Monofrequent oscillations of a non-linear model of a suspended cable

Angelo Luongo, Giuseppe Rega, Fabrizio Vestroni

MONOFREQUENT OSCILLATIONS OF A NON-LINEAR MODEL OF A SUSPENDED CABLE

A. LUONGO

Istituto di Scienza delle Costruzioni, Università di Roma, Roma 00184, *Italy*

AND

G. REGA AND F. VESTRONI

Istituto di Scienza delle Costruzioni, Università dell'Aquila, L'Aquila 67100, *Italy*

A two-degree-of-freedom non-linear elastic model is considered to analyze the effects of non-linearities on the free motion of a suspended cable. The discretized model is obtained by referring to simplified kinematics of the cable; the equations of motion which show quadratic and cubic non-linearities are solved through the multiple time scale perturbation technique. The monofrequent oscillations of the system are studied in order to analyze the modifications of frequency and motion amplitude of the modal oscillations due to geometric non-linearities in the absence of internal resonance. The possibility that effects arise due to non-linear coupling is examined. A numerical analysis is made for the first symmetric mode for different amplitudes of motion by parametrically varying the geometric and mechanical properties of the cable. The correction of frequency for the in-plane oscillation varies appreciably with the cable properties due to prevalence of either the quadratic or cubic term. In the out-of-plane monofrequent oscillation non-linearities establish a coupling between the two components of motion which strongly influences the frequency correction.

1. INTRODUCTION

The motion of mechanical systems is usually governed by ordinary or partial non-linear differential equations. While in many cases a sufficiently broad and reliable description of the dynamic phenomenon can be obtained through a linear analysis of the problem, in other cases the actual non-linearity produces a very wide variety of dynamic phenomena not disclosed by linear analysis [1, 2].

With the Lagrangian strain assumed to be the strain measure, the free motion of a suspended cable about a deformed initial configuration of static equilibrium is governed by a system of partial differential equations which contain non-linear terms of second and third order in the displacement function [3].

The main characteristics of the dynamic behaviour of a linear cable have been analyzed in previous works [4, 5]. However, in the linearized theory coupling among the displacement components in the equations of motion is disregarded, so that in-plane and out-of-plane oscillations are uncoupled. Coupling can be expected in the actual behaviour of cables since in the linear free oscillations both in-plane and out-of-plane frequencies can be commensurable for certain values of the geometric and mechanical properties of the system.

Coupling phenomena occur in the dynamic behaviour of many mechanical systems. In particular, for the non-linear free motions of beams, considerable attention has been given in the literature to the non-linear resonance between the in-plane and out-of-plane oscillation modes of the system [6–8]. As far as the free vibrations of a suspended cable are concerned, the motion under conditions of internal resonance has been partially dealt with in references [9, 10].

In the absence of internal resonance the non-linearities still establish a relationship between the frequency of natural oscillation of each mode and the motion amplitude which is influenced by the non-linear coupling among the modes. In this respect the study of the monofrequent oscillations of the system, as defined by Bogolioubov and Mitropolski [11], is of special interest. They occur if the natural frequencies of the linearized system are incommensurable and are therefore stable. One of the normal displacement co-ordinates prevails over the others in these oscillations and all the points move with a frequency that is equal to or a multiple of the non-linear frequency of one co-ordinate. From this point of view monofrequent oscillations are analogous, to a certain extent, to the modal oscillations of linear dynamics and their study permits one to determine the modifications of frequency and motion amplitude of these latter due to non-linearities.

In this paper a two degree-of-freedom non-linear elastic model is considered to analyze the effects of non-linearities on the free motion of a suspended cable in the absence of internal resonance. Two ordinary differential equations are obtained for the in-plane and the out-of-plane displacement components which are similar to those derived in the study of non-linear vibrations of many structural problems (see references [6–8, 12, 13]). The present equations, however, differ somewhat since not only cubic but also quadratic non-linearities occur. The conditions under which in-plane and out-of-plane monofrequent oscillations exist are examined as well as those for which effects arise due to non-linear coupling. A numerical evaluation of the modifications of the frequency and of the motion amplitude is performed for the first symmetric mode of the cable, with parametric variation of its geometric and mechanical characteristics and consideration of different amplitudes of the motion.

2. EQUATIONS OF MOTION OF THE SYSTEM

Consider a heavy elastic cable suspended between two fixed supports at the same level; the static equilibrium configuration of the cable—subsequently adopted as the reference configuration of length l_c—lies in the xy plane and is represented by the function $y(s)$.

The varied configuration is described through the components of displacement $\tilde{q}_1(s, t)$, $\tilde{q}_2(s, t)$, $\tilde{q}_3(s, t)$ of a given point $P(s)$ as shown in Figure 1. A simple two-parameter model, governed by uncoupled equations in the linear theory, but able to represent the

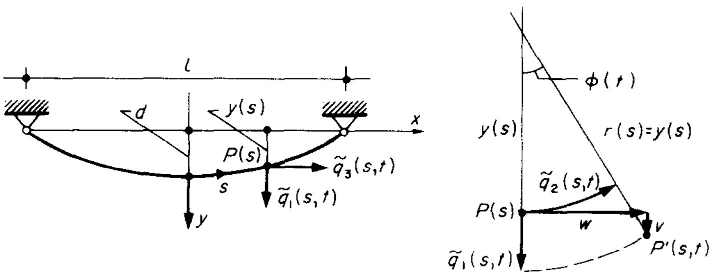

Figure 1. Cable configuration.

main kinematical aspects of both the in-plane modes and the out-of-plane first symmetric mode, is obtained by assuming that

$$\tilde{q}_1(s,t)=q_1(t)f(s),$$
$$\tilde{q}_2(s,t)=\varphi(t)y(s)=[\tilde{q}_2(0,t)/y(0)]y(s)=[q_2(t)/y(0)]y(s),$$
$$\tilde{q}_3(s,t)=0, \tag{1}$$

$f(s)$ being the in-plane displacement shape function. This means that the cable kinematics are assumed to be such that during the motion the cable always remains in a plane whose position with respect to the xy plane is defined by the angle $\varphi(t)$ while the deformed in-plane configuration is described only by the variation $\tilde{q}_1(s,t)$ of the distance $r(s)\equiv y(s)$ from the x axis. With these assumptions, and with Lagrangian strain used as the strain measure, the extensional strain of the axis of the cable is

$$\varepsilon(s,t)=(\mathrm{d}\tilde{q}_1/\mathrm{d}s)(\mathrm{d}y/\mathrm{d}s)+\tfrac{1}{2}(\mathrm{d}\tilde{q}_1/\mathrm{d}s)^2. \tag{2}$$

The expressions for the strain energy U, the kinetic energy K and the potential energy W are as follows:

$$U=U^I+\int_{l_c}(T^I\varepsilon+\tfrac{1}{2}EA\varepsilon^2)\,\mathrm{d}s,\qquad K=\int_{l_c}\tfrac{1}{2}m[\dot{\tilde{q}}_1^2+(y+\tilde{q}_1)^2\dot{\varphi}^2]\,\mathrm{d}s, \tag{3, 4}$$

$$W=W^I-\int_{l_c}mgv\,\mathrm{d}s, \tag{5}$$

where U^I, W^I are the values in the initial configuration, l_c, E, A and m are length of the cable, elastic modulus, cross-sectional area and mass per unit length, respectively, and T^I is the tension in the initial configuration. By using equations (1) and (2) and the relationship among vertical displacement v and components q_1 and φ, assumed to be of the same order,

$$v(s,t)=-y+[y+\tilde{q}_1]\cos\varphi=\tilde{q}_1-(y/2)\varphi^2-\tfrac{1}{2}\tilde{q}_1\varphi^2+(y/24)\varphi^4+o(\varphi^4), \tag{6}$$

the quantities U, K and W can be expressed in terms of just the two parameters $q_1(t)$ and $q_2(t)$.

Use of Lagrange equations leads to the following equations of motion:

$$\begin{aligned}ml_cI_{ff}\ddot{q}_1+\{mg(l_c/d)I_T+EA(d^2/l_c^3)I_v\}q_1=\{&-\tfrac{3}{2}EA(d/l_c^3)I_{vv}q_1^2\\&-(mg/2)(l_c/d^2)I_fq_2^2+m(l_c/d)I_{cf}\dot{q}_2^2\}\\&+\{m(l_c/d^2)I_{ff}q_1\dot{q}_2^2-\tfrac{1}{2}(EA/l_c^3)I_{vvv}q_1^3\},\\ md^2I_{cc}\ddot{q}_2+mgdI_cq_2=\{&-2mdI_{cf}q_1\ddot{q}_2-2mdI_{cf}\dot{q}_1\dot{q}_2-mgI_fq_1q_2\}\\&+\{(mg/6d)I_cq_2^3-2mI_{ff}q_1\dot{q}_1\dot{q}_2-mI_{ff}q_1^2\ddot{q}_2\}.\end{aligned} \tag{7}$$

The geometric and mechanical constants of the cable l_c, $d\equiv y(0)$, m, E and A, and the dimensionless coefficients I (see the Appendix), which depend on the initial configuration and the displacement interpolation function adopted, appear in these equations. To obtain an approximate solution to system (7) through a perturbation method, a dimensionless form of the equations of motion can be deduced after the substitutions

$$q_1=\varepsilon du_1,\qquad q_2=\varepsilon du_2, \tag{8}$$

where ε is a perturbation parameter of the order of the motion amplitude and the variables u_i are of order $O(1)$. Upon introducing the dimensionless time $\tau=\omega_2 t$ and letting $\lambda = \omega_1/\omega_2$, where ω_1 and ω_2 are the frequencies of the associated linear problem, equations (7) become

$$\ddot{u}_1 + \lambda^2 u_1 = \varepsilon(c_1 u_1^2 + c_2 u_2^2 + c_3 \dot{u}_2^2) + \varepsilon^2(c_7 u_1 \dot{u}_2^2 + c_8 u_1^3),$$
$$\ddot{u}_2 + u_2 = \varepsilon(c_4 u_1 \ddot{u}_2 + c_5 \dot{u}_1 \dot{u}_2 + c_6 u_1 u_2) + \varepsilon^2(c_9 u_2^3 + c_{10} u_1 \dot{u}_1 \dot{u}_2 + c_{11} u_1^2 \ddot{u}_2), \quad (9)$$

where the dot indicates $d/d\tau$. The c_i coefficients (see the Appendix), functions of the I integrals mentioned above, are all of order $O(1)$; c_1 and c_8 also contain the parameter $\rho^2 = (EA/mgl_c)(d/l_c)^3$.

Equations (9) govern the free oscillations of the cable within the approximations made. They are uncoupled only in the linear part and contain quadratic and cubic non-linearities of both geometrical and inertial nature. As is to be expected, an analogy occurs with the equations of motion of a pendulum with an elastic spring for which large elongations and rotations are considered.

The multiple scale method [2, 14] can be adopted to obtain the solution to system (9). This is accomplished by considering the displacements to be functions of a sequence of independent variables, or time scales, $T_0, T_1, \ldots, T_n, \ldots$, which are related to τ by the relations

$$T_n = \varepsilon^n \tau. \quad (10)$$

Expansion of the unknown functions u_i in power of ε up to second order terms, which needs introduction of three time scales, gives

$$u_i = u_{i0}(T_0, T_1, T_2) + \varepsilon u_{i1}(T_0, T_1, T_2) + \varepsilon^2 u_{i2}(T_0, T_1, T_2) + O(\varepsilon^3 \tau), \quad (11)$$

where u_{i0} is the solution to the linear problem (generating solution). By expressing the time derivatives in terms of the T_n variables and substituting equations (11) in equations (9) a system of two partial differential equations with the unknowns $u_{ij} (i = 1, 2; j = 0, 1, 2)$ is obtained. By equating coefficients of like powers of ε, a sequence of linear systems is obtained in each of which the non-linear part is known from the lower-order solutions: of order ε^0:

$$D_{00}^2 u_{10} + \lambda^2 u_{10} = 0, \qquad D_{00}^2 u_{20} + u_{20} = 0; \quad (12)$$

of order ε:

$$D_{00}^2 u_{11} + \lambda^2 u_{11} = -2D_{01}^2 u_{10} + c_1 u_{10}^2 + c_2 u_{20}^2 + c_3 (D_0 u_{20})^2,$$
$$D_{00}^2 u_{21} + u_{21} = -2D_{01}^2 u_{20} + c_4 u_{10} D_{00}^2 u_{20} + c_5 D_0 u_{10} D_0 u_{20} + c_6 u_{10} u_{20}; \quad (13)$$

of order ε^2:

$$D_{00}^2 u_{12} + \lambda^2 u_{12} = -2D_{02}^2 u_{10} - D_{11}^2 u_{10} - 2D_{01}^2 u_{11} + 2c_1 u_{10} u_{11} + 2c_2 u_{20} u_{21}$$
$$+ 2c_3 (D_0 u_{20} D_0 u_{21} + D_0 u_{20} D_1 u_{20}) + c_7 u_{10} (D_0 u_{20})^2 + c_8 u_{10}^3,$$
$$D_{00}^2 u_{22} + u_{22} = -2D_{02}^2 u_{20} - D_{11}^2 u_{20} - 2D_{01}^2 u_{21} + c_4 (u_{11} D_{00}^2 u_{20} + u_{10} D_{00}^2 u_{21} + 2u_{10} D_{01}^2 u_{20})$$
$$+ c_5 (D_0 u_{10} D_0 u_{21} + D_0 u_{11} D_0 u_{20} + D_0 u_{10} D_1 u_{20} + D_1 u_{10} D_0 u_{20})$$
$$+ c_6 (u_{11} u_{20} + u_{10} u_{21}) + c_9 u_{20}^3 + c_{10} u_{10} D_0 u_{10} D_0 u_{20} + c_{11} u_{10}^2 D_{00}^2 u_{20}. \quad (14)$$

Here, for the sake of simplicity the notations $D_i = \partial/\partial T_i$ and $D_{ij}^2 = \partial^2/\partial T_i \partial T_j$ have been used. The problem is completed with the initial conditions

$$u_i(0) = \bar{u}_i, \qquad \dot{u}_i(0) = \bar{\dot{u}}_i \quad (15)$$

to be imposed on equations (11).

By means of the perturbation method adopted, the general solution of the equations of motion can be found. Here only the steady motions of the system in absence of internal resonance are treated, with attention focussed on the monofrequent oscillations. The problem consists of ascertaining under what conditions the motion described by the series expansion (11), in which $u_{i0}(i=1,2)$ is a single-component periodic solution of the linear system, is still periodic with the non-linear period of the generating non-zero component. With ε assumed to be small enough to guarantee asymptotic convergence of the expansion (11), a solution to this problem is generally possible only for given initial conditions on the non-generating component of motion.

In the following sections the monofrequent oscillations obtained by assuming as generating solutions both that for in-plane motion and that for out-of-plane motion are studied. Each oscillation represents a two-parameter family of stable particular solutions; their superimposition does not provide, of course, the general solution of the non-linear system.

The study of monofrequent oscillations does not exhaust the analysis of all periodic oscillations; in particular such oscillations may also exist with two-component generating solutions for rational values of λ, and with the occurrence of internal resonance phenomena not excluded [9, 10].

3. EXTENSIONAL (IN-PLANE) OSCILLATION

The periodic solution

$$u_{10}=A(T_1, T_2)\,e^{i\lambda T_0}+\text{c.c.}, \qquad u_{20}=0 \tag{16}$$

is adopted as the solution to system (12). In equations (16) and the following, c.c. and the overbar (e.g., $\bar{A}$) indicate the complex conjugate. Substituting in system (13) gives

$$D_{00}^2u_{11}+\lambda^2u_{11}=-2i\lambda D_1A\,e^{i\lambda T_0}+c_1A^2\,e^{2i\lambda T_0}+c_1A\bar{A}+\text{c.c.}, \qquad D_{00}^2u_{21}+u_{21}=0. \tag{17}$$

For the series expansion (11) to be valid for times up to $O(\varepsilon^{-2})$ (uniformly valid), $u_{ij}/u_{ij-1}(i, j=1,2)$ must be limited for each τ; therefore there must be no secular terms like $\tau\sin\lambda\tau$ and $\tau\cos\lambda\tau$.

By eliminating these terms in the equation of u_{11} it is found that A is constant over the time scale T_1 and no frequency correction occurs at this order. Solving equations (17) gives the solution at the improved first order† of ε as

$$u_1=A\,e^{i\lambda T_0}-\varepsilon(c_1/\lambda^2)(\tfrac{1}{3}A^2\,e^{2i\lambda T_0}-A\bar{A})+\text{c.c.}+O(\varepsilon^2), \quad u_2=O(\varepsilon^2). \tag{18}$$

It is evident that at the ε order approximation, motion is still planar and periodic of frequency λ but it is no longer simply sinusoidal.

For the second order approximation, system (14) gives

$$\begin{aligned} D_{00}^2u_{12}+\lambda^2u_{12}&=\{-2i\lambda D_2A+[(10c_1^2/3\lambda^2)+3c_8]A^2\bar{A}\}\,e^{i\lambda T_0}\\ &\quad+(c_8-2c_1^2/3\lambda^2)A^3\,e^{3i\lambda T_0}+\text{c.c.},\\ D_{00}^2u_{22}+u_{22}&=0. \end{aligned} \tag{19}$$

† According to reference [11], this wording implies that in addition to the zeroing of the secular terms at the order ε^n $(n=0,1,2)$ the solution of the linear system at that order is also calculated.

By introducing the polar form $A(T_2)=a(T_2)\,e^{i\beta(T_2)}$ the zeroing of the secular terms in equations (19) provides the relations

$$a=a_0, \qquad \beta=\hat{\beta}T_2+\beta_0, \tag{20}$$

wherein

$$\hat{\beta}=-K_0a_0^2, \qquad K_0=\tfrac{5}{3}(c_1^2/\lambda^3)+(3/2\lambda)c_8, \tag{21}$$

and $a_0=a(0)$ and $\beta_0=\beta(0)$. At the second order of ε, therefore, the solution is given by

$$u_1=2a_0\cos[(\lambda+\varepsilon^2\hat{\beta})\tau+\beta_0]-(2c_1/\lambda^2)\varepsilon a_0^2\{\tfrac{1}{3}\cos[2(\lambda+\varepsilon^2\hat{\beta})\tau+2\beta_0]-1\}+O(\varepsilon^2),$$
$$u_2=O(\varepsilon^3). \tag{22}$$

The following points may now be made.

(a) The monofrequent oscillation of extensional type can exist for any value of λ, and also for ω_1 a multiple of ω_2. The explanation for this is that the non-linear terms containing u_1 in the second equation of motion are always multiplied by u_2 which is identically zero up to ε^2 order, so that no modal coupling arises. Nevertheless it must be noticed that for $\lambda=2$ (internal resonance conditions) the extensional oscillation becomes unstable; the resonance region is studied in reference [10].

(b) The motion is still planar, since u_{21} and u_{22} are zero identically.

(c) The in-plane motion is not sinusoidal owing to the superposition, compared with the linear case, of a double frequency harmonic and of a constant term. These contributions, associated with the non-linearities of even order in the first of equations (9), reveal how the oscillation occurs about a position different from the initial configuration. In the problem considered such a displacement is negative since the non-symmetrical relationship f_1-u_1 of the equivalent non-linear spring shows softening behaviour for $u_1<0$ (see Figure 2).

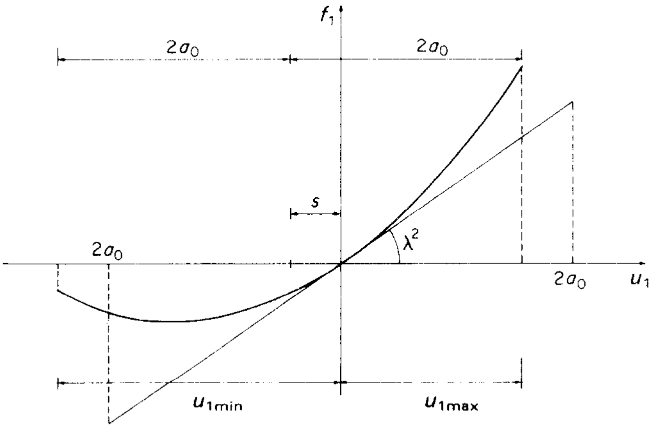

Figure 2. Non-symmetrical relationship f_1-u_1 of the in-plane motion (quadratic non-linearities). $u_{1\max}=2a_0-s$, $u_{1\min}=2a_0+s$; $f_1=\lambda^2u_1-\varepsilon c_1u_1^2$, $c_1<0$, $s=2a_0^2\varepsilon c_1/\lambda^2$.

(d) There is a frequency correction ($\varepsilon^2\hat{\beta}$) dependent on the square of the amplitude, due to the contribution of the quadratic and cubic non-linearities; it appears only at the ε^2 order, since the non-linearities of system (9) at the ε order are of even order. According to expressions (21) quadratic non-linearities produce a reduction of the fundamental frequency irrespective of the c_1 sign; this behaviour always occurs for one degree-of-freedom systems with quadratic non-linearities since the midpoint of the motion moves towards the soft range and the frequency of the system decreases. Cubic non-linearities which are of hardening type for the cable instead give rise to an increase of frequency.

In seeking solutions belonging to a two-parameter family a_0 and β_0, it is possible to impose only two arbitrary initial conditions. The imposition of these on the non-zero co-ordinate of the generating solution,

$$2a_0 \cos \beta_0 - (2c_1/\lambda^2)\varepsilon a_0^2[\tfrac{1}{3}\cos 2\beta_0 - 1] + O(\varepsilon^2) = \bar{u}_1,$$
$$-2a_0\lambda \sin \beta_0 + (4c_1/3\lambda)\varepsilon a_0^2 \sin 2\beta_0 + O(\varepsilon^2) = \bar{\dot{u}}_1, \tag{23}$$

provides the values of these parameters. The conditions on the other co-ordinate are automatically determined in terms of these and in this case are all zero; under such conditions the extensional monofrequent oscillation can exist.

4. PENDULUM (OUT-OF-PLANE) OSCILLATION

For the case of pendulum (out-of-plane) oscillation the generating solution is

$$u_{10} = 0, \qquad u_{20} = A(T_1, T_2)\,\mathrm{e}^{\mathrm{i}T_0} + \text{c.c.}, \tag{24}$$

which, when substituted in system (13) gives

$$D_{00}^2 u_{11} + \lambda^2 u_{11} = (c_2 - c_3)A^2\,\mathrm{e}^{2\mathrm{i}T_0} + (c_2 + c_3)A\bar{A} + \text{c.c.},$$
$$D_{00}^2 u_{21} + u_{21} = -2\mathrm{i}D_1 A\,\mathrm{e}^{\mathrm{i}T_0} + \text{c.c.} \tag{25}$$

It is observed that for $\lambda = 2$ (i.e., at a multiple of the fundamental frequency) there is internal resonance: i.e., there exists no uniformly valid expansion for the selected generating solution. Under such conditions a two-component generating solution must be adopted.

By eliminating the secular terms in the equation for u_{21} and solving system (25) the following expression is derived for the improved first order solution:

$$u_1 = \varepsilon\left\{\frac{c_2 - c_3}{\lambda^2 - 4}A^2\,\mathrm{e}^{2\mathrm{i}T_0} + (c_2 + c_3)\frac{A\bar{A}}{\lambda^2} + \text{c.c.}\right\} + O(\varepsilon^2), \quad u_2 = A\,\mathrm{e}^{\mathrm{i}T_0} + \text{c.c.} + O(\varepsilon^2), \tag{26}$$

where A is constant over T_1. This still represents a periodic oscillation of frequency equal to one; the out-of-plane vibration is still sinusoidal as per the linear theory, but it is no longer uncoupled from the in-plane motion which is excited, at ε order, with a multiple frequency of the spatial motion.

The second-order approximation is found by proceeding in the same way as in the case of the extensional type oscillation:

$$u_1 = 2[(c_2 - c_3)/(\lambda^2 - 4)]\varepsilon a_0^2 \cos[2(1 + \varepsilon^2\hat{\beta})\tau + 2\beta_0] + 2[(c_2 + c_3)/\lambda^2]\varepsilon a_0^2 + O(\varepsilon^3),$$
$$u_2 = 2a_0 \cos[(1 + \varepsilon^2\hat{\beta})\tau + \beta_0] + O(\varepsilon^2), \tag{27}$$

in which $\hat{\beta}$ is still given by the first of equations (21) and expressions similar to equations (20) hold good with

$$K_0 = [(2c_5 + c_6 - c_4)(c_2 - c_3)/2(\lambda^2 - 4)] + [(c_6 - c_4)(c_2 + c_3)/\lambda^2] + \tfrac{3}{2}c_9. \tag{28}$$

The following points may be made.

(a) Monofrequent oscillations of the pendulum type can exist only for $\lambda \neq 2$; for $\lambda = 2$ of necessity $u_{10} \neq 0$.

(b) Out-of-plane oscillation forces in-plane oscillation of ε order and frequency twice the fundamental, the amplitude of which depends on the amplitude of the u_2 component and on λ; as in the extensional oscillation a constant term arises which is due in the present case to the forced quadratic terms in the first of equations (9).

(c) The spatial motion is still sinusoidal since only mixed quadratic terms occur in the second of equations (9) which are zero identically at ε order. The superposition of a triple frequency harmonic appears at the improved second order.

(d) There is a frequency correction depending on the square of the amplitude and due to the contribution of both quadratic and cubic non-linearities.

It is worthwhile to observe that for values of λ less or greater than the resonant value $\lambda = 2$ an increase or a reduction of frequency occurs respectively, since the first term in equation (28) prevails over the others.

This different behaviour is bound up with the motion of the system which arises in the two cases owing to the different sign of the initial amplitude of the in-plane component, as shown in Figure 3. In case (a) ($\lambda < 2$) the in-plane motion is opposite in phase with

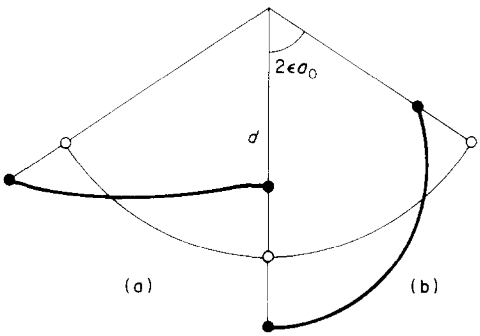

Figure 3. Diagram of the motion (thick line) of the midpoint of the cable for (a) $\lambda < 2$ and (b) $\lambda > 2$ (constant term in first of equations (27) disregarded).

respect to the forced terms due to the out-of-plane component which appear in the first of equations (9), while in case (b) ($\lambda > 2$) it is in phase. The corresponding modification of the law $f_2 - u_2$ with respect to the linear case due to the quadratic terms in the second of equations (9) (see Figure 4) is such that for $\lambda < 2$ the stiffness of the system in the

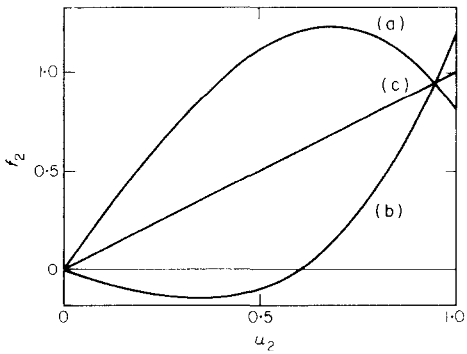

Figure 4. Relationship $f_2 - u_2$ in the non-linear case for (a) $\lambda < 2$ and (b) $\lambda > 2$ with respect to the (c) linear case.

pendulum oscillation is greater than the linear one and therefore the frequency increases; instead for $\lambda > 2$ the stiffness is lower and the frequency decreases.

The monofrequent oscillation thus determined can exist only if the initial conditions on u_1 are such that

$$\bar{u}_1 = 2\varepsilon a_0^2\left(\frac{c_2+c_3}{\lambda^2}+\frac{c_2-c_3}{\lambda^2-4}\cos 2\beta_0\right)+O(\varepsilon^3), \qquad \bar{\dot{u}}_1 = -4\varepsilon a_0^2\frac{c_2-c_3}{\lambda^2-4}\sin 2\beta_0+O(\varepsilon^2), \tag{29}$$

where a_0 and β_0 are obtained in terms of $\bar{u}_2$ and $\bar{\dot{u}}_2$.

5. NUMERICAL RESULTS

In the linear analysis of free in-plane oscillations of a suspended cable it was found that the dynamic behaviour of the cable is completely described by Irvine's elasto-geometric parameter $\Lambda^2 = (8d/l)^3 EA/mgl[1+8(d/l)^2]$, which determines the shape of the first mode, and therefore of higher modes [4, 5].

By means of the model presented in this paper a numerical investigation has been made of the non-linear in-plane and out-of-plane oscillations of a cable associated with the first symmetric mode having zero nodal points, which occurs only with values of Λ^2 less than $4\pi^2$: i.e., with low values of the d/l ratio when technical values of the cable mechanical properties are considered.

In this case the displacement $f(s)$ of the first in-plane symmetric mode is adequately described by a cosine function and the assumption $\tilde{q}_3 = 0$ appears reliable enough. The initial configuration is described by the inextensible catenary. In the range of low values of the d/l ratio the parameter ρ^2 which appears in the first of equations (9) is nearly the same as Λ^2, apart from a constant. Based on the assumed kinematics of the cable, the value of the linear frequency of the in-plane mode is governed for the most part by ρ^2 and to a little extent by the d/l ratio as well; differences of some percent with respect to the *exact* theory occur as regards ω_1—which however decrease as the suspended cable approaches the taut string—while good agreement is obtained as regards ω_2.

A numerical evaluation of the corrections for frequencies and amplitudes of motion for the monofrequent oscillations considered has been made by parametrically varying the characteristics of the cable through ρ^2 in the range between 0·07, which corresponds to $\Lambda^2 = 4\pi^2$, and a lower value which approaches the taut string; with technical values of the mechanical properties, the d/l ratio varies between about 1/12 and 1/250. The influence of non-linear terms has been analyzed by varying the motion amplitude through the parameter ε.

For the extensional oscillation the ratio between the non-linear and linear frequencies versus the amplitude of the linear part of the motion is shown in Figure 5 for different values of ρ^2.

Each curve is drawn as a solid line up to the value of amplitude for which the non-linear terms in the first of equations (9) are of the same order as the linear term. As mentioned in section 3, the effect of non-linearities is quite different depending on the predominance of either the quadratic or the cubic term. The former prevails for higher values of ρ^2 and produces a reduction of the frequency. As ρ^2 decreases a lower reduction of the frequency occurs up to a certain value of ρ^2 below which the cubic non-linearity prevails and produces an increase of the frequency. Considerable corrections both positive and negative occur.

The modification of the amplitude of motion is illustrated in Figure 6 for a given value of ρ^2, putting into evidence the occurrence of a drift of the midpoint of the oscillation,

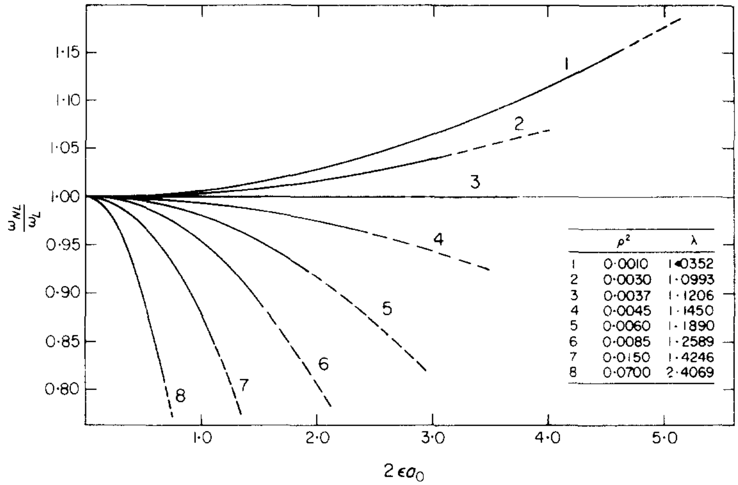

Figure 5. Non-linear–linear frequencies ratio ω_{1NL}/ω_{1L} *vs.* the linear amplitude of motion (extensional oscillation).

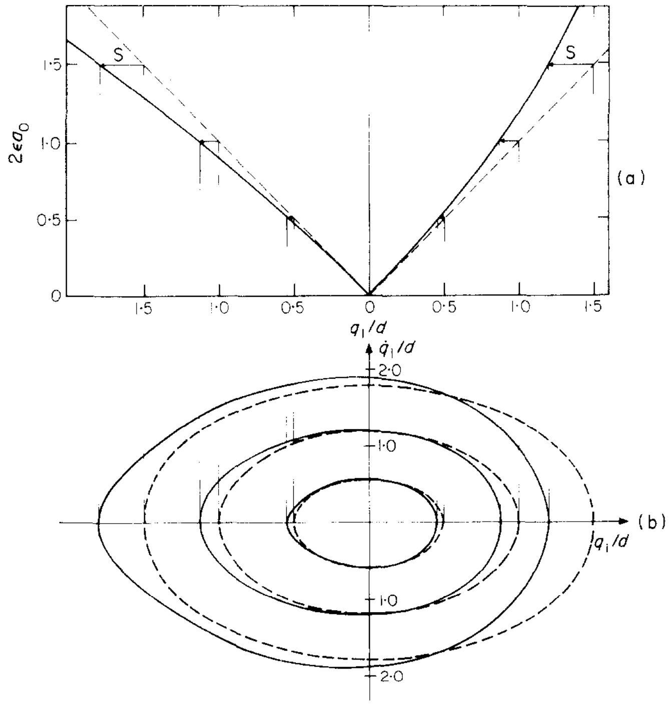

Figure 6. (a) Amplitude of non-linear motion *vs.* the linear amplitude; (b) trajectories of linear (- - -) and non-linear (——) motion (extensional oscillation).

which is an important aspect of the non-linear dynamic behaviour; the trajectories of the motion in the phase plane are shown in Figure 6(b), the linear ones plotted as broken curves and the non-linear as solid ones.

For the pendulum oscillation the ratio between the non-linear and linear frequency versus the amplitude of motion is shown in Figure 7 for different values of ρ^2; these correspond to a range of λ which contains the resonant value $\lambda = 2$. This value separates

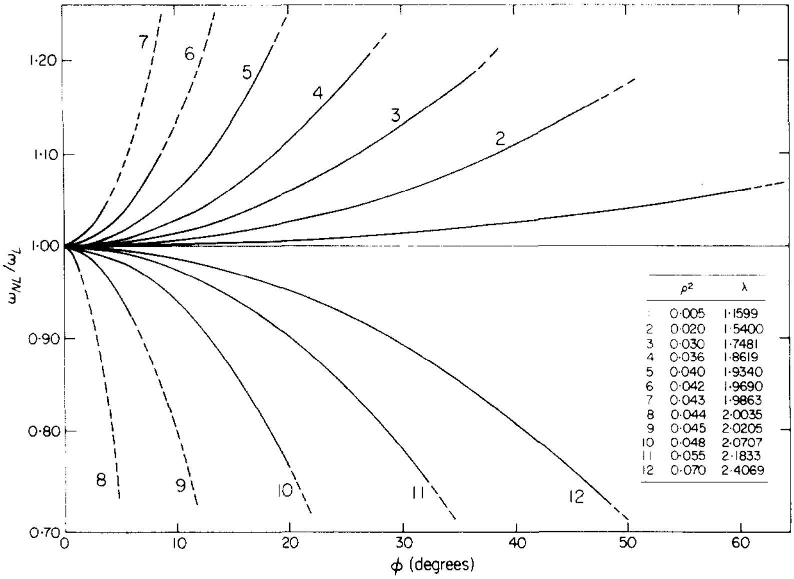

Figure 7. Non-linear–linear frequencies ratio ω_{2NL}/ω_{2L} *vs.* the amplitude of motion (pendulum oscillation).

the range of ρ^2 values (i.e., for instance, of the d/l ratio values for a cable with given mechanical properties) into two parts where either decrease or increase of frequency occurs. In this monofrequent oscillation the correction of frequency depends on the non-linear terms in the pendulum equation which contain the in-plane forced component; it follows that the curves are steeper as λ approaches the resonant value and u_1 grows largely.

Each curve is drawn in solid line up to the value of amplitude for which the non-linear terms in the pendulum equation are the same order as the linear term and in the neighbourhood of $\lambda = 2$ up to the value of amplitude for which the more restrictive condition $u_1 \leq O(1)$ is satisfied. It is interesting to observe that as ρ^2 increases (e.g., d/l

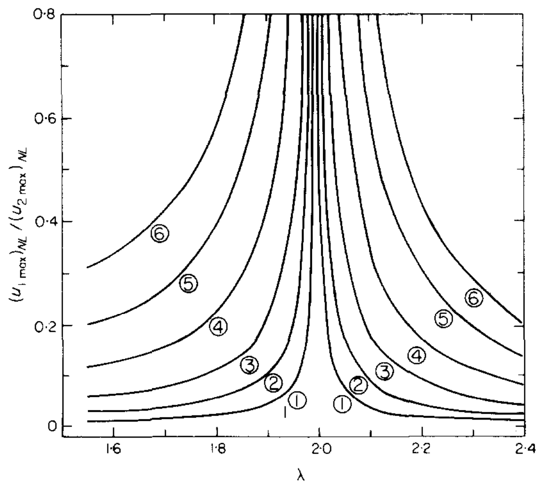

Figure 8. In-plane to out-of-plane amplitudes ratio u_{1NL}/u_{2NL} *vs.* λ (pendulum oscillation). ϕ_{max}: ① 1°08′; ② 2°51′; ③ 5°44′; ④ 11°27′; ⑤ 20°00′; ⑥ 30°00′.

increases for a given cable) the maximum correction of frequency of the pendulum oscillation increases as well up to a value of about 30%.

In the amplitude of motion there is no variation with respect to the linear theory as far as the fundamental mode u_2 is concerned, while the forced in-plane oscillation u_1 can grow up to considerable values depending on the λ-value. This is shown in Figure 8 where the ratio between the maximum values of the two co-ordinates is plotted *vs.* λ for different φ_{max} values; for each motion amplitude the neighbourhood of the resonant λ-value to be excluded in order that $u_1 \leqslant O(1)$ can be easily obtained.

6. CONCLUSIONS

The modification of the vibration modes of hyperelastic cable suspended between two supports due to geometric non-linearities has been analyzed in this paper. For cases of simplified kinematics a two degree-of-freedom model of the cable has been obtained. The system of two differential equations of motion, which contain quadratic and cubic non-linearities, has been solved by the multiple time scale perturbation method. Attention has been focused on the monofrequent oscillations whose solution has been determined up to the second order; extensional in-plane and pendulum out-of-plane monofrequent oscillations have been studied.

Oscillation of the first kind exists if the initial conditions of the out-of-plane motion are zero; the motion is always contained within the plane and can occur for any value of the ratio between the linear frequencies of the system. Oscillation of the second kind, instead, can occur only for particular initial conditions of the in-plane displacement component, which in any case is present in the oscillation, forced by the out-of-plane component; it cannot occur when the in-plane frequency is twice the out-of-plane one, since internal resonance exists. In both cases the non-linear terms produce an amplitude-dependent frequency correction.

Numerical results for the modification of the amplitude of motion and the correction of frequency of the first symmetric mode have been obtained by varying the perturbation parameter and a parameter which accounts for the geometric and mechanical properties of the cable. For the in-plane oscillation, drift of the midpoint of the motion occurs; the amount of the correction of frequency varies considerably with the cable properties, being either negative or positive, due to predominance of either the quadratic or cubic term. In the pendulum oscillation non-linearities establish a coupling between the two components of motion so that the forced in-plane component grows considerably in the neighbourhood of the resonant value; the correction of frequency strongly depends on the ratio between the natural frequencies of the system, being positive or negative respectively when the ratio is lower or higher than the resonant value.

ACKNOWLEDGMENT

This research was partially supported by the Consiglio Nazionale delle Ricerche (Italian Research Council), under Grant No. 80.02216.07.

REFERENCES

1. N. MINORSKI 1962 *Nonlinear Oscillations.* Princeton, New Jersey: Van Nostrand Company.
2. A. H. NAYFEH and D. T. MOOK 1979 *Nonlinear Oscillations.* New York: J. Wiley & Sons.
3. G. REGA and A. LUONGO 1978 *Istituto di Scienza delle Costruzioni, University of Rome, Report II–236.* Free oscillations of suspended cables (in Italian).

4. H. M. IRVINE and T. K. CAUGHEY 1974 *Proceedings of the Royal Society* **A341**, 299–315. The linear theory of free vibrations of a suspended cable.
5. G. REGA and A. LUONGO 1980 *Computers and Structures* **12**, 65–75. Natural vibrations of suspended cables with flexible supports.
6. C. H. HO, R. A. SCOTT and J. G. EISLEY 1976 *Journal of Sound and Vibration* **47**, 333–339. Non-planar, non-linear oscillations of a beam II. Free motions.
7. M. R. M. CRESPO DA SILVA and C. C. GLYNN 1979 *International Journal of Solids and Structures* **15**, 209–219. Non-linear non-planar resonant oscillations in fixed-free beams with support asymmetry.
8. M. R. M. CRESPO DA SILVA 1980 *Journal of Applied Mechanics* **47**, 409–414. Nonlinear resonances in a column subjected to a constant end force.
9. A. LUONGO, G. REGA and F. VESTRONI 1980 in *Recent Advances in Structural Dynamics* (Editor M. Petyt), **2**, 597–609, Institute of Sound and Vibration Research. Free oscillations of a nonlinear simple model of a suspended cable.
10. A. LUONGO, G. REGA and F. VESTRONI 1980 *Proceedings of the 5th Italian Congress of Theoretical and Applied Mechanics* **2**, 225–236. Perturbation analysis of free nonlinear oscillations of a two-parameter model of a suspended cable (in Italian).
11. N. BOGOLIOUBOV and I. MITROPOLSKI 1962 *Les Méthodes Asymptotiques en Théorie des Oscillations Non-Linéaries.* Paris: Gauthier-Villars Editeur.
12. A. H. NAYFEH, D. T. MOOK and D. W. LOBITZ 1974 *American Institute of Aeronautics and Astronautics Journal* **12**, 1222–1228. Numerical-perturbation method for the nonlinear analysis of structural vibrations.
13. M. W. HYER 1980 in *Recent Advances in Structural Dynamics* (Editor M. Petyt) **2**, 621–630, Institute of Sound and Vibration Research. Nonplanar motions of a base-excited cantilever.
14. A. H. NAYFEH 1973 *Perturbation Methods.* New York: J. Wiley & Sons.

APPENDIX: EXPRESSIONS FOR THE DIMENSIONLESS COEFFICIENTS I AND c_i

$$I_T = \frac{d}{mgl_c}\int_{l_c} T^I \left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)^2 \mathrm{d}s, \qquad I_c = \frac{1}{dl_c}\int_{l_c} y\,\mathrm{d}s, \qquad I_v = \frac{l_c^3}{d^2}\int_{l_c}\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)^2 \mathrm{d}s,$$

$$I_{cc} = \frac{1}{d^2 l_c}\int_{l_c} y^2\,\mathrm{d}s, \qquad I_{vv} = \frac{l_c^3}{d}\int_{l_c}\frac{\mathrm{d}y}{\mathrm{d}s}\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)^3 \mathrm{d}s, \qquad I_{ff} = \frac{1}{l_c}\int_{l_c} f^2\,\mathrm{d}s,$$

$$I_{vvv} = l_c^3\int_{l_c}\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)^4 \mathrm{d}s, \qquad I_f = \frac{1}{l_c}\int_{l_c} f\,\mathrm{d}s, \qquad I_{cf} = \frac{1}{dl_c}\int_{l_c} yf\,\mathrm{d}s.$$

$$c_1 = -\tfrac{3}{2}\rho^2 I_{vv}I_{cc}/I_{ff}I_c, \qquad c_2 = -\tfrac{1}{2}I_f I_{cc}/I_{ff}I_c, \qquad c_3 = I_{cf}/I_{ff},$$

$$c_4 = c_5 = -2I_{cf}/I_{cc}, \qquad c_6 = -I_f/I_c, \qquad c_7 = 1, \qquad c_8 = -\tfrac{1}{2}\rho^2 I_{vvv}I_{cc}/I_{ff}I_c,$$

$$c_9 = 1/6, \qquad c_{10} = -2I_{ff}/I_{cc}, \qquad c_{11} = c_{10}/2.$$

Here $\rho^2 = (EA/mgl_c)(d/l_c)^3$.