $\mathbb{A}$, satisfying

$$\forall n, m, \qquad n \leq m \Longrightarrow F^m(A) \subset F^n(A) \quad (resp.\, n \leq m \Longrightarrow F_n(A) \subset F_m(A))$$

The pair (A, F) of an object of $\mathbb{A}$ and a decreasing (resp. increasing) filtration will be called a filtered object.

If F is a decreasing filtration (resp. W an increasing filtration), a shifted filtration $F[n]$ by an integer n is defined as

$$(F[n])^p(A) = F^{n+p}(A) \quad (W[n])_p(A) = W_{n-p}(A)$$

Decreasing filtrations F are considered for a general study. Statements for increasing filtrations W are deduced by the change of indices $W_n(A) = F^{-n}(A)$. A filtration is finite if there exist integers n and m such that $F^n(A) = A$ and $F^m(A) = 0$.

A morphism of filtered objects $(A, F) \xrightarrow{f} (B, F)$ is a morphism $A \xrightarrow{f} B$ satisfying $f(F^n(A)) \subset F^n(B)$ for all $n \in \mathbb{Z}$. Filtered objects (resp. of finite filtration) form an additive category with existence of *kernel* and *cokernel* of a morphism with natural induced filtrations as well image and coimage, however the image and co-image will not be necessarily filtered-isomorphic, which is the main obstruction to obtain an abelian category.

To lift this obstruction, we are lead to define the notion of strictness for compatible morphims (see 3.1.3 below).

Definition 3.6. The graded object associated to (A, F) is defined as:

$$Gr_F(A) = \oplus_{n\in\mathbb{Z}} Gr_F^n(A) \quad \text{where} \quad Gr_F^n(A) = F^n(A)/F^{n+1}(A).$$

3.1.1. *Induced filtration.* A filtered object (A, F) induces a filtration on a sub-object $i : B \to A$ of A, defined by $F^n(B) = B \cap F^n(A)$. Dually, the quotient filtration on A/B is defined by:

$$F^n(A/B) = p(F^n(A)) = (B + F^n(A))/B \simeq F^n(A)/(B \cap F^n(A)),$$

where $p : A \to A/B$ is the projection.

3.1.2. The cohomology of a sequence of filtered morphisms:

$$(A, F) \xrightarrow{f} (B, F) \xrightarrow{g} (C, F)$$

satisfying $g \circ f = 0$ is defined as $H = Ker\, g/Im\, f$; it is *filtered* and endowed with the quotient filtration of the induced filtration on $Ker\, g$. It is equal to the induced filtration on H by the quotient filtration on $B/Im\, f$, since $H \subset (B/Im\, f)$.

3.1.3. *Strictness.* For filtered modules over a ring, a morphism of filtered objects $f : (A, F) \to (B, F)$ is called strict if the relation:

$$f(F^n(A)) = f(A) \cap F^n(B)$$

is satisfied; that is, any element $b \in F^n(B) \cap ImA$ is already in $Im\, F^n(A)$.

This is not the case for any morphism.

A filtered morphism in an additive category $f : (A, F) \to (B, F)$ is called strict if it induces a filtered isomorphism $(Coim(f), F) \to (Im(f), F)$ from the coimage to the image of f with their induced filtrations.

This concept is basic to the theory, so we mention the following criteria:

Proposition 3.7. i) A filtered morphism $f : (A, F) \to (B, F)$ of objects with finite filtrations is strict if and only if the following exact sequence of graded objects is exact:

$$0 \to Gr_F(Kerf) \to Gr_F(A) \to Gr_F(B) \to Gr_F(Cokerf) \to 0$$

ii) Let $S : (A, F) \xrightarrow{f} (B, F) \xrightarrow{g} (C, F)$ be a sequence S of strict morphisms such that $g \circ f = 0$, then the cohomology H with induced filtration satisfies:

$$H(Gr_F(S)) \simeq Gr_F(H(S)).$$

3.1.4. *Two filtrations.* Let A be an object of $\mathbb{A}$ with two filtrations F and G. By definition, $Gr_F^n(A)$ is a quotient of a sub-object of A, and as such, it is endowed with an induced filtration by G. Its associated graded object defines a bigraded object $Gr_G^n Gr_F^m(A)_{n,m\in\mathbb{Z}}$. We refer to [6] for:

Lemma 3.8 (Zassenhaus' lemma)**.** *The objects $Gr_G^n Gr_F^m(A)$ and $Gr_F^m Gr_G^n(A)$ are isomorphic.*

Remark 3.9. Let H be a third filtration on A. It induces a filtration on $Gr_F(A)$, hence on $Gr_G Gr_F(A)$. It induces also a filtration on $Gr_F Gr_G(A)$. These filtrations do not correspond in general under the above isomorphism. In the formula $Gr_H Gr_G Gr_F(A)$, G et H have symmetric role, but not F and G.

3.1.5. *Hom and tensor functors.* If A and B are two filtered objects of $\mathbb{A}$, we define a filtration on the left exact functor Hom:

$$F^k Hom(A, B) = \{f : A \to B : \forall n, f(F^n(A)) \subset F^{n+k}(B)\}$$

Hence:

$$Hom((A, F), (B, F)) = F^0(Hom(A, B)).$$

If A and B are modules on some ring, we define:

$$F^k(A \otimes B) = \sum_m Im(F^m(A) \otimes F^{k-m}(B) \to A \otimes B)$$

3.1.6. *Multifunctor.* In general if $H : \mathbb{A}_1 \times \ldots \times \mathbb{A}_n \to \mathbb{B}$ is a right exact multi-additive functor, we define:

$$F^k(H(A_1, ..., A_n)) = \sum_{\sum k_i = k} Im(H(F^{k_1}A_1, ..., F^{k_n}A_n) \to H(A_1, ..., A_n))$$

and dually if H is left exact:

$$F^k(H(A_1, ..., A_n) = \bigcap_{\sum k_i = k} Ker(H(A_1, ..., A_n) \to H(A_1/F^{k_1}A_1, ..., A_n/F^{k_n}A_n))$$

If H is exact, both definitions are equivalent.

3.2. **Opposite filtrations.** We review here the definition of Hodge Structure in terms of filtrations, as it is adapted to its generalization to Mixed Hodge Structure. For any A-module H_A, the complex conjugation extends to a conjugation on the space $H_{\mathbb{C}} = H_A \otimes_A \mathbb{C}$. A filtration F on $H_{\mathbb{C}}$ has a conjugate filtration $\overline{F}$ such that $(\overline{F})^j H_{\mathbb{C}} = \overline{F^j H_{\mathbb{C}}}$.

Definition 3.10. ([HS1]) An A-Hodge structure H of weight n consists of:
i) an A-module of finite type H_A,
ii) a finite filtration F on $H_{\mathbb{C}}$ (the Hodge filtration) such that F and its conjugate $\overline{F}$ satisfy the relation:

$$Gr_F^p Gr_{\overline{F}}^q(H_{\mathbb{C}}) = 0, \quad \text{for } p+q \neq n$$

The module H_A, or its image in $H_{\mathbb{Q}}$ is called the lattice. The Hodge Structure is real when $A = \mathbb{R}$, rational when $A = \mathbb{Q}$ and integral when $A = \mathbb{Z}$.

3.2.1. *Opposite filtrations.* The linear algebra of Mixed Hodge Structures applies to an abelian category $\mathbb{A}$ if we use the following definition where no conjugation map appears.
Two finite filtrations F and G on an object A of $\mathbb{A}$ are n-opposite if:

$$Gr_F^p Gr_G^q(A) = 0 \quad \text{for } p+q \neq n.$$

hence the Hodge filtration F on a Hodge Sructure of weight n is n-opposite to its conjugate $\overline{F}$. The following constructions will define an equivalence of categories between objects of $\mathbb{A}$ with two n-opposite filtrations and bigraded objects of $\mathbb{A}$ of the following type.

Example 3.11. Let $A^{p,q}$ be a bigraded object of $\mathbb{A}$ such that $A^{p,q} = 0$, for all but a finite number of pairs (p,q) and $A^{p,q} = 0$ for $p+q \neq n$; then we define two n-opposite filtrations on $A = \sum_{p,q} A^{p,q}$

$$F^p(A) = \sum_{p' \geq p} A^{p',q'}, \quad G^q(A) = \sum_{q' \geq q} A^{p',q'}$$

We have $Gr_F^p Gr_G^q(A) = A^{p,q}$.

We state reciprocally [6]:

Proposition 3.12. i) Two finite filtrations F and G on an object A are n-opposite, if and only if:

$$\forall\, p, q, \quad p+q = n+1 \Rightarrow F^p(A) \oplus G^q(A) \simeq A.$$

ii) If F and G are n-opposite, and if we put $A^{p,q} = F^p(A) \cap G^q(A)$, for $p+q = n$, $A^{p,q} = 0$ for $p+q \neq n$, then A is a direct sum of $A^{p,q}$.

Moreover, F and G can be deduced from the bigraded object $A^{p,q}$ of A by the above procedure. We recall the previous equivalent definition of Hodge Structure:

Definition 3.13. ([HS2]) An A-Hodge Structure on H of weight n is a pair of a finite A-module H_A and a decomposition into a direct sum on $H_{\mathbb{C}} = H_A \otimes_A \mathbb{C}$:

$$H_{\mathbb{C}} = \oplus_{p+q=n} H^{p,q} \quad \text{such that} \quad \overline{H}^{p,q} = H^{q,p}$$

The relation with the previous definition is given by $H^{p,q} = F^p(H_{\mathbb{C}}) \cap \overline{F}^q(H_{\mathbb{C}})$ for $p+q = n$.

3.2.2. *Complex Hodge Structures.* For some arguments in analysis, we don't need to know that $\overline{F}$ is the conjugate of F.

Definition 3.14. A complex Hodge Structure of weight n on a complex vector space H is given by a pair of $n-$opposite filtrations F and $\overline{F}$, hence a decomposition into a direct sum of subspaces:

$$H = \oplus_{p+q=n} H^{p,q}, \quad \text{where } H^{p,q} = F^p \cap \overline{F}^q$$

The two $n-$opposite filtrations F and $\overline{F}$ on a complex Hodge Structure of weight n can be recovered from the decomposition by the formula:

$$F^p = \oplus_{i \geq p} H^{i,n-i} \quad \overline{F}^q = \oplus_{i \leq n-q} H^{i,n-i}$$

Here we don't assume the existence of conjugation although we keep the notation $\overline{F}$. An A-Hodge Structure of weight n defines a complex Hodge Structure on $H = H_{\mathbb{C}}$.

To define polarization, we recall that the conjugate space $\overline{H}$ of a complex vector space H, is the same group H with a different complex structure, such that the identity map on the group H defines a real linear map $\sigma : H \to \overline{H}$ and the product by scalars satisfy the relation $\forall \lambda \in \mathbb{C}, v \in H,\ \lambda \times_{\overline{H}} \sigma(v) := \sigma(\overline{\lambda} \times_H v)$, then the complex structure on $\overline{H}$ is unique. On the other hand a complex linear morphism $f : V \to V'$ defines a complex linear conjugate morphism $\overline{f} : \overline{V} \to \overline{V}'$ satisfying $\overline{f}(\sigma(v)) = \sigma(f(v))$.

Definition 3.15. A polarization of a complex Hodge Structure of weight n is a bilinear morphism $S : H \otimes \overline{H} \to \mathbb{C}$ such that:

$$S(x, \sigma(y)) = (-1)^n \overline{S(y, \sigma(x))} \text{ for } x, y \in L \text{ and } S(F^p, \sigma(\overline{F}^q)) = 0 \text{ for } p + q > n.$$

and moreover $S(C(H)u, \sigma(v))$ is a positive definite Hermitian form on H where $C(H)$ denotes the Weil action on H.

Example 3.16. A complex Hodge Structure of weight 0 on a complex vector space H is given by a decomposition into a direct sum of subspaces $H = \oplus_{p \in \mathbb{Z}} H^p$, then $H^{i,j} = 0$ for $i + j \neq 0$, $F^p = \oplus_{i \geq p} H^p$ and $\overline{F}^q = \oplus_{i \leq -q} H^i$.

A polarization is an Hermitian form on H for which the decomposition is orthogonal and whose restriction to H^p is definite for p even and negative definite for odd p.

3.2.3. *Examples of Mixed Hodge Structure.* 1) A Hodge Structure H of weight n, is a Mixed Hodge Structure with weight filtration:

$$W_i(H_{\mathbb{Q}}) = 0 \quad \text{for} \quad i < n \qquad \text{and} \quad W_i(H_{\mathbb{Q}}) = H_{\mathbb{Q}} \quad \text{for} \quad i \geq n.$$

2) Let (H^i, F_i) be a finite family of A-Hodge Structures of weight $i \in \mathbb{Z}$; then $H = \oplus_i H^i$ is endowed with the following Mixed Hodge Structure:

$$H_A = \oplus_i H^i_A, \quad W_n = \oplus_{i \leq n} H^i_A \otimes \mathbb{Q}, \quad F^p = \oplus_i F^p_i.$$

3) Let $H_{\mathbb{Z}} = i\mathbb{Z}^n \subset \mathbb{C}^n$, then we consider the isomorphism $H_{\mathbb{Z}} \otimes \mathbb{C} \simeq \mathbb{C}^n$ defined with respect to the canonical basis e_j of $\mathbb{C}^n$ by:

$$H_{\mathbb{C}} \xrightarrow{\sim} \mathbb{C}^n : ie_j \otimes (a_j + ib_j) \mapsto i(a_j + ib_j)e_j = (-b_j + ia_j)e_j$$

hence the conjugation $\sigma(ie_j \otimes (a_j + ib_j)) = ie_j \otimes (a_j - ib_j)$ on $H_{\mathbb{C}}$, corresponds to the following conjugation on $\mathbb{C}^n$: $\sigma(-b_j + ia_j)e_j = (b_j + ia_j)e_j$.

4) Let $H = (H_{\mathbb{Z}}, F, W)$ be a Mixed Hodge Structure; its $m-$twist is a Mixed Hodge Structure denoted by $H(m)$ and defined by:

$H(m)_{\mathbb{Z}} := H_{\mathbb{Z}} \otimes (2i\pi)^m \mathbb{Z}$, $W_r H(m) := (W_{r+2m} H_{\mathbb{Q}}) \otimes (2i\pi)^m \mathbb{Q}$, $F_r H(m) := F^{r+m} H_{\mathbb{C}}$.

3.2.4. *Tensor product and Hom.* Let H and H' be two Mixed Hodge Structures.
1) We define their Mixed Hodge Structure tensor product $H \otimes H'$, by applying the above general rules of filtrations, as follows:
i) $(H \otimes H')_{\mathbb{Z}} = H_{\mathbb{Z}} \otimes H'_{\mathbb{Z}}$
ii) $W^r(H \otimes H')_{\mathbb{Q}} := \sum_{p+p'=r} W^p H_{\mathbb{Q}} \otimes W^{p'} H'_{\mathbb{Q}}$
iii) $F^r(H \otimes H')_{\mathbb{C}} := \sum_{p+p'=r} F^p H_{\mathbb{C}} \otimes F^{p'} H'_{\mathbb{C}}$.
2) The Mixed Hodge Structure on $Hom(H, H')$ is defined as follows:
i) $Hom(H, H')_{\mathbb{Z}} := Hom_{\mathbb{Z}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})$
ii) $W_r Hom(H, H')_{\mathbb{Q}} := \{f : Hom_{\mathbb{Q}}(H_{\mathbb{Q}}, H'_{\mathbb{Q}}) : \forall n, f(W_n H) \subset W_{n+r} H'\}$
iii) $F^r Hom(H, H')_{\mathbb{C}} := \{f : Hom_{\mathbb{C}}(H_{\mathbb{C}}, H'_{\mathbb{C}}) : \forall n, f(F^n H) \subset F^{n+r} H'\}$.
In particular the dual H^* to H is a Mixed Hodge Structure.

3.3. **Proof of the canonical decomposition of the weight filtration.** *Injectivity of φ.* Let $m = p + q$, then we have the inclusion modulo W_{m-1} of the image $\varphi(I^{p,q}) \subset H^{p,q} = (F^p \cap \overline{F^q})(Gr^W_m H)$. Let $v \in I^{p,q}$ such that $\varphi(v) = 0$, then $v \in F^p \cap W_{m-1}$ and the class $cl(v)$ in $(F^p \cap \overline{F^q})(Gr^W_{m-1} H) = 0$ since $p + q > m - 1$, hence $cl(v)$ must vanish; so we deduce that $v \in F^p \cap W_{m-2}$. This is a step in an inductive argument based on the formula $F^p \oplus \overline{F^{q-r+1}} \simeq Gr^W_{m-r} H$. We want to prove $v \in F^p \cap W_{m-r}$ for all $r > 0$. We just proved this for $r = 2$. Hence we write

$$v \in \overline{F^q} \cap W_m + \sum_{r-1 \geq i \geq 1} \overline{F^{q-i}} \cap W_{m-i-1} + \sum_{i > r-1 \geq 1} \overline{F^{q-i}} \cap W_{m-i-1}$$

where the right term is in W_{m-r-1}, since $W_{m-i-1} \subset W_{m-r-1}$ for $i > r - 1$, hence $v \in F^p \cap \overline{F^{q-r+1}} \cap W_{m-r}$ modulo W_{m-r-1} since $\overline{F}$ is decreasing. As $(F^p \cap \overline{F^{q-r+1}}) Gr^W_{m-r} H = 0$ for $r > 0$, the class $cl(v) = 0 \in Gr^W_{m-r} H$, then $v \in F^p \cap W_{m-r-1}$. Finally, as $W_{m-r} = 0$ for large r, we deduce $v = 0$.
Surjectivity. Let $\alpha \in H^{p,q}$; there exists $v_0 \in F^p \cap W_m$ (resp. $u_0 \in F^q \cap W_m$) such that $\varphi(v_0) = \alpha = \varphi(\overline{u}_0)$, hence $v_0 = \overline{u}_0 + w_0$ with $w_0 \in W_{m-1}$. Applying the formula $F^p \oplus \overline{F^q} \simeq Gr^W_{m-1}$, the class of w_0 decomposes as $cl(w_0) = cl(v') + cl(\overline{u}')$ with $v' \in F^p \cap W_{m-1}$ and $u' \in F^q \cap W_{m-1}$; hence there exists $w_1 \in W_{m-2}$ such that $v_0 = \overline{u}_0 + v' + \overline{u}' + w_1$. Let $v_1 := v_0 - v'$ and $u_1 = u_0 + u'$, then

$$v_1 = \overline{u}_1 + w_1, \quad \text{where } u_1 \in F^q \cap W_m, F^p \cap W_m, w_1 \in W_{m-2}.$$

By an increasing inductive argument on k, we apply the formula:

$$F^p \oplus \overline{F^{q-k+1}} \simeq Gr^W_{m-k}$$

to find vectors v_k, u_k, w_k satisfying:

$$v_k \in F^p \cap W_m,\ w_k \in W_{m-1-k},\ \varphi(v_k) = \alpha,\ v_k = \overline{u}_k + w_k$$
$$u_k \in F^q \cap W_m + F^{q-1} \cap W_{m-2} + F^{q-2} \cap W_{n-3} + ... + F^{q+1-k} \cap W_{m-k}$$

then we decompose the class of w_k in $Gr^W_{m-k-1} H$ in the inductive step as above. For large k, $W_{m-1-k} = 0$ and we represent α in $I^{p,q}$.
Moreover $W_n = W_{n-1} \oplus (\oplus_{p+q=n} I^{p,q})$, hence by induction we decompose W_n as direct sum of $I^{p,q}$ for $p + q \geq n$.
Next we suppose, by induction, the formula for F^p satisfied for all $v \in W_{n-1} \cap F^p$.

The image of an element $v \in F^p \cap W_n$ in $Gr_n^W H$ decomposes into Hodge components of type $(i, n-i)$ with $i \geq p$ since $v \in F^p \cap W_n$. Hence the decomposition of v may be writen as $v = v_1 + v_2$ with $v_1 \in \oplus_{i<p} I^{i,n-i}$ and $v_2 \in \oplus_{i \geq p} I^{i,n-i}$ with $v_1 \in W_{n-1}$ since its image vanish in $Gr_n^W H$. Hence, the formula for F^p follows by the inductive step.

3.3.1. *Proof of the abelianness of the category of MHS and strictness.* The definition of Mixed Hodge Structure has a surprising strong property, since any morphism of Mixed Hodge Structures is necessarily strict for each filtration W and F. In consequence, the category is abelian.

Lemma 3.17. *The kernel (resp. cokernel) of a morphism f of Mixed Hodge Structures: $H \to H'$ is a Mixed Hodge Structure K with underlying module K_A equal to the kernel (resp. cokernel) of $f_A : H_A \to {H'}_A$; moreover $K_A \otimes \mathbb{Q}$ and $K_A \otimes \mathbb{C}$ are endowed with induced filtrations (resp. quotient filtrations) by W on $H_{A\otimes\mathbb{Q}}$ (resp. ${H'}_{A\otimes\mathbb{Q}}$) and F on $H_{\mathbb{C}}$ (resp. ${H'}_{\mathbb{C}}$).*

Proof. A morphism compatible with the filtrations is necessarily compatible with the canonical decomposition of the Mixed Hodge Structure into $\oplus I^{p,q}$. It is enough to check the statement on $K_{\mathbb{C}}$, hence we drop the index by $\mathbb{C}$. We consider on $K = Ker(f)$ the induced filtrations from H. The morphism $Gr^W K \to Gr^W H$ is injective, since it is injective on the corresponding terms $I^{p,q}$; moreover, the filtration F (resp. $\overline{F}$) of K induces on $Gr^W K$ the inverse image of the filtration F (resp. $\overline{F}$) on $Gr^W H$:

$$Gr^W K = \oplus_{p,q}(Gr^W K) \cap H^{p,q}(Gr^W H) \text{ and } H^{p,q}(Gr^W K) = (Gr^W K) \cap H^{p,q}(Gr^W H)$$

Hence the filtrations W, F on K define a Mixed Hodge Structure on K which is a kernel of f in the category of Mixed Hodge Structures. The statement on the cokernel follows by duality. □

We still need to prove that for a morphism f of Mixed Hodge Structures, the canonical morphism $Coim(f) \to Im(f)$ is an isomorphism of Mixed Hodge Structures. Since by the above lemma $Coim(f)$ and $Im(f)$ are endowed with natural Mixed Hodge Structure, the result follows from the statement:
A morphism of Mixed Hodge Structures which induces an isomorphism on the lattices, is an isomorphism of Mixed Hodge Structures.

Corollary 3.18. i) Each morphism $f : H \to H'$ is strictly compatible with the filtrations W on $H_{A\otimes\mathbb{Q}}$ and ${H'}_{A\otimes\mathbb{Q}}$ as well the filtrations F on $H_{\mathbb{C}}$ and ${H'}_{\mathbb{C}}$. It induces morphisms $Gr_n^W(f) : Gr_n^W(H_{A\otimes\mathbb{Q}}) \to Gr_n^W({H'}_{A\otimes\mathbb{Q}})$ compatible with the $A \otimes \mathbb{Q}$-Hodge Structures, and morphisms $Gr_F^p(f) : Gr_F^p(H_{\mathbb{C}}) \to Gr_F^p({H'}_{\mathbb{C}})$ strictly compatible with the induced filtrations by $W_{\mathbb{C}}$.
ii) The functor Gr_n^W from the category of Mixed Hodge Structures to the category $A \otimes \mathbb{Q}$-Hodge Structures of weight n is exact and the functor Gr_F^p is also exact.

Remark 3.19. The above result shows that any exact sequence of Mixed Hodge Structures gives rise to various exact sequences which, in the case of Mixed Hodge Structures on cohomology of algebraic varieties that we are going to construct, have in general interesting geometrical interpretation, since we deduce from any long exact sequence of Mixed Hodge Structures:

$$H'^i \to H^i \to H''^i \to H'^{i+1}$$

various exact sequences:

$$Gr_n^W H'^i \to Gr_n^W H^i \to Gr_n^W H''^i \to Gr_n^W H'^{i+1}$$

for $\mathbb{Q}$ (resp. $\mathbb{C}$) coefficients, and:

$$Gr_F^n H'^i \to Gr_F^n H^i \to Gr_F^n H''^i \to Gr_F^n H'^{i+1}$$

$$Gr_F^m Gr_n^W H'^i \to Gr_F^m Gr_n^W H^i \to Gr_F^m Gr_n^W H''^i \to Gr_F^m Gr_n^W H'^{i+1}.$$

3.3.2. *Hodge numbers.* Let H be a Mixed Hodge Structure and set:

$$H^{pq} = Gr_F^p Gr_{\overline{F}}^q Gr_{p+q}^W H_{\mathbb{C}} = (Gr_{p+q}^W H_{\mathbb{C}})^{p,q}.$$

The Hodge numbers of H are the integers $h^{pq} = \dim_{\mathbb{C}} H^{pq}$, that is the Hodge numbers h^{pq} of the Hodge Structure $Gr_{p+q}^W H$.

3.3.3. *Opposite filtrations.* Most of the proofs on the algebraic structure of Mixed Hodge Structure may be carried for three filtrations in an abelian category defined as follows [6]

Definition 3.20 (Opposite filtrations)**.** Three finite filtrations W (increasing), F and G on an object A of $\mathbb{A}$ are opposite if

$$Gr_F^p Gr_G^q Gr_n^W(A) = 0 \quad \text{for } p+q \neq n.$$

This condition is symmetric in F and G. It means that F and G induce on $W_n(A)/W_{n-1}(A)$ two n-opposite filtrations, then $Gr_n^W(A)$ is bigraded

$$W_n(A)/W_{n-1}(A) = \oplus_{p+q=n} A^{p,q} \quad \text{where } A^{p,q} = Gr_F^p Gr_G^q Gr_{p+q}^W(A)$$

Example 3.21. i) A bigraded object $A = \oplus A^{p,q}$ of finite bigrading has the following three opposite filtrations

$$W_n = \oplus_{p+q\leq n} A^{p,q} \quad , F^p = \oplus_{p'\geq p} A^{p'q'} \quad , G^q = \oplus_{q'\geq q} A^{p'q'}$$

ii) In the definition of an A-Mixed Hodge Structure, the filtration $W_{\mathbb{C}}$ on $H_{\mathbb{C}}$ deduced from W by saclar extension, the filtration F and its complex conjugate , form a system $(W_{\mathbb{C}}, F, \overline{F})$ of three opposite filtrations.

Theorem 3.22 (Deligne)**.** *Let $\mathbb{A}$ be an abelian category and $\mathbb{A}'$ the category of objects of $\mathbb{A}$ endowed with three opposite filtrations W (increasing), F and G. The morphisms of $\mathbb{A}'$ are morphisms in $\mathbb{A}$ compatible with the three filtrations.*
i) $\mathbb{A}'$ is an abelian category.
ii) The kernel (resp. cokernel) of a morphism $f : A \to B$ in $\mathbb{A}'$ is the kernel (resp. cokernel) of f in $\mathbb{A}$, endowed with the three induced filtrations from A (resp. quotient of the filtrations on B).
iii) Any morphism $f : A \to B$ in $\mathbb{A}'$ is strictly compatible with the filtrations W, F and G; the morphism $Gr^W(f)$ is compatible with the bigradings of $Gr^W(A)$ and $Gr^W(B)$; the morphisms $Gr_F(f)$ and $Gr_G(f)$ are strictly compatible with the induced filtration by W.
iv) The forget the filtrations functors, as well Gr^W, Gr_F, Gr_G, $Gr^W Gr_F \simeq Gr_F Gr^W$, $Gr_F Gr_G Gr^W$ and $Gr_G Gr^W \simeq Gr^W Gr_G$ of $\mathbb{A}'$ in $\mathbb{A}$ are exact.

3.4. **Complex Mixed Hodge Structures.** Although the cohomology of algebraic varieties carries a Mixed Hodge Structure defined over $\mathbb{Z}$, we may need to work in analysis without such structure over $\mathbb{Z}$.

Definition 3.23. A complex Mixed Hodge Structure of weight n on a complex vector space H is given by an increasing filtration W and two decreasing filtrations F and G such that $(Gr_k^W H, F, G)$, with the induced filtrations, is a complex HS of weight $n+k$.

For $n=0$, we say a complex Mixed Hodge Structure. The definition of complex Hodge Structure of weight n is obtained in the particular case when $W_n = H$ and $W_{n-1} = 0$.

3.4.1. *Variation of complex Mixed Hodge Structures.* The structure which appears in deformation theory on the cohomology of the fibers of a morphism of algebraic varieties leads one to introduce the concept of variation of Mixed Hodge Structure.

Definition 3.24. i) A variation (VHS) of complex Hodge Structures on a complex manifold X of weight n is given by a data $(\mathcal{H}, F, \overline{F})$ where $\mathcal{H}$ is a complex local system, F (resp. $\overline{F}$) is a decreasing filtration varying holomorphically (resp. anti-holomorphically) by sub-bundles of the vector bundle $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{H}$ (resp. $\mathcal{O}_{\overline{X}} \otimes_{\mathbb{C}} \mathcal{H}$ on the conjugate variety $\overline{X}$ with anti-holomorphic structural sheaf) such that for each point $x \in X$, data $(\mathcal{H}(x), F(x), \overline{F}(x))$ form a Hodge Structure of weight n. Moreover, the connection ∇ defined by the local system satisfies Griffiths tranversality: for tangent vectors v holomorphic and $\overline{u}$ anti-holomorphic

$$(\nabla_v F^p) \subset F^{p-1}, \quad (\nabla_{\overline{u}} \overline{F}^p) \subset \overline{F}^{p-1}$$

ii) A variation (VMHS) of complex Mixed Hodge Structures of weight n on X is given by the following data

$$(\mathcal{H}, W, F, \overline{F})$$

where $\mathcal{H}$ is a complex local system, W an increasing filtration by sub-local systems (resp. $\overline{F}$) is a decreasing filtration varying holomorphically (resp. anti-holomorphically) satisfying Griffiths tranversality

$$(\nabla_v F^p) \subset F^{p-1}, \quad (\nabla_{\overline{u}} \overline{F}^p) \subset \overline{F}^{p-1}$$

such that $(Gr_k^W \mathcal{H}, F, \overline{F})$, with the induced filtrations, is a complex VHS of weight $n+k$.

For $n=0$ we just say a complex Variation of Mixed Hodge Structures. Let $\overline{\mathcal{H}}$ be the conjugate local system of $\mathcal{L}$. A linear morphism $S : \mathcal{H} \otimes_{\mathbb{C}} \overline{\mathcal{H}} \to \mathbb{C}_X$ defines a polarization of a VHS if it defines a polarization at each point $x \in X$. A complex Mixed Hodge Structure of weight n is graded polarisable if $(Gr_k^W \mathcal{H}, F, \overline{F})$ is a polarized Variation Hodge Structure.

4. Hypercohomology and spectral sequence of a Filtered Complex

The abstract definition of Mixed Hodge Structures is intended to describe global topological and geometrical properties of algebraic varieties and will be established by constructing special complexes of sheaves K endowed with two filtrations W and F. As in the definition of Mixed Hodge Structures, the filtration W must be defined on the rational level, while F exists only on the complex level.

The topological techniques used to construct W on the rational level are different from the geometrical techniques represented by de Rham complex used to construct the filtration F on the complex level.

Comparison morphisms between the rational and complex levels must be added in order to obtain a satisfactory functorial theory of Mixed Hodge Structures with respect to algebraic morphisms. This functoriality on the level of complexes cannot be obtained in the actual abelian category of complexes of sheaves but in a modified category called the derived category [35], [36], [28], that we describe in this section.

In this section we are going to introduce the proper terminology.

In the next section we use this language in order to state rigorously the hypotheses on the weight filtration W and Hodge filtration F on K needed to induce a Mixed Hodge Structure on the ith-cohomology $(H^i(K), W, F)$ of the complex K.

The proof of the existence of the Mixed Hodge Structure is based on the use of the spectral sequence of the filtered complex (K, W). In this section we recall what a spectral sequence is and discuss its behavior with respect to morphisms of filtered complexes.

The filtration F induces different filtrations on the spectral sequence in various ways. A detailed study in the next section will show that these filtrations coincide under adequate hypotheses and define a Hodge Structure on the terms of the spectral sequence.

4.0.2. *Spectral sequence defined by a filtered complex (K, F) in an abelian category.* We consider decreasing filtrations. A change of the sign of the indices transforms an increasing filtration into a decreasing one, hence any result has a meaning for both filtrations.

Definition 4.1. Let K be a complex of objects of an abelian category $\mathbb{A}$, with a decreasing filtration by subcomplexes F. It induces a filtration F on the cohomology $H^*(K)$, defined by:

$$F^iH^j(K) = Im(H^j(F^iK) \to H^j(K)), \quad \forall i, j \in \mathbb{Z}.$$

The spectral sequence defined by the filtered complex (K, F) is a method to compute the graded object $Gr_F^*H^*(K)$. It consists of indexed objects of $\mathbb{A}$ endowed with differentials (see 4.3 for explicit definitions):

(1) terms E_r^{pq} for $r > 0, p, q \in \mathbb{Z}$,
(2) differentials $d_r : E_r^{pq} \to E_r^{p+r,q-r+1}$ such that $d_r \circ d_r = 0$,
(3) isomorphisms:

$$E_{r+1}^{pq} \simeq H(E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{pq} \xrightarrow{d_r} E_r^{p+r,q-r+1})$$

of the (p, q)-term of index $r+1$ with the corresponding cohomology of the sequence with index r. To avoid ambiguity we may write ${}_FE_r^{pq}$ or $E_r^{pq}(K, F)$. The first term is defined as:

$$E_1^{pq} = H^{p+q}(Gr_F^p(K)).$$

The aim of the spectral sequence is to compute the terms:

$$E_\infty^{pq} := Gr_F^p(H^{p+q}(K))$$

The spectral sequence is said to degenerate if:

$$\forall\, p, q,\ \exists\ r_0(p, q) \ \text{ such that } \ \forall r \geq r_0, \quad E_r^{pq} \simeq E_\infty^{pq} := Gr_p^F H^{p+q}(K).$$

It degenerates at rank r if the differentials d_i of E_i^{pq} vanish for $i \geq r$.
In Deligne-Hodge theory, the spectral sequences of the next section degenerates at rank less than 2. Most known applications are in the case of degenerate spectral sequences [18], for example we assume the filtration biregular, that is finite in each degree of K. It is often convenient to locate the terms on the coordinates (p, q) in some region of the plan $\mathbb{R}^2$; it degenerates for example when for r increasing, the differential d_r has source or target outside the region of non vanishing terms.

Formulas for the terms $r > 1$ are mentioned later in this section, but we will work hard to introduce sufficient conditions to imply specifically in our case that $d_r = 0$ for $r > 1$, hence the terms E_r^{pq} are identical for all $r > 1$ and isomorphic to the cohomology of the sequence $E_1^{p-1,q} \xrightarrow{d_1} E_1^{p,q} \xrightarrow{d_1} E_1^{p+1,q}$.

Morphisms of spectral sequences. A morphism of filtered complexes $f : (K, F) \to (K', F')$ compatible with the filtration induces a morphism of the corresponding spectral sequences.

Filtered resolutions. In presence of two filtrations by subcomplexes F and W on a complex K of objects of an abelian category $\mathbb{A}$, the filtration F induces by restriction a new filtration F on the terms W^iK, which also induces a quotient filtration F on Gr_W^iK. We define in this way the graded complexes Gr_FK, Gr_WK and Gr_FGr_WK.

Definition 4.2. i) A filtration F on a complex K is called biregular if it is finite in each degree of K.
ii) A morphism $f : X \xrightarrow{\approx} Y$ of complexes of objects of $\mathbb{A}$ is a quasi-isomorphism denoted by $\approx$ if the induced morphisms on cohomology $H^*(f) : H^*(X) \xrightarrow{\sim} H^*(Y)$ are isomorphisms for all degrees.
iii) A morphism $f : (K, F) \xrightarrow{\approx} (K', F)$ of complexes with biregular filtrations is a filtered quasi-isomorphism if it is compatible with the filtrations and induces a quasi-isomorphism on the graded object $Gr_F^*(f) : Gr_F^*(K) \xrightarrow{\approx} Gr_F^*(K')$.
iv) A morphism $f : (K, F, W) \xrightarrow{\approx} (K', F, W)$ of complexes with biregular filtrations F and W is a bi-filtered quasi-isomorphism if $Gr_F^*Gr_W^*(f)$ is a quasi-isomorphism.

In the case iii) we call (K', F) a filtered resolution of (K, F) and in iv) we say a bi-filtered resolution, while in i) it is just a resolution.

Proposition 4.3. Let $f : (K, F) \to (K', F')$ be a filtered morphism with biregular filtrations, then the following assertions are equivalent:
i) f is a filtered quasi-isomorphism.
ii) $E_1^{pq}(f) : E_1^{pq}(K, F) \to E_1^{pq}(K', F')$ is an isomorphism for all p, q.
iii) $E_r^{pq}(f) : E_r^{pq}(K, F) \to E_r^{pq}(K', F')$ is an isomorphism for all $r \geq 1$ and all p, q.

By definition of the terms E_1^{pq}, (ii) is equivalent to (i). We deduce iii) from ii) by induction. If we suppose the isomorphism in iii) satisfied for $r \leq r_0$, the isomorphism for $r_0 + 1$ follows since $E_{r_0}^{pq}(f)$ is compatible with d_{r_0}.

Proposition 4.4. (Prop. 1.3.2 [6]) Let K be a complex with a biregular filtration F. The following conditions are equivalent:
i) The spectral sequence defined by F degenerates at rank 1 ($E_1 = E_\infty$)
ii) The differentials $d : K^i \to K^{i+1}$ are strictly compatible with the filtrations.

4.0.3. *Diagrams of morphisms.* In the next section, we call morphism of filtered complexes $f : (K_1, F_1) \to (K_2, F_2)$, a class of diagrams of morphisms:

$$(K_1, F_1) \xleftarrow{g_1 \approx} (K_1', F_1') \xrightarrow{f_1} (K_2, F_2) \quad , \quad (K_1, F_1) \xrightarrow{f_2} (K_2', , F_2') \xleftarrow{g_2 \approx} (K_2, , F_2)$$

where g_1 and g_2 are filtered quasi-isomorphisms. We think of f as $f_1 \circ g_1^{-1}$ (resp. $f = g_2^{-1} \circ f_2$) by adding the inverse of a quasi-isomorphism to the morphisms in the category. This formal construction is explained below.

It follows that a diagram of filtered morphisms of complexes induces a morphism of the corresponding spectral sequences, but the reciprocal statement is not true: the existence of a quasi-isomorphism is stronger than the existence of an isomorphism of spectral sequences.

The idea to add the inverse of quasi-isomorphisms to the morphisms of complexes is due to Grothendieck (inspired by the construction of the inverse of a multiplicative system in a ring). It is an advantage to use such diagrams on the complex level, rather than to work with morphisms on the cohomology level, as the theory of derived category will show. The derived filtered categories described below have been used extensively in the theory of perverse sheaves and developments of Hodge theory in an essential way.

4.1. **Derived filtered category.** We will use in the next section the language of *filtered and bi-filtered derived categories*, to define Mixed Hodge Structure with compatibility between the various data and functoriality compatible with the filtrations.

4.1.1. *Background on derived category.* The idea of Grothendieck is to construct a new category where the class of all quasi-isomorphisms become isomorphisms in the new category. We follow here the construction given by Verdier [35] in two steps. In the first step we construct the homotopy category where the morphisms are classes defined up to homotopy ([35], see [28],[1]), and in the second step, a process of inverting all quasi-isomorphisms called localization is carried by a calculus of fractions similar to the process of inverting a multiplicative system in a ring, although in this case the system of quasi-isomorphisms is not commutative so that a set of diagram relations must be carefully added in the definition of a multiplicative system ([28],[1]).

4.1.2. *The homotopy category* $K(\mathbb{A})$. Let $\mathbb{A}$ be an abelian category and let $C(\mathbb{A})$ (resp. $C^+(\mathbb{A})$, $C^-(\mathbb{A})$, $C^b(\mathbb{A})$) denotes the abelian category of complexes of objects in $\mathbb{A}$ (resp. complexes X^* satisfying $X^j = 0$, for $j << 0$ and for $j >> 0$, i.e., for j outside a finite interval).
An homotopy between two morphisms of complexes $f, g : X^* \to Y^*$ is a family of morphisms $h^j : X^j \to Y^{j-1}$ in $\mathbb{A}$ satisfying $f^j - g^j = d_Y^{j-1} \circ h^j + h^{j+1} \circ d_X^j$. Homotopy defines an equivalence relation on the additive group $Hom_{C(\mathbb{A})}(X^*, Y^*)$.

Definition 4.5. The category $K(\mathbb{A})$ has the same object as the category of complexes $C(\mathbb{A})$, while the group of morphisms $Hom_{K(\mathbb{A})}(X^*, Y^*)$ is the group of morphisms of the two complexes of $\mathbb{A}$ modulo the homotopy equivalence relation.

4.1.3. *Injective resolutions.* . An abelian category $\mathbb{A}$ is said to have enough injectives if each object $A \in \mathbb{A}$ is embedded in an injective object of $\mathbb{A}$. In this case we can give another description of $Hom_{K(\mathbb{A})}(X^*, Y^*)$.

Any complex X of $\mathbb{A}$ bounded below is quasi-isomorphic to a complex of injective objects $I^*(X)$ called its injective resolution [28].

Proposition 4.6. Given a morphism $f : A_1 \to A_2$ in $C^+(\mathbb{A})$ and two injective resolutions $A_i \xrightarrow{\approx} I^*(A_i)$ of A_i, there exists an extension of f as a morphism of resolutions $I^*(f) : I^*(A_1) \to I^*(A_2)$; moreover two extensions of f are homotopic. In particular $Hom_{K^+(\mathbb{A})}(A_1, A_2) \simeq Hom_{K^+(\mathbb{A})}(I^*(A_1), I^*(A_2))$.

See lemma 4.4 in [24]. Hence, an injective resolution of an object in $\mathbb{A}$ becomes unique up to an isomorphism and functorial in the category $K^+(\mathbb{A})$.
The category $K^+(\mathbb{A})$ is only additive, even if $\mathbb{A}$ is abelian. Although we keep the same objects as in $C^+(\mathbb{A})$, the transformation on Hom is an important change in the category since an homotopy equivalence between two complexes (i.e $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ (resp.$f \circ g$) is homotopic to the identity) becomes an isomorphism.

Remark 4.7. The ith cohomology of a sheaf $\mathcal{F}$ on a topological space V, is defined up to an isomorphism as the cohomology of the global section of an injective resolution $H^i(I^*(\mathcal{F})(V))$. The complex of global sections $I^*(\mathcal{F})(V)$ is defined up to an homotopy in the category of groups $C^+(\mathbb{Z})$, while it is defined up to an isomorphism in the homotopy category of groups $K^+(\mathbb{Z})$, and called the higher direct image of $\mathcal{F}$ by the global section functor Γ or the image $R\Gamma(V, \mathcal{F})$ by the derived functor.

4.1.4. *The derived category* $D(\mathbb{A})$. The resolutions of a given complex are quasi-isomorphic. If we want to consider all resolutions as isomorphic, we must invert quasi-isomorphisms of complexes. We construct now a new category $D(\mathbb{A})$ with the same objects as $K(\mathbb{A})$ but with a different additive group of morphisms of two objects $Hom_{D(\mathbb{A})}(X, Y)$ that we describe.
Let I_Y denotes the category whose objects are quasi-isomorphisms $s' : Y \xrightarrow{\approx} Y'$ in $K(\mathbb{A})$. Let $s'' : Y \xrightarrow{\approx} Y''$ be another object. A morphism $Y' \xrightarrow{h} Y''$ satisfying $h \circ s' = s''$ defines a morphism $h : s' \to s''$. The key property is that we can take limits in $K(\mathbb{A})$:

$$Hom_{D(\mathbb{A})}(X, Y) := \lim_{\longrightarrow I_Y} Hom_{K(\mathbb{A})}(X, Y')$$

By definition, $X \xrightarrow{f} Y' \xleftarrow{s'} Y$ is equal to $X \xrightarrow{g} Y'' \xleftarrow{s''} Y$ in the inductive limit if and only if there exists a diagram $Y' \xrightarrow{u} Y''' \xleftarrow{v} Y''$ such that $u \circ s' = v \circ s''$ is a quasi isomorphism $s''' : Y \to Y'''$ and $u \circ f = v \circ g$ is the same morphism $X \to Y'''$. In this case, an element of the group at right may be represented by a symbol ${s'}^{-1} \circ f$ and this representation is not unique since in the above limit ${s'}^{-1} \circ f = {s''}^{-1} \circ g$.
We summarize what we need to know here by few remarks:
1) A morphism $f : X \to Y$ in $D(\mathbb{A})$ is represented by a diagram of morphisms:

$$X \xleftarrow{u\approx} Z \xrightarrow{g} Y \quad \text{or} \quad X \xleftarrow{g} Z \xrightarrow{u\approx} Y$$

where u is a quasi-isomorphism in $K(\mathbb{A})$ or by a sequence of such diagrams.
2) When there are enough injectives, the Hom of two objects A_1, A_2 in $D^+(\mathbb{A})$ is defined by their injective resolutions (lemma 4.5, Prop. 4.7 in [24]):

$$Hom_{D^+(\mathbb{A})}(A_1, A_2) \simeq Hom_{D^+(\mathbb{A})}(I^*(A_1), I^*(A_2)) \simeq Hom_{K^+(\mathbb{A})}(I^*(A_1), I^*(A_2))$$

In particular, all resolutions of a complex are isomorphic in the derived category.

4.1.5. *Triangles.* We define the shift of a complex (K, d_K), denoted by TK or $K[1]$, by shifting the degrees:

$$(TK)^i = K^{i+1}, \quad d_{TK} = -d_K$$

Let $u : K \to K'$ be a morphism of $C^+\mathbb{A}$, the cone $C(u)$ is the complex $TK \oplus K'$ with the differential $\begin{pmatrix} -d_K & 0 \\ u & d_{K'} \end{pmatrix}$. The exact sequence associated to $C(u)$ is:

$$0 \to K' \to C(u) \to TK \to 0$$

Let h denotes an homotopy from a morphism $u : K \to K'$ to u', we define an isomorphism $I_h : C(u) \xrightarrow{\sim} C(u')$ by the matrix $\begin{pmatrix} Id & 0 \\ h & Id \end{pmatrix}$ acting on $TK \oplus K'$, which commute with the injections of K' in $C(u)$ and $C(u')$, and with the projections on TK.
Let h and h' be two homotopies of u to u'. A second homotopy of h to h', that is a family of morphisms $k^{j+2} : K^{j+2} \to K'^j$ for $j \in \mathbb{Z}$, satisfying $h - h' = d_{K'} \circ k - k \circ d_K$, defines an homotopy of I_h to $I_{h'}$.
A distinguished (or exact) triangle in $K(\mathbb{A})$ is a sequence of complexes isomorphic to the image of an exact sequence associated to a cone in $C(\mathbb{A})$. We remark:
1) The cone over the identity morphism of a complex X is homotopic to zero.
2) Using the construction of the mapping cylinder complex over a morphism of complexes $u : X \to Y$, one can transform u, up to an homotopy equivalence into an injective morphism of complexes [28]).
3) The derived category is a triangulated category (that is endowed with a class of distinguished triangles). A distinguished triangle in $D(\mathbb{A})$ is a sequence of complexes isomorphic to the image of a distinguished triangle in $K(\mathbb{A})$. Long exact sequences of cohomologies are associated to triangles.

4.1.6. *Derived functor.* Let $T : \mathbb{A} \to \mathbb{B}$ be a functor of abelian categories. We denote also by $T : C^+\mathbb{A} \to C^+\mathbb{B}$ the corresponding functor on complexes, and by $can : C^+\mathbb{A} \to D^+\mathbb{A}$ the canonical functor, then we construct a derived functor:

$$RT : D^+\mathbb{A} \to D^+\mathbb{B}$$

satisfying $RT \circ can = can \circ T$ under the following conditions:
1) the functor $T : \mathbb{A} \to \mathbb{B}$ is left exact;
2) the category $\mathbb{A}$ has enough injective objects.

The construction is dual with the following condition: T is right exact and there exists enough projective.
a) Given a complex K in $D^+(\mathbb{A})$, we start by choosing an injective resolution of K, that is a quasi-isomorphism $i : K \xrightarrow{\approx} I(K)$ where the components of I are injectives in each degree (see [24] Lemma 4.6 p. 42).
b) We define $RT(K) = T(I(K))$.
c) A morphism $f : K \to K'$ gives rise to a morphism $RT(K) \to RT(K')$ functorially, since f can be extended to a morphism $I(f) : T(I(K)) \to T(I(K'))$, defined on the injective resolutions uniquely up to homotopy.
In particular, for a different choice of an injective resolution $J(K)$ of K, we have an isomorphism $T(I(K)) \simeq T(J(K))$ in $D^+(\mathbb{B})$.

Definition 4.8. i) The cohomology $H^j(RT(K))$ is called the hypercohomology $R^jT(K)$ of T at K.
ii) An object $A \in \mathbb{A}$ is T-acyclic if $R^jT(A) = 0$ for $j > 0$.

Remark 4.9. i) If $K \xrightarrow{\approx} K'$ is a quasi-isomorphism of complexes, TK and TK' are not quasi-isomorphisc in general, while $RT(K)$ and $RT(K')$ must be quasi-isomorphic since the image of an isomorphism in the derived category must be an isomorphism.
ii) It is important to know that we can use acyclic objects to compute RT: for any resolution $A(K)$ of a complex $K : K \xrightarrow{\approx} A(K))$, by acyclic objects, $TA(K)$ is isomorphic to the complex $RT(K)$.

Example 4.10. 1) The hypercohomology of the global section functor Γ in the case of sheaves on a topological space, is equal to the cohomology defined via flasque resolutions or any "acyclic " resolution.
2) *Extension groups.* The group of morphisms of two complexes $Hom_{D(\mathbb{A})}(X^*, Y^*)$ obtained in the new category $D(\mathbb{A})$ called derived category has a significant interpretation as an extension group:

$$Hom_{D(\mathbb{A})}(X^*, Y^*[n]) = Ext^n(X^*, Y^*)$$

In presence of enough injectives, these groups are derived from the Hom functor. In general the group $Hom_{D(\mathbb{A})}(A, B[n])$ of two objects in $\mathbb{A}$ may be still interpreted as the Yoneda $n-$extension group [28].

4.1.7. *Filtered homotopy categories* $K^+F(\mathbb{A}), K^+F_2(\mathbb{A})$. For an abelian category $\mathbb{A}$, let $F\mathbb{A}$ (resp. $F_2\mathbb{A}$) denotes the category of filtered objects (resp. bi-filtered) of $\mathbb{A}$ with finite filtration(s), $C^+F\mathbb{A}$ (resp. $C^+F_2\mathbb{A}$) the category of complexes of $F\mathbb{A}$ (resp. $F_2\mathbb{A}$) bounded on the left (zero in degrees near $-\infty$) with morphisms of complexes respecting the filtration(s).
Two morphisms $u, u' : (K, F, W) \to (K', F, W)$ to are homotopic if there exists an homotopy from u to u' compatible with the filtrations, then it induces an homotopy on each term $k^{i+1} : F^jK^{i+1} \to F'^jK'^i$ (resp. for W) and in particular $Gr_F(u - u')$ (resp. $Gr_FGr_W(u - u')$) is homotopic to 0.
The homotopy category whose objects are bounded below complexes of filtered (resp. bi-filtered) objects of $\mathbb{A}$, and whose morphisms are equivalence classes modulo homotopy compatible with the filtration(s) is denoted by $K^+F\mathbb{A}$ (resp. $K^+F_2\mathbb{A}$).

4.1.8. *Derived filtered categories* $D^+F(\mathbb{A}), D^+F_2(\mathbb{A})$. They are deduced from $K^+F\mathbb{A}$ (resp. $K^+F_2\mathbb{A}$) by inverting the filtered quasi-isomorphisms (resp. bi-filtered quasi-isomorphisms). The objects of $D^+F\mathbb{A}$ (resp.$D^+F_2\mathbb{A}$) are complexes of filtered objects of $\mathbb{A}$ as of $K^+F\mathbb{A}$ (resp. $K^+F_2\mathbb{A}$). Hence, the morphisms are represented by diagrams with filtered (resp. bi-filtered) quasi-isomorphisms.

4.1.9. *Triangles.* The complex $T(K, F, W)$ and the cone $C(u)$ of a morphism $u : (K, F, W) \to (K', F, W)$ are endowed naturally with filtrations F and W. The exact sequence associated to $C(u)$ is compatible with the filtrations. A filtered homotopy h of morphisms u and u' defines a filtered isomorphism of cones $I_h : C(u) \xrightarrow{\sim} C(u')$. Distinguished (or exact) triangles are defined similarly in $K^+F\mathbb{A}$ and $K^+F_2\mathbb{A}$ as well in $D^+F\mathbb{A}$ and $D^+F_2\mathbb{A}$. Long filtered (resp. bi- filtered) exact sequences of cohomologies are associated to triangles.

4.2. **Derived functor on a filtered complex.** Let $T : \mathbb{A} \to \mathbb{B}$ be a left exact functor of abelian categories with enough injectives in $\mathbb{A}$. We want to construct a derived functor $RT : D^+F\mathbb{A} \to D^+F\mathbb{B}$ (resp. $RT : D^+F_2\mathbb{A} \to D^+F_2\mathbb{B}$). Given a filtered complex with biregular filtration(s) we define first the image of the filtrations

via acyclic filtered resolutions. Then, we remark that the image of a filtered quasi-isomorphism is a filtered quasi-isomorphism, hence the construction factors by RT through the derived filtered category.

We need to introduce the concept of T-acyclic filtered resolutions.

4.2.1. *Image of a filtration by a left exact functor.* Let (A, F) be a filtered object in $\mathbb{A}$, with a finite filtration. Since T is left exact, a filtration TF of TA is defined by the sub-objects $TF^p(A)$.

If $Gr_F(A)$ is T-acyclic, then the objects $F^p(A)$ are T-acyclic as successive extensions of T-acyclic objects. Hence, the image by T of the sequence of acyclic objects:

$$0 \to F^{p+1}(A) \to F^p(A) \to Gr_F^p(A) \to 0$$

is exact; then:

Lemma 4.11. *If $Gr_F A$ is a $T-$acyclic object, we have $Gr_{TF}TA \simeq TGr_F A$.*

4.2.2. Let A be an object with two finite filtrations F and W such that $Gr_F Gr_W A$ is *T-acyclic*, then the objects $Gr_F A$ and $Gr_W A$ are T-acyclic, as well $F^q(A) \cap W^p(A)$. As a consequence of acyclicity, the sequences:

$$0 \to T(F^q \cap W^{p+1}) \to T(F^q \cap W^p) \to T((F^q \cap W^p)/(F^q \cap W^{p+1})) \to 0$$

are exact, and $T(F^q(Gr_W^p A))$ is the image in $T(Gr_W^p(A)$ of $T(F^q \cap W^p)$. Moreover, the isomorphism $Gr_{TW}TA \simeq TGr_W A$ transforms the filtration $Gr_{TW}(TF)$ on $Gr_{TW}TA$ into the filtration $T(Gr_W(F))$ on $TGr_W A$.

4.2.3. $RT : D^+F(\mathbb{A}) \to D^+F(\mathbb{B})$.

Let F be a biregular filtration of K.

A filtered T-acyclic resolution of K is given by a filtered quasi-isomorphism $i\colon (K, F) \to (K', F')$ to a complex with a biregular filtration such that $Gr_F^p(K^n)$ are acyclic for T for all p and n.

Lemma 4.12 (Filtered derived functor of a left exact functor $T : \mathbb{A} \to \mathbb{B}$)**.** *Suppose we are given functorially for each filtered complex (K, F) a filtered T-acyclic resolution $i : (K, F) \to (K', F')$, we define $T' : C^+F(\mathbb{A}) \to D^+F(\mathbb{B})$ by the formula $T'(K, F) = (TK', TF')$. A filtered quasi-isomorphism $f : (K_1, F_1) \to (K_2, F_2)$ induces an isomorphism $T'(f) : T(K_1, F_1) \simeq T(K_2, F_2)$ in $D^+F(\mathbb{B})$, hence T' factors through a derived functor $RT : D^+F(\mathbb{A}) \to D^+F(\mathbb{B})$ such that $RT(K, F) = (TK', TF')$, and we have $Gr_F RT(K) \simeq RT(Gr_F K)$.*

In particular for a different choice (K'', F'') of (K', F') we have an isomorphism $(TK'', TF'') \simeq (TK', TF')$ in $D^+F(\mathbb{B})$ and

$$RT(Gr_F K) \simeq Gr_{TF'}T(K') \simeq Gr_{TF''}T(K'').$$

Example 4.13. In the particular case of interest, where $\mathbb{A}$ is the category of sheaves of A-modules on a topological space X, and where T is the global section functor Γ of $\mathbb{A}$ to the category of modules over the ring A, an example of filtered T-acyclic resolution of K is the simple complex $\mathcal{G}^*(K)$, associated to the double complex defined by Godement resolution ([28] Chap.II, §3.6 p.95 or [15] Chap.II, §4.3 p.167) $\mathcal{G}^*$ in each degree of K, filtered by $\mathcal{G}^*(F^p K)$.

This example will apply to the next result for bi-filtered complexes (K, W, F) and the resolution $(\mathcal{G}^*K, \mathcal{G}^*W, \mathcal{G}^*F)$ satisfying

$$Gr_{\mathcal{G}^*F}Gr_{\mathcal{G}^*W}(\mathcal{G}^*K) \simeq \mathcal{G}^*(Gr_F Gr_W K)$$

4.2.4. $RT : D^+F_2(\mathbb{A}) \to D^+F_2(\mathbb{B})$.

Let F and W be two biregular filtrations of K.

A bi-filtered T-acyclic resolution of K is a bi-filtered quasi- isomorphism $i : (K, W, F) \to (K', W', F')$ of (K, W, F) to a bi-filtered complex biregular for each filtration such that $Gr_F^p Gr_W^q(K'^n)$ are acyclic for T for all p, q and n.

Lemma 4.14. *Suppose we are given functorially for each bi-filtered complex (K, F, W) a bi-filtered T-acyclic resolution $i : (K, F, W) \to (K', F', W')$, we define $T' : C^+F(\mathbb{A}) \to D^+F(\mathbb{B})$ by the formula $T'(K, F, W) = (TK', TF', TW')$.*
A bi-filtered quasi-isomorphism $f : (K_1, F_1, W_1) \to (K_2, F_2, W_2)$ induces an isomorphism $T'(f) : T'(K_1, F_1, W_1) \simeq T'(K_2, F_2, W_2)$ in $D^+F_2(\mathbb{B})$, hence T' factors through a derived functor $RT : D^+F_2(\mathbb{A}) \to D^+F_2(\mathbb{B})$ such that $RT(K, F, W) = (TK', TF', TW')$, and we have $Gr_F Gr_W RT(K) \simeq RT(Gr_F Gr_W K)$.

In particular for a different choice (K'', F'', W'') of (K', F', W') we have an isomorphism $(TK'', TF'', TW'') \simeq (TK', TF', TW'')$ in $D^+F_2(\mathbb{B})$ and

$$RT(Gr_F Gr_W K) \simeq Gr_{TF'} Gr_{TW'} T(K') \simeq Gr_{TF''} T(K'')$$

4.3. **The spectral sequence defined by a filtered complex.** A decreasing filtration F of a complex K by sub-complexes induces a filtration still denoted F on its cohomology $H^*(K)$.

The aim of a spectral sequence is to compute the associated graded cohomology $Gr_F^* H^*(K)$ of the filtered group $(H^*(K), F)$, out of the cohomology $H^*(F^iK/F^jK)$ of the various indices of the filtration. The spectral sequence $E_r^{p,q}(K, F)$ associated to F ([4], [6]) leads for large r and under mild conditions, to such graded cohomology defined by the filtration.

A morphism in the derived filtered category defines a natural morphism of associated spectral sequences; in particular a quasi-isomorphism defines an isomorphism.

Later we shall study Mixed Hodge Complex where the weight spectral sequence becomes interesting and meaningful since it contains geometrical information and degenerates at rank 2. Now, we give the definition of the terms of the spectral sequence and some examples.
To define the spectral terms $E_r^{pq}(K, F)$ or ${}_F E_r^{pq}$ or simply E_r^{pq} with respect to F, we put for $r > 0$ and $p, q \in \mathbb{Z}$:

$$Z_r^{pq} = Ker(d : F^pK^{p+q} \to K^{p+q+1}/F^{p+r}K^{p+q+1})$$
$$B_r^{pq} = F^{p+1}K^{p+q} + d(F^{p-r+1}K^{p+q-1})$$

Such formula still makes sense for $r = \infty$ if we set, for a filtered object (A, F), $F^{-\infty}(A) = A$ and $F^{\infty}(A) = 0$:

$$Z_\infty^{pq} = Ker(d : F^pK^{p+q} \to K^{p+q+1})$$
$$B_\infty^{pq} = F^{p+1}K^{p+q} + d(K^{p+q+1})$$

We set by definition:

$$E_r^{pq} = Z_r^{pq}/(B_r^{pq} \cap Z_r^{pq}) \quad E_\infty^{pq} = Z_\infty^{pq}/B_\infty^{pq} \cap Z_\infty^{pq}$$

The notations are similar to [6] but different from [15].

Remark 4.15. In our case, in order to obtain some arguments by duality, we note the following equivalent dual definitions to Z_r^{pq} and Z_∞^{pq}:

$$K^{p+q}/B_r^{pq} = coker(d : F^{p-r+1}K^{p+q-1} \to K^{p+q}/F^{p+1}(K^{p+q}))$$
$$K^{p+q}/B_\infty^{pq} = coker(d : K^{p+q+1} \to K^{p+q}/F^{p+1}K^{p+q})$$
$$E_r^{pq} = Im(Z_r^{pq} \to K^{p+q}/B_r^{pq}) = Ker(K^{p+q}/B_r^{pq} \to K^{p+q}/(Z_r^{pq} + B_r^{pq})).$$

The term $Gr_F^p(H^{p+q}(K))$ is said to be the limit of the spectral sequence. If the filtration is biregular the terms E_r^{pq} compute this limit after a finite number of steps; for each p, q there exists r_0 such that:

$$Z_r^{pq} = Z_\infty^{pq}, \quad B_r^{pq} = B_\infty^{pq}, \quad E_r^{pq} = E_\infty^{pq}, \quad \forall r > r_0$$

Note that in some cases, it is not satisfactory to get only the graded cohomology and this is one motivation to be not happy with spectral sequences and prefer to keep the complex as in the derived category.

Lemma 4.16. *For each r, there exists a differential d_r on the terms E_r^{pq} with the property that its cohomology is exactly E_{r+1}^{pq}:*

$$E_{r+1}^{pq} = H(E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{pq} \xrightarrow{d_r} E_r^{p+r,q-r+1}).$$

where d_r is induced by the differential $d : Z_r^{pq} \to Z_r^{p+r,q-r+1}$.

We define d_r later with new notations for an increasing filtration.
For $r < \infty$, the terms E_r form a complex of objects, graded by the degree $p-r(p+q)$ (or a direct sum of complexes with index $p-r(p+q)$). The first term may be written as:

$$E_1^{pq} = H^{p+q}(Gr_F^p(K))$$

so that the differentials d_1 are obtained as connecting morphisms defined by the short exact sequences of complexes

$$0 \to Gr_F^{p+1}K \to F^pK/F^{p+2}K \to Gr_F^pK \to 0.$$

It will be convenient to set for $r = 0$, $E_0^{pq} = Gr_F^p(K^{p+q})$.
The spectral sequence $E_i^{p,q}(K, F)$ is said to degenerate at rank r if the differentials d_i are zero for $i \geq r$ independently of p, q, then we have in this case

$$E_r^{pq} = E_i^{pq} = E_\infty^{pq} \quad for\ i \geq r.$$

There is no easy general construction for (E_r, d_r) for $r > 0$; however we will see that in the case of Mixed Hodge Structures of interest to us the terms with respect to the weight filtration W have a geometric meaning and degenerate at rank 2: ${}_WE_2^{pq} = {}_WE_\infty^{pq}$.

4.3.1. *Equivalent notations for increasing filtrations.* For an increasing filtration W on K, the precedent formulas are transformed by the usual change of indices to pass from W to a decreasing filtration F, that is $F^i = W_{-i}$.

Let W be an increasing filtration on K. We set for all j, $n \leq m$ and $n \leq i \leq m$:

$$W_iH^j(W_mK/W_nK) = Im(H^j(W_iK/W_nK) \to H^j(W_mK/W_nK)$$

then we adopt the following new notations for the terms, for all $r \geq 1, p$ and q:

Lemma 4.17. *The terms of the spectral sequence for (K, W) are equal to:*

$$E_r^{pq}(K,W)) = Gr_{-p}^W H^{p+q}(W_{-p+r-1}K/W_{-p-r}K).$$

Proof. Let (K_r^p, W) denotes the quotient complex $K_r^p := W_{-p+r-1}K/W_{-p-r}K$ with the induced filtration by subcomplexes W; we put:

$$Z_\infty^{pq}(K_r^p, W) := Ker\,(d : (W_{-p}K^{p+q}/W_{-p-r}K^{p+q}) \to (W_{-p+r-1}K^{p+q+1}/W_{-p-r}K^{p+q+1}))$$
$$B_\infty^{pq}(K_r^p, W) := (W_{-p-1}K^{p+q} + dW_{-p+r-1}K^{p+q-1})/W_{-p-r}K^{p+q}$$

which coincide, *up to the quotient by* $W_{-p-r}K^{p+q}$, with $Z_r^{pq}(K,W)$ (resp. $B_r^{pq}(K,W)$) with:

$$Z_r^{p,q} := Ker\,(d : W_{-p}K^{p+q} \to K^{p+q+1}/W_{-p-r}K^{p+q+1})$$
$$B_r^{p,q} := W_{-p-1}K^{p+q} + dW_{-p+r-1}K^{p+q-1}$$

then, we define:

$$E_\infty^{pq}(K_r^p, W) = \frac{Z_\infty^{pq}(K_r^p, W)}{B_\infty^{pq}(K_r^p, W) \cap Z_\infty^{pq}(K_r^p, W)} = Gr_{-p}^W H^{pq}(W_{-p+r-1}K/W_{-p-r}K)$$

and find:

$$E_r^{pq}(K,W) = Z_r^{pq}/(B_r^{pq} \cap Z_r^{pq}) = Z_\infty^{pq}(K_r^p, W)/(B_\infty^{pq}(K_r^p, W) \cap Z_\infty^{pq}(K_r^p, W))$$
$$= E_\infty^{pq}(K_r^p, W) = Gr_{-p}^W H^{pq}(W_{-p+r-1}K/W_{-p-r}K)$$

□

To define the differential d_r, we consider the exact sequence:

$$0 \to W_{-p-r}K/W_{-p-2r}K \to W_{-p+r-1}K/W_{-p-2r}K \to W_{-p+r-1}K/W_{-p-r}K \to 0$$

and the connecting morphism:

$$H^{p+q}(W_{-p+r-1}K/W_{-p-r}K) \xrightarrow{\partial} H^{p+q+1}(W_{-p-r}K/W_{-p-2r}K)$$

the injection $W_{-p-r}K \to W_{-p-1}K$ induces a morphism:

$$\varphi : H^{p+q+1}(W_{-p-r}K/W_{-p-2r}K) \to W_{-p-r}H^{p+q+1}(W_{-p-1}K/W_{-p-2r}K)$$

. Let π denote the projection on the right term below, equal to $E_r^{p+r,q-r+1}$:

$$W_{-p-r}H^{p+q+1}(W_{-p-1}K/W_{-p-2r}K) \xrightarrow{\pi} Gr_{-p-r}^W H^{p+q+1}(W_{-p-1}K/W_{-p-2r}K)$$

the composition of morphisms $\pi \circ \varphi \circ \partial$ restricted to $W_{-p}H^{p+q}(W_{-p+r-1}K/W_{-p-r}K)$ induces the differential:

$$d_r : E_r^{pq} \to E_r^{p+r,q-r+1}$$

while the injection $W_{-p+r-1} \to W_{-p+r}K$ induces the isomorphism:

$$H(E_r^{pq}, d_r) \xrightarrow{\sim} E_{r+1}^{pq} = Gr_{-p}^W H^{p+q}(W_{-p+r}K/W_{-p-r-1}K).$$

4.3.2. *Hypercohomology spectral sequence.* Let $T : \mathbb{A} \to \mathbb{B}$ be a left exact functor of abelian categories, and (K, F) an object of $D^+F\mathbb{A}$ and $RT(K, F) : D^+F\mathbb{A} \to D^+F\mathbb{B}$ its derived functor. The spectral sequence defined by the complex $RT(K, F)$ is written as:

$$_FE_1^{p,q} = R^{p+q}T(Gr_F^p) \Rightarrow Gr_F^pR^{p+q}T(K)$$

This is the hypercohomology spectral sequence of the filtered complex K. For an increasing filtration W on K, we have:

$$_WE_1^{p,q} = R^{p+q}T(Gr_{-p}^W) \Rightarrow Gr_{-p}^WR^{p+q}T(K)$$

It depends functorially on K and a filtered quasi-isomorphism induces an isomorphism of spectral sequences. The differentials d_1 of this spectral sequence are the image by T of the connecting morphisms defined by the short exact sequences:

$$0 \to Gr_F^{p+1}K \to F^pK/F^{p+2}K \to Gr_F^pK \to 0.$$

4.3.3. *Examples.* 1) Let K be a complex, the canonical filtration τ is the increasing filtration by sub-complexes:

$$\tau_{\leq p} = (\cdots \to K^{p-1} \to Ker d \to 0 \cdots \to 0)$$

then:

$$Gr_{\leq p}^\tau K \xrightarrow{\approx} H^p(K)[-p], \quad H^i(\tau_{\leq p}(K)) = H^i(K) \text{ if } i \leq p, \text{ and } 0 \text{ if } i > p.$$

2) The sub-complexes of K:

$$\sigma_{\geq p}K := K^{*\geq p} = (0 \to \cdots \to 0 \to K^p \to K^{p+1} \to \cdots)$$

define a decreasing biregular filtration, the trivial filtration of K such that $Gr_\sigma^pK = K^p[-p]$, i.e., it coincides with the Hodge filtration on de Rham complex.

A quasi-isomorphism $f : K \to K'$ is necessarily a filtered quasi-isomorphism for both, the trivial and the canonical filtrations. The hypercohomology spectral sequences of a left exact functor attached to the trivial and canonical filtrations of K are the two natural hypercohomology spectral sequences of K.

3) Let $f : X \to Y$ be a continued map of topological spaces. Let $\mathcal{F}$ be an abelian sheaf on X and $\mathcal{F}^*$ a resolution of $\mathcal{F}$ by f_*-acyclic sheaves, then $R^if_*\mathcal{F} \cong H^i(f_*\mathcal{F}^*)$. The hypercohomology spectral sequence of the global section functor $\Gamma(Y, *)$ of the complex $Rf_*\mathcal{F}^*$ with its canonical filtration, is:

$$_\tau E_1^{pq} = \mathbb{H}^{p+q}(Y, R^{-p}f_*\mathcal{F}[p]) \simeq H^{2p+q}(Y, R^{-p}f_*\mathcal{F}) \Rightarrow Gr_{-p}^\tau H^{p+q}(X, \mathcal{F}).$$

This formula illustrates basic difference in Deligne's notation: the sheaf $R^{-p}f_*\mathcal{F}[p]$ is in degree $-p$.

In classical notations, Leray's spectral sequence for f and $\mathcal{F}$ starts at:

$$E_2^{pq} = H^p(Y, R^qf_*\mathcal{F})$$

To relate both notations we need to renumber the classical term $E_{r+1}^{2p+q,-p}$ into the new term E_r^{pq}.

5. Mixed Hodge Complex (MHC)

In this section, we give the definition of a Mixed Hodge Complex (MHC) and prove Deligne's fundamental theorem that the cohomology of a Mixed Hodge Complex is endowed with a Mixed Hodge Structure.

First, on an algebraic variety V, we define a cohomological version of a Mixed Hodge Complex, that we call Cohomological Mixed Hodge Complex (CMHC), which is defined essentially by a bi-filtered complex of sheaves $(K_{\mathbb{C}}, F, W)$ where the filtration W is rationally defined and satisfies precise conditions sufficient to define a Mixed Hodge Complex structure on the global section functor $R\Gamma(V, K)$. The results proved by Deligne are technically difficult and so strong that the theory is reduced to constructing such a Cohomological Mixed Hodge Complex for all algebraic varieties in the remaining sections; hence the theoretical path to construct a Mixed Hodge Structure on a variety follows the pattern:

$$\text{CMHC} \Rightarrow \text{ MHC} \Rightarrow \text{MHS}$$

It is true that a direct study of the logarithmic complex by Griffiths and Schmid [19] is very attractive, but the initial work of Deligne is easy to apply, flexible and helps to go beyond this case towards a general theory.

The de Rham complex of a smooth compact complex variety is a special case of a Mixed Hodge Complex, called a Hodge complex (HC) with the characteristic property that it induces a Hodge Structure on its hypercohomology. We start by rewriting the Hodge theory that we know, with terminology that is fitted to our generalization.

Let A denote $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ and $A \otimes \mathbb{Q}$ the field $\mathbb{Q}$ or $\mathbb{R}$ accordingly as in section 3, $D^+(\mathbb{Z}))$ (resp. $D^+(\mathbb{C}))$, $D^+(V, \mathbb{Z}))$, $D^+(V, \mathbb{C})))$ denotes the derived category of $\mathbb{Z}-$modules (resp. $\mathbb{C}-$vector spaces and corresponding sheaves on V).

Definition 5.1 (Hodge Complex (HC))**.** A Hodge A-complex K of weight n consists of:
i) A complex K_A of A-modules, such that $H^k(K_A)$ is an A-module of finite type for all k;
ii) A filtered complex $(K_{\mathbb{C}}, F)$ of $\mathbb{C}-$vector spaces;
iii) An isomorphism $\alpha : K_A \otimes \mathbb{C} \simeq K_{\mathbb{C}}$ in $D^+(\mathbb{C})$.
The following axioms must be satisfied:
(HC 1) the differential d of $K_{\mathbb{C}}$ is strictly compatible with the filtration F, i.e., $d^i : (K_A^i, F) \to (K_A^{i+1}, F)$ is strict, for all i;
(HC 2) for all k, the filtration F on $H^k(K_{\mathbb{C}}) \simeq H^k(K_A) \otimes \mathbb{C}$ defines an A-Hodge Structure of weight $n + k$ on $H^k(K_A)$.

Equivalently, in (HC1) the spectral sequence defined by $(K_{\mathbb{C}}, F)$ degenerates at $E_1 (E_1 = E_\infty)$ (see 4.4 or [6]), and in (HC2) the filtration F is $(n + k)$-opposed to its complex conjugate (conjugation makes sense since $A \subset \mathbb{R}$).

Definition 5.2. Let X be a topological space. An A-Cohomological Hodge Complex (CHC) K of weight n on X, consists of:
i) A complex of sheaves K_A of $A-$modules on X;
ii) A filtered complex of sheaves $(K_{\mathbb{C}}, F)$ of $\mathbb{C}$-vector spaces on X;
iii) An isomorphism $\alpha : K_A \otimes \mathbb{C} \xrightarrow{\approx} K_{\mathbb{C}}$ in $D^+(X, \mathbb{C})$ of $\mathbb{C}-$sheaves on X.
Moreover, the following axiom must be satisfied:
(CHC) The triple $(R\Gamma(K_A), R\Gamma(K_{\mathbb{C}}, F), R\Gamma(\alpha))$ is a Hodge Complex of weight n.

If (K, F) is a Hodge Complex (resp. Cohomological Hodge Complex) of weight n, then $(K[m], F[p])$ is a Hodge Complex (resp. Cohomological Hodge Complex) of weight $n + m - 2p$.

The following statement is a new version of Hodge decomposition Theorem:

Theorem 5.3. *Let X be a compact complex algebraic manifold and consider:*
i) $K_{\mathbb{Z}}$ the complex reduced to a constant sheaf $\mathbb{Z}$ on X in degree zero;
ii) $K_{\mathbb{C}}$ the analytic de Rham complex Ω_X^* with its trivial filtration $F^p = \Omega_X^{*\geq p}$ by subcomplexes:

$$F^p\Omega_X^* := 0 \to 0 \cdots 0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots \to \Omega_X^n \to 0;$$

iii) The quasi-isomorphism $\alpha\colon K_{\mathbb{Z}} \otimes \mathbb{C} \xrightarrow{\approx} \Omega_X^*$ (Poincaré lemma).
Then $(K_{\mathbb{Z}}, (K_{\mathbb{C}}, F), \alpha)$ is a Cohomological Hodge Complex of weight 0; its hypercohomology on X, isomorphic to the cohomology of X, carries a functorial Hodge Structure for morphisms of algebraic varieties.

The new idea here is to observe the degeneracy of the spectral sequence of (Ω_X^*, F) and deduce the definition of the Hodge filtration from the trivial filtration F on the de Rham complex without any reference to harmonic forms, although the proof of the decomposition is given via a reduction to the case of a projective variety, hence a compact Kähler manifold:

Definition 5.4. The Hodge filtration F is defined on the cohomology as follows:

$$F^pH^i(X, \mathbb{C}) := F^p\mathbb{H}^i(X, \Omega_X^*) := Im(\mathbb{H}^i(X, F^p\Omega_X^*) \to \mathbb{H}^i(X, \Omega_X^*)),$$

where the first equality follows from holomorphic Poincaré lemma on the resolution of the constant sheaf $\mathbb{C}$ by the analytic de Rham complex Ω_X^*.

Proposition 5.5 (Deligne [5])**.** Let X be a smooth compact complex algebraic variety, then the filtration F induced on cohomology by the filtration F on the de Rham complex is a Hodge filtration.

The proof is based on the degeneration at rank one of the spectral sequence with respect to F defined as follows:

$${}_FE_1^{p,q} := \mathbb{H}^{p+q}(X, Gr_F^p\Omega_X^*) \simeq H^q(X, \Omega_X^p) \Rightarrow Gr_F^p\mathbb{H}^{p+q}(X, \Omega_X^*).$$

The degeneration at rank one may be deduced from classical Hodge decomposition, but it has been also obtained by direct algebraic methods by Deligne and Illusie in [9] for the algebraic de Rham complex with respect to Zariski topology on which the filtration F is also defined.

The isomorphism on complex smooth algebraic varieties between analytic and algebraic de Rham hypercohomology defined respectively with analytic and algebraic de Rham complexes is given by Grothendieck's comparison theorem (see [21]).

Now, we define the structure including two filtrations by weight W and F needed on a complex, in order to define a Mixed Hodge Structure on its cohomology.

Definition 5.6 (MHC)**.** An A-Mixed Hodge Complex (MHC) K consists of:
i) A complex K_A of $A-$modules such that $H^k(K_A)$ is an A-module of finite type for all k;
ii) A filtered complex $(K_{A\otimes\mathbb{Q}}, W)$ of $(A \otimes \mathbb{Q})-$vector spaces with an increasing filtration W;
iii) An isomorphism $K_A \otimes \mathbb{Q} \xrightarrow{\sim} K_{A\otimes\mathbb{Q}}$ in $D^+(A \otimes \mathbb{Q})$;

iv) A bi-filtered complex $(K_{\mathbb{C}}, W, F)$ of $\mathbb{C}-$vector spaces with an increasing (resp. decreasing) filtration W (resp. F) and an isomorphism:

$$\alpha : (K_{A\otimes\mathbb{Q}}, W)\otimes\mathbb{C} \xrightarrow{\sim} (K_{\mathbb{C}}, W)$$

in $D^+F(\mathbb{C})$.
Moreover, the following axiom is satisfied:
(MHC) For all n, the system consisting of
- the complex $Gr_n^W(K_{A\otimes\mathbb{Q}})$ of $(A\otimes\mathbb{Q})-$vector spaces,
- the complex $Gr_n^W(K_{\mathbb{C}}, F)$ of $\mathbb{C}-$vector spaces with induced F and
- the isomorphism $Gr_n^W(\alpha) : Gr_n^W(K_{A\otimes\mathbb{Q}})\otimes\mathbb{C} \xrightarrow{\sim} Gr_n^W(K_{\mathbb{C}})$,
is an $A\otimes\mathbb{Q}$-Hodge Complex of weight n.

The above structure has a corresponding structure on a complex of sheaves on X called a Cohomological Mixed Hodge Complex:

Definition 5.7 (CMHC)**.** An A-Cohomological Mixed Hodge Complex K (CMHC) on a topological space X consists of:
i) A complex of sheaves K_A of sheaves of A-modules on X such that $H^k(X, K_A)$ are A-modules of finite type;
ii) A filtered complex $(K_{A\otimes\mathbb{Q}}, W)$ of sheaves of $(A\otimes\mathbb{Q})-$vector spaces on X with an increasing filtration W and an isomorphism $K_A\otimes\mathbb{Q}\simeq K_{A\otimes\mathbb{Q}}$ in $D^+(X, A\otimes\mathbb{Q})$;
iii) A bi-filitered complex of sheaves $(K_{\mathbb{C}}, W, F)$ of $\mathbb{C}-$vector spaces on X with an increasing (resp. decreasing) filtration W (resp. F) and an isomorphism:

$$\alpha : (K_{A\otimes\mathbb{Q}}, W)\otimes\mathbb{C} \xrightarrow{\sim} (K_{\mathbb{C}}, W)$$

in $D^+F(X, \mathbb{C})$.
Moreover, the following axiom is satisfied:
(CMHC) For all n, the system consisting of:
- the complex $Gr_n^W(K_{A\otimes\mathbb{Q}})$ of sheaves of $(A\otimes\mathbb{Q})$-vector spaces on X;
- the complex $Gr_n^W(K_{\mathbb{C}}, F)$ of sheaves of $\mathbb{C}$-vector spaces on X with induced F;
- the isomorphism $Gr_n^W(\alpha) : Gr_n^W(K_{A\otimes\mathbb{Q}})\otimes\mathbb{C} \xrightarrow{\sim} Gr_n^W(K_{\mathbb{C}})$,
is an $A\otimes\mathbb{Q}$-Cohomological Hodge Complex of weight n.

If (K, W, F) is a Mixed Hodge Complex (resp. Cohomological Mixed Hodge Complex), then for all m and $n\in\mathbb{Z}, (K[m], W[m-2n], F[n])$ is a Mixed Hodge Complex (resp. Cohomological Mixed Hodge Complex).

In fact, any Hodge Complex or Mixed Hodge Complex described here is obtained from de Rham complexes with modifications (at infinity) as the logarithmic complex described later in the next section. A new construction of Hodge Complex has been later introduced with the theory of differential modules and perverse sheaves [1] and [32] but will not be covered in these lectures.

Now we describe how we deduce first the Mixed Hodge Complex from a Cohomological Mixed Hodge Complex, then a Mixed Hodge Structure from a Mixed Hodge Complex.

Proposition 5.8. If $K = (K_A, (K_{A\otimes\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F))$ and the isomorphism α is an A-Cohomological Mixed Hodge Complex then:

$$R\Gamma K = (R\Gamma K_A, R\Gamma(K_{A\otimes\mathbb{Q}}, W), R\Gamma(K_{\mathbb{C}}, W, F))$$

with $R\Gamma(\alpha)$ is an A-Mixed Hodge Complex.

The main result of Deligne in [6] and [7] is algebraic and states in short:

Theorem 5.9 (Deligne)**.** *The cohomology of a Mixed Hodge Complex carries a Mixed Hodge Structure.*

The proof of this result requires a detailed study of spectral sequences and will occupy the rest of this section. We give here, before the proof, the properties of the various spectral sequences which may be of interest as independent results.

Precisely, the weight spectral sequence of a Mixed Hodge complex is in the category of Hodge Structure.

So, the Mixed Hodge Structure on cohomology is approached step by step by Hodge Structures on the terms of the weight spectral sequence ${}_WE_r^{pq}$ of $(K_{\mathbb{C}}, W)$. However, the big surprise is that the spectral sequence degenerates quickly, at rank two for W and at rank one for F: this is all which is needed.

A careful study shows that there are various induced filtrations by F on the weight spectral sequence ${}_WE_r^{pq}$ of $(K_{\mathbb{C}}, W)$. Two direct filtrations F_d and F_{d^*} and one recurrent F_{rec} are induced by F on the terms ${}_WE_r^{pq}$. Under the conditions imposed on the two filtrations W and F in the definition of a Mixed Hodge Complex, the three filtrations coincide, the spectral sequence with respect to W degenerates at rank 2 and the induced filtration by F on the first terms $E_1^{p,q}$ define a Hodge Structure of weight q (lemma on two filtrations). The filtration d_1 is a morphism of HS, hence the terms $E_2^{p,q}$ carry a Hodge Structures of weight q. The proof consists to show that d_r is compatible with the induced Hodge Structure, but for $r > 1$ it is a morphism between two Hodge Structures of different weight, hence it must vanish.

Proposition 5.10 (MHS on the cohomology of a MHC)**.** Let K be an A-MHC.
i)The filtration $W[n]$ of $H^n(K_A) \otimes \mathbb{Q} \simeq H^n(K_{A\otimes\mathbb{Q}})$:

$$(W[n])_q(H^n(K_{A\otimes\mathbb{Q}}) := Im\,(H^n(W_{q-n}K_{A\otimes\mathbb{Q}}) \to H^n(K_{A\otimes\mathbb{Q}}))$$

and the filtration F on $H^n(K_{\mathbb{C}}) \simeq H^n(K_A) \otimes_A \mathbb{C}$:

$$F^p(H^n(K_{\mathbb{C}}) := Im\,(H^n(F^pK_{\mathbb{C}}) \to H^n(K_{\mathbb{C}}))$$

define on $H^n(K)$ an A-Mixed Hodge Structure, i.e.:

$$(H^n(K_A), (H^n(K_{A\otimes\mathbb{Q}}), W), (H^n(K_{\mathbb{C}}), W, F))$$

is an A-Mixed Hodge Structure.
ii) On the terms ${}_WE_r^{pq}(K_{\mathbb{C}}, W)$, the recurrent filtration and the two direct filtrations coincide $F_d = F_{rec} = F_{d^*}$ and define the Hodge filtration F of a Hodge Sructure of weight q and d_r is compatible with F.
iii) The morphisms $d_1 : {}_WE_1^{p,q} \to {}_WE_1^{p+1,q}$ are strictly compatible with F.
iv) The spectral sequence of ($K_{A\otimes\mathbb{Q}}$, W) degenerates at rank 2 (${}_WE_2 = {}_WE_\infty$).
v) The spectral sequence of Hodge Structures of $(K_{\mathbb{C}}, F)$ degenerates at rank 1 (${}_FE_1 = {}_FE_\infty$).
vi) The spectral sequence of the complex $Gr_F^p(K_{\mathbb{C}})$, with the induced filtration W, degenerates at rank 2.

Notice that the index of the weight is not given by the index of the induced filtration W on cohomology, but shifted by $[n]$. One should recall that the weight of the Hodge Structure on the terms ${}_WE_r^{p,q}$ is always q, hence the weight of $Gr_{-p}^W H^{p+q}(K)$ is q, that is $(W[p+q])_q = W_{-p}$.

Now, we are going to give the proofs of the statements of this introduction.

5.1. **HS on the cohomology of a smooth compact algebraic variety.**
Proof of the proposition (5.5). i) For X projective, we use the Dolbeault resolution of the sub-complex $F^p\Omega_X^*$ by the sub-complex $F^p\mathcal{E}_X^*$ defined as a simple complex associated to a double complex:

$$F^p\mathcal{E}_X^* = s((\mathcal{E}_X^{r,*}, \overline{\partial}), \partial)_{r\geq p}$$

This is a fine resolution (see 1.17), which compute de Rham hypercohomology:

$$\mathbb{H}^n(X, F^p\Omega_X^*) \simeq H^n(X, F^p\mathcal{E}^*).$$

Then we can identify the terms $E_1^{pq} = H^q(X, \Omega_X^p)$ with the harmonic forms of type (p, q) to deduce the degeneration of the spectral sequence $({}_FE_r^{p,q}, d_r)$ of the filtered de Rham complex at rank 1 from classical Hodge theory, and moreover we obtain the Hodge decomposition on X.

ii) If X is not projective, by Chow's lemma (see [33] p.69), there exists a projective variety and a projective birational morphism $f : X' \to X$. By Hironaka's desingularization ([26]) we can suppose X' smooth, hence X' is a Kähler manifold. Then, by Grothendieck's duality theory ([5] § 4 or [24]), there exists for all integers p a trace map $Tr(f) : Rf_*\Omega_{X'}^p \to \Omega_X^p$ inducing a map on cohomology $Tr(f) : H^q(X', \Omega_{X'}^p) \to H^q(X, \Omega_X^p)$, because $\mathbb{H}^q(X, Rf_*\Omega_{X'}^p) \simeq H^q(X', \Omega_{X'}^p)$, such that the composition with the canonical reciprocal morphism $f^* : H^q(X, \Omega_X^p) \to H^q(X', \Omega_{X'}^p)$ is the identity:

$$Tr(f) \circ f^* = Id : H^q(X, \Omega_X^p) \to H^q(X', \Omega_{X'}^p) \to H^q(X, \Omega_X^p)$$

In particular we deduce that f^* is injective. Since the map:

$$f^* : f^*(\Omega_X^p, F) \to (\Omega_{X'}^p, F)$$

is compatible with filtrations, we deduce a map of spectral sequences:

$$E_1^{pq} = H^q(X, \Omega_X^p) \xrightarrow{f^*} E_1'^{pq} = H^q(X', \Omega_{X'}^p), \quad f^* : E_r^{pq}(X) \hookrightarrow E_r^{pq}(X')$$

which is injective on all terms and commute with the differentials d_r. The proof is by induction on r as follows. It is true for $r = 1$, as we have just noticed. The differential d_1 vanishes on X'; it must vanish on X, then the terms for $r = 2$ coincide with the terms for $r = 1$ and we can repeat the argument for all r.

The degeneration of the Hodge spectral sequence on X at rank 1 follows, and it is equivalent to the isomorphism:

$$\mathbb{H}^n(X, F^p\Omega_X^*) \xrightarrow{\sim} F^p\mathbb{H}^n(X, \Omega_X^*).$$

Equivalently, the dimension of the hypercohomology of the de Rham complex $\mathbb{H}^j(X, \Omega_X^*)$ is equal to the dimension of Hodge cohomology $\oplus_{p+q=j}H^q(X, \Omega_X^p)$.

However, Hodge theory tells more. From the Hodge filtration, we deduce the definition of the subspaces:

$$H^{p,q}(X) := F^pH^i(X, \mathbb{C}) \cap \overline{F^q}H^i(X, \mathbb{C}), \quad \text{for } p + q = i$$

satisfying $H^{p,q}(X) = \overline{H^{q,p}(X)}$. We must check the decomposition:

$$H^i(X, \mathbb{C}) = \oplus_{p+q=i}H^{p,q}(X),$$

and moreover, we will deduce $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$. We have:

$$F^pH^n(X) \cap \overline{F^{n-p+1}}H^n(X) \subset F^pH^n(X') \cap \overline{F^{n-p+1}}H^n(X') = 0.$$

This shows that $F^pH^n(X)+\overline{F^{n-p+1}}H^n(X)$ is a direct sum. We want to prove that this sum equals $H^n(X)$.

Let $h^{p,q} = \dim H^{p,q}(X)$, since the spectral sequence degenerates at rank 1, we have:

$$\dim F^pH^n(X) = \sum_{i\geq p} h^{i,n-i},\ \dim\overline{F^{n-p+1}}H^n(X) = \sum_{i\geq n-p+1} h^{i,n-i},$$

then:

$$\sum_{i\geq p} h^{i,n-i} + \sum_{i\geq n-p+1} h^{i,n-i} \leq \dim H^n(X) = \sum_i h^{i,n-i}$$

from which we deduce $\sum_{i\geq p} h^{i,n-i} \leq \sum_{i\leq n-p} h^{i,n-i}$.

By Serre duality on X of dimension N, we have $h^{i,j} = h^{N-i,N-j}$, which transforms the inequality into: $\sum_{N-i\leq N-p} h^{N-i,N-n+i} \leq \sum_{N-i\geq N-n+p} h^{N-i,N-n+i}$, from which we deduce on $H^m(X)$ for $m = 2N-n$, the opposite inequality by setting $j = N-i, q = N-n+p$ shows that, for all q and m:

$$\sum_{j\geq q} h^{j,m-j} \geq \sum_{j\leq m-q} h^{j,m-j},$$

In particular:

$$\sum_{i\geq p} h^{i,n-i} \geq \sum_{i\leq n-p} h^{i,n-i} \quad \text{hence} \quad \sum_{i\geq p} h^{i,n-i} = \sum_{i\leq n-p} h^{i,n-i}.$$

This implies $\dim F^p + \dim\overline{F^{n-p+1}} = \dim H^n(X)$. Hence:

$$H^n(X) = F^pH^n(X) \oplus \overline{F^{n-p+1}}H^n(X)$$

which, in particular, induces a decomposition:

$$F^{n-1}H^n(X) = F^pH^n(X) \oplus H^{n-1,n-p+1}(X).$$

Remark 5.11. If we use distinct notations for X with Zariski topology and X^{an} for the analytic associated manifold, then the filtration F is defined on the algebraic de Rham hypercohomology groups and the comparison theorem (see [21]) is compatible with the filtrations: $\mathbb{H}^i(X, F^p\Omega_X^*) \simeq \mathbb{H}^i(X^{an}, F^p\Omega_{X^{an}}^*)$.

5.1.1. Let $\mathcal{L}$ be a rational local system with a polarization, rationally defined, on the associated local system $\mathcal{L}_{\mathbb{C}} = \mathcal{L}\otimes_{\mathbb{Q}}\mathbb{C}$, then the spectral sequence defined by the Hodge filtration on de Rham complex with coefficients $\Omega_X^*\otimes_{\mathbb{C}}\mathcal{L}$ degenerates at rank 1:

$$E_1^{pq} = H^q(X, \Omega_X^p(\mathcal{L})) \Rightarrow H^{p+q}(X, \mathcal{L}_{\mathbb{C}}), \quad E_1^{pq} = E_\infty^{pq}$$

and the induced filtration by F on cohomology defines a Hodge Structure. The proof by Deligne proceeds as in Hodge theory.

5.2. **MHS on the cohomology of a Mixed Hodge Complex.** The proof is based on a delicate study of the induced filtration W and F on the cohomology. To explain the difficulty, imagine for a moment that we want to give a proof by induction on the length of W. Suppose that the weights of a Mixed Hodge Complex: (K, W, F) vary from $W_0 = 0$ to $W_l = K$ and suppose we did construct the Mixed Hodge Structure on cohomology for $l-1$, then we consider the long exact sequence of cohomology:

$$H^{i-1}(Gr_l^W K) \to H^i(W_{l-1}K) \to H^i(K) \to H^i(Gr_l^W K) \to H^{i+1}(W_{l-1}K)$$

If the result was established then the morphisms of the sequence are strict, hence the difficulty is a question of relative positions of the subspaces W_p and F^q on $H^i(K)$ with respect to $\mathrm{Im}H^i(W_{l-1}K)$ and the projection on $H^i(Gr_l^W K)$. This study is known as the two filtrations lemma.

5.2.1. *Two filtrations.* This section relates results on various induced filtrations on terms of a spectral sequence, contained in [6] and [7] (lemma on two filtrations). A filtration W of a complex by subcomplexes define a spectral sequence $E_r(K,W)$. The filtration F induces in various ways filtrations on $E_r(K,W)$, different in general. We discuss here axioms on W and F, at the level of complexes, in order to get compatibility of the various induced filtrations by F on the spectral sequence of (K,W). What we have in mind is to find axioms leading to define the Mixed Hodge Structure with induced filtrations W and F on cohomology. The proof is technical, hence we emphasize here the main ideas as a guide to Deligne's proof.

5.2.2. Let (K,F,W) be a bi-filtered complex of objects of an abelian category, bounded below. The filtration F, assumed to be biregular, induces on the terms E_r^{pq} of the spectral sequence $E(K,W)$ various filtrations as follows:

Definition 5.12 (F_d, F_{d^*})**.** The first direct filtration on $E_r(K,W)$ is the filtration F_d defined for r finite or $r=\infty$, by the image:

$$F_d^p(E_r(K,W)) = Im(E_r(F^pK,W) \to E_r(K,W)).$$

Dually, the second direct filtration F_{d^*} on $E_r(K,W)$ is defined by the kernel:

$$F_{d^*}^p(E_r(K,W)) = Ker(E_r(K,W) \to E_r(K/F^pK,W)).$$

This definition is temporarily convenient for the next computation, since the filtrations F_d, F_{d^*} are naturally induced by F, hence compatible with the differentials d_r.

Since for $r=0$, 1, $B_r^{pq} \subset Z_r^{pq}$,

Lemma 5.13. *We have* $F_d = F_{d^*}$ *on* $E_0^{pq} = Gr_F^p(K^{p+q})$ *and* $E_1^{pq} = H^{p+q}(Gr_p^W K)$.

Definition 5.14 (F_{rec})**.** The recurrent filtration F_{rec} on E_r^{pq} is defined as follows
i) On E_0^{pq}, $F_{rec} = F_d = F_{d^*}$.
ii) The recurrent filtration F_{rec} on E_r^{pq} induces a filtration on $\ker d_r$, which induces on its turn the recurrent filtration F_{rec} on E_{r+1}^{pq} as a quotient of $\ker d_r$.

5.2.3. The precedent definitions of direct filtrations apply to E_∞^{pq} as well and they are compatible with the isomorphism $E_r^{pq} \simeq E_\infty^{pq}$ for large r. We deduce, via this isomorphism a recurrent filtration F_{rec} on E_∞^{pq}. The filtrations F and W induce each a filtration on $H^{p+q}(K)$. We want to prove that the isomorphism $E_\infty^{pq} \simeq Gr_W^{-p} H^{p+q}(K)$ is compatible with F at right and F_{rec} on E_∞^{pq}.

5.2.4. *Comparison of* F_d, F_{rec}, F_{d^*}. In general we have only the inclusions

Proposition 5.15. i) On E_r^{pq}, we have the inclusions

$$F_d(E_r^{pq}) \subset F_{rec}(E_r^{pq}) \subset F_{d^*}(E_r^{pq})$$

ii) On E_∞^{pq}, the filtration induced by the filtration F on $H^*(K)$ satisfy

$$F_d(E_\infty^{pq}) \subset F(E_\infty^{pq}) \subset F_{d^*}(E_\infty^{pq}).$$

iii) The differential d_r is compatible with F_d and F_{d^*}.

We want to show that these three filtrations coincide under the conditions on the Mixed Hodge Complex, for this we need to know the compatibility of d_r with F_{rec} which will be the Hodge filtration of a Hodge Structure. In fact, Deligne proves an intermediary statement, that will apply inductively for Mixed Hodge Complexes.

Theorem 5.16 (Deligne)**.** *Let K be a complex with two filtrations W and F, where W is biregular and suppose that for each non negative integer $r < r_0$, the differentials d_r of the graded complex $E_r(K, W)$ are strictly compatible with F_{rec}, then*
i) For $r \leq r_0$ the sequence

$$0 \to E_r(F^pK, W) \to E_r(K, W) \to E_r(K/F^pK, W) \to 0$$

is exact, and for $r = r_0 + 1$, the sequence

$$E_r(F^pK, W) \to E_r(K, W) \to E_r(K/F^pK, W)$$

is exact. In particular for $r \leq r_0 + 1$, the two direct and the recurrent filtration on $E_r(K, W)$ coincide $F_d = F_{rec} = F_{d^*}$.
ii) For $a < b$ and $r < r_0$, the differentials d_r of the graded complex $E_r(F^aK/F^bK, W)$ are strictly compatible with F_{rec}.
iii) If the above condition is satisfied for all r, then the spectral sequence $E(K, F)$ degenerates at E_1 ($E_1 = E_\infty$) and the filtrations F_d, F_{rec}, F_{d^*} coincide into a unique filtration called F. Moreover, the isomorphism $E_\infty^{pq} \simeq Gr_W^{-p} H^{p+q}(K)$ is compatible with the filtrations F and we have an isomorphism of spectral sequences

$$Gr_F^p E_r(K, W) \simeq E_r(Gr_F^p K, W)$$

Proof. This surprising statement looks natural only if we have in mind the degeneration of $E(K, F)$ at rank 1 and the strictness in the category of Mixed Hodge Structures.

For fixed p, we consider the following property:

(P_r) $E_i(F^pK, W)$ injects into $E_i(K, W)$ for $i \leq r$ and its image is F^p_{rec} for $i \leq r+1$.

We already noted that (P_0) is satisfied. The proof by induction on r will apply as long as r remains $\leq r_0$. Suppose $r < r_0$ and (P_s) true for all $s \leq r$, we prove (P_{r+1}). The sequence:

$$E_r(F^pK, W) \xrightarrow{d_r} E_r(F^pK, W) \xrightarrow{d_r} E_r(F^pK, W)$$

injects into:

$$E_r(K, W) \xrightarrow{d_r} E_r(K, W) \xrightarrow{d_r} E_r(K, W)$$

with image $F_d = F_{rec}$, then, the image of F^p_{rec} in E_{r+1}:

$$F^p_{rec}E_{r+1} = Im[Ker(F^p_{rec}E_r(K, W) \xrightarrow{d_r} E_r(K, W)) \to E_{r+1}(K, W)]$$

coincides with the image of F^p_d which is by definition $Im[E_{r+1}(F^pK, W) \to E_{r+1}(K, W)]$.

Since d_r is strictly compatible with F_{rec}, we have:

$$d_rE_r(K, W) \cap E_r(F^pK, W) = d_rE_r(F^pK, W)$$

which means that $E_{r+1}(F^pK, W)$ injects into $E_{r+1}(K, W)$, hence we deduce the injectivity for $r + 1$. Since $\ker d_r$ on F^p_{rec} is equal to $\ker d_r$ on $E_{r+1}(F^pK, W)$, we deduce $F^p_{rec} = F^p_d$ on $E_{r+2}(K, W)$, which proves (P_{r+1}).

Then, i) follows from a dual statement applied to F_{d^*} and ii) follows, because $E_r(F^bK, W) \hookrightarrow E_r(F^aK, W)$ is injective as they are both in $E_r(K, W)$.

iii) We deduce from i) the first exact sequence followed by its dual:

$$0 \to E_r(F^{p+1}K, W) \to E_r(F^pK, W) \to E_r(Gr_F^pK, W) \to 0$$
$$0 \leftarrow E_r(K/F^pK, W) \leftarrow E_r(F^{p+1}K, W) \leftarrow E_r(Gr_F^pK, W) \leftarrow 0.$$

In view of the injections in i) and the coincidence of $F_d = F_{rec} = F_{d^*}$ we have a unique filtration F, the quotient of the first two terms in the first exact sequence above is isomorphic to $Gr_F^pE_r(K, W)$, hence we deduce an isomorphism

$$Gr_F^pE_r(K, W) \simeq E_r(Gr_F^pK, W)$$

compatible with d_r and autodual. If the hypothesis is now true for all r, we deduce an exact sequence:

$$0 \to E_\infty(F^pK, W) \to E_\infty(K, W) \to E_\infty(K/F^pK, W) \to 0$$

which is identical to:

$$0 \to Gr_W H^*(F^pK) \to Gr_W H^*(K) \to Gr_W H^*(K/F^pK) \to 0$$

from which we deduce, for all i:

$$0 \to H^i(F^pK) \to H^i(K) \to H^i(K/F^pK) \to 0$$

and that the filtrations W induced on $H^i(F^pK)$ from (F^pK, W) and from $(H^i(K), W)$ coincide. □

5.2.5. *Proof of the existence of a MHS on cohomology of a MHC.* The above theorem 5.16 applies to Mixed Hodge Complexes, since the hypothesis of the induction on r in the theorem will be satisfied as follows. If we assume that the filtrations $F_d = F_{rec} = F_{d^*}$ coincide for $r < r_0$ and moreover define the same Hodge filtration F of a Hodge Structure of weight q on $E_r^{p,q}(K, W)$ and $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is compatible with such Hodge Structure, then in particular d_r is strictly compatible with F, hence the induction apply.

Lemma 5.17. *For $r \geq 1$, the differentials d_r of the spectral sequence ${}_WE_r$ are strictly compatible to the recurrent filtration $F = F_{rec}$. For $r \geq 2$, they vanish.*

The initial statement applies for $r = 1$ by definition of a Mixed Hodge Complex since the complex Gr_{-p}^WK is a Hodge Complex of weight $-p$. Hence, the two direct filtrations and the recurrent filtration F_{rec} coincide with the Hodge filtration F on ${}_WE_1^{pq} = H^{p+q}(Gr_{-p}^WK)$. The differential d_1 is compatible with the direct filtrations, hence with F_{rec}, and commutes with complex conjugation since it is defined on $A \otimes \mathbb{Q}$, hence it is compatible with $\overline{F}_{rec}$. Then it is strictly compatible with the Hodge filtration $F = F_{rec}$.

The filtration F_{rec} defined in this way is q-opposed to its complex conjugate and defines a Hodge Structure of weight q on ${}_WE_2^{pq}$.

We suppose by induction that the two direct filtrations and the recurrent filtration coincide on ${}_WE_s(s \leq r+1) : F_d = F_{rec} = F_{d^*}$ and ${}_WE_r = {}_WE_2$. On ${}_WE_2^{p,q} = {}_WE_r^{p,q}$, the filtration $F_{rec} := F$ is compatible with d_r and q-opposed to its complex conjugate. Hence the morphism $d_r : {}_WE_r^{p,q} \to {}_WE_r^{p+r,q-r+1}$ is a morphism of a Hodge Structure of weight q to a Hodge Structure of weight $q-r+1$ and must vanish for $r > 1$. In particular, we deduce that the weight spectral sequence degenerates at rank 2.

The filtration on ${}_WE_\infty^{p,q}$ induced by the filtration F on $H^{p+q}(K)$ coincides with the filtration F_{rec} on ${}_WE_2^{p,q}$.

6. Logarithmic complex, normal crossing divisor and the mixed cone

In the next two sections, we show how the previous abstract algebraic study applies to algebraic varieties by constructing explicitly the Cohomological Mixed Hodge Complex. First, on a smooth complex variety X containing a Normal Crossing Divisor (NCD) Y with smooth irreducible components, we introduce the complex of sheaves of differential forms with logarithmic singularities along Y denoted $\Omega_X^*(LogY)$ and sometimes $\Omega_X^* < Y >$. Its hypercohomology is isomorphic to the cohomology of $X - Y$ with coefficients in $\mathbb{C}$ and it is naturally endowed with two filtrations W and F. It is only when X is compact that the bi-filtered complex $(\Omega_X^*(LogY), W, F)$ underlies a Cohomological Mixed Hodge complex which defines a Mixed Hodge Structure on the cohomology of $X - Y$.

The Mixed Hodge Structure of a smooth variety V depends on the properties at infinity of V, i.e., we have to consider a compactification of V by a compact algebraic variety X, which is always possible by a result of Nagata [30]. Moreover, by Hironaka's desingularization theorem [26], we can suppose X is smooth and the complement $Y = X - V$ is a Normal Crossing Divisor (NCD) with smooth irreducible components. Hence we can use $(\Omega_X^*(LogY), W, F)$ to define a Mixed Hodge Structure on the cohomology of $X - Y$ and then carry this Mixed Hodge Structure onto the cohomology of V. It is not difficult to show that such a Mixed Hodge Structure does not depend on the compactification X and will be referred to as the Mixed Hodge Structure on V. We stress that the weight W of the cohomology $H^i(V)$ is $\geq i$, to be precise $W_{i-1} = 0, W_{2i} = H^i(V)$.

The dual of the logarithmic complex [11] is more natural to construct and is an example of the general construction in the next section. Similar to the basic Mayer-Vietoris construction in cohomology, in the case of the normal crossing divisor Y, we define a resolution of the complex $\mathbb{Z}_Y$ by a complex $\mathbb{Z}_{Y_*}$ defined in degree $j-1$ by the constant sheaf $\mathbb{Z}$ on the disjoint sum Y_j of the intersections of j distinct components of Y. The interest in such resolution is its construction from the spaces Y_j forming what we call a strict simplicial resolution of Y. In such case we will develop a natural procedure to deduce a canonical Mixed Hodge Complex on Y. This construction is relatively easy to understand in this case, hence it is the best illustration of the construction in the following section. The weight W of the cohomology $H^j(Y)$ is $\leq j$, to be precise $W_{-1} = 0, W_j = H^j(V)$.

We associate to the natural morphism $\mathbb{Z}_X \to \mathbb{Z}_{Y_*}$ a complex called the cone defining the relative cohomology isomorphic to the cohomology with compact supports of the complement $X - Y$, Poincaré dual to the cohomology of $X - Y$.

The combination of the two cases, when we consider a sub-normal Crossing Divisor Z, a union of some components of Y, and consider the complement $Y - Z$, we obtain an open Normal Crossing Divisor which will be a model for the general case with the most general type of Mixed Hodge Structure on the cohomology $H^j(Y - Z)$ of weights varying from 0 to $2j$.

Finally we discuss the technique of the mixed cone which associates a new Mixed Hodge Complex to a morphism of Mixed Hodge Complexes at the level of the homotopy category, which leads to a Mixed Hodge Structure on the relative cohomology and a long exact sequence of Mixed Hodge Structures. However there exists an ambiguity on the Mixed Hodge Structure obtained since it depends on an homotopy

between various resolutions [11]. In this case the ambiguity is related to the embedding of the rational cohomology into the complex cohomology and equivalently related to the problem of extension of Mixed Hodge Structure in general.

6.1. **MHS on the cohomology of smooth varieties.** Now we construct the Mixed Hodge Structure on the cohomology of a smooth complex algebraic variety V. The algebraicity of V allows us to use a result of Nagata [30] to embed V as an open Zariski subset of a complete variety Z. Then the singularities of Z are included in $D := Z - V$. Since Hironaka's desingularization process in characteristic zero (see [26]) is carried out by blowing up smooth centers above D, there exists a variety $X \to Z$ above Z such that the inverse image of D is a normal crossing divisor Y with smooth components in X such that $X - Y \simeq Z - D$.

Hence we may start with the hypothesis that $V = X^* := X - Y$ is the complement of a normal crossing divisor Y in a smooth proper variety X, so the construction of the Mixed Hodge Structure is reduced to this situation, under the condition that we prove its independence of the choice of X.

We introduce now the logarithmic complex and prove that it underlies a Cohomological Mixed Hodge Complex on X and computes the cohomology of V.

6.1.1. *Logarithmic complex.* Let X be a complex manifold and Y be a Normal Crossing Divisor in X. We denote by $j : X^* \to X$ the embedding of $X^* := X - Y$ into X. A meromorphic form ω has a pole of order at most 1 along Y if at each point y, $f\omega$ is holomorphic for some local equation f of Y at y. Let $\Omega^*_X(*Y)$ denote the sub-complex of $j_*\Omega^*_{X^*}$ defined by meromorphic forms along Y, holomorphic on X^*.

Definition 6.1. The logarithmic de Rham complex of X along a Normal Crossing Divisor Y is the subcomplex $\Omega^*_X(Log\, Y)$ of the complex $\Omega^*_X(*Y)$ defined by the sections ω such that ω and $d\omega$ both have a pole of order at most 1 along Y.

By definition, at each point $y \in Y$, there exist local coordinates $(z_i)_{i\in[1,n]}$ on X and $I \subset [1,n]$ such that Y is defined at y by the equation $\Pi_{i\in I} z_i = 0$. Then ω has logarithmic poles along Y if and only if it can be written locally as:

$$\omega = \sum_{i_1,\cdots,i_r} \omega_{i_1,\cdots,i_r} \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_r}}{z_{i_r}} \quad \text{where} \quad \omega_{i_1,\cdots,i_r} \text{ is holomorphic.}$$

This formula may be used as a definition but then we have to prove its independence of the choice of coordinates, that is ω may be written in this form with respect to any set of local coordinates at y.

The $\mathcal{O}_X$-module $\Omega^1_X(Log\, Y)$ is locally free with basis $(dz_i/z_i)_{i\in I}$ and $(dz_j)_{j\in[1,n]-I}$ and $\Omega^p_X(Log\, Y) \simeq \wedge^p \Omega^1_X(Log\, Y)$.

Let $f : X_1 \to X_2$ be a morphism of complex manifolds, with normal crossing divisors Y_i in X_i for $i = 1, 2$, such that $f^{-1}(Y_2) = Y_1$. Then, the reciprocal morphism $f^* : f^*(j_{2*}\Omega^*_{X_2^*}) \to j_{1*}\Omega^*_{X_1^*}$ induces a morphism on logarithmic complexes:

$$f^* : f^*\Omega^*_{X_2}(Log\, Y_2) \to \Omega^*_{X_1}(Log\, Y_1).$$

6.1.2. *Weight filtration W.* Let $Y = \cup_{i\in I} Y_i$ be the union of smooth irreducible divisors. Let S^q denotes the set of strictly increasing sequences $\sigma = (\sigma_1, ..., \sigma_q)$ in the set of indices I, $Y_\sigma = Y_{\sigma_1 \ldots \sigma_q} = Y_{\sigma_1} \cap ... \cap Y_{\sigma_q}$, $Y^q = \coprod_{\sigma\in S^q} Y_\sigma$ the disjoint

union of Y_σ. Set $Y^0 = X$ and $\Pi : Y^q \to Y$ the canonical projection. An increasing filtration W, called the weight, is defined as follows:

$$W_m(\Omega^p_X(Log\, Y)) = \sum_{\sigma\in S^m} \Omega^{p-m}_X \wedge dz_{\sigma_1}/z_{\sigma_1} \wedge ... \wedge dz_{\sigma_m}/z_{\sigma_m}$$

The sub-$\mathcal{O}_X$-module $W_m(\Omega^p_X(Log\, Y)) \subset \Omega^p_X(Log\, Y)$ is the smallest sub-module stable by exterior multiplication with local sections of Ω^*_X and containing the products $dz_{i_1}/z_{i_1} \wedge ... \wedge dz_{i_k}/z_{i_k}$ for $k \leq m$ for local equations z_j of the components of Y.

6.1.3. *The Residue isomorphism.* To define the Poincaré residue along a component of an intersection of Y_i, we need to fix and order a set of hypersurfaces $Y_{i_1}, \ldots, Y_{i_m}$ to intersect. The following composed Poincaré residue is defined on $Gr^W_m \Omega^*_X(Log\, Y)$:

$$Res : Gr^W_m(\Omega^p_X(Log\, Y)) \to \Pi_* \Omega^{p-m}_{Y^m}$$

given by $Res([\alpha \wedge (dz_{i_1}/z_{i_1} \wedge ... \wedge (dz_{i_m}/z_{i_m})]) = \alpha/Y_{i_1,\ldots,i_m}$.

In the case $m = 1$, it defines:

$$Res : \Omega^*_X(Log\, Y) \to \Pi_* \Omega^*_{Y^1}[-1]$$

where Y^1 is the disjoint union of the irreducible components of Y. In general it is a composition map of such residue in the one codimensional case up to a sign. We need to prove it is well defined, independent of the coordinates, compatible with the differentials and induces an isomorphism:

$$Res : Gr^W_m(\Omega^*_X(Log\, Y)) \xrightarrow{\sim} \Pi_* \Omega^*_{Y^m}[-m]$$

We construct its inverse. Consider, for each sequence of indices $\sigma = (i_1, \ldots, i_m)$ in S^m, the morphism $\rho_\sigma : \Omega^p_X \to Gr^W_m(\Omega^{p+m}_X(Log\, Y))$, defined locally as:

$$\rho_\sigma(\alpha) = \alpha \wedge dz_{\sigma_1}/z_{\sigma_1} \wedge ... \wedge dz_{i_m}/z_{i_m}$$

It does not depend on the choice of z_i, since for another choice of coordinates z'_i, z_i/z'_i are holomorphic and the difference $(dz_i/z_i) - (dz'_i/z'_i) = d(z_i/z'_i)/(z_i/z'_i)$ is holomorphic.

Then $\rho_\sigma(\alpha) - \alpha \wedge dz'_{i_1}/z'_{i_1} \wedge ... \wedge dz'_{i_m}/z'_{i_m} \in W_{m-1}\Omega^{p+m}_X(LogY)$, and successively $\rho_\sigma(\alpha) - \rho'_\sigma(\alpha) \in W_{m-1}\Omega^{p+m}_X(Log\, Y)$. We have $\rho_\sigma(z_{i_j} \cdot \beta) = 0$ and $\rho_\sigma(dz_{i_j} \wedge \beta') = 0$ for sections β of Ω^p_X and β' of Ω^{p-1}_X; hence ρ_σ factors by $\overline{\rho}_\sigma$ on $\Pi_* \Omega^p_{Y_\sigma}$ defined locally and glue globally into a morphism of complexes on X:

$$\overline{\rho}_\sigma : \Pi_* \Omega^p_{Y_\sigma} \to Gr^W_m(\Omega^{p+m}_X(Log\, Y)), \quad \overline{\rho} : \Pi_* \Omega^*_{Y^m}[-m] \to Gr^W_m \Omega^*_X(Log\, Y).$$

Lemma 6.2. *We have the following isomorphisms of sheaves:*
i) $H^i(Gr^W_m \Omega^*_X(Log\, Y)) \simeq \Pi_* \mathbf{C}_{Y^m}$ *for* $i = m$ *and* 0 *for* $i \neq m$,
ii) $H^i(W_r \Omega^*_X(Log\, Y)) \simeq \Pi_* \mathbf{C}_{Y^i}$ *for* $i \leq r$ *and* $H^i(W_r \Omega^*_X(Log\, Y)) = 0$ *for* $i > r$,
and in particular $H^i(\Omega^*_X(Log\, Y)) \simeq \Pi_* \mathbb{C}_{Y^i}$.

Proof. The statement in i) follows from the residue isomorphism.

The statement in ii) follows easily by induction on r, from i) and the long exact sequence associated to the short exact sequence $0 \to W_r \to W_{r+1} \to Gr^W_{r+1} \to 0$, written as

$$H^i(W_r) \to H^i(W_{r+1}) \to H^i(Gr^W_{r+1}) \to H^{i+1}(W_r)$$

□

Proposition 6.3 (Weight filtration W)**.** The morphisms of filtered complexes:

$$(\Omega_X^*(Log\,Y), W) \overset{\alpha}{\leftarrow} (\Omega_X^*(Log\,Y), \tau) \overset{\beta}{\rightarrow} (j_*\Omega_{X^*}^*, \tau)$$

are filtered quasi-isomorphisms.

Proof. The quasi-isomorphism α follows from the lemma.

The morphism j is Stein, since for each polydisc $U(y)$ in X centered at a point $y \in Y$, the inverse image $X^* \cap U(y)$ is Stein as the complement of an hypersurface, hence j is acyclic for coherent sheaves, that is $Rj_*\Omega_{X^*}^* \simeq j_*\Omega_{X^*}^*$. By Poincaré lemma $\underline{\mathbb{C}}_{X^*} \simeq \Omega_{X^*}^*$, so that $Rj_*\underline{\mathbb{C}}_{X^*} \simeq j_*\Omega_{X^*}^*$, hence it is enough to prove $Gr_i^\tau Rj_*\mathbb{C}_{X^*} \simeq \Pi_*\mathbb{C}_{Y^i}$, which is a local statement.

For each polydisc $U(y) = U$ with $U^* = U - U \cap Y = (D^*)^m \times D^{n-m}$, the cohomology $\mathbb{H}^i(U, Rj_*\underline{\mathbb{C}}_{X^*}) = H^i(U^*, \mathbb{C})$ can be computed by Künneth formula and is equal to $\wedge^i H^1(U^*, \mathbb{C}) \simeq \wedge^i \Gamma(U, \Omega_U^1(Log\,Y)))$ where $dz_i/z_i, i \in [1, m]$ form a basis dual to the homology basis since $(D^*)^m \times D^{n-m}$ is homotopic to an $i-$dimensional torus $(S^1)^m$. □

Corollary 6.4. The weight filtration is rationally defined.

The main point here is that the τ filtration is defined with rational coefficients as $(Rj_*\underline{\mathbb{Q}}_{X^*}, \tau) \otimes \mathbb{C}$, which gives the rational definition for W.

6.1.4. *Hodge filtration* F. It is defined by the formula $F^p = \Omega_X^{*\geq p}(Log\,Y)$, which includes all forms of type (p', q') with $p' \geq p$. We have:

$$Res : F^p(Gr_m^W \Omega_X^*(Log\,Y)) \simeq \Pi_* F^{p-m}\Omega_{Y^m}^*[-m]$$

hence a filtered isomorphism:

$$Res : (Gr_m^W \Omega_X^*(Log\,Y), F) \simeq (\Pi_*\Omega_{Y^m}^*[-m], F[-m]).$$

Corollary 6.5. The system $\mathbf{K}$:

(1) $(\mathbf{K}^{\mathbb{Q}}, W) = (Rj_*\underline{\mathbb{Q}}_{X^*}, \tau) \in ObD^+F(X, \mathbb{Q})$
(2) $(\mathbf{K}^{\mathbb{C}}, W, F) = (\Omega_X^*(Log\,Y), W, F) \in ObD^+F_2(X, \mathbb{C})$
(3) The isomorphism $(\mathbf{K}^{\mathbb{Q}}, W) \otimes \mathbb{C} \simeq (\mathbf{K}^{\mathbb{C}}, W)$ in $D^+F(X, \mathbb{C})$

is a Cohomological Mixed Hodge Complex on X.

Theorem 6.6 (Deligne)**.** *The system $K = R\Gamma(X, \mathbf{K})$ is a Mixed Hodge Complex. It endows the cohomology of $X^* = X - Y$ with a canonical Mixed Hodge Structure.*

Proof. The result follows directly from the general theory of Cohomological Mixed Hodge Complex.

Nevertheless, it is interesting to understand what is needed for a direct proof and compute the weight spectral sequence at rank 1:

$$_WE_1^{pq}(R\Gamma(X, \Omega_X^*(Log\,Y)) = \mathbb{H}^{p+q}(X, Gr_{-p}^W\Omega_X^*(LogY)) \simeq \mathbb{H}^{p+q}(X, \Pi_*\Omega_{Y^{-p}}^*[p])$$
$$\simeq H^{2p+q}(Y^{-p}, \mathbb{C}) \Rightarrow Gr_q^W H^{p+q}(X^*, \mathbb{C}).$$

where the double arrow means that the spectral sequence degenerates to the cohomology graded with respect to the filtration W induced by the weight on the complex level. In fact, we recall the proof up to rank 2. The differential d_1:

$$d_1 = \sum_{j=1}^{-p}(-1)^{j+1}G(\lambda_{j,-p}) = G : H^{2p+q}(Y^{-p}, \mathbb{C}) \longrightarrow H^{2p+q+2}(Y^{-p-1}, \mathbb{C})$$

is equal to an alternate Gysin morphism, Poincaré dual to the alternate restriction morphism:

$$\rho = \sum_{j=1}^{-p}(-1)^{j+1}\lambda^*_{j,-p} : H^{2n-q}(Y^{-p-1},\mathbb{C}) \to H^{2n-q}(Y^{-p},\mathbb{C})$$

hence the first term:

$$({}_WE_1^{pq}, d_1)_{p\in\mathbb{Z}} = (H^{2p+q}(Y^{-p},\mathbb{C}), d_1)_{p\in\mathbb{Z}}$$

is viewed as a complex in the category of Hodge Structures of weight q. It follows that the terms:

$${}_WE_2^{pq} = H^p({}_WE_1^{*,q}, d_1)$$

are endowed with a Hodge Structure of weight q.

We need to prove that the differential d_2 is compatible with the induced Hodge filtration. For this we introduced the direct filtrations compatible with d_2 and proved that they coincide with the induced Hodge filtration. The differential d_2 is necessarily zero since it is a morphism of Hodge Structure of different weights: the Hodge Structure of weight q on E_2^{pq} and the Hodge Structure of weight $q-1$ on $E_2^{p+2,q-1}$. The proof is the same as any Mixed Hodge Complex and consists of a recurrent argument to show in this way that the differentials d_i for $i \geq 2$ are zero. □

6.1.5. *Independence of the compactification and functoriality.* Let U be a complex smooth variety, X (resp. X') a compactification of U by a Normal Crossing Divisor Y (resp. Y') at infinity, $j : U \to X$ (resp. $j' : U \to X'$) the open embedding; then $j\times j' : U \to X\times X'$ is a locally closed embedding, with closure V. By desingularizing V outside the image of U, we are reduced to the case where we have a smooth variety $X'' \xrightarrow{f} X$ such that $Y'' := f^{-1}(Y)$ is a Normal Crossing Divisor and $U \simeq X'' - Y''$, then we have an induced morphism f^* on the corresponding logarithmic complexes, compatible with the structure of Mixed Hodge Complex. It follows that the induced morphism f^* on hypercohomology is compatible with the Mixed Hodge Structure and is an isomorphism on the hypercohomology groups, hence it is an isomorphism of Mixed Hodge Structure.
Functoriality. Let $f : U \to V$ be a morphism of smooth varieties, X (resp. Z) smooth compactifications of U (resp. V) by Normal Crossing Divisor at infinity, then taking the closure of the graph of f in $X \times Z$ and desingularizing, we are reduced to the case where there exists a compactification X with an extension $\overline{f} : X \to Z$ inducing f on U. The induced morphism $\overline{f}^*$ on the corresponding logarithmic complexes is compatible with the filtrations W and F and with the structure of Mixed Hodge Complex, hence it is compatible with the Mixed Hodge Structure on hypercohomology.

Proposition 6.7. Let U be a smooth complex algebraic variety.
i) The Hodge numbers $h^{p,q} := \dim.H^{p,q}(Gr^W_{p+q}H^i(U))$ vanish for $p, q \notin [0, i]$. In particular, the weight of the cohomology $H^i(U)$ vary from i to $2i$.
ii) let X be a smooth compactification of U, then:

$$W_iH^i(U) = \text{Im } (H^i(X) \to H^i(U)).$$

Proof. i) The space $Gr^W_rH^i(U)$ is isomorphic to the term $E_2^{i-r,r}$ of the spectral sequence with a Hodge Structure of weight r, hence it is a sub-quotient of $E_1^{i-r,r} =$

$H^{2i-r}(Y^{r-i})$ twisted by $\mathbb{Q}(i-r)$. Hence, we have $h^{p,q}(H^{2i-r}(Y^{r-i})) = 0$, for $p+q=r$, unless $r-i \geq 0, 2i-r \geq 0$ (i.e $i \leq r \leq 2i$) and $p \in [0, 2r-i]$, $h^{p,q}(E_1^{i-r,r}) = 0$ unless $p \in [i-r, r]$.

ii) Suppose U is the complement of a Normal Crossing Divisor. Denote $j : U \to X$ the inclusion. By definition $W_i H^i(U) = \text{Im } (\mathbb{H}^i(X, \tau_{\leq 0} Rj_* \mathbb{Q}_U) \to H^i(U, \mathbb{Q}))$, hence it is equal to the image of $H^i(X, \mathbb{Q})$ since $\mathbb{Q}$ is quasi-isomorphic to $\tau_{\leq 0} Rj_* \mathbb{Q}_U$. If $X - U$ is not a Normal Crossing Divisor, there exists a desingularization $\pi : X' \to X$ with an embedding of U in X' as the complement of a Normal Crossing Divisor, then we use the trace map $Tr\,\pi : H^i(X', \mathbb{Q}) \to H^i(X, \mathbb{Q})$ satisfying $(Tr\pi) \circ \pi^* = Id$ and compatible with Hodge Structures. In fact the trace map is defined as a morphism of sheaves $R\pi_* \mathbb{Q}_{X'} \to \mathbb{Q}_X$ [36], hence commutes with the restriction to U. In particular, the images of both cohomology groups coincide in $H^i(U)$. □

Exercise 6.8 (Riemann Surface)**.** Let $\overline{C}$ be a connected compact Riemann surface of genus g, $Y = \{x_1, \ldots, x_m\}$ a subset of m points, and $C = \overline{C} - Y$ the open surface with m points in $\overline{C}$ deleted. Consider the long exact sequence:

$$0 \to H^1(\overline{C}, \mathbb{Z}) \to H^1(C, \mathbb{Z}) \to H^2_Y(\overline{C}, \mathbb{Z}) = \oplus_{i=1}^{i=m} \mathbb{Z} \to H^2(\overline{C}, \mathbb{Z}) \simeq \mathbb{Z} \to H^2(C, \mathbb{Z}) = 0$$

then:

$$0 \to H^1(\overline{C}, \mathbb{Z}) \to H^1(C, \mathbb{Z}) \to \mathbb{Z}^{m-1} \simeq Ker(\oplus_{i=1}^{i=m} \mathbb{Z} \to \mathbb{Z}) \to 0$$

is a short exact sequence of Mixed Hodge Structures where $H^1(C, \mathbb{Z}) = W_2 H^1(C, \mathbb{Z})$ is an extension of two different weights: $W_1 H^1(C, \mathbb{Z}) = H^1(\overline{C}, \mathbb{Z})$ of rank $2g$ and $Gr_2^W H^1(C, \mathbb{Z}) \simeq \mathbb{Z}^{m-1}$.

The Hodge filtration is given by $F^0 H^1(C, \mathbb{C}) = H^1(C, \mathbb{C})$, $F^2 H^1(C, \mathbb{C}) = 0$, while:

$$F^1 H^1(C, \mathbb{C}) \simeq \mathbb{H}^1(\overline{C}, (0 \to \Omega^1_{\overline{C}}(Log\{x_1, \ldots, x_m\}) \simeq H^0(\overline{C}, \Omega^1_{\overline{C}}(Log\{x_1, \ldots, x_m\})$$

is of rank $g + m - 1$ and fits into the exact sequence determined by the residue morphism:

$$0 \to \Omega^1_{\overline{C}} \to \Omega^1_{\overline{C}}(Log\{x_1, \ldots, x_m\}) \to \mathcal{O}_{\{x_1, \ldots, x_m\}} \to 0.$$

Exercise 6.9 (Hypersurfaces)**.** Let $i : Y \hookrightarrow P$ be a smooth hypersurface in a projective variety P.

1) To describe the cohomology of the affine open set $U = P - Y$ we may use, by Grothendieck's result on algebraic de Rham cohomology, algebraic forms on P regular on U denoted $\Omega^*(U) = \Omega_P(*Y)$, or meromorphic forms along Y, holomorphic on U denoted $\Omega_{P^{an}}(*Y)$, where the Hodge filtration is described by the order of the pole ([6] Prop. 3.1.11) (but not the trivial filtration F on $\Omega^*(U)$ and by Deligne's resulton the logarithmic complex with its trivial filtration F.

2) For example, in the case of a curve Y in a plane $\mathbb{P}^2$, global holomorphic forms are all rational by Serre's result on cohomology of coherent sheaves. We have an exact sequence of sheaves:

$$0 \to \Omega^2_P \to i_* \Omega^2_P(LogY) \to \Omega^1_Y \to 0.$$

Since $h^{2,0} = h^{2,1} = 0$, $H^0(P, \Omega^2_P) = H^1(P, \Omega^2_P) = 0$, hence we deduce from the associated long exact sequence, the isomorphism:

$$Res : H^0(P, \Omega^2_P(LogY)) \xrightarrow{\sim} H^0(Y, \Omega^1_Y),$$

that is 1-forms on Y are residues of rational $2-$forms on P with simple pole along the curve.

In homogeneous coordinates, let $F = 0$ be the homogeneous equation of Y. We take the residue along Y of the rational form:

$$\frac{A(z_0dz_1 \wedge dz_2 - z_1dz_0 \wedge dz_2 + z_2dz_0 \wedge dz_1)}{F}$$

where A is homogeneous of degree $d-3$ if F has degree d ([3] example 3.2.8).
3) For P projective again, we consider the exact sequence for relative cohomology (or cohomology with support in Y):

$$H^{k-1}(U) \xrightarrow{\partial} H^k_Y(P) \to H^k(P) \xrightarrow{j^*} H^k(U)$$

which reduces via Thom's isomorphism, to:

$$H^{k-1}(U) \xrightarrow{r} H^{k-2}(Y) \xrightarrow{i_*} H^k(P) \xrightarrow{j^*} H^k(U)$$

where r is the topological Leray's residue map dual to the tube over cycle map $\tau : H_{k-2}(Y) \to H_{k-1}(U)$ associating to a cycle c the boundary in U of a tube over c, and i_* is Gysin map, Poincaré dual to the map i^* in cohomology.

For $P = \mathbb{P}^{n+1}$ and n odd, the map r is an isomorphism:

$$H^{n-1}(Y) \simeq H^{n+1}(P) \to H^{n+1}(U) \xrightarrow{r} H^n(Y) \xrightarrow{i_*} H^{n+2}(P) = 0 \xrightarrow{j^*} H^{n+2}(U)$$

and for n even the map r is injective:

$$H^{n+1}(P) = 0 \to H^{n+1}(U) \xrightarrow{r} H^n(Y) \xrightarrow{i_*} H^{n+2}(P) = \mathbb{Q} \xrightarrow{j^*} H^{n+2}(U)$$

then r is surjective onto the primitive cohomology:

$$r : H^{n+1}(U) \xrightarrow{\mathrm{Im}} H^n_{\mathrm{prim}}(X)$$

6.2. **MHS of a normal crossing divisor (NCD).** Let Y be a Normal Crossing Divisor in a proper complex smooth algebraic variety. We suppose the irreducible components $(Y_i)_{i\in I}$ of Y smooth and ordered.

6.2.1. *Mayer-Vietoris resolution.* Let S_q denotes the set of strictly increasing sequences $\sigma = (\sigma_0, ..., \sigma_q)$ on the ordered set of indices I, $Y_\sigma = Y_{\sigma_0} \cap ... \cap Y_{\sigma_q}$, $Y_{\underline{q}} = \coprod_{\sigma\in S_q} Y_\sigma$ is the disjoint union, and for all $j \in [0,q], q \geq 1$ let $\lambda_{j,\underline{q}} : Y_{\underline{q}} \to Y_{\underline{q}-1}$ denotes a map inducing for each σ the embedding $\lambda_{j,\sigma} : Y_\sigma \to Y_{\sigma(\widehat{j})}$ where $\sigma(\widehat{j}) = (\sigma_0, ..., \widehat{\sigma_j}, ..., \sigma_q)$ is obtained by deleting σ_j. Let $\Pi_q : Y_{\underline{q}} \to Y$ (or simply Π) denotes the canonical projection and $\lambda^*_{j,\underline{q}} : \Pi_*\mathbf{Z}_{Y_{\underline{q-1}}} \to \Pi_*\mathbf{Z}_{Y_{\underline{q}}}$ the restriction map defined by $\lambda_{j,\underline{q}}$ for $j \in [0,q]$.

Definition 6.10 (Mayer-Vietoris resolution of $\mathbb{Z}_Y$)**.** It is defined by the following complex of sheaves $\Pi_*\mathbb{Z}_{Y_\cdot}$:

$$0 \to \Pi_*\mathbb{Z}_{Y_{\underline{0}}} \to \Pi_*\mathbb{Z}_{Y_{\underline{1}}} \to \cdots \to \Pi_*\mathbb{Z}_{Y_{\underline{q-1}}} \xrightarrow{\delta_{q-1}} \Pi_*\mathbb{Z}_{Y_{\underline{q}}} \to \cdots$$

where $\delta_{q-1} = \sum_{j\in[0,q]}(-1)^j\lambda^*_{j,\underline{q}}$.

This resolution is associated to an hypercovering of Y by topological spaces in the following sense. Consider the diagram of spaces over Y:

$$Y_\cdot = (Y_{\underline{0}} \;\begin{matrix}\leftarrow\\ \leftarrow\\ \leftarrow\end{matrix}\; Y_{\underline{1}} \;\begin{matrix}\leftarrow\\ \leftarrow\\ \leftarrow\end{matrix}\; \cdots \quad Y_{\underline{q-1}} \;\begin{matrix}\xleftarrow{\lambda_{j,\underline{q}}}\\ \vdots\\ \longleftarrow\end{matrix}\; Y_{\underline{q}} \cdots) \xrightarrow{\Pi} Y$$

This diagram is the strict simplicial scheme associated in [6] to the normal crossing divisor Y, called here after Mayer-Vietoris. The Mayer-Vietoris complex is canonically associated as direct image by Π of the sheaf $\mathbb{Z}_{Y_{\underline{*}}}$ equal to $\mathbb{Z}_{Y_{\underline{i}}}$ on $Y_{\underline{i}}$. The generalization of such resolution is the basis of the later general construction of Mixed Hodge Structure using simplicial covering of an algebraic variety.

6.2.2. *The cohomological mixed Hodge complex of a NCD.* The weight filtration W on $\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}$ is defined by:

$$W_{-q}(\Pi_*\mathbb{Q}_{Y_{\underline{*}}}) = \sigma_{\cdot\geq q}\Pi_*\mathbb{Q}_{Y_{\underline{*}}} = \Pi_*\sigma_{\cdot\geq q}\mathbb{Q}_{Y_{\underline{*}}}, \quad Gr^W_{-q}(\Pi_*\mathbb{Q}_{Y_{\underline{*}}}) \simeq \Pi_*\mathbb{Q}_{Y_{\underline{q}}}[-q]$$

We introduce the complexes $\Omega^*_{Y_{\underline{i}}}$ of differential forms on $Y_{\underline{i}}$. The simple complex $s(\Omega^*_{Y_{\underline{*}}})$ is associated to the double complex $\Pi_*\Omega^*_{Y_{\underline{*}}}$ with the exterior differential d of forms and the differential δ_* defined by $\delta_{q-1} = \sum_{j\in[0,q]}(-1)^j\lambda^*_{j,\underline{q}}$ on $\Pi_*\Omega^*_{Y_{\underline{q-1}}}$. The weight W, and Hodge F filtrations are defined as:

$$W_{-q} = s(\sigma_{\cdot\geq q}\Omega^*_{Y_{\underline{*}}}) = s(0\to\cdots 0\to\Pi_*\Omega^*_{Y_{\underline{q}}}\to\Pi_*\Omega^*_{Y_{\underline{q+1}}}\to\cdots)$$

$$F^p = s(\sigma_{*\geq p}\Omega^*_{Y_{\underline{*}}}) = s(0\to\cdots 0\to\Pi_*\Omega^p_{Y_{\underline{*}}}\to\Pi_*\Omega^{p+1}_{Y_{\underline{*}}}\to\cdots)$$

We have a filtered isomorphism in the filtered derived category of sheaves of abelian groups $D^+F(Y,\mathbb{C})$ on Y:

$$(Gr^W_{-q}s(\Omega^*_{Y_{\underline{*}}}), F) \simeq (\Pi_*\Omega^*_{Y_{\underline{q}}}[-q], F) \quad \text{in} \quad D^+F(Y,\mathbb{C}).$$

inducing isomorphisms in $D^+(Y,\mathbb{C})$:

$$(\Pi_*\mathbb{Q}_{Y_{\underline{*}}}, W)\otimes\mathbb{C} = (\mathbb{C}_{Y_{\underline{\cdot}}}, W) \xrightarrow{\alpha\sim} (s(\Omega^*_{Y_{\underline{*}}}), W)$$

$$Gr^W_{-q}(\Pi_*\mathbb{C}_{Y_{\underline{*}}}) \simeq \Pi_*\mathbb{C}_{Y_{\underline{q}}}[-q]\tilde{\to}\Pi_*\Omega^*_{Y_{\underline{q}}}[-q] \simeq Gr^W_{-q}s(\Omega^*_{Y_{\underline{*}}})$$

Let $\mathbf{K}$ be the system consisting of:

$$(\Pi_*\mathbb{Q}_{Y_{\underline{*}}}, W), \mathbb{Q}_Y \xrightarrow{\sim} \Pi_*\mathbb{Q}_{Y_{\underline{*}}}, (s(\Omega^*_{Y_{\underline{*}}}), W, F), \quad (\Pi_*\mathbb{Q}_{Y_{\underline{*}}}, W)\otimes\mathbb{C} \xrightarrow{\sim} (s(\Omega^*_{Y_{\underline{*}}}), W)$$

Proposition 6.11. The system $\mathbf{K}$ associated to a normal crossing divisor Y with smooth proper irreducible components, is a Cohomological Mixed Hodge Complex on Y. It defines a functorial Mixed Hodge Structure on the cohomology $H^i(Y,\mathbb{Q})$, with weights varying between 0 and i.

In terms of Dolbeault resolutions : $(s(\mathcal{E}^{*,*}_{Y_{\underline{\cdot}}}), W, F)$, the statement means that the complex of global sections $\Gamma(Y, s(\mathcal{E}^{*,*}_{Y_{\underline{*}}}), W, F) := (\mathbb{R}\Gamma(Y,\mathbb{C}), W, F)$ is a Mixed Hodge Complex in the following sense:

$$(Gr^W_{-i}(\mathbb{R}\Gamma(Y,\mathbb{C}), F) := (\Gamma(Y, W_{-i}s(\mathcal{E}^{*,*}_{Y_{\underline{*}}})/\Gamma(Y, W_{-i-1}s(\mathcal{E}^{*,*}_{Y_{\underline{*}}}), F)$$
$$\simeq (\Gamma(Y, Gr^W_{-i}s(\mathcal{E}^{*,*}_{Y_{\underline{*}}}), F) \simeq (\mathbb{R}\Gamma(Y_i, \Omega^*_{Y_{\underline{i}}}[-i]), F)$$

is a Hodge Complex of weight $-i$ in the sense that:

$$(H^n(Gr^W_{-i}\mathbb{R}\Gamma(Y,\mathbb{C})), F) \simeq (H^{n-i}(Y_{\underline{i}},\mathbb{C}), F)$$

is a Hodge Structure of weight $n-i$.
The terms of the spectral sequence $E_1(K,W)$ of (K,W) are written as:

$${}_WE_1^{pq} = \mathbb{H}^{p+q}(Y, Gr^W_{-p}(s\Omega^*_{Y_{\underline{*}}})) \simeq \mathbb{H}^{p+q}(Y, \Pi_*\Omega^*_{Y_{\underline{p}}}[-p]) \simeq H^q(Y_{\underline{p}},\mathbb{C})$$

They carry the Hodge Structure of the space $Y_{\underline{p}}$. The differential is a combinatorial restriction map inducing a morphism of Hodge Structures:

$$d_1 = \sum_{j\leq p+1} (-1)^j \lambda^*_{j,p+1} : H^q(Y_{\underline{p}}, \mathbb{C}) \to H^q(Y_{\underline{p+1}}, \mathbb{C}).$$

The spectral sequence degenerates at E_2 ($E_2 = E_\infty$).

Corollary 6.12. The Hodge Structure on $Gr^W_q H^{p+q}(Y, \mathbb{C})$ is the cohomology of the complex of Hodge Structure defined by $(H^q(Y_{\underline{*}}, \mathbb{C}), d_1)$ equal to $H^q(Y_{\underline{p}}, \mathbb{C})$ in degree $p \geq 0$:

$$(Gr^W_q H^{p+q}(Y, \mathbb{C}), F) \simeq ((H^p(H^q(Y_{\underline{*}}, \mathbb{C}), d_1), F).$$

In particular, the weight of $H^i(Y, \mathbb{C})$ vary in the interval $[0, i]$ ($Gr^W_q H^i(Y, \mathbb{C}) = 0$ for $q \notin [0, i]$).

We will see that the last condition on the weight is true for all complete varieties.

6.3. **Relative cohomology and the mixed cone.** The notion of morphism of Mixed Hodge Complex involves compatibility between the rational level and complex level in the derived category, hence up to quasi-isomorphism.

To define a Mixed Hodge Structure on the relative cohomology, we define the notion of mixed cone with respect to a representative of the morphism on the level of complexes, hence depending on the representative.

The isomorphism between two structures obtained for two representatives depends on the choice of an homotopy, hence it is not naturally defined. Nevertheless this notion is interesting in applications. Later, one solution is to consider Mixed Hodge Complex on a category of diagrams, then the diagonal filtration is a natural example of such construction applied for example on simplicial varieties.

6.3.1. A morphism $u : K \to K'$ of Mixed Hodge Complex (resp. CMHC) consists of morphisms:

$u_A : K_A \to K'_A$ in $D^+A(\text{resp.})D^+(X, A))$,

$u_{A\otimes\mathbb{Q}} : (K_{A\otimes\mathbb{Q}}, W) \to (K'_{A\otimes\mathbb{Q}}, W)$ in $D^+F(A \otimes \mathbb{Q})(\text{ resp. } D^+F(X, A \otimes \mathbb{Q}))$,

$u_{\mathbb{C}} : (K_{\mathbb{C}}, W, F) \to (K'_{\mathbb{C}}, W, F)$ in $D^+F_2\mathbb{C}(\text{ resp. } D^+F_2(X, \mathbb{C}).)$

and commutative diagrams:

$$\begin{array}{ccc} K_{A\otimes\mathbb{Q}} & \xrightarrow{u_{A\otimes\mathbb{Q}}} & K'_{A\otimes\mathbb{Q}} \\ \wr\downarrow\alpha & & \wr\downarrow\alpha' \\ K_A\otimes\mathbb{Q} & \xrightarrow{u_A\otimes\mathbb{Q}} & K'_A\otimes\mathbb{Q} \end{array} \qquad \begin{array}{ccc} K_{A\otimes\mathbb{Q}}\otimes\mathbb{C} & \xrightarrow{u_{A\otimes\mathbb{Q}}\otimes\mathbb{C}} & K'_{A\otimes\mathbb{Q}}\otimes\mathbb{C} \\ \wr\downarrow\beta & & \wr\downarrow\beta' \\ K_{\mathbb{C}} & \xrightarrow{u_{\mathbb{C}}} & K'_{\mathbb{C}} \end{array}$$

in $D^+(A \otimes \mathbb{Q})$ (resp. $D^+(X, A \otimes \mathbb{Q})$ and in $D^+F(\mathbb{C})$ (resp. $D^+F(X, \mathbb{C})$) with respect to W.

6.3.2. Let (K, W) be a complex of objects of an abelian category $\mathbb{A}$ with an increasing filtration W. We denote by $(T_M K, W)$ or $(K[1], W[1])$ the mixed shifted complex with a translation on degrees of K and W: $W_n(T_M K^i) = W_{n-1}K^{i+1}$.

Definition 6.13 (Mixed cone)**.** Let $u : K \to K'$ be a morphism of complexes in $C^+F(\mathbb{A})$ with an increasing filtration. The mixed cone $C_M(u)$ is defined by the complex $T_M K \oplus K'$ with the differential of the cone $C(u)$.

6.3.3. Let $u : K \to K'$ be a morphism of a Mixed Hodge Complex. There exists a quasi-isomorphism $v = (v_A, v_{A\otimes\mathbb{Q}}, v_{\mathbb{C}}) : \tilde{K} \xrightarrow{\approx} K$ and a morphism $\tilde{u} = (\tilde{u}_A, \tilde{u}_{A\otimes\mathbb{Q}}, \tilde{u}_{\mathbb{C}}) : \tilde{K} \to K'$ of Mixed Hodge Complexes such that v and $\tilde{u}$ are defined successively in C^+A, $C^+F(A\otimes\mathbb{Q})$ and $C^+F_2\mathbb{C}$, i.e., we can find, by definition, diagrams:

$$K_A \xleftarrow{\approx} \tilde{K}_A \to {K'}_A, \quad K_{A\otimes\mathbb{Q}} \xleftarrow{\approx} \tilde{K}_{A\otimes\mathbb{Q}} \to {K'}_{A\otimes\mathbb{Q}}, \quad K_{\mathbb{C}} \xleftarrow{\approx} \tilde{K}_{\mathbb{C}} \to {K'}_{\mathbb{C}},$$

or in short $K \xleftarrow{\approx v} \tilde{K} \xrightarrow{\tilde{u}} K'$ (or equivalently $K \xrightarrow{\tilde{u}} \tilde{K}' \xleftarrow{\approx v} K'$) representing u.

6.3.4. *Dependence on homotopy.* Consider a morphism $u\colon K \to K'$ of Mixed Hodge Complexes, represented by a morphism of complexes $\tilde{u} : \tilde{K} \to K'$.
To define the mixed cone $C_M(\tilde{u})$ out of:
(i) the cones $C(\tilde{u}_A) \in C^+(A)$, $C_M(\tilde{u}_{A\otimes\mathbb{Q}}) \in C^+F(A\otimes\mathbb{Q})$, $C_M(\tilde{u}_{\mathbb{C}}) \in C^+F_2(\mathbb{C})$, we still need to define compatibility isomorphisms:

$$\gamma_1 : C_M(\tilde{u}_{A\otimes\mathbb{Q}}) \simeq C(\tilde{u}_A)\otimes\mathbb{Q}, \quad \gamma_2 : (C_M(\tilde{u}_{\mathbb{C}}, W) \simeq (C_M(\tilde{u}_{A\otimes\mathbb{Q}}), W)\otimes\mathbb{C}$$

successively in $D^+(A\otimes\mathbb{Q})$ and $D^+F(\mathbb{C})$. With the notations of 6.3.1 the choice of isomorphisms $C_M(\tilde{\alpha}, \tilde{\alpha}')$ and $C_M(\tilde{\beta}, \tilde{\beta}')$ from the compatibility isomorphisms in K and K' does not define compatibility isomorphisms for the cone since the diagrams of compatibility are commutative only up to homotopy, that is there exists homotopies h_1 and h_2 such that:

$$\tilde{\alpha}' \circ (\tilde{u}_{A\otimes\mathbb{Q}}) - (\tilde{u}_A \otimes \mathbb{Q}) \circ \tilde{\alpha} = h_1 \circ d + d \circ h_1,$$

and:

$$\tilde{\beta}' \circ \tilde{u}_{\mathbb{C}} - (\tilde{u}_{A\otimes\mathbb{Q}} \otimes \mathbb{C}) \circ \tilde{\beta} = h_2 \circ d + d \circ h_2.$$

ii) Then we can define the compatibility isomorphism as:

$$C_M(\tilde{\alpha}, \tilde{\alpha}', h_1) := \begin{pmatrix} \tilde{\alpha} & 0 \\ h_1 & \tilde{\alpha}' \end{pmatrix} : C_M(\tilde{u}_{A\otimes\mathbb{Q}}) \xrightarrow{\sim} C(\tilde{u}_A)\otimes\mathbb{Q}$$

and a similar formula for $C_M(\tilde{\beta}, \tilde{\beta}', h_2)$.

Definition 6.14. Let $u\colon K \to K'$ be a morphism of Mixed Hodge Complexes. The mixed cone $C_M(\tilde{u}, h_1, h_2)$ constructed above depends on the choices of the homotopies (h_1, h_2) and the choice of a representative $\tilde{u}$ of u, such that:

$$Gr_n^W(C_M(\tilde{u}), F) \simeq (Gr_{n-1}^W(T\tilde{K}), F) \oplus (Gr_n^W K', F)$$

is a HC of weight n; hence $C_M(\tilde{u}, h_1, h_2)$ is a Mixed Hodge Complex.

7. MHS on the cohomology of a complex algebraic variety

The aim of this section is to prove:

Theorem 7.1 (Deligne)**.** *The cohomology of a complex algebraic variety carries a natural Mixed Hodge Structure.*

The uniqueness and the functoriality will follow easily, once we have fixed the case of Normal Crossing Divisors and the smooth case.

We need a construction based on a diagram of algebraic varieties:

$$X_* = (X_0 \;\begin{matrix}\leftarrow\\ \leftarrow\\ \leftarrow\end{matrix}\; X_1 \;\begin{matrix}\leftarrow\\ \leftarrow\\ \leftarrow\end{matrix}\; \cdots \quad X_{q-1} \;\begin{matrix}\xleftarrow{X_*(\delta_i)}\\ \vdots\\ \longleftarrow\end{matrix}\; X_q \cdots)$$

similar to the model in the case of a Normal Crossing Divisor. Here the $X_*(\delta_i)$ are called the face maps (see below 7.2 for the definition), one for each $i \in [0, q]$.

Moreover, we still need to state commutativity relations when we compose the face maps. When we consider diagrams of complexes of sheaves, we need resolutions of such sheaves, with compatibility with respect to the maps $X_*(\delta_i)$ so as to avoid the dependence on homotopy that we met in the mixed cone construction.

The language of the simplicial category gives a rigorous setting to state the compatibility relations needed, and leads to the construction in [7] of a simplicial hypercovering of an algebraic variety X, using a general simplicial technique combined with desingularization at the various steps.

We will take this topological construction here as granted and concentrate on the construction of a cohomological mixed Hodge complex on the variety X. This construction is based on a diagonal process, out of various logarithmic complexes on the terms of the simplicial hypercovering, as in the previous case of normal crossing divisor, without the ambiguity of the choice of homotopy, because we carry resolutions of simplicial complexes of sheaves, hence functorial in the simplicial derived category.

In particular, we should view the simplicial category as a set of diagrams and the construction is carried out in a "category of diagrams". In fact, there exists another construction based on the "category of diagrams" of cubical schemes [31], and an alternative construction with diagrams with only four edges [11] for embedded varieties.

In all cases, the Mixed Hodge Structure is constructed first for smooth varieties and normal crossing divisors, then it is deduced for general varieties. The uniqueness follows from the compatibility of the Mixed Hodge Structure with Poincaré duality and classical exact sequences on cohomology.

7.1. **MHS on cohomology of simplicial varieties.** To construct a natural Mixed Hodge Structure on the cohomology of an algebraic variety S, not necessarily smooth or compact, Deligne considers a simplicial smooth variety $\pi : U_* \to S$ which is a cohomological resolution of the original variety in the sense that the direct image $\pi_*\mathbb{Z}_{U_*}$ is a resolution of $\mathbb{Z}_S$ (descent theorem, see Theorem 7.7).

On each term of the simplicial resolution, which consists of the complement of a normal crossing divisor in a smooth compact complex variety, the various logarithmic complexes are connected by functorial relations and form a simplicial Cohomological Mixed Hodge Complex giving rise to the Cohomological Mixed Hodge Complex defining the Mixed Hodge Structure we are looking for on the cohomology of S. Although such a construction is technically elaborate, the above abstract development of Mixed Hodge Complexes leads easily to the result without further difficulty.

7.1.1. *Simplicial category.* The simplicial category Δ is defined by its objects, its morphisms and the composition of morphisms.
i) The objects of Δ are the subsets of integers $\Delta_n := \{0, 1, \ldots, n\}$ for $n \in \mathbf{N}$,
ii) The set of morphisms of Δ are the sets $H_{p,q}$ of increasing mappings from Δ_p to Δ_q for integers $p, q \geq 0$, with the natural composition of mappings : $H_{pq} \times H_{qr} \to H_{pr}$. Notice that $f : \Delta_p \to \Delta_q$ is increasing in the non-strict sense $\forall i < j$, $f(i) \leq f(j)$.

Definition 7.2. We define for $0 \leq i \leq n+1$ the $i-$th face map as the unique strictly increasing mapping such that $i \notin \delta_i(\Delta_n)$: $\delta_i \colon \Delta_n \to \Delta_{n+1}$, $\quad i \notin \delta_i(\Delta_n) := Im\, \delta_i$.

The semi-simplicial category $\Delta_>$ is obtained when we consider only the strictly increasing morphisms in Δ. In what follows we could restrict the constructions to semi-simplicial spaces which underly the simplicial spaces and work only with semi-simplicial spaces, since we use only the face maps.

Definition 7.3. A simplicial (resp. co-simplicial) object $X_* := (X_n)_{n\in\mathbb{N}}$ of a category $\mathcal{C}$ is a contravariant (resp. covariant) functor from Δ to $\mathcal{C}$.
A morphism $a : X_* \to Y_*$ of simplicial (resp. co-simplicial) objects is defined by its components $a_n : X_n \to Y_n$ compatible with the various maps image by the functor of simplicial morphisms in H_{pq} for all $p, q \in \mathbb{N}$.

The functor $\Gamma : \Delta \to \mathcal{C}$ is defined by $\Gamma(\Delta_n) := X_n$ and for each $f : \Delta_p \to \Delta_q$, by $\Gamma(f) : X_q \to X_p$ (resp. $\Gamma(f) : X_p \to X_q$); $\Gamma(f)$ will be denoted by $X_*(f)$.

7.1.2. *Sheaves on a simplicial space.* If $\mathcal{C}$ is the category of topological spaces, we can define simplicial topological spaces. A sheaf F^* on a simplicial topological space X_* is defined by:
1) A family of sheaves F^n on X_n,
2) For each $f : \Delta_n \to \Delta_m$ with $X_*(f) : X_m \to X_n$, an $X_*(f)-$morphism $F_*(f)$ from F^n to F^m, that is $X_*(f)^*F^n \to F^m$ on X_m satisfying for all $g : \Delta_r \to \Delta_n$, $F_*(f \circ g) = F_*(f) \circ F_*(g)$.
A morphism $u : F^* \to G^*$ is a family of morphisms $u^n : F^n \to G^n$ such that for all $f : \Delta_n \to \Delta_m$, $u^m F^*(f) = G^*(f)u^n$ where the left term is:

$$X_*(f)^*F^n \to F^m \xrightarrow{u^m} G^m$$

and the right term is:

$$X_*(f)^*F^n \xrightarrow{X_*(f)^*(u_n)} X_*(f)^*G^n \to G^m.$$

The image of the $i-$th face map by a functor is also denoted abusively by the same symbol $\delta_i \colon X_{n+1} \to X_n$.
For a ring A, we will consider the derived category of cosimplicial sheaves of $A-$modules.

7.1.3. *Derived category on a simplicial space.* The definition of a complex of sheaves K on a simplicial topological space X_* follows from the definition of sheaves, it has two degrees $K := K^{p,q}$ where p is the degree of the complex and q is the simplicial degree, hence for each p, $K^{p,*}$ is a simplicial sheaf and for each q, $K^{*,q}$ is a complex on X_q.

A quasi-isomorphism (resp. filtered, bi-filtered) $\gamma : K \to K'$ (resp. with filtrations) of simplicial complexes on X_*, is a morphism of simplicial complexes inducing a quasi-isomorphism $\gamma^{*,q} : K^{*,q} \to K'^{*,q}$ (resp. filtered, bi-filtered) for each space X_q.

The definition of the derived category (resp. filtered, bi-filtered) of the abelian category of abelian sheaves of groups (resp. vector spaces) on a simplicial space is obtained by inverting the quasi-isomorphisms (resp. filtered, bi-filtered).

7.1.4. A topological space S defines a simplicial constant space S_* such that $S_n = S$ for all n and $S_*(f) = Id$ for all $f \in H^{p,q}$.

An augmented simplicial space $\pi : X_* \to S$ is defined by a family of maps $\pi_n : X_n \to S_n = S$ defining a morphism of simplicial spaces.

7.1.5. If the target category $\mathcal{C}$ is the category of complex analytic spaces, we define a simplicial complex analytic space. The structural sheaves $\mathcal{O}_{X_n}$ of a simplicial complex analytic space form a simplicial sheaf of rings. Let $\pi : X_* \to S$ be an augmentation to a complex analytic space S, the de Rham complex of sheaves $\Omega^*_{X_n/S}$ for various n form a complex of sheaves on X_* denoted $\Omega^*_{X_*/S}$.

A simplicial sheaf F^* on a constant simplicial space S_* defined by S corresponds to a co-simplicial sheaf on S; hence if F^* is abelian, it defines a complex via the face maps, with:

$$d = \sum_i (-1)^i \delta_i : F^n \to F^{n+1}.$$

A complex of abelian sheaves K on S_*, denoted by $K^{n,m}$ with m the cosimplicial degree, defines a simple complex sK:

$$(sK)^n := \oplus_{p+q=n} K^{pq}; \quad d(x^{pq}) = d_K(x^{pq}) + \sum_i (-1)^i \delta_i x^{pq}.$$

The filtration L with respect to the second degree will be useful:

$$L^r(sK) = s(K^{pq})_{q \geq r}.$$

7.1.6. *Direct image in the derived category of abelian sheaves (resp. filtered, bi-filtered).* For an augmented simplicial space $a : X_* \to S$, we define a functor denoted Ra_* on complexes K (resp. filtered (K, F), bi-filtered (K, F, W))) of abelian sheaves on X_*. We may view S as a constant simplicial scheme S_* and a as a morphism $a_* : X_* \to S_*$. In the first step we construct a complex I (resp. (I, F), (I, F, W))of acyclic (for example flabby) sheaves, quasi-isomorphic (resp. filtered, bi-filtered) to K (resp. (K, F), (K, F, W))); we can always take Godement resolutions ([28] Chap. II, §3.6 p. 95 or [15] Chap. II, §4.3 p.167) for example, then in each degree p, $(a_q)_* I^p$ on $S_q = S$ defines for varying q a cosimplicial sheaf on S denoted $(a_*)_* I^p$, and a differential graded complex for varying p, which is a double complex whose associated simple complex is denoted $s(a_*)_* I := Ra_* K$:

$$(Ra_* K)^n := \oplus_{p+q=n} (a_q)_* I^{p,q}; \quad dx^{pq} = d_I(x^{pq}) + (-1)^p \sum_{i=0}^{q+1} (-1)^i \delta_i x^{pq} \in (Ra_* K)^{n+1}$$

where q is the simplicial index ($\delta_i(x^{pq}) \in I^{p,q+1}$ and p is the degree. In particular for S a point we define the hypercohomology of K:

$$R\Gamma(X_*, K) := sR\Gamma^*(X_*, K); \quad \mathbb{H}^i(X_*, K) := H^i(R\Gamma(X_*, K)).$$

Respectively, the definition of $Ra_*(K, F)$ and $Ra_*(K, F, W)$is similar.

The filtration L on $s(a_*)_* I := Ra_* K$ defines a spectral sequence:

$$E_1^{pq} = R^q(a_p)_*(K_{|X_p}) := H^q(R(a_p)_*(K_{|X_p})) \Rightarrow H^{p+q}(Ra_* K) := R^{p+q} a_* K$$

Remark 7.4 (Topological realization). Recall that a morphism of simplices $f : \Delta_n \to \Delta_m$ has a geometric realization $|f| : |\Delta_n| \to |\Delta_m|$ as the affine map defined when we identify a simplex Δ_n with the vertices of its affine realization in $\mathbb{R}^{\Delta_n}$. We construct the topological realization of a topological semi-simplicial space X_* as the quotient of the topological space $Y = \coprod_{n \geq 0} X_n \times |\Delta_n|$ by the equivalence relation $\mathcal{R}$ generated by the identifications:

$$\forall f : \Delta_n \to \Delta_m, x \in X_m, a \in |\Delta_n|, \quad (x, |f|(a)) \equiv (X_*(f)(x), a)$$

The topological realization $|X_*|$ is the quotient space of Y, modulo the relation $\mathcal{R}$, with its quotient topology. The construction above of the cohomology amounts to the computation of the cohomology of the topological space $|X_*|$ with coefficient in an abelian group A:

$$H^i(X_*, A) \simeq H^i(|X_*|, A).$$

7.1.7. *Cohomological descent.* Let $a : X_* \to S$ be an augmented simplicial scheme; any abelian sheaf F on S, lifts to a sheaf a^*F on X_* and we have a natural morphism:

$$\varphi(a) : F \to Ra_*a^*F \text{ in } D^+(S).$$

Definition 7.5 (cohomological descent)**.** The morphism $a : X_* \to S$ is of cohomological descent if the natural morphism $\varphi(a)$ is an isomorphism in $D^+(S)$ for all abelian sheaves F on S.

The definition amounts to the following conditions:

$$F \xrightarrow{\sim} Ker(a_{0*}a_0^*F \xrightarrow{\delta_1-\delta_0} a_1^*F); \quad R^ia_*a^*F = 0 \text{ for } i > 0.$$

In this case for all complexes K in $D^+(S)$:

$$R\Gamma(S, K) \simeq R\Gamma(X_*, a^*K)$$

and we have a spectral sequence:

$$E_1^{pq} = \mathbb{H}^q(X_p, a_p^*K) \Rightarrow \mathbb{H}^{p+q}(S, K), \quad d_1 = \sum_i (-1)^i \delta_i : E_1^{p,q} \to E_1^{p+1,q}.$$

7.1.8. *MHS on cohomology of algebraic varieties.* A simplicial complex variety X_* is smooth (resp. proper) if every X_n is smooth (resp. compact).

Definition 7.6 (NCD)**.** A simplicial Normal Crossing Divisor is a family $Y_n \subset X_n$ of normal crossing divisors such that the family of open subsets $U_n := X_n - Y_n$ form a simplicial subvariety U_* of X_*, hence the family of filtered logarithmic complexes $(\Omega^*_{X_n}(LogY_n))_{n\geq 0}, W)$ form a filtered complex on X_*.

The following theorem is admitted here:

Theorem 7.7. *([7] 6.2.8) For each separated complex variety S,*
i) There exist a simplicial variety proper and smooth X_* over $\mathbb{C}$, containing a normal crossing divisor Y_* in X_* and an augmentation $a : U_* = (X_* - Y_*) \to S$ satisfying the cohomological descent property.
Hence for all abelian sheaves F on S, we have an isomorphism $F \xrightarrow{\sim} Ra_*a^*F$.
ii) Moreover, for each morphism $f : S \to S'$, there exists a morphism $f_* : X_* \to X'_*$ of simplicial varieties proper and smooth with normal crossing divisors Y_* and Y'_* and augmented complements $a : U_* \to S$ and $a' : U'_* \to S'$ satisfying the cohomological descent property, with $f(U_*) \subset U'_*$ and $a' \circ f = a$.

The proof is based on Hironaka's desingularisation theorem and on a general contruction of hypercoverings described briefly by Deligne in [7] after preliminaries on the general theory of hypercoverings. The desingularisation is carried at each step of the construction by induction.

Remark 7.8. We can assume the Normal Crossing Divisor with smooth irreducible components.

7.1.9. An A-Cohomological Mixed Hodge Complex K (CMHC) on a topological simplicial space X_* consists of:
i) A complex K_A of sheaves of $A-$modules on X_* such that $\mathbb{H}^k(X_*, K_A)$ are A-modules of finite type, and $\mathbb{H}^*(X_*, K_A) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{H}^*(X_*, K_A \otimes \mathbb{Q})$,
ii) A filtered complex $(K_{A\otimes\mathbb{Q}}, W)$ of filtered sheaves of $A \otimes \mathbb{Q}$ modules on X_* with an increasing filtration W and an isomorphism $K_{A\otimes\mathbb{Q}} \simeq K_A \otimes \mathbb{Q}$ in the derived category on X_*,
iii) A bi-filitered complex $(K_{\mathbb{C}}, W, F)$ of sheaves of complex vector spaces on X_* with an increasing (resp. decreasing) filtration W (resp. F) and an isomorphism $\alpha : (K_{A\otimes\mathbb{Q}}, W) \otimes \mathbb{C} \xrightarrow{\sim} (K_{\mathbb{C}}, W)$ in the derived category on X_*.
Moreover, the following axiom is satisfied
(CMHC) The restriction of K to each X_n is an A-Cohomological Mixed Hodge Complex .

7.1.10. Let X_* be a simplicial complex compact smooth algebraic variety with Y_* a Normal Crossing Divisor in X_* such that $j : U_* = (X_* - Y_*) \to X_*$ is an open simplicial embedding, then

$$(Rj_*\mathbb{Z}, (Rj_*\mathbb{Q}, \tau_{\leq}), (\Omega^*_{X_*}(LogY_*), W, F))$$

is a Cohomological Mixed Hodge Complex on X_*.

7.1.11. If we apply the global section functor to an A-Cohomological Mixed Hodge Complex K on X_*, we get an A-cosimplicial Mixed Hodge Complex defined as follows:
1) A cosimplicial complex $R\Gamma^* K_A$ in the derived category of cosimplicial $A-$modules,
2) A filtered cosimplicial complex $R\Gamma^*(K_{A\otimes\mathbb{Q}}, W)$ in the derived category of filtered cosimplicial vector spaces, and an isomorphism $(R\Gamma^* K_A) \otimes \mathbb{Q} \simeq R\Gamma^*(K_{A\otimes\mathbb{Q}})$.
3) A bi-filtered cosimplicial complex $R\Gamma^*(K_{\mathbb{C}}, W, F)$ in the derived category of bi-filtered cosimplicial vector spaces,
4) An isomorphism $R\Gamma^*(K_{A\otimes\mathbb{Q}}, W) \otimes \mathbb{C} \simeq R\Gamma^*(K_{\mathbb{C}}, W)$ in the derived category of filtered cosimplicial vector spaces.
The hypercohomology of a cosimplicial A-cohomological mixed Hodge complex on X_* is such a complex.

7.1.12. *Diagonal filtration.* To a cosimplicial mixed Hodge complex K, we associate here a differential graded complex which is viewed as a double complex whose associated simple complex is denoted sK. We put on sK a weight filtration by a diagonal process.

Definition 7.9 (Differential graded A-MHC)**.** A differential graded DG^+ (or a complex of graded objects) is a bounded below complex with two degrees, the first defined by the complex and the second by the gradings. It can be viewed as a double complex.
A differential graded A-Mixed Hodge Complex is defined by a system of DG^+-complex (resp. filtered, bi-filtered):

$$K_A, (K_{A\otimes\mathbb{Q}}, W), K_A \otimes \mathbb{Q} \simeq K_{A\otimes\mathbb{Q}}, (K_{\mathbb{C}}, W, F), (K_{A\otimes\mathbb{Q}}, W) \otimes \mathbb{C} \simeq (K_{\mathbb{C}}, W)$$

such that for each degree n of the grading, the component $(K^{*,n}_{\mathbb{C}}, W, F)$ underlies an A-Mixed Hodge Complex.

A cosimplicial Mixed Hodge Complex (K, W, F) defines a DG^+-A-Mixed Hodge Complex

$$sK_A, (sK_{A\otimes\mathbb{Q}}, W), sK_A \otimes \mathbb{Q} \simeq sK_{A\otimes\mathbb{Q}}, (sK_{\mathbb{C}}, W, F), (sK_{A\otimes\mathbb{Q}}, W) \otimes \mathbb{C} \simeq (sK_{\mathbb{C}}, W)$$

the degree of the grading is the cosimplicial degree.

Definition 7.10 (Diagonal filtration)**.** The diagonal filtration $\delta(W, L)$ of W and L on sK is defined by:

$$\delta(W, L)_n(sK)^i := \oplus_{p+q=i} W_{n+q}K^{p,q}.$$

where $L^r(sK) = s(K^{p,q})_{q\geq r}$. For a bi-filtered complex (K, W, F) with a decreasing F, the sum over F is natural (not diagonal).

7.1.13. *Properties.* We have:

$$Gr_n^{\delta(W,L)}(sK) \simeq \oplus_p Gr^W_{n+p}K^{*,p}[-p]$$

In the case of a DG^+-complex defined as the hypercohomology of a complex (K, W) on a simplicial space X_*, we have:

$$Gr_n^{\delta(W,L)}R\Gamma K \simeq \oplus_p R\Gamma(X_p, Gr^W_{n+p}K^{*,p})[-p].$$

and for a bi-filtered complex with a decreasing F:

$$Gr_n^{\delta(W,L)}R(\Gamma K, F) \simeq \oplus_p R\Gamma(X_p, (Gr^W_{n+p}K, F))[-p].$$

Next we remark:

Lemma 7.11. *If $H = (H_A, W, F)$ is an A-Mixed Hodge Structure, a filtration L of H_A is a filtration of Mixed Hodge Structure, if and only if, for all n,*

$$(Gr^n_L H_A, Gr^n_L(W), Gr^n_L(F))$$

is an A-mixed Hodge structure.

Theorem 7.12 (Deligne ([7] thm. 8.1.15))**.** *Let K be a graded differential A-Mixed Hodge Complex (for example, defined by a cosimplicial A-mixed Hodge complex).*
i) Then, $(sK, \delta(W, L), F)$ is an A-Mixed Hodge Complex.
The first terms of the weight spectral sequence:

$${}_{\delta(W,L)}E_1^{pq}(sK \otimes \mathbb{Q}) = \oplus_n H^{q-n}(Gr^W_n K^{*,p+n})$$

form the simple complex $({}_{\delta(W,L)}E_1^{pq}, d_1)$ of $A \otimes \mathbb{Q}$-Hodge structures of weight q associated to the double complex where $m = n + p$ and E_1^{pq} is represented by the sum of the terms on the diagonal :

$$\begin{array}{ccccc} H^{q-(n+1)}(Gr^W_{n+1}K^{*,m+1}) & \xrightarrow{\partial} & H^{q-n}(Gr^W_n K^{*,m+1}) & \xrightarrow{\partial} & H^{q-(n-1)}(Gr^W_{n-1}K^{*,m+1}) \\ d'' \uparrow & & d'' \uparrow & & d'' \uparrow \\ H^{q-(n+1)}(Gr^W_{n+1}K^{*,m}) & \xrightarrow{\partial} & H^{q-n}(Gr^W_n K^{*,m}) & \xrightarrow{\partial} & H^{q-(n-1)}(Gr^W_{n-1}K^{*,m}) \end{array}$$

where ∂ is a connecting morphism and d'' is simplicial.
ii) The terms ${}_LE_r$ for $r > 0$, of the spectral sequence defined by $(sK_{A\otimes\mathbb{Q}}, L)$ are endowed with a natural A-Mixed Hodge Structure, with differentials d_r compatible with such structures.
iii) The filtration L on $H^*(sK)$ is a filtration in the category of Mixed Hodge Structures and:

$$Gr^p_L(H^{p+q}((sK), \delta(W, L)[p + q], F) = ({}_LE^{pq}_\infty, W, F).$$

7.1.14. In the case of a smooth simplicial variety complement of a normal crossing divisor at infinity, the cohomology groups $H^n(U_*,\mathbb{Z})$ are endowed with the Mixed Hodge Structure defined by the following Mixed Hodge Complex:

$$R\Gamma(U_*,\mathbb{Z}),\ R\Gamma(U_*,\mathbb{Q}),\delta(W,L)),R\Gamma(U_*,\Omega^*_{X_*}(LogY_*)),\delta(W,L)),F)$$

with natural compatibility isomorphisms, satisfying:

$$Gr_n^{\delta(W,L)}R\Gamma(U_*,\mathbb{Q})\simeq\oplus_m Gr^W_{n+m}R\Gamma(U_m,,\mathbb{Q}))[-m]\simeq\oplus_m R\Gamma(Y_m^{n+m},\mathbb{Q})[-n-2m]$$

where the first isomorphism corresponds to the diagonal filtration and the second to the logarithmic complex for the open set U_m; recall that Y_m^{n+m} denotes the disjoint union of intersections of $n+m$ components of the normal crossing divisor Y_m of simplicial degree m. Moreover:

$$_{\delta(W,L)}E_1^{p,q}=\oplus_n H^{q-2n}(Y^n_{n+p},Q)\Rightarrow H^{p+q}(U_*,\mathbb{Q})$$

The filtration F induces on $_{\delta(W,L)}E_1^{p,q}$ a Hodge Structure of weight b and the differentials d_1 are compatible with the Hodge Structures. The term E_1 is the simple complex associated to the double complex of Hodge Structure of weight q where G is an alternating Gysin map:

$$\begin{array}{ccccc} H^{q-(2n+2)}(Y^{n+1}_{p+n+1},\mathbb{Q}) & \xrightarrow{G} & H^{q-2n}(Y^n_{p+n+1},\mathbb{Q}) & \xrightarrow{G} & H^{q-2(n-2)}(Y^{n-1}_{p+n+1},\mathbb{Q}) \\ \sum_i(-1)^i\delta_i\uparrow & & \sum_i(-1)^i\delta_i\uparrow & & \sum_i(-1)^i\delta_i\uparrow \\ H^{q-(2n+2)}(Y^{n+1}_{p+n},\mathbb{Q}) & \xrightarrow{G} & H^{q-2n}(Y^n_{p+n},\mathbb{Q}) & \xrightarrow{G} & H^{q-2(n-2)}(Y^{n-1}_{p+n},\mathbb{Q}) \end{array}$$

where the Hodge Structure on the columns are twisted respectively by $(-n-1),(-n),(-n+1)$, the lines are defined by the logarithmic complex, while the vertical differentials are simplicial. We deduce from the general theory:

Proposition 7.13. i) The Mixed Hodge Structure on $H^n(U_*,\mathbb{Z})$ is defined by the graded differential mixed Hodge complex associated to the simplicial Mixed Hodge Complex defined by the logarithmic complex on each term of X_* and it is functorial in the couple (U_*,X_*),
ii) The rational weight spectral sequence degenerates at rank 2 and the Hodge Structure on E_2 induced by E_1 is isomorphic to the Hodge Structure on $Gr_WH^n(U_*,\mathbb{Q})$.
iii)The Hodge numbers h^{pq} of $H^n(U_*,\mathbb{Q})$ vanish for $p\notin[0,n]$ or $q\notin[0,n]$.
iv) For $Y_*=\emptyset$, the Hodge numbers h^{pq} of $H^n(X_*,\mathbb{Q})$ vanish for $p\notin[0,n]$ or $q\notin[0,n]$ or $p+q>n$.

Definition 7.14. The Mixed Hodge Structure on the cohomology of a complex algebraic variety $H^n(X,\mathbb{Z})$ is defined by any logarithmic simplicial resolution of X via the isomorphism with $H^n(U_*,\mathbb{Z})$ defined by the augmentation $a:U.\to X$.

The Mixed Hodge Structure just defined does not depend on the resolution and is functorial in X since we can reduce to the case where a morphism $f:X\to Z$ is covered by a morphism of hypercoverings.

7.1.15. *Problems.* 1) Let $i:Y\to X$ be a closed subvariety of X and $j:U:=X-Y\to X$ the embedding of the complement. Then the two long exact sequences of cohomology:

$$\ldots\to H^i(X,X-Y,\mathbb{Z})\to H^i(X,\mathbb{Z})\to H^i(Y,\mathbb{Z})\to H_Y^{i+1}(X,\mathbb{Z})\to\ldots$$
$$\ldots\to H^i_Y(X,\mathbb{Z})\to H^i(X,\mathbb{Z})\to H^i(X-Y,\mathbb{Z})\to H_Y^{i+1}(X,\mathbb{Z})\to\ldots$$

underly exact sequences of Mixed Hodge Structure.

The idea is to use a simplicial hypercovering of the morphism i in order to define two Mixed Hodge Complexes: $K(Y)$ on Y and $K(X)$ on X with a well defined morphism on the level of complexes $i^* : K(X) \to K(Y)$ (resp. $j^* : K(X) \to K(X - Y)$), then the long exact sequence is associated to the mixed cone $C_M(i^*)$(resp. $C_M(j^*)$).

In particular, one deduce associated long exact sequences by taking the graded spaces with respect to the filtrations F and W.

2) *Künneth formula* [31]. Let X and Y be two algebraic varieties, then the isomorphisms of cohomology vector spaces:

$$H^i(X \times Y, \mathbb{C}) \simeq \oplus_{r+s=i} H^r(X, \mathbb{C}) \otimes H^s(Y, \mathbb{C})$$

underly isomorphisms of $\mathbb{Q}$-mixed Hodge structure. The answer is in two steps:

i) Consider the tensor product of two mixed Hodge complex defining the mixed Hodge structure of X and Y and deduce the right term, direct sum of tensor product of mixed Hodge structures.

ii) Construct a quasi-isomorphism of the tensor product with a mixed Hodge complex defining the mixed Hodge structure of $X \times Y$.

iii) Deduce that the cup product on the cohomology of an algebraic variety is compatible with mixed Hodge structure.

7.2. **MHS on the cohomology of a complete embedded algebraic variety.** For embedded varieties into smooth varieties, the mixed Hodge structure on cohomology can be deduced by a simple method using exact sequences, once the mixed Hodge structure for normal crossing divisor has been constructed, which should easily convince of the natural aspect of this theory. The technical ingredients consist of Poincaré duality and its dual the trace (or Gysin) morphism.

Let $p : X' \to X$ be a proper morphism of complex smooth varieties of same dimension, Y a closed subvariety of X and $Y' = p^{-1}(Y)$. We suppose that Y' is a Normal Crossing Divisor in X' and the restriction of p induces an isomorphism $p_{/X'-Y'} : X' - Y' \xrightarrow{\sim} X - Y$:

$$\begin{array}{ccccc} Y' & \xrightarrow{i'} & X' & \xleftarrow{j'} & X' - Y' \\ \downarrow p_Y & & \downarrow p & & \downarrow p_{X'-Y'} \\ Y & \xrightarrow{i} & X & \xleftarrow{j} & X - Y \end{array}$$

The trace morphism Trp is defined as Poincaré dual to the inverse image p^* on cohomology, hence the Trp is compatible with the Hodge Structures. It can be defined at the level of sheaf resolutions of $\mathbb{Z}_{X'}$ and $\mathbb{Z}_X$ as constructed by Verdier [36], that is in derived category $Tr\, p$: $Rp_*\mathbb{Z}_{X'} \to \mathbb{Z}_X$ hence we deduce by restriction morphisms depending on the embeddings of Y and Y' into X'.

$(Tr\, p)/_Y : Rp_*\mathbb{Z}_{Y'} \to \mathbb{Z}_Y$, $(Tr\, p)/_Y : H^i(Y', \mathbb{Z}) \to H^i(Y, \mathbb{Z})$, and $Trp : H^i_c(Y', \mathbb{Z}) \to H^i_c(Y, \mathbb{Z})$.

Remark 7.15. Let U be a neighbourhood of Y in X, retract by deformation onto Y such that $U' = p^{-1}(U)$ is a retract by deformation onto Y'; this is the case if Y is a sub-variety of X. Then the morphism $(Tr\, p)/_Y$ is deduced from $Tr(p/_U)$ in the diagram:

$$\begin{array}{ccc} H^i(Y', \mathbb{Z}) & \xleftarrow{\sim} & H^i(U', \mathbb{Z}) \\ \downarrow (Tr\, p)/_Y & & \downarrow Tr(p/_U) \\ H^i(Y, \mathbb{Z}) & \xleftarrow{\sim} & H^i(U, \mathbb{Z}) \end{array}$$

Consider now the diagram:

$$\begin{array}{ccccc} R\Gamma_c(X'-Y',\mathbb{Z}) & \xrightarrow{j'_*} & R\Gamma(X',\mathbb{Z}) & \xrightarrow{i'^*} & R\Gamma(Y',\mathbb{Z}) \\ Trp\downarrow & & \downarrow Trp & & \downarrow (Trp)_{|Y} \\ R\Gamma_c(X-Y,\mathbb{Z}) & \xrightarrow{j_*} & R\Gamma(X,\mathbb{Z}) & \xrightarrow{i^*} & R\Gamma(Y,\mathbb{Z}) \end{array}$$

Proposition 7.16. [11] i) The morphism $p_Y^*\colon H^i(Y,\mathbb{Z}) \to H^i(Y',\mathbb{Z})$ is injective with retraction $(Tr\,p)_{/Y}$.
ii) We have a quasi-isomorphism of $i_*\mathbb{Z}_Y$ with the cone $C(i'^*-Tr\,p)$ of the morphism $i'^* - Tr\,p$. The long exact sequence associated to the cone splits into short exact sequences:

$$0 \to H^i(X',\mathbb{Z}) \xrightarrow{i'^*-Tr\,p} H^i(Y',\mathbb{Z}) \oplus H^i(X,\mathbb{Z}) \xrightarrow{(Tr\,p)_{/Y}+i^*} H^i(Y,\mathbb{Z}) \to 0.$$

Moreover $i'^* - Tr\,p$ is a morphism of mixed Hodge structures. In particular, the weight of $H^i(Y,\mathbb{C})$ vary in the interval $[0,i]$ since this is true for Y' and X.

Definition 7.17. The mixed Hodge structure of Y is defined as cokernel of $i'^*-Tr\,p$ via its isomorphism with $H^i(Y,\mathbb{Z})$, induced by $(Tr\,p)_{/Y} + i^*$. It coincides with Deligne's mixed Hodge structure.

This result shows the uniqueness of the theory of mixed Hodge structure, once the Mixed Hodge Structure of the normal crossing divisor Y' has been constructed. The above technique consists in the realization of the Mixed Hodge Structure on the cohomology of Y as relative cohomology with Mixed Hodge Structures on X, X' and Y' all smooth proper or normal crossing divisor. Notice that the Mixed Hodge Structure on $H^i(Y,\mathbb{Z})$ is realized as a quotient and not as an extension.

Proposition 7.18. Let $p : X' \to X$ be a desingularization of a complete variety X, then for all integers i, we have

$$W_{i-1}H^i(X,\mathbb{Q}) = Ker\,(H^i(X,\mathbb{Q}) \xrightarrow{p^*} H^i(X',\mathbb{Q})$$

Let $i : Y \to X$ be the subvariety of singular points in X and let $Y' := p^{-1}(Y)$, $i' : Y' \to X'$, then we have a long exact sequence:

$$H^{i-1}(Y',\mathbb{Q}) \to H^i(X,\mathbb{Q}) \xrightarrow{(p^*,-i^*)} H^i(X',\mathbb{Q}) \oplus H^i(Y,\mathbb{Q}) \xrightarrow{i'^*+p^*_{|Y'}} H^i(Y',\mathbb{Q}) \dots$$

Since the weight of $H^{i-1}(Y',\mathbb{Q})$ is $\leq i-1$, we deduce an injective morphism $Gr_i^W H^i(X,\mathbb{Q}) \xrightarrow{(p^*,-i^*)} Gr_i^W H^i(X',\mathbb{Q}) \oplus Gr_i^W H^i(Y,\mathbb{Q})$. It is enough to prove for any element $a \in Gr_i^W H^i(X,\mathbb{Q})$ such that $Gr_i^W(p^*)(a) = 0$, we have $Gr_i^W(i^*)(a) = 0$; which follows from $Gr_i^W(p^*_{|Y'} \circ i^*)(a)) = 0$ if we prove $Gr_i^W(p^*_{|Y'})$ is injective. By induction on dim.Y, we may assume the injectivity for a resolution $\tilde{Y} \to Y$. There exists a subvariety $Z \subset Y'$generically covering Y, then a desingularization $\tilde{Z}$ of $Z' \times_Y \tilde{Y}$ is a ramified covering of $\tilde{Y}$, hence $H^i(\tilde{Y},\mathbb{Q})$ injects into $H^i(\tilde{Z},\mathbb{Q})$. We deduce that $H^i(Y,\mathbb{Q}) \to H^i(\tilde{Z},\mathbb{Q})$ is injective and in particular the factor $H^i(Y,\mathbb{Q}) \to H^i(Y',\mathbb{Q})$ is also injective.

Remark 7.19 (Mixed Hodge Structure on the cohomology of an embedded algebraic variety). The construction still apply for non proper varieties if we construct the Mixed Hodge Structure of an open normal crossing divisor.
Hypothesis. Let $i_Z : Z \to X$ a closed embedding and $i_X : X \to P$ a closed

embedding in a projective space (or any proper smooth complex algebraic variety). By Hironaka desingularization we construct a diagram:

$$\begin{array}{ccccc} Z'' & \to & X'' & \to & P'' \\ \downarrow & & \downarrow & & \downarrow \\ Z' & \to & X' & \to & P' \\ \downarrow & & \downarrow & & \downarrow \\ Z & \to & X & \to & P \end{array}$$

first by blowing up centers over Z so to obtain a smooth space $p : P' \to P$ such that $Z' := p^{-1}(Z)$ is a normal crossing divisor; set $X' := p^{-1}(X)$, then:

$$p| : X' - Z' \xrightarrow{\sim} X - Z, \quad p| : P' - Z' \xrightarrow{\sim} P - Z$$

are isomorphisms since the modifications are all over Z. Next, by blowing up centers over X' we obtain a smooth space $q : P'' \to P'$ such that $X'' := q^{-1}(X')$ and $Z'' := q^{-1}(Z')$ are normal crossing divisor, and $q| : P'' - X'' \xrightarrow{\sim} P' - X'$. Then, we deduce the diagram:

$$\begin{array}{ccccc} X'' - Z'' & \xrightarrow{i''_X} & P'' - Z'' & \xleftarrow{j''} & P'' - X'' \\ q_X \downarrow & & q \downarrow & & q \downarrow \wr \\ X' - Z' & \xrightarrow{i'_X} & P' - Z' & \xleftarrow{j'} & P' - X' \end{array}$$

Since all modification are above X', we still have an isomorphism induced by q at right. For $\dim.P = d$ and all integers i, the morphism $q^* : H_c^{2d-i}(P'' - Z'', \mathbb{Q}) \to H_c^{2d-i}(P' - Z', \mathbb{Q})$ is well defined on cohomology with compact support since q is proper; its Poincaré dual is called the trace morphism $Trq : H^i(P'' - Z'', \mathbb{Q}) \to H^i(P' - Z', \mathbb{Q})$ and satisfy the relation $Trq \circ q^* = Id$. Moreover, the trace morphism is defined as a morphism of sheaves $q_*\mathbb{Z}_{P''-Z''} \to \mathbb{Z}_{P'-Z'}$ [36], hence an induced trace morphism $(Trq)|(X'' - Z'') : H^i(X'' - Z'', \mathbb{Q}) \to H^i(X' - Z', \mathbb{Q})$ is well defined.

Proposition 7.20. With the notations of the above diagram, we have short exact sequences:

$$0 \to H^i(P'' - Z'', \mathbb{Q}) \xrightarrow{(i''_X)^* - Trq} H^i(X'' - Z'', \mathbb{Q}) \oplus H^i(P' - Z', \mathbb{Q}) \xrightarrow{(i'_X)^* - (Trq)|(X''-Z'')} H^i(X' - Z', \mathbb{Q}) \to 0$$

Since we have a vertical isomorphism q at right of the above diagram, we deduce a long exact sequence of cohomology spaces containing the sequences of the proposition; the injectivity of $(i''_X)^* - Trq$ and the surjectivity of $(i'_X)^* - (Trq)|X'')$ are deduced from $Trq \circ q^* = Id$ and $(Trq)|(X'' - Z'') \circ q^*|(X' - Z') = Id$, hence the long exact sequence splits into short exact sequences.

Corollary 7.21. The cohomology $H^i(X - Z, \mathbb{Z})$ is isomorphic to $H^i(X' - Z', \mathbb{Z})$ since $X - Z \simeq X' - Z'$ and then carries the Mixed Hodge Structure isomorphic to the cokernel of $(i''_X)^* - Trq$ acting as a morphism of Mixed Hodge Structures.

The left term carry a Mixed Hodge Structure as the special case of the complementary of the normal crossing divisor: Z'' into the smooth proper variety P'' , while the middle term is the complementary of the normal crossing divisor: Z'' into the normal crossing divisor: X''. Both cases can be treated by the above special cases without the general theory of simplicial varieties. This shows that the Mixed

Hodge Structure is uniquely defined by the construction on an open normal crossing divisor and the logarithmic case for a smooth variety.

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